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Numerical Simulation of Nonoptimal Dynamic Equilibrium Models*

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Abstract

In this paper we present a recursive method for the numerical simulation of non-optimal dynamic equilibrium models. This method builds upon a convergent operator over an expanded set of state variables. The fixed point of this operator defines the set of all Markovian equilibria. We study approximation properties of the operator. We also apply our numerical algorithm to various models with heterogeneous agents, incomplete financial markets, exogenous and endogenous borrowing constraints, taxes, and money.

Keywords: Heterogeneous agents, taxes, externalities, financial frictions, competitive equilibrium, computation, simulation.

JEL Codes: C6, D5,E2.

1 Introduction

In this paper we present a recursive method for the numerical simulation of non-optimal dynamic equilibrium models and study its convergence and accuracy properties. Computation of these models is usually a formidable task because of various technical issues that preclude direct application of standard dynamic programming techniques. We apply our numerical algorithm to various models with heterogeneous agents and real and financial frictions. Computation of these models is critical to advance our understanding in several basic areas of macroeconomics and finance. We simulate a variant of the two-agent model of Kehoe and Levine (2001) to assess the influence of exogenous and endogenous borrowing constraints on the volatility of asset prices and consumption. We simulate various versions of the two-country model of Kehoe and Perri (2002) to assess the influence of borrowing constraints, incomplete markets, preference shocks, and taxes on international risk sharing and investment. We study the overlapping generations (OLG) economy of Kubler and Polemarchakis (2004), and introduce money.

Standard solution methods search for a continuous equilibrium function over a natural state space. However, since the seminal work of Kydland and Prescott (1980), we know that equilibria of non-optimal economies may not have a recursive representation over the natural state space. These authors consider a game of optimal taxation with a representative household, but similar technical problems are observed for competitive economies with market frictions because Pareto-optimality may fail to hold. Kydland and Prescott rewrite their model in a recursive form by appending a set of Lagrange multipliers to the original state space so as to characterize the exact solution. Unfortunately, their methods are not directly suited for the computation of decentralized economies with heterogeneous agents and market frictions. We certainly lack reliable algorithms for the simulation of these economies.

Ignoring these technical issues and proceeding with standard solution methods may result in substantial biases. We simulate some simple versions of the aforementioned OLG

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economies by established algorithms using continuous equilibrium functions. The ensuing computed solutions present large approximation errors, and fail to mimic the true dynamics. In spite of these large approximation errors, traditional algorithms may be quite deceptive as they can produce small Euler equation residuals or may do well under some other independent accuracy checks. Peralta-Alva and Santos (2010) find similar biases from standard algorithms when applied to a growth model with distortionary taxation.

Positive results on existence of a continuous equilibrium over a natural state space rely upon certain monotonicity properties of the equilibrium dynamics [e.g., see Bizer and Judd (1989), Coleman (1991), and Datta, Mirman and Reffett (2002)]. For the canonical one-sector growth model with taxes and externalities, monotone dynamics follow from fairly mild restrictions on the primitives. But monotone dynamics are much harder to obtain in multi-sector models with heterogeneous agents and incomplete financial markets.

Duffie *et al.* (1994) search for general representations of stationary equilibria over an expanded state space that includes all endogenous variables such as asset prices and individual consumptions. Unfortunately, they offer no algorithm to find equilibria. Building upon these methods, Kubler and Schmedders (2003) show existence of a Markovian equilibrium for a class of financial economies with collateral requirements. Their computations are based on a projection-type algorithm iterating over functions, and thus cannot offer any guarantee of convergence. Marcet and Marimon (2010) study a general class of contracting problems with incentive constraints. Following Kydland and Prescott (1980) these authors enlarge the state space with a vector of weights for the utility of each agent, and compute a transition for such weights from the shadow values of the agents' participation constraints. They assume that equilibrium solutions can be characterized by convex social planning problems. By construction this method cannot capture multiple equilibria, but seems to be more operative for the computation of some dynamic incentive problems written in a Pareto-welfare form.

Our work is closest to Kubler and Schmedders (2003), but we consider a broader set

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of economies with exogenous and endogenous borrowing constraints. For computational reasons, we expand the state space with a different set of variables: The shadow values of investment. We also include continuation utility values to deal with endogenous borrowing constraints. Abreu, Pierce, and Stacchetti (1990) introduce continuation utility values to find a recursive representation of sequential equilibria for dynamic games. This additional state vector – continuation utilities – is not sufficient for the computation of our economies. For the characterization of a recursive competitive equilibrium we need to consider both continuation utilities and shadow values of investment. Finally, unlike Kubler and Schmedders (2003), in the numerical implementation we iterate over candidate equilibrium sets – rather than functions – to preserve convergence properties of our algorithm. We can thus compute the set of all competitive equilibria. Our algorithm can successfully be applied to various types of models, and it seems particularly useful for models with market distortions and exogenous and endogenous borrowing constraints.

We start in section 2 with our general framework of analysis. Section 3 studies the numerical implementation of our algorithm and its convergence properties. Then, we apply these numerical procedures to various types of models. Sections 4 and 5 are devoted to simulation of equilibria for economies with borrowing constraints, and section 6 replicates these methods for an OLG economy. We conclude in section 7.

2   General Theory

In this section, we first set out a general analytical framework that encompasses various economic models. We then present our numerical approach and main results on existence and global convergence to the Markovian equilibrium correspondence.

