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MULTIVARIATE AFFINE GENERALIZED HYPERBOLIC DISTRIBUTIONS: AN EMPIRICAL INVESTIGATION

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Multivariate Affine Generalized Hyperbolic Distributions: An Empirical Investigation

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Resumo

The aim of this paper is to estimate the Multivariate Affine Generalized distributions (MAGH) using market data. We use Ibovespa, CAC, DAX, FTSE, NIKKEI and S&P500 indexes . We estimate the univariate distributions, the bi-variate distributions and the 6-dimensional distribution. Then, we asses their goodness of fit using Kolmogorov distances.

Key words: Generalized Hyperbolic Distributions, Multivariate distributions, Affine transformation, Fat tails.

1 Introduction

In the last decade a class of distributions called Generalized Hyperbolic distributions (GH) have been suggested to fit financial data. The development of theses distributions is due to Barndorff-Nielsen (1977). He applied the Hyperbolic subclass to fit grain size of sand subjected to continuous wind blow. Further, in Barndorff-Nielsen (1978), the extension to the Multivariate Generalized Hyperbolic Distributions (MGH) was introduced. This class of MGH were used in different fields of knowledge like physics, biology, agronomy and others (see Blæsild & Sørensen (1992)).

Eberlein & Keller (1995) were the first to apply these distributions to finance. They used univariate Hyperbolic subclasses to fit German data. Keller (1997) studied derivative pricing with GH and Prause (1999) extended Eberlein & Keller (1995) results by fitting financial data using the MGH class. He also prices derivatives and measures Value at Risk. Using GH class we can capture fat tails and the skewness observed on asset returns.

Blæsild & Sørensen (1992) were the first to develop a computer program, called *Hyp*, to estimate the parameter of the Hyperbolic subclass up to three dimensions. Prause (1999) developed a

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program to estimate the MGH class.

More recently, Schmidt *et al.* (2006) introduced the Multivariate Affine Generalized Hyperbolic Distributions (MAGH). Their main goal was to develop a multivariate distributions that can capture fat tails, skewness and tail dependence. Such that at the same time were simple to estimate. Since, the parameter estimation procedure of MGH is computational intense and have some shortcomings.

Some applications to the Brazilian market have been carried on to analyze the use of GH. Using the *Hyp* software Fajardo *et al.* (2001) studied the goodness of fit Hyperbolic subclass. Fajardo & Farias (2004) and Fajardo *et al.* (2005) extended that results and price some derivatives using the GH class.

In this paper we generalize Fajardo & Farias (2004) using MAGH. We asses the goodness of fit of MAGH with international financial data and the Brazilian index Ibovespa.

The paper is organized as follows: Section 2 presents the Generalized Hyperbolic Distributions. In Section 3, we present Multivariate Affine Generalized Distributions. In Section 4, we describe our sample. Section 5 describes the MAGH estimation procedures. In Section 6, we present the empirical results and in the last sections we have the conclusions and an appendix.

2 Generalized Hyperbolic Distributions

The probability density function of the one dimensional GH is defined by:

$$gh(x;\alpha,\beta,\delta,\mu,\lambda) = a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{\frac{(\lambda-\frac{1}{2})}{2}}K(\lambda,\alpha,\beta,\delta,\mu)$$

where μ is a location parameter, δ is a scale factor, compared to Gaussian σ in Eberlein (2000), α and β determine the distribution shape, λ defines the tail fatness (Barndorff-Nielsen & Blæsild 1981)), therefore the subclasses of GH, and

$$K(\lambda,\alpha,\beta,\delta,\mu) = K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu))$$

where,

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\alpha^{(\lambda-\frac{1}{2})}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}$$

is a norming factor to make the curve area totals 1 and

$$K_{\lambda}(x) = \frac{1}{2}\int_0^{\infty} y^{\lambda-1}exp\left(-\frac{1}{2}x\left(y+y^{-1}\right)\right)dy$$

is the modified Bessel function [1] of third kind with index λ. The domains of the parameters are:

$$\mu, \lambda \in \mathbb{R}$$
$$-\alpha < \beta < \alpha$$
$$\delta, \alpha > 0.$$

The GH have semi-heavy tails, this name due to the fact that their tails are heavier than Gaussian's, but they have finite variance, which is observed by the following approximation:

$$gh(x; \lambda, \alpha, \beta, \delta) \sim\mid x \mid^{\lambda-1} \exp\left((\mp\alpha + \beta)x\right) \text{ as } x \to \pm\infty$$

Many distributions can be obtained as subclasses of or limiting distributions of GH. We cite as examples the Gaussian distribution, Student's T and Normal Inverse Gaussian. We refer to Barndorff-Nielsen (1978) and Prause (1999) for a detailed description.

We can, alternatively, write the GH density as an affine transformation of a canonical form, with scale 1 and position 0

Proposition 1 *We can write $GH(x; \omega, \delta^2, \mu)$ as an affine transformation of a canonical GH: $GH(x; \omega, 1, 0)$, where $\omega := (\tilde{\alpha}, \tilde{\beta}, \lambda)$.*

3 Multivariate Affine Generalized Hyperbolic Distributions

The n-dimensional MAGH consists in the following stochastic representation:

$$X \stackrel{d}{=} A'Y + M$$

where A is an upper triangular matrix $\in \mathbb{R}^{nxn}$ such that $A'A = \Sigma$ is an positive definite and the random vector $Y \in \mathbb{R}^n$ consists of n mutually independent one-dimensional canonical $GH(\omega, 1, 0)$ (to more details see Schmidt *et al.* (2006)). This definition is responsible for easying the estimation procedure. M is the location parameter and Σ is an scaling factor. The family of n-dimensional Multivariate Affine Generalized Hyperbolic distributions is denoted by $MAGH_n(\omega, \Sigma, M)$, where $\omega := (\omega_1, \ldots, \omega_n)$ and $\omega_i := (\lambda_i, \alpha_i, \beta_i)$, $\imath = 1, \ldots, n$.