2.1   General Framework

Time is discrete $t = 0, 1, 2, \cdots$. The economy is perturbed every period by a vector of exogenous shocks. This vector follows a Markov chain $(z_t)_{t \geq 0}$ over a finite set $Z = \{1, 2, ..., Z\}$

characterized by positive transition probabilities $\pi\left(z'|z\right)$ for all $z, z' \in Z$. The initial state, $z_0 \in Z$, is known to all agents in the economy. Then $z^t = (z_0, z_1, z_2, ..., z_t) \in Z^{t+1}$ is a history of shocks, often called a date-event or node. Endogenous predetermined variables are denoted by $x \in X$, $X \subset \mathbb{R}^N$, and may include agents' holdings of physical capital, human capital, and financial assets. All other endogenous variables are denoted by y, with $y \in Y$, $Y \subset \mathbb{R}^J$. This latter vector may include equilibrium prices and choice variables such as consumption and investment. We let m denote the vector of shadow values of investment for all assets and all agents, with $m \in M$, $M \subset \mathbb{R}^K$, and $p \in \mathbb{R}^I$ the vector of continuation utility values.

Our framework encompasses some models where agents have the choice of default. It is thus necessary to specify the payoff of default, which in our case implies permanent exclusion from the market. Default carries a lifetime utility that may depend on the shocks, and on endogenous predetermined variables. To preserve the convexity of the feasible set, however, it will be important that individual actions are perceived as not affecting the payoff of default. Of course, in equilibrium they may turn out to have such an impact. Hence, each of the I agents in this economy confronts an expected discounted lifetime utility given by $P^{aut} : \mathbb{R}^N \times Z \to \mathbb{R}$ in case of default. As will become clear later on, it is useful to keep track of the set of possible expected discounted lifetime utilities that each individual may obtain if she does not default. These values will be captured by correspondence $P : X \times Z \times M \rightrightarrows \mathbb{R}^I$.

We focus on computation of sequential competitive equilibria (SCE) described by infinite sequences $\{x(z^t), y(z^t)\}_{t \geq 0}$ in which endogenous constraints may be binding and so default does not actually occur at any node. Our analysis applies to models where SCE is fully characterized by Euler equations and aggregate resource constraints. Specifically, the law of motion of the state vector x is conformed by a system of non-linear equations:

$$\varphi\left(x_{t+1}, x_t, y_t, z_t\right) = 0. \tag{1}$$

Function φ may embed technological constraints as well as individual budget constraints. For some models we can explicitly solve for x_{t+1} as a function of (x_t, y_t, z_t). But in some

other applications such as in models with adjustment costs, x_{t+1} may not admit an analytical solution. Second, the vector of shadow values of investment must be a continuous function of current variables:

$$m_t = \Psi\left(x_t, y_t, z_t\right). \tag{2}$$

This is usually a vacuous assumption under continuously differentiable production and utility functions. Finally, $\{x(z^t), y(z^t), m(z^t)\}_{t \geq 0}$ must satisfy

$$\Phi\left(\lambda_t, x_t, y_t, z_t, E_t\left[m_{t+1}\right]\right) = 0, \tag{3}$$

together with the participation constraints

$$p(z^t) \geq P^{aut}(x(z^t), z_t) \tag{4}$$

for at least one $p(z^t) \in P(x(z^t), z_t, m(z^t))$. Function Φ may describe individual optimality conditions (such as Euler equations), market-clearing conditions, various types of budget restrictions, and resource constraints. The vector of Lagrange multipliers that include the possibly binding individual rationality constraints is denoted by λ. In this paper, in which we only consider time-separable utilities and a constant discount factor $0 < \beta < 1$, the discounted life-time utility of participation can be represented as

$$p = u(y) + \beta E p', \tag{5}$$

where $u(y)$ denotes current utility, and p' denotes the continuation utility starting in the next period.

In our simple framework equilibrium vectors lie on a compact set so that the Euler equations are necessary and sufficient conditions to characterize individual optimal solutions for concave programs. We provide a set of methods that validate this assumption for the model economies we evaluate quantitatively in later sections of the paper. Indeed, in all those economies the domain of each economic variable will be explicitly laid down.

As is well known, even if the exogenous shock z_t is driven by a Markov process, the set of SCE may not admit a recursive representation over the natural state space, (x, z). We show

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instead existence of the Markovian property over the "enlarged" state space (x, z, m, p). That is, to the natural state space we append the shadow values of investment m, and continuation utilities p.

2.2 Recursive Equilibrium Theory

In order to compute the set of SCE we define the equilibrium correspondence $V^* : (x, z) \longmapsto V^*(x, z) \subseteq M$, as the set of equilibrium vectors of shadow values of investment m for any given state. Similarly, we define $P^* : (x, z, m) \longmapsto P^*(x, z, m)$ as the set of continuation utility values $p \in P^*(x, z, m)$, which satisfy $p \geq P^{aut}(x, z)$ for every (x, z), $m \in V^*(x, z)$. The theoretical underpinnings of our algorithm combine ideas from Kydland and Prescott (1980) and Abreu, Pierce and Stacchetti (1990). We iterate over correspondences of shadow values and participation satisfying feasibility and endogenous constraints starting from some well chosen pair (V_0, P_0). This iterative procedure is based on a monotone operator, B, that generates sequences of non-empty compact sets (V_n, P_n) that shrink to the equilibrium pair (V^*, P^*). Operator B embodies all temporary equilibrium conditions such as agents' Euler equations, exogenous and endogenous constraints, and market-clearing conditions from any initial value z to all immediate successor nodes z_+. This operator is analogous to the expectations correspondence of Duffie $et\ al.$ (1994), albeit it may contain a smaller set of endogenous variables. Using operator B, we can derive the set of all SCE from a well-chosen decreasing sequence of correspondences.