The mean and variance of the MAGH can be easily calculated:

$$E[X] = E[A'Y + M] = A'E[Y] + M$$

where E[Y] by independence, is a vector containing at each row the mean of the univariate $GH(\omega, 1, 0)$.

[1] For more details about Bessel functions, see Abramowitz & Stegun (1968).

$$VAR[X] = VAR[A'Y + M] = \Sigma VAR[Y]$$

where VAR[Y] by independence, is a vector containing at each row the variance of the univariate $GH(\omega, 1, 0)$.

This distribution is extremely flexible because the parameters λ and α can be defined to each margin, improving fitness even if the margins have extremely different tail fatness, because, λ and α are directly responsible for that phenomena. Furthermore, if Σ is a diagonal matrix the margins are independent, which is important in some scenarios.

The easiness on estimation is due to a simple procedure that allows, instead of a simultaneous parameters acquiring process, estimation using n univariate estimations, where n represents the number of dimensions.

4 Sample

The empirical evaluation uses the Ibovespa, CAC 40, Dax 100, FTSE 100, Nikkei 225 and Standard and Poors 500 indexes. The data consisted of the daily log-returns which were calculated using:

$$R_{i,t} = \ln\left(\frac{P_{i,t}}{P_{i,t-1}}\right)$$

The samples with their tickers and respective periods are in table 1. The chosen starting date was the date Brazil implemented its currency stabilization plan (Real plan), that brought some stability to the prices avoiding daily correction of asset prices. Since we are talking about different countries we interpolated the data when some date wasn't a trade date in all countries. We didn't exclude any trade date, not even the September 11th.

Tabela 1. Sample

Asset	***Ticker***	***Start***	***End***
Bovespa	BVSP	08/01/1994	10/20/2005
Cac40	CAC	08/01/1994	10/20/2005
Dax	DAX	08/01/1994	10/20/2005
FTSE	FTSE	08/01/1994	10/20/2005
Nikkei	NIKK	08/01/1994	10/20/2005
Standard and Poors	SP500	08/01/1994	10/20/2005

In table 2 we give the main descriptive statistics of the data, and in table 3 we have the correlation matrix of the data. Using these two tables we can see important features of this database:

- High-correlated data. CACxDAX have 0.7902 correlation coefficient;
- Almost uncorrelated data. BVSPxNIKK have 0.1136 correlation coefficient;
- High amplitude data. BVSP has a minimum of -17.2082 and a maximum of 28.8325, a lot greater the other indexes.
- High kurtosis data. Following the last item, BVSP has a kurtosis of 16.9409, implying in a much heavy distribution tail.

Tabela 2. Descriptive Statistics (%)

Index	***Mean***	***Std Deviation***	***Skewness***	***Kurtosis***	***Min***	***Max***
BVSP	0.0668	2.4107	0.5901	16.9409	-17.2082	28.8325
CAC	0.0256	1.3766	-0.0988	5.7326	-7.6781	7.0023
DAX	0.0277	1.4967	-0.1415	5.6244	-6.4999	7.5527
FTSE	0.0174	1.0680	-0.1238	5.9075	-5.5888	5.9038
NIKK	-0.0147	1.3625	-0.0997	5.1872	-7.2340	7.6553
SP500	0.0322	1.0737	-0.1002	6.4608	-7.1127	5.5744

Tabela 3. Correlation Matrix (%)

	BVSP	CAC	DAX	FTSE	NIKK	SP500
BVSP	1.0000	0.2682	0.2756	0.2735	0.1136	0.4277
CAC	0.2682	1.0000	0.7902	0.7875	0.2434	0.4366
DAX	0.2756	0.7902	1.0000	0.7215	0.2315	0.4983
FTSE	0.2735	0.7875	0.7215	1.0000	0.2551	0.4311
NIKK	0.1136	0.2434	0.2315	0.2551	1.0000	0.1138
SP500	0.4277	0.4366	0.4983	0.4311	0.1138	1.0000

Analyzing the correlation matrix, we can see that the more correlated data are Dax, CAC and FTSE, what was expected since all of them are European markets (not EURO zone). The BVSP and NIKK, following SP500 and NIKK are the less correlated data, which also was expected since they are from completely different continents.

5 Estimation Algorithm

In order to estimate GH parameters we used a slight modification into Fajardo & Farias (2004) algorithm in order to estimate the affine transformation form of $GH(\omega, \delta^2, \mu)$.That algorithm was implemented in Matlab and uses maximum likelihood estimation. Freund (2004); Lagarias *et al.* (1998); Neumaier (2004) shows properties of restricted optimization and Baritompal & Hendrix (2005); Björkman & Holmström (1999); Hart (1994); Iwaarden (1996); Mendivii *et al.* (1999); Stützle & Hrycej (2002a) discuss, also ways to implement global optimization. Based on them, in order to improve performance and get more reliable estimates we transformed the restricted parameters to unrestricted parameters:

$$\tilde{\alpha}_u = \ln(\tilde{\alpha}) \tag{1}$$
$$\delta_u = \ln(\delta) \tag{2}$$
$$\tilde{\beta} = (1 - \exp(-\tilde{\beta}_u \times sign(\tilde{\beta}_u))) \times sign(\tilde{\beta}_u), \tag{3}$$

To estimate MAGH parameters we used some propositions. In order to simplify the procedure and improve efficiency in the estimation. This approach was used by Schmidt *et al.* (2006), Stützle & Hrycej (2001), Stützle & Hrycej (2002a), Stützle & Hrycej (2002b) and Stützle & Hrycej (2005), applied in many other distributions, included MAGH.