More precisely, $(V', P') = B(V, P)$ is defined (pointwise) as follows: Pick a vector (x, z). Then, for any $m \in V(x, z), p \in P(x, z, m)$ we have $m \in V'(x, z)$ and $p \in P'(x, z, m)$ if there is $y, x_+,\ m_+(z_+)$, and $p_+(z_+)$, for all $z_+ \in Z$, with $m_+(z_+) \in V(x_+, z_+), p_+(z_+) \in P(x_+, z_+, m_+)$ such that technological and individual constraints are satisfied, $\varphi(x_+, x, y, z) = 0$, as well as intertemporal optimality together with additional equilibrium constraints:

$$\Phi\left(\lambda, x, y, z, \sum_{z_+ \in Z} \pi(z_+|z) m_+(z_+)\right) = 0,$$

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subject to (4-5). For models where a SCE exists, operator $B(V, P)$ will be non-empty since all equilibria must satisfy the required conditions. Further, operator B is monotone: If $V \subset \hat{V}$ and $P \subset \hat{P}$ then $B(V, P) \subset B(\hat{V}, \hat{P})$.[1] Also, if (V, P) has a closed graph then $B(V, P)$ will also have a closed graph as the above functions φ, Φ, Ψ are continuous.

Assumption 2.1 *Operator B preserves compactness: If V and P are compact valued then $B(V, P)$ is also compact valued.*

We now show existence of a fixed-point solution (V^*, P^*) and a general form of global convergence.

Theorem 2.1 *(convergence) Let V_0, P_0 be compact-valued correspondences such that $V_0 \supset V^*$, $P_0 \supset P^*$. Let $(V_n, P_n) = B(V_{n-1}, P_{n-1})$ for all $n \geq 1$. Then, operator B has a fixed-point solution, i.e., $(V^*, P^*) = B(V^*, P^*)$, where $V^* = \lim_{n \to \infty} V_n$, and $P^* = \lim_{n \to \infty} P_n$. Moreover, (V^*, P^*) is the largest fixed point of operator B, i.e., $(V, P) = B(V, P)$ implies $(V, P) \subset (V^*, P^*)$.*

We would like to remark that operator B iterates over sets rather than functions. Hence, if there are multiple equilibria we can find all of them. By construction, for any $(x, z, m) \in graph(V^*)$, and $(x, z, m, p) \in graph(P^*)$, under the action of operator B we can generate a new vector (x_+, z_+, y, m_+, p_+) that can be extended into a SCE $\{x(z^t), y(z^t)\}_{t \geq 0}$. Since the fixed point of operator B is an upper semicontinuous correspondence, it is possible to select a measurable policy function $y = g^y(x, z, m)$, transition functions $m_+(z_+) = g^m(x, z, m, p; z_+)$ and $p(z_+) = g^p(x, z, m, p; z_+)$, and continuation values for the endogenous predetermined variables from $\varphi(x_+, x, y, z) = 0$. Let us summarize these future equilibrium values over the extended state space as $g(x, z, m, p; z_+) = (x_+, z_+, m_+, p_+)$. Then, g is an equilibrium selection, and provides a Markovian characterization of a subset of dynamic equilibria.[2]

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[1] For correspondences $V, \hat{V}$ we say that $V \subset \hat{V}$ if $V(x, z) \subset \hat{V}(x, z)$ for all (x, z). We shall use the usual notion of distance over sets given by the Hausdorff metric.

[2] It should be clear that $g(\cdot; z_+)$ denotes a coordinate function of $g(\cdot)$ corresponding to the successor

3   Numerical Implementation

In this section we develop a numerical implementation of operator B and study its convergence and accuracy properties. We show that iterations of the numerical algorithm must converge to a fixed point that contains the equilibrium correspondence. Given that we approximate the image of the correspondence directly, our results imply that in models where a unique equilibrium exists the accuracy of the numerical approximation is of the same order of magnitude as the size of our discretization. For models where the equilibrium correspondence is multivalued we show that as the size of our discretizations converge to zero then the fixed points of the numerical algorithm *converge uniformly* to the equilibrium correspondence.

Judd, Yeltekin and Conklin (2003), and Judd and Yeltekin (2010), develop methods for computing equilibria of dynamic games. Essentially, their approximation strategy relies on the convexity of the sets being approximated, which may be achieved via a public randomization device. Randomization over the original set of strategies seems quite appealing and natural in game theoretic settings. However, it is difficult to interpret such ex post convexifications in competitive economies. Indeed, in our numerical implementation of operator B convexity may be lost as more iterations are performed. Therefore, in our model economies this convexity requirement seems very strong. Instead, we build our analysis on monotonicity of the operator, and compactness of the state space and of the image of the relevant correspondences involved.

We proceeds as follows. First, we partition the state space into a finite set of J simplices with mesh size h. Compatible with this partition we consider a sequence of *step correspondences*, which take constant set-values on each simplex. Step correspondences are the analog of step functions and have good approximation properties. We also introduce a finite-dimensional outer approximation over the image of these correspondences; this outer approximation is made up of N cubes or finite-dimensional elements. Then, us-

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node $z_+|z$.

ing these approximations we obtain a computable operator $B^{h,N}$ with accuracy parameters (h, N). We show that the sequence of correspondences defined by iterations of operator $B^{h,N}$ converges to a fixed point conformed by the equilibrium correspondence (V^*, P^*). Finally, we study accuracy properties of the algorithm.

3.1 The Numerical Algorithm

Assume that all equilibrium state vectors (x, z, m, p) belong to some set S, which is a subset of the product space $S = X \times Z \times M \times P$. Let $\{X^j\}_{j=1}^J$ be a finite family of simplices with non-empty interior such that $\cup_j X^j = X$ and $int(X^j) \cap int(X^{j'})$ is empty for every pair $X^j, X^{j'}$. Let $\{M^i\}_{i=1}^I$ be the corresponding family for the values of the shadow values of investment. Define the *mesh size h* of this discretization as

$$h = \max_{j,i} diam \left\{ X^j, M^i \right\}.$$

Consider any given correspondence $V : X \times Z \to 2^M$, where 2^M denotes the subsets of vectors for space M containing the shadow values of investment m. An approximation V^h compatible with the partition $\{X^j\}$ takes on constant set-values $V^h(x, z)$ on each simplex X^j. We then build the step correspondence:

$$V^h(x, z) = \cup_{x \in X^j} V(x, z), \text{ for each given } z \text{ and all } x \in X^j. \tag{6}$$

A symmetric construction defines the representative step correspondence for the values of participation:

$$P^h(x, z, m) = \cup_{p \in P^i} P(x, z, m), \text{ for each given } z \text{ and all } x \in X^j, m \in M^i.$$

Now, we construct an operator $B^h(V, P)$ between step correspondences as $B^h(V, P)(x, z, m) = \cup_{x \in X^j, m \in M^i} B(V, P)(x, z, m)$, for each given z. By similar arguments as above, we can prove that B^h has a fixed-point solution. To obtain a computable representation of these correspondences we also discretize the image space. For a given set V we say that $\mathcal{C}^N(V) \supseteq V$ is an N-element outer approximation of V if $\mathcal{C}^N(V)$ can be generated by N elements. We

require this numerical representation to preserve monotonicity: $V \subset \hat{V}$ implies $\mathcal{C}^N(V) \subset \mathcal{C}^N(\hat{V})$. This is essential to guarantee monotonicity of a computable version of operator B. We also require $\lim_{N \to \infty} \mathcal{C}^N(V) = V$.

Using these approximations, we can construct a new operator $B^{h,N}$ that starts by computing a step correspondence $B^h(V, P)$ of $B(V, P)$. Each set-value is then adjusted by the N-element outer approximation to get $\mathcal{C}^N\left(B^h(V, P)\right)$.

In conclusion, the output of our numerical algorithm would be summarized by some correspondences $(V_n^{h,N}, P_n^{h,N})$ under the action of an operator $B^{h,N}$. From the application of operator $B^{h,N}$ on $(V_n^{h,N}, P_n^{h,N})$, we can choose an approximate policy function $y = g_n^{y,h,N}(x, z, m)$, and transition functions $m_+(z_+) = g_n^{m,h,N}(x, z, m; z_+)$, $p_+(z_+) = g_n^{p,h,N}(x, z, m, p; z_+)$. From these approximate equilibrium functions we can generate SCE paths $\{x_t(z^t), y_t(z^t)\}_{t=0}^{\infty}$. Sections 4 to 6 illustrate examples of such operators, and their application to different dynamic models.

3.2 Convergence and Accuracy Properties

We start by showing that our discretized operator $B^{h,N}$ has good convergence properties: The fixed point of this operator $(V^{*,h,N}, P^{*,h,N})$ contains (V^*, P^*) and it approaches this limit point as we refine the approximations. The proof of this result extends the convergence arguments of Beer (1980) to a dynamic setting.

Theorem 3.1 *For given h, N, let $V_0 \supseteq V^*$ and $P_0 \supseteq P^*$. Let $(V_n^{h,N}, P_n^{h,N}) = B^{h,N}(V_{n-1}^{h,N}, P_{n-1}^{h,N})$ for all $n \geq 1$. Then, (i) $V_n^{h,N} \supseteq V^*$ and $P_n^{h,N} \supseteq P^*$ for all n; (ii) $V_n^{h,N} \to V^{*,h,N}$, $P_n^{h,N} \to P^{*,h,N}$ as $n \to \infty$; and (iii) $V^{*,h,N} \to V^*$, $P^{*,h,N} \to P^*$, as $h \to 0$ and $N \to \infty$.*

There are three points to emphasize from these results. First, the set of numerical solutions generates the set of all competitive equilibria. Second, our algorithm is globally convergent. And third, as we refine these approximations the fixed-points of our algorithm shrink to the set of exact recursive equilibria competitive equilibria.

Regarding accuracy properties of the algorithm, in our next theorem we establish that such convergence is uniform. Hence, the approximation error is directly correlated with the mesh size of the discretization. This important approximation result comes directly from Theorem 3.1 and the fact that equilibria lie in compact domains. For correspondences $(V_n^{h,N}, P_n^{h,N})$ and (V, P), let us denote the distance between sets $d(graph(V_n^{h,N}, P_n^{h,N}), graph(V, P))$, where d represents the Hausdorff metric.