Proposition 2 *If $X \sim MAGH_n(\omega, \Sigma, M)$ then $W = BX$ is a set of n independent $GH(\omega, \delta, \mu)$ distributions, where B is the inverse Cholesky factorization of X covariance matrix.*

By definition if $X \sim MAGH_n(\omega, \Sigma, M)$ then we can state that:

$X \stackrel{d}{=} A'Y + M$

for some upper triangular matrix A such that $A'A = \Sigma$ is positive-definite and the random vector $Y = (Y_1, ..., Y_n)'$ consists of mutually independent random variables $Y_i \sim GH(\omega_i, 1, 0)$.

So the dependence structure of the $MAGH$ is due to the A matrix. Let S be he covariance matrix of X, so we can use Cholesky to factorize it as $S = \tilde{B}'\tilde{B}$.

Applying the inverse, we have:

$$S^{-1} = \tilde{B}^{-1}(\tilde{B}^{-1})'$$

Calling $(\tilde{B}^{-1})' = B$ we get $S^{-1} = B'B$

So, when we let:

$$W = BX$$

we indeed transform the correlated X to an uncorrelated W ((Horn & Johnson. 1985; Press *et al.* 1992)). The question is: Is $W \in MAGH$?

$$W = BX = B(A'Y + M) = BA'Y + BM$$

$BA'AB'$ is clearly positive definite, so $W \in MAGH$ and as stated is a set of independent $GH(\omega, \delta, \mu)$ distributions. $\square$

Proposition 3 *We can estimate X by a two step procedure*

In above proposition we stated that W is a vector of independent distributions, so we can estimate W by its conditional distributions W_i.

After we estimate all W_i we can recover the original X parameters:

Each W_i can be written as $W_i \stackrel{d}{=} \delta Y_i + \mu_i$, so:

$$W = BX \rightarrow (W_1, W_2, ..., W_n)' = BX \rightarrow X = B^{-1}W$$

As stated before

$B^{-1}W \sim MAGH(\omega, \Sigma, M)$ so, $A'Y + M = B^{-1}(DY + \mu)$

Thus:

$A' = B^{-1}D$ and $M = B^{-1}\mu$.

Where D is the diagonal matrix containing the δ_i of marginal distributions, and μ is the vector of μ_i. $\square$

So we use the following steps:

(1) Find B as the Cholesky series factorization of the inverse sample covariance matrix.
(2) Get $W = BX$, that is a set of independent $GH(\omega_i, \delta^2, \mu)$.
(3) Estimate the univariate GHs.
(4) Translate the univariate parameters into the multivariate parameters using Proposition 3.

This procedure leads to a less computational effort, since it consists of n univariate estimations, acquiring 5 parameters at each, instead of 1 multivariate estimation of $4n+n(n+1)/2$ parameters at once.

6 Empirical Results

6.1 Unidimensional Estimation

Table 4 presents the results of the unidimensional estimation of GH distributions and Normal Inverse Gaussian (NIG) and Hyperbolic (Hyp) subclasses. The GH estimation has to be taken care, since the presence of λ as a free-parameter may lead to multiple local minimums.

Tabela 4. GH and its subclasses estimated parameters.

Index		$\tilde{\alpha}$	$\tilde{\beta}$	λ	δ	μ	LogLike
BVSP	GH	0.6350	-0.0998	-1.0149	2.3911	0.2131	-6387.0
	NIG	0.8862	-0.0957	-0.5000	2.1473	0.2693	-6392.8
	HYP	0.4917	-0.0437	1.0000	0.7420	0.2162	-6408.0
CAC	GH	1.0065	-0.0819	-0.9964	1.6317	0.1194	-4919.8
	NIG	1.0410	-0.0091	-0.5000	1.3874	0.0520	-4920.6
	HYP	1.0511	-0.0041	1.0000	0.8504	0.0372	-4927.0
DAX	GH	0.9814	-0.0863	-0.0100	1.2092	0.1776	-5141.6
	NIG	0.9726	-0.1003	-0.5000	1.4476	0.1733	-5142.2
	HYP	1.0531	-0.0057	1.0000	0.9167	0.0454	-5153.9
FTSE	GH	0.9851	-0.0804	-0.4989	1.0388	0.0975	-4153.2
	NIG	0.9851	-0.0804	-0.5000	1.0388	0.0975	-4153.2
	HYP	1.0543	-0.0075	1.0000	0.6541	0.0318	-4163.7
NIKK	GH	1.0405	-0.0055	0.5046	0.9902	0.0056	-4935.9
	NIG	1.0376	0.0000	-0.5000	1.4066	-0.0067	-4937.7
	HYP	1.0470	0.0000	1.0000	0.8489	-0.0085	-4936.8
SP500	GH	1.0444	-0.0075	0.0026	0.9030	0.0471	-4180.3
	NIG	1.0414	-0.0076	-0.5000	1.0751	0.0477	-4180.4
	HYP	1.0529	-0.0053	1.0000	0.6600	0.0424	-4186.7

The subclasses Hyp and NIG are specially important because the first one (Hyperbolic) is easier to estimate, since the Bessel function (the most computer demanding) is evaluated only once for log-likelihood evaluation [2] while the others need at least n times, where n is the sample size.

The second one (Normal Inverse Gaussian) is more often desirable specially in derivative pricing, since it is closed under convolution, characteristic not applicable to others subclasses.