Theorem 3.2 *Under the conditions of Theorem 3.1, for any given $\epsilon > 0$ there are $\hat{N}, \hat{h}, \hat{n}$ such that the distance $d(graph(V_n^{h,N}, P_n^{h,N}), graph(V^*, P^*)) \leq \epsilon$ for all $N \geq \hat{N}, h \leq \hat{h}, n \geq \hat{n}$.*

Hence, for any given (x, z, m) and a sufficiently close approximation N, h, n, there exists (x', z, m') such that $d(graph(V_n^{h,N}(x, z), P_n^{h,N}(x, z, m)), graph(V^*(x', z), P^*(x', z, m'))) \leq \epsilon$.

4 Asset Pricing Models with Market Frictions

An important family of macroeconomic models incorporates financial frictions in the form of sequentially incomplete markets, borrowing constraints, transactions costs, cash-in-advance constraints, and margin and collateral requirements.[3] Fairly general conditions rule out the existence of financial bubbles in these economies, and hence equilibrium asset prices are determined by the expected value of future dividends [Santos and Woodford (1997)]. There is, however, no reliable algorithm for the numerical approximation and simulation of these economies. Here, we illustrate the workings of our algorithm in the economy of Kehoe and Levine (2001). These authors provide a characterization of steady-state equilibria for an economy with idiosyncratic risk under exogenous and endogenous borrowing constraints. We complement their qualitative analysis with numerical simulation to appraise quantitatively the effects of both borrowing constraints on consumption and asset prices.

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[3]For instance, see Campbell (1999), Heaton and Lucas (1996), Huggett (1993), Krebs and Wilson (2004), Mankiw (1986), and Telmer (1993).

4.1 Economic Environment

There are two states of uncertainty: Each household may be getting a current high endowment e_h or a low endowment e_l. There is no aggregate risk: One household gets the high endowment whilst the other one gets the low endowment at every date. There is only one asset, a Lucas tree with a constant dividend, d. Shares of ownership are normalized to equal one.

For each agent $i = 1, 2$, preferences are represented by the intertemporal objective

$$E\left[\sum_{t=0}^{\infty} \beta^t u^i\left(c_t^i\right)\right]. \tag{7}$$

The discount factor is $\beta \in (0, 1)$, and E is the expectations operator. The one-period utility function u^i depends on the quantity consumed c_t^i. This function is assumed to satisfy standard properties.

Let $q_t(z^t)$ be the price of a unit share of the real asset. Then, every agent i faces the following sequence of budget constraints:

$$c_t^i(z^t) + \theta_{t+1}^i(z^t)q_t(z^t) = e_t^i(z^t) + \theta_t^i(z^{t-1})(q_t(z^t) + d). \tag{8}$$

There are two scenarios for the modeling of financial markets. In the first scenario, each household has to satisfy the participation constraint

$$E_{z^t} \sum_{\tau=t}^{\infty} \beta^\tau u^i\left(c_\tau^i\right) \geq V^{i, aut}(z^t), \text{ for all } i \text{ and } z^t, \tag{9}$$

and is allowed to trade contingent shares of the tree that are honored depending upon the state of uncertainty. Here, $V^{i, aut}(z^t)$ denotes the expected discounted value of consuming the endowment allocation from the time t of default into the infinite future. In the second scenario, each household can trade uncontingent shares of the Lucas tree subject to the exogenous borrowing constraint

$$\theta_t^i \geq 0, \text{ for all } z^t. \tag{10}$$

Kehoe and Levine refer to the fist scenario as the debt constrained economy, and to the second scenario as the liquidity constrained economy.

For each scenario a sequential competitive equilibrium (SCE) is a collection of stochastic paths of individual consumptions, asset holdings, and asset prices such that (i) Individual consumption and asset holding allocations solve the constrained-utility maximization problem of each household $i = 1, 2$, and (ii) Goods and financial markets clear.

4.2 Quantitative Experiments

For convenience of the presentation, we consider the baseline case of Kehoe and Levine (2001). The equilibrium for the debt constrained economy takes on a simple form. Roughly, consumption is high for the household that gets the high endowment, and consumption is low for the other household for whom the limited enforcement constraint is binding. For the liquidity constrained economy, the solution does not take on such a simple form, and needs to be computed. Basically, the ergodic set comprises the whole domain of capital holdings, and allocations depend on the shock and the distribution of asset holdings.

Let π be a transition probability of switching endowments. Both households share the same Bernoulli utility function $u(c) = \log(c)$, and the same discount factor. Our computations center upon the following baseline values

$$\beta = 1/2; e_l = 9; e_h = 24; d = 1; \pi = 1/2.$$

The Equilibrium Correspondence

Note that in equilibrium $\theta^1 = 1 - \theta^2$. Hence, in the sequel we let θ be the share holdings of household 1, and e_s be the endowment of household 1, for $s = l, h$. Then, the equilibrium correspondence $V^*(\theta, e_s)$ is a map from the space of possible values for share holdings and endowments for agent 1 into the set of possible equilibrium shadow values of investment for each agent (m^1, m^2).

For the economy with exogenous constraints, both θ, q are scalars. For this latter economy the shadow values of investment are defined as follows:

$$m^1(\theta, e^1) = \frac{1}{e^1 + \theta(d+q) - \theta_+ q}[d+q], \tag{11}$$

$$m^2(\theta, e^2) = \frac{1}{e^2 + (1-\theta)(d+q) - (1-\theta_+)q}[d+q], \tag{12}$$

where $(e^1, e^2) = (e_l, e_h)$, or $(e^1, e^2) = (e_h, e_l)$. For any pair of equilibrium shadow values of investment $(m^1, m^2) \in V^*(\theta, e_s)$, there must be share prices q, multipliers λ, tomorrow's share holdings θ_+, and shadow values of investment $(m^1_+, m^1_+) \in V^*(\theta_+, e_+)$ such that

$$qDu^i(e^i + \theta^i(d+q) - \theta^i_+ q) = \lambda^i + \beta^i E m^i_+.$$

Here $\lambda^i \geq 0$, with strict inequality if today's borrowing constraint binds. As before, E is the expectations operator.

Analogously, for the economy with endogenous constraints we must have

$$qDu^i(e^i + \theta^i(d+q) - \theta^i \cdot q) = \lambda^i \beta^i \pi[e^i_+ | e^i] m^i_+.$$

Note that in this economy with endogenous debt constraints, asset holdings and prices are state contingent and thus both θ, q are vectors in $\mathbb{R}^2$. Also, in the Euler equation above $\lambda^i \geq 1$ is a ratio of multipliers corresponding to the participation constraints. That is, $\lambda^i = \frac{1+\mu^i+\mu^i_+}{1+\mu^i}$, where $\mu^i \geq 0$ is a multiplier associated with today's participation constraint, and $\mu^i_+ \geq 0$ is a multiplier associated with tomorrow's participation constraint at state $e^i_+ | e^i$. Therefore, $\lambda^i > 1$ only if tomorrow's participation constraint is binding.[4]

Our Algorithm

Our method proceeds as follows. We start with a correspondence V_0 such that $V_0(\theta, e_s) \supseteq V^*(\theta, e_s)$ for all (θ, e_s) with $s = l, h$. It is easy to come up with the initial candidate V_0, since the low endowment e_l is a lower bound for consumption, and the marginal utility of consumption can be used to bound asset prices as discounted values of dividends. It is also straightforward to derive bounds for the value of participation P_0.

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[4]Note that this is the Euler equation for $t = 0$ to build operator B. Using the envelope theorem for the value function J^i defined below, we can derive the Euler equation for any future date $t \geq 1$.

For the purposes of presentation, let us first consider the scenario of the exogenous borrowing constraint (10) where correspondence P is not really operative. Our mapping B dictates that for $(m^1, m^2) \in BV_n(\theta, e_s)$ it must be possible to find continuation shadow values of investment $(m^1_+, m^2_+) \in V_n(\theta_+, e_{s_+})$, a bond price q, and multipliers (λ^1, λ^2), such that the individual's intertemporal optimality conditions are satisfied

$$\frac{q}{e^1 + \theta(d+q) - \theta_+ q} = \lambda^1 + \beta E m^1_+,$$
$$\frac{q}{e^2 + (1-\theta)(d+q) - (1-\theta_+)q} = \lambda^2 + \beta E m^2_+.$$

If we cannot find values that satisfy the previous conditions, then $(m^1, m^2) \notin BV_n(\theta, e_s)$. A new correspondence $V_{n+1} = B(V_n)$ is defined after proceeding with these computations over every possible value (θ, e_s) for $s = l, h$.

For the scenario with the limited enforcement constraint (9), our algorithm requires iterating over candidate values that preclude default. In the present model the reservation value of default is autarky. Let $V^{i,aut}(e_s)$ be the expected utility value of consuming the endowment allocation starting from e_s for $s = l, h$. Iterations of operator B result in new candidate values for the shadow values of investment, and new candidate values for participation. Specifically, given (θ, e_s), $(m^1, m^2) \in V_n(\theta, e_s)$, and $(p^1, p^2) \in P_n(\theta, e_s, m^1, m^2)$ we have that $(m^1, m^2) \in V_{n+1}(\theta, e_s)$, and $(p^1, p^2) \in P_{n+1}(\theta, e_s, m^1, m^2)$ iff we can find portfolio holdings for next period, θ_+, a bond price q, multipliers (λ^1, λ^2), continuation shadow values of investment $(m^1_+, m^2_+) \in V_n(\theta_+, e_{s_+})$, and continuation utilities $(p^1_+, p^2_+) \in P_n(\theta_+, e_{s_+}, m^1_+, m^2_+)$ such that the individual's intertemporal optimality conditions are satisfied, and are consistent with the definition of promised utilities and with participation constraints

$$p^i = u(c^i) + \beta E p^i_+$$

$$p^i \geq V^{i,aut}(e_s).$$

Our algorithm can then be used to generate a sequence of approximations to the equilibrium correspondence via the recursion $(V_{n+1}, P_{n+1}) = B(V_n, P_n)$.