Due to this, we did likelihood ratio tests checking for the possibility of restricting GH parameters to NIG or Hyp subclasses. The tests statistics such as their p-values are in table 5.

Tabela 5. Log-likelihood ratio test

	NIG		Hyp	
Index	Stats	P-Value	Stats	P-Value
BVSP	11.5260	6.86E-04	41.9750	9.24E-11
CAC	1.7523	0.1856	14.4540	1.44E-04
DAX	1.2142	0.2705	24.4810	7.50E-07
FTSE	0.0005	0.9820	20.9120	4.81E-06
NIKK	3.7139	0.0540	1.9122	0.1667
SP500	0.2018	0.6533	12.7990	3.47E-04

We can see in table 5 that the developed countries stocks markets can be modelled with NIG instead of GH, not remaining true for Brazilian market. This happens because high kurtosis and standard deviation of Brazilian stock market, characteristic already explored in Fajardo & Farias (2004).

The same table shows that only for NIKK we can restrict estimation to Hyp subclass, being the null hypothesis rejected for all other indexes.

6.2 Unidimensional Goodness of Fit

In order to evaluate the goodness of fit we show some figures and calculated distances. Figures 1-4 shows the estimated x empirical distributions of indexes BVSP and NIKK. We can see in PDF graphics that the GH distribution fits well the kurtosis of the distribution, reenforced by the log-density, showing that the tails are also well fitted too.

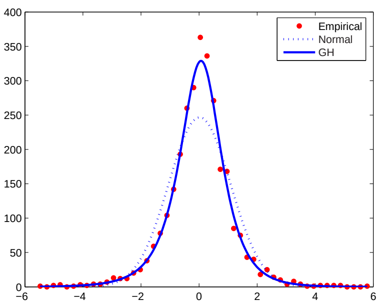

Figura 1. NIKK PDF.

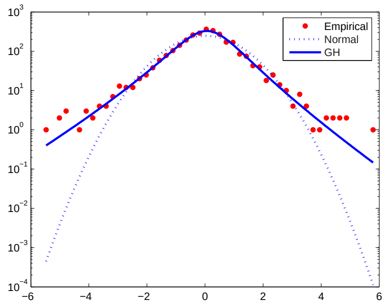

Figura 2. NIKK log-PDF.

[2] To explore GH subclasses see Barndorff-Nielsen (1977) and Barndorff-Nielsen (1978)

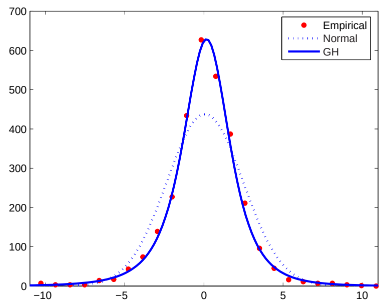

Figura 3. BVSP PDF.

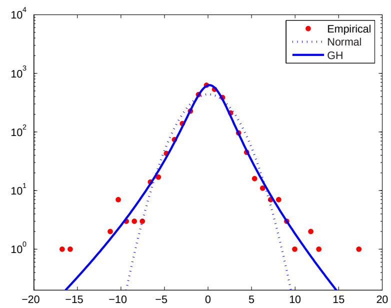

Figura 4. BVSP log-PDF.

Table 6 lists all Kolmogorov distances and the respective p-values (for details see Fajardo & Farias (2004)). The GH and NIG don't reject the null hypothesis that the empirical distribution is GH/NIG distributed. In HYP case, only DAX is rejected.

Tabela 6. Kolmogorov-Smirnov Tests.

	Normal		GH		NIG		Hyp	
Index	KS	pValue	KS	pValue	KS	pValue	KS	pValue
BVSP	0.0683	2.7E-12	0.0129	0.7134	0.0115	0.8318	0.0141	0.6049
CAC	0.0525	1.9E-07	0.0098	0.9414	0.0085	0.9841	0.0139	0.6244
DAX	0.0680	3.4E-12	0.0167	0.3871	0.0173	0.3413	0.0307	0.0080
FTSE	0.0571	9.9E-09	0.0143	0.5875	0.0142	0.5917	0.0219	0.1199
NIKK	0.0486	2.0E-06	0.0112	0.8576	0.0107	0.8902	0.0122	0.7744
SP500	0.0549	4.2E-08	0.0149	0.5360	0.0144	0.5731	0.0200	0.1924

Just as an comparison exercise, we post in table 7 the Anderson & Darling distance (for more details see Fajardo & Farias (2004)). This distance show mainly the difference in the tails of the distribution. In this case the Normal distribution shows terrible performance, being the worst a 50077 distance against 0.0339 of GH (BVSP) and in the best performance 0.6159 against 0.0288 of GH (DAX).

Tabela 7. Anderson-Darling distances.

	Normal	GH	NIG	Hyp
Bovespa	50077.4824	0.0470	0.1080	0.3071
CAC40	8.3298	0.0355	0.0530	0.0786
Dax	0.6159	0.0374	0.0375	0.0858
FTSE	2.8125	0.0416	0.0418	0.0868
Nikkei	3.2669	0.0388	0.0351	0.0563
SP500	334.6462	0.1211	0.0991	0.2245

The question is: how can we model multivariate data, considering the dependence among them? So in next section we provide the Multivariate Affine Generalized Hyperbolic, an attempt to solve this without intensive computational effort.

6.3 2-Dimension MAGH Estimation

Even though the Univariate estimates present desirable goodness of fit measures, the correlation between the assets is not negligible, so if we want to model the joint distribution of the assets, for V@R necessities or even derivative pricing, we have to consider Multivariate distributions.

Now we present the results of MAGH estimation. In order to assess more easily the results, we first present the estimation of 2 by 2 combinations of the sample, then we present the full sample treated as one multivariate distribution.

Table 8 has the estimates of the two assets with higher correlation (CAC and DAX) and the two assets with lower correlation (BVSP and NIKK).

Tabela 8. MAGH and its subclasses estimated parameters

Assets		$\tilde{\alpha}$	$\tilde{\beta}$	λ	M	Σ		LogLike
CAC	GH	1.0441	0.0037	-0.5036	0.1301	1.4936	1.0866	-3939.5
X		1.0028	-0.0878	-0.0100	0.1773	1.0866	1.4952	-3965.0
DAX	Nig	1.0441	0.0037	-0.5	0.1310	1.8239	1.5411	-3939.5
		0.9842	-0.1058	-0.5	0.1785	1.5411	2.1206	-3965.4
	Hyp	1.0574	0.0024	1	0.1412	0.6857	0.5771	-3949.4
		1.0191	-0.0699	1	0.1912	0.5771	0.7940	-3972.8
BVSP	GH	1.0377	-0.0157	-1.0173	0.1208	7.3805	0.1977	-3820.7
X		1.0409	-2E-05	0.5022	-0.0011	0.1977	0.9834	-4033.1
NIKK	Nig	1.0474	-0.0151	-0.5	0.1258	5.3607	0.3984	-3823.1
		1.0386	-0.0056	-0.5	0.0010	0.3984	1.9816	-4034.6
	Hyp	1.0759	-0.0077	1	0.0975	2.0753	0.1449	-3844.8
		1.0470	-1.5E-07	1	-0.0033	0.1449	0.7206	-4033.8

Again, we may be interested in particular subclasses (MANig and MAHyp). Table 9 shows the estimates concerning the restriction of MAGH to one of its main subclasses. The results are quite similar to the univariate case. When BVSP is one of the distributions, only BVSP x CAC can be restricted to MANig, but all other two-index combination can be restricted. Once again the high volatility and kurtosis of BVSP distribution contributes to this. In the MAHyp case, we cannot restrict any of the samples.

Tabela 9. Log-Likelihood ratio tests.

	NIG		Hyp	
Assets	Stats	P-Value	Stats	P-Value
BVSP x CAC	3.1351	0.2086	57.2950	0.0000
BVSP x DAX	25.1630	0.0000	86.1360	0.0000
BVSP x FTSE	5.4250	0.0664	73.2390	0.0000
BVSP x NIKK	7.8421	0.0198	49.7060	0.0000
BVSP x SP500	7.9218	0.0190	59.6940	0.0000
CAC x DAX	0.8372	0.6580	35.4220	0.0000
CAC x FTSE	0.8429	0.6561	30.2700	0.0000
CAC x NIKK	3.0086	0.2222	12.1060	0.0024
CAC x SP500	0.2348	0.8892	25.3790	0.0000
DAX x FTSE	0.8951	0.6392	29.7720	0.0000
DAX x NIKK	3.4144	0.1814	18.5980	0.0001
DAX x SP500	0.4480	0.7993	16.9600	0.0002
FTSE x NIKK	3.0308	0.2197	16.3950	0.0003
FTSE x SP500	0.2363	0.8886	34.0040	0.0000
NIKK x SP500	3.3853	0.1840	14.8280	0.0006