For the numerical implementation of the algorithm, let us just consider the debt constrained economy. We assume a pre-specified interval of share holdings $[\theta_l, \theta_h]$, which in this case is $[0, 1]$. We then partition the state space by selecting a set of vertex points with grid size h. The step correspondence approximating V_0 at θ over a simplex $[\theta_i, \theta_{i+1}]$ can be defined as

$$V_0^h(\theta, e_h) = \cup_{\theta^1 \in [\theta_i, \theta_{i+1}]} V_0(\theta, e_h)$$

$$V_0^h(\theta, e_l) = \cup_{\theta^1 \in [\theta_i, \theta_{i+1}]} V_0(\theta, e_l).$$

The image of this correspondence are the shadow values of investment (m^1, m^2). Hence, a simple outer approximation $\mathcal{C}^N \left(B^h(V) \right)$ would be a finite collection of squares for vectors (m^1, m^2). This completes the numerical implementation of operator $B^{h,N}$, defined over computable step correspondences. The various tasks involved in this process can adequately be performed by parallel computing.

Quantitative Results

We now compare the quantitative implications for consumption volatility and asset prices for the two different scenarios. The debt constrained economy inherits a simple dynamic structure with two steady-state values for consumption. The liquidity constrained economy, however, presents richer dynamics. The ergodic set is made up of the whole distribution of shares $\theta \in [0, 1]$ as agent 1 buys shares of the asset in the presence of the good shock, and sells shares of the asset in the presence of the bad shock.

Table 2 below reports sample statistics for equilibrium time series from both economies. In this table, q refers to the price of a state uncontingent share. This is the price of the asset for the liquidity constrained economy and the sum of the two current prices of the asset for the debt constrained economy.

Model	mean(q)	std(q)	mean(c_1)	stdev(c_1)
Exogenous constraint	2.11	1.23	16.91	7.36
Endogenous constraint	1.07	0.00	17.00	4.52

Table 2: Simulated moments for the debt and liquidity constrained economies – mean and standard deviation (stdev).

Perfect risk sharing would require constant consumption across states equal to 17. The endogenous participation constraint prevents perfect risk sharing and consumption displays some volatility in the debt constrained economy. Since the unique equilibrium is a symmetric stochastic steady state and the agent with the good shock (who is unconstrained) determines the price of the asset, the price of a state uncontingent share is constant. As is well understood, however, the volatility of the pricing kernel of this economy is higher than that of a complete markets economy but we do not report state contingent prices. The economy with exogenous borrowing constraints and uncontingent trading displays asset price volatility together with almost twice as much volatility in consumption. The basic reason for this higher volatility is market incompleteness coupled with a higher variability of asset holdings. Indeed, the borrowing constraint binds less than 4 percent of the time. This is an interesting result for asset pricing that may be extended to more general finance scenarios. In conclusion, the liquidity constrained economy generates more asset price volatility and almost twice as much volatility in consumption. Differences in the absolute price of the asset comes from the behavior of the interest rate [cf., Kehoe and Levine (2001)].

These are economies with purely idiosyncratic risk. The introduction of aggregate risk may have considerable effects on the volatility of asset prices. The numerical solution of models with aggregate risk has proven quite challenging but can be readily integrated into our computational method.

5   International Risk Sharing

A growing literature has developed to explore the performance of business cycle models under limited risk sharing because of market imperfections. As documented in various papers [e.g., Backus, Kehoe and Kydland (1992)] standard versions of the neoclassical growth model cannot account for certain co-movements of macroeconomic aggregates. We now show that our reliable algorithm can naturally be applied to the computation of two-country models with real and financial frictions.

5.1   Economic Environment

We just outline an extended version of the economy of Kehoe and Perri (2002) in which we include shocks on preferences and taxes. Consider a two-country model with a representative household in each country. There is a unique aggregate good. Total factor productivity (TFP) of each country is affected by a vector of shocks z that follow a Markov chain. There is a constant returns to scale technology. Labor and capital stocks cannot be moved across countries, but limited international borrowing is possible. Assets include physical capital and bonds.

The representative household of country $i = 1, 2$ has preferences over stochastic sequences of consumption and labor given by the utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u^i\left(c_t^i, l_t^i, z_t\right)\right]. \tag{13}$$

Function $u^i(\cdot, \cdot, z_t) : \mathbb{R}^2 \to \mathbb{R}$ is increasing in $c^i \geq 0$ and decreasing in $l^i \in [0, 1]$, strictly concave, and twice continuously differentiable. Stochastic consumption plans $\left(c_t^i\right)_{t \geq 0}$ are financed by commodity endowments, after-tax capital returns, labor income, and lump-sum transfers. These values are expressed in terms of the single good, which is taken as the numeraire commodity of the system at each date-event. For a given rental rate r_t^i and wage w_t^i in country i, the representative household offers $k_t^i(z^{t-1}) \geq 0$ units of capital accumulated from the previous period, and supplies $l_t^i(z^t)$ units of labor.

One-period bonds can be traded at all times. Let $b^i(z^t, \xi^l_{t+1}(z^t))$ denote bond holdings of agent i, where $\xi^l_{t+1}(z^t)$ is a representative element of a given partition of the possible successors $z^{t+1}|z^t$. Hence, $\cup_l \xi^l_{t+1}(z^t)$ equals the set of all $z^{t+1}|z^t$, and $\xi^{l'}_{t+1}(z^t) \cap \xi^l_{t+1}(z^t) = 0$ whenever $l' \neq l$. A bond is a promise to deliver 1 unit of the consumption good whenever $z^{t+1} \in \xi^l_{t+1}(z^t)$, and 0 otherwise. This specification allows for a full set of state contingent bonds if $\xi^l_{t+1}(z^t)$ is a unique element for each l. An uncontingent bond pays one unit of the good for any possible future state. Let $q(z^t, \xi^l_{t+1}(z^t))$ be the price of a bond issued at z^t.