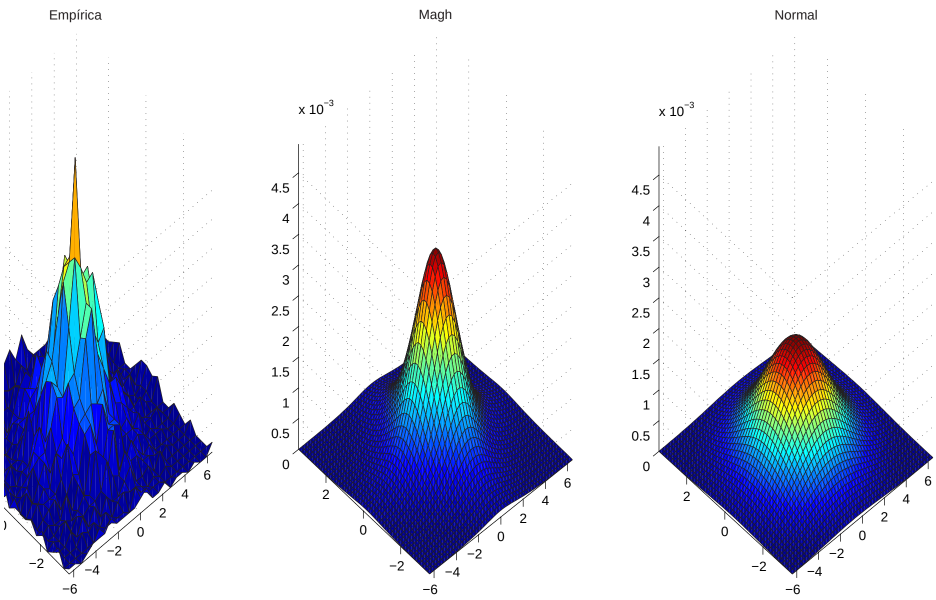

Figura 5. BVSP x NIKK PDF. ρ=0.11

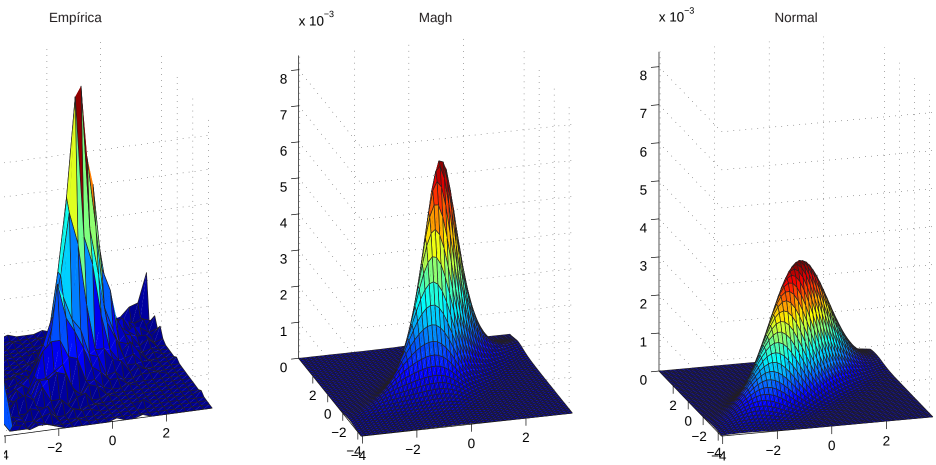

Figura 6. CAC x DAX PDF. ρ=0.79

6.4 2-Dimension Goodness of Fit

We present two kinds of goodness of fit evaluations. First we show the multivariate distribution of the more correlated pair of assets (CACxDAX) and the less correlated one (BVSPxNIKK), then we calculate two-dimensional Kolmogorov distances.

Figures 5 and 6 represents the empirical, MAGH fit and Normal fit to BVSP x NIKK (correlation = 0.1136) and CAC x DAX (correlation = 0.7902). Once again the kurtosis of the series are better captured with MAGH distributions.

Furthermore we need to calculate Kolmogorov distance to the multidimensional case. We use the approach of Fasano & Franceschini (1987) and Peacock (1983) that calculates the maximum distance in all possible directions (in this case 4). The number of sample points used in distance calculus was 100 for each margin, totaling 10000 points.

Tabela 10. Kolmogorov distances for 2-dimensional estimations.

	Normal	GH	Nig	Hyp
BVSP x CAC	0.0065	0.0016	0.0017	0.0022
BVSP x DAX	0.0072	0.0022	0.0024	0.0028
BVSP x FTSE	0.0054	0.0028	0.0011	0.0021
BVSP x NIKK	0.0062	0.0016	0.0017	0.0017
BVSP x SP500	0.0059	0.0023	0.0028	0.0025
CAC x DAX	0.0023	0.0013	0.0008	0.0009
CAC x FTSE	0.0017	0.0010	0.0006	0.0006
CAC x NIKK	0.0018	0.0006	0.0005	0.0005
CAC x SP500	0.0017	0.0005	0.0006	0.0007
DAX x FTSE	0.0019	0.0008	0.0006	0.0009
DAX x NIKK	0.0019	0.0006	0.0005	0.0007
DAX x SP500	0.0019	0.0007	0.0010	0.0012
FTSE x NIKK	0.0015	0.0005	0.0005	0.0005
FTSE x SP500	0.0014	0.0004	0.0005	0.0006
NIKK x SP500	0.0016	0.0004	0.0004	0.0005

The results of table 10 leads us to conclude that the MAGH really provides better fit to the data. Consistently the MAGH distributions and its subclasses have less distance between theoretical distribution and empirical.

6.5 6-Dimension MAGH Estimation

In this section we present the results of the joint estimation of the assets. Table 11, 12 and 13 gives us the results concerning respectively for MAGH, MANig and MAHyp. The estimation procedure explained in section 3 turns more easy this estimation.

Tabela 11. MAGH 6-dimensional estimations.

	BVSP	CAC	DAX	FTSE	NIKK	SP500
$\tilde{\alpha}$	1.0420	1.0430	1.0423	1.0432	1.0419	1.0447
$\tilde{\beta}$	-0.0142	0.0017	-0.0065	-0.0066	0.0000	-0.0079
λ	-1.0157	-0.5017	-1.0146	-0.4980	0.5028	0.0030
M	0.1304	0.0510	0.0634	0.0374	0.0011	0.0480
Σ	6.9439	0.6938	0.7598	0.5495	0.2256	0.7831
	0.6938	1.8567	1.6821	1.0513	0.2575	0.4564
	0.7598	1.6821	2.5637	1.0277	0.2697	0.5665
	0.5495	1.0513	1.0277	1.0582	0.2086	0.3496
	0.2256	0.2575	0.2697	0.2086	0.9832	0.1178
	0.7831	0.4564	0.5665	0.3496	0.1178	0.8155
Log-Like	-3846.8	-3972.3	-4041.2	-3963.3	-4028.8	-3972.8

Following our script, we show in table 14 the log-likelihood ratio test for subclasses restriction. They show that we can restrict to MANig subclass but not for MAHyp subclass, reaffirming the previous results (less dimensions).

Tabela 12. MANig 6-dimensional estimations.

	BVSP	CAC	DAX	FTSE	NIKK	SP500
$\tilde{\alpha}$	1.0135	1.0430	1.0415	1.0432	1.0382	1.0417
$\tilde{\beta}$	-0.0727	0.0017	-0.0070	-0.0066	0.0000	-0.0077
λ	-0.5000	-0.5000	-0.5000	-0.5000	-0.5000	-0.5000
M	0.2329	0.0512	0.0637	0.0375	0.0012	0.0483
	5.3897	0.8962	1.0013	0.7083	0.3862	1.1103
	0.8962	1.9261	1.6716	1.1646	0.4781	0.6472
Σ	1.0013	1.6716	2.3335	1.1601	0.4933	0.8032
	0.7083	1.1646	1.1601	1.1469	0.3889	0.4958
	0.3862	0.4781	0.4933	0.3889	1.9640	0.1670
	1.1103	0.6472	0.8032	0.4958	0.1670	1.1563
Log-Like	-3848.3	-3972.3	-4042.7	-3963.3	-4030.3	-3972.9

Tabela 13. MAHyp 6-dimensional estimations.

	BVSP	CAC	DAX	FTSE	NIKK	SP500
$\tilde{\alpha}$	1.0726	0.5985	1.0451	1.0543	1.0470	1.0530
$\tilde{\beta}$	-0.0072	-0.0013	0.0000	-0.0047	0.0000	-0.0060
λ	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
M	0.1050	0.0460	0.0535	0.0311	-0.0036	0.0433
	2.1198	0.3279	0.3764	0.2664	0.1432	0.4181
	0.3279	0.5989	0.6220	0.4382	0.1762	0.2437
Σ	0.3764	0.6220	0.8617	0.4365	0.1820	0.3025
	0.2664	0.4382	0.4365	0.4316	0.1433	0.1867
	0.1432	0.1762	0.1820	0.1433	0.7203	0.0629
	0.4181	0.2437	0.3025	0.1867	0.0629	0.4354
Log-Like	-3867.9	-3974.6	-4041.4	-3972.2	-4029.8	-3979.2

Tabela 14. Log-Likelihood Ratio tests.

NIG		Hyp	
Stats	P-Value	Stats	P-Value
9.4108	0.1518	80.072	3.4E-15

6.6 6-Dimension Goodness of fit

We felt challenged to give some measure of goodness of fit to a 6-dimension data, since all multidimensional Kolmogorov distances in literature goes up to 4 dimensions. We implemented the algorithm mentioned in the 2-Dimension case, that calculates the Kolmogorov distance if all possible accumulation directions.

In a 6 dimension problem, it leads do $2^6 = 64$ possible directions. We used 20 point to evaluate the data in each marginal, giving a total of 64,000,000 evaluation points in each accumulation. Table 15 shows the results, and once again the MAGH distributions obtain better fit.

Tabela 15. 6-Dimension Kolmogorov Distances.

Distribution	Distance
Normal	0.2394
GH	0.1773
NIG	0.1742
Hyp	0.2312

The size of the distances were influenced by the number of data points in each marginal, but the main result remains valid.

Another way to show the goodness of fit is showing the behavior of the fit in each marginal. Figures 7-12 show a visual intuition of the fit in each one of the marginals. We can infer that the MAGH distributions have a good fit performance.

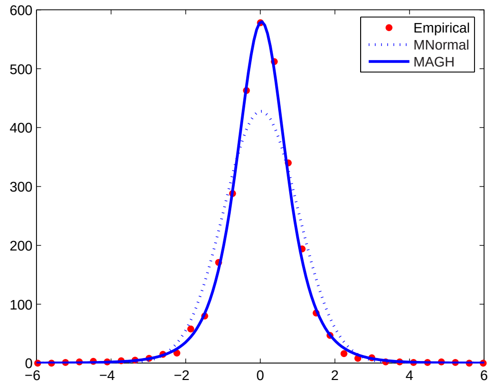

Figura 7. BVSP margin PDF.

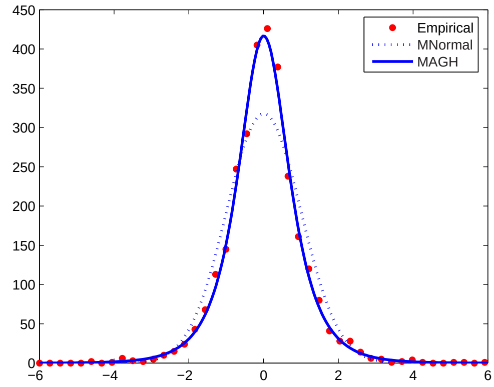

Figura 8. CAC margin PDF.

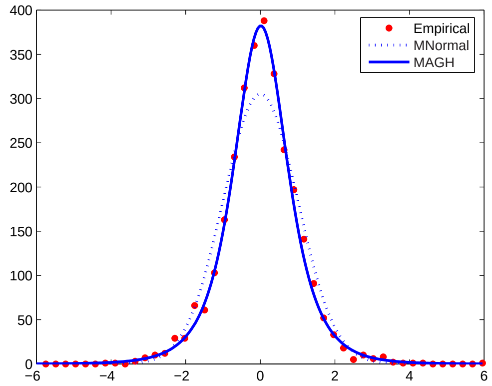

Figura 9. DAX margin PDF.

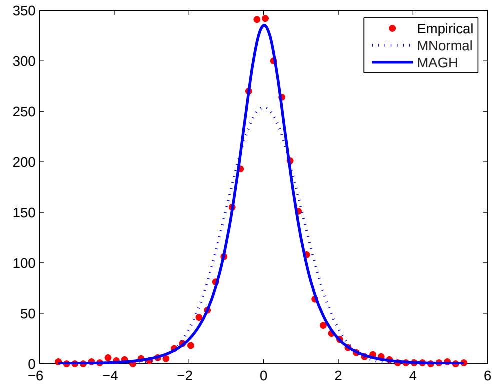

Figura 10. FTSE margin PDF.

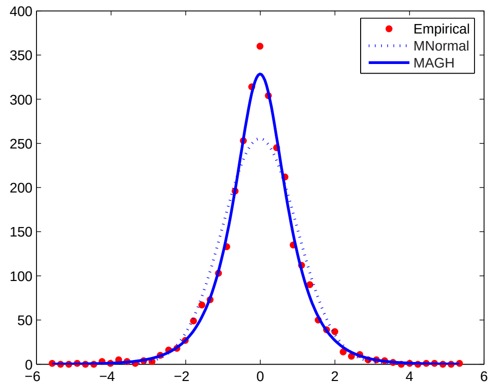

Figura 11. NIKK margin PDF.