The representative household of country i is subject to the following sequence of budget constraints:

$$c^i_t\left(z^t\right) + k^i_{t+1}\left(z^t\right) + b^i(z^t, \xi^l_{t+1}(z^t))q(z^t, \xi^l_{t+1}(z^t)) = w^i_t\left(z^t\right) l^i_t\left(z^t\right) + \quad (14)$$

$$(1 - \tau^i_k(K^i))r^i_t\left(z^t\right) k^i_t\left(z^{t-1}\right) + (1 - \delta) k^i_t\left(z^{t-1}\right) + e^i_t\left(z^t\right) + b^i(z^{t-1}, \xi^l_t(z^{t-1})) + T^i_t\left(z^t\right),$$

$$\text{for all } z^t, \ t \geq 0, \ \text{given } k^i_0.$$

Endowments $e^i_t(z^t)$ are strictly positive and depend on the current realization z_t. Capital income is taxed according to function τ_k, which may depend on the aggregate capital stock, K^i_t, or some other state variables. This tax function is assumed to be positive, continuous, and bounded away from 1. Tax revenues are rebated back to the representative consumer as lump-sum transfers $T^i_t\left(z^t\right) = \tau^i_k(K^i)r^i_t\left(z^t\right) K^i_t\left(z^t\right)$.

As in the preceding section we consider two scenarios. In the debt constrained economy consumers have a complete menu of contingent bonds. There are therefore complete financial markets, but debt repudiation entails permanent exclusion from financial markets. Then, to prevent default as an equilibrium outcome the following individually rational debt constraint must always be satisfied.

$$E_{z^t} \sum_{\tau=t}^{\infty} \left(\beta^i\right)^\tau u^i\left(c^i_\tau, l^i_\tau, z_\tau\right) \geq V^{i,aut}(K^i_{t-1}(z^t), z^t), \text{ for all } t \geq 0. \quad (15)$$

Here, $V^{i,aut}$ is the expected discounted utility value for autarky as a result of zero bond

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trading for country i at all dates after z^t. Hence, $K_{t-1}^i(z^t)$ is the average level of physical capital of country i starting at node z^t.

In the liquidity constrained economy, households can trade quantities $b^i(z^t)$ of a single uncontingent bond that yields one unit of the commodity for all states, subject to the following exogenous constraint:

$$b^i(z^t) \geq -\Omega^i, \tag{16}$$

where Ω^i is some positive number.

In each country $i = 1, 2$, the production sector is made up of a continuum of identical units that have access to a constant returns to scale technology in individual factors. Thus, without loss of generality we shall focus on the problem of a representative firm. After observing the current shock z the firm rents K^i units of capital and hires L^i units of labor. The total quantity produced of the single aggregate good is given by a production function $A_t^i F\left(K_t^i, L_t^i\right)$, where A_t^i is a TFP index and $F\left(K_t^i, L_t^i\right)$ is the direct contribution of the firm's inputs to the production of the aggregate good. At every date-event z^t, factors of production are demanded by the firm to the point in which the marginal productivity of capital equals the rental rate r_t^i and the marginal productivity of labor equals the wage w_t^i. We shall maintain the following standard conditions on production function F. Let $D_1 F\left(K, L\right)$ be the derivative of F with respect to K.

Assumption 5.1 $F : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing, concave, continuous, and linearly homogeneous. This function is continuously differentiable at each interior point (K, L); moreover,* $\lim_{K \to \infty} D_1 F\left(K, L\right) = 0$ *for all $L > 0$.*

5.2 Competitive Equilibrium

Definition 5.1 *: A SCE is a tax function $\tau_k^i(K)$, and a collection of vectors*
$\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2}, q(z^t, \xi_{t+1}^l(z^t))\right)_{t \geq 0}$
that satisfy the following conditions:

(i) Constrained-utility maximization: For $i = 1, 2$ the sequence $\{c_t^i, l_t^i, k_{t+1}^i, b_t^i\}_{t \geq 0}$ solves the maximization problem for the objective (13) subject to the sequence of budget constraints (14), as well as constraint (15) for the debt constrained economy, and constraint (16) for the liquidity constrained economy.

(ii) Market clearing in the goods, capital, labor, and bond markets.

We are just extending the definition of SCE of Kehoe and Perri (2002) for an international economy with taxes. Note that in this economy international borrowing allows for imports of the aggregate good produced abroad – available for consumption and investment – but the representative firm can only hire local inputs – capital and labor. There does not seem to be a general proof of existence of competitive equilibria for infinite-horizon economies with distortions. We are aware of a related contribution by Jones and Manuelli (1999), but their analysis is not directly applicable to models with incomplete markets or externalities. Hence, the appendix outlines a proof of the following result.

Proposition 5.2 *A SCE exists.*

5.3 Bounds on Equilibrium Allocations and Prices

The appendix shows existence of positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $\{k_{t+1}^i(z^t))\}_{t \geq 0}$ if $K^{max} \geq \sum_{i=1}^{2} k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^{2} k_{t+1}^i(z^t) \geq K^{min}$ for all z^t. Hence, in what follows the domain of aggregate capital will be restricted to the interval $[K^{min}, K^{max}]$. We also show that every equilibrium sequence of factor prices $\{r_t^i(z^t), w_t^i(z^t)\}_{t \geq 0}$ is bounded