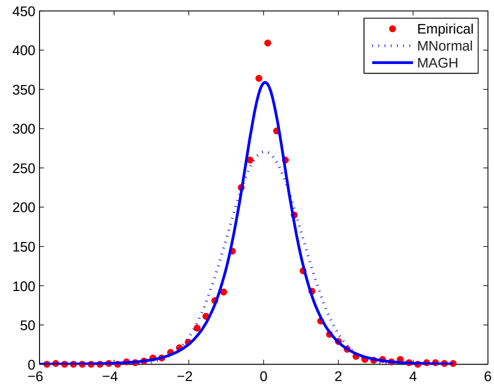

Figura 12. SP500 margin PDF.

7 Conclusions

In this paper we evaluated the goodness of fit of Multivariate Affine Generalized Hyperbolic Distributions to various assets return and showed that they present a very good fit. They can improve multivariate derivative pricing since they capture in a better way the data kurtosis.

The main limitations of the model were the computational effort to parameter estimation, although simpler than MGH estimation it is quite intensive, plus the utilization of numerical calculus that requires attention in precision determination. It is important to observe the trade-off between the use of a subclass or the Generalized family.

8 Appendix

Proof 1 (Proposition 1) *Let X be a $GH(x;\omega,\delta^2,\mu)$. Define Y as:*

$$Y \stackrel{d}{=} \frac{X-\mu}{\delta}, \tag{4}$$

this leads to:

$$P(Y \leq y) = P\left(\frac{X-\mu}{\delta} \leq y\right) = P(X \leq \delta y + \mu), \tag{5}$$

then,

$$F_Y(y) = F_X(\delta y + \mu) \tag{6}$$

Deriving both sides w.r.t y, we have:

$$f_Y(y) = f_X(\delta y + \mu)\delta \tag{7}$$

SO using the definition of the GH density , we have:

$$f_Y(y) = \left[\frac{(\alpha^2-\beta^2)^{\lambda/2}(\delta^2+(\delta y)^2)^{(\lambda-\frac{1}{2})/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(\delta y)^2}\right) e^{\beta\delta y}\right]\delta \tag{8}$$

Now doing a simple parameter transformation:

$$\alpha = \frac{\tilde{\alpha}}{\delta} \quad \text{and} \quad \beta = \frac{\tilde{\alpha}\tilde{\beta}}{\delta} \tag{9}$$

And replacing in 9 and 8, we obtain:

$$
\begin{aligned}
f_Y(y) &= \frac{\left(\frac{\tilde{\alpha}^2}{\delta^2} - \frac{\tilde{\alpha}^2\tilde{\beta}^2}{\delta^2}\right)^{\frac{\lambda}{2}} (\delta^2 + \delta^2 y^2)^{(\lambda-\frac{1}{2})/2}}{\sqrt{2\pi}\left(\frac{\tilde{\alpha}}{\delta}\right)^{\lambda-1/2} \delta^\lambda K_\lambda\left(\delta\sqrt{\frac{\tilde{\alpha}^2}{\delta^2} - \frac{\tilde{\alpha}^2\tilde{\beta}^2}{\delta^2}}\right)} K_{\lambda-\frac{1}{2}}\left(\frac{\tilde{\alpha}}{\delta}\sqrt{\delta^2 + \delta^2 y^2}\right) e^{\frac{\tilde{\alpha}\tilde{\beta}\delta y}{\delta}}\delta \\
&= \frac{\delta(\tilde{\alpha}^2(1-\tilde{\beta}^2))^{\frac{\lambda}{2}}\delta^{-\lambda}(1+y^2)^{(\lambda-\frac{1}{2})/2}\delta^{\lambda-\frac{1}{2}}}{\sqrt{2\pi}\tilde{\alpha}^{\lambda-\frac{1}{2}}\delta^{-\lambda+\frac{1}{2}}\delta^\lambda K_\lambda\left(\delta\sqrt{(\tilde{\alpha}^2(1-\tilde{\beta}^2))\delta^{-2}}\right)} K_{\lambda-\frac{1}{2}}\left(\tilde{\alpha}\delta^{-1}\sqrt{\delta^2(1+y^2)}\right) e^{\tilde{\alpha}\tilde{\beta}y} \\
&= \frac{\delta\tilde{\alpha}^\lambda(1-\tilde{\beta}^2)^{\frac{\lambda}{2}}\delta^{-\lambda}(1+y^2)^{(\frac{\lambda}{2}-\frac{1}{4})}\ \delta^{\lambda-1/2}}{\sqrt{2\pi}\tilde{\alpha}^{\lambda-1/2}\delta^{-\lambda+1/2}\delta^\lambda K_\lambda\left(\tilde{\alpha}\sqrt{1-\tilde{\beta}^2}\right)} K_{\lambda-\frac{1}{2}}\left(\tilde{\alpha}\sqrt{1+y^2}\right) e^{\tilde{\alpha}\tilde{\beta}y} \\
&= \frac{\tilde{\alpha}^{1/2}(1-\tilde{\beta}^2)^{\frac{\lambda}{2}}(1+y^2)^{(\frac{\lambda}{2}-\frac{1}{4})}}{\sqrt{2\pi}K_\lambda\left(\tilde{\alpha}\sqrt{1-\tilde{\beta}^2}\right)} K_{\lambda-\frac{1}{2}}\left(\tilde{\alpha}\sqrt{1+y^2}\right) e^{\tilde{\alpha}\tilde{\beta}y} \qquad (10)
\end{aligned}
$$

We got an expression similar to Schmidt et al. *(2006).* □

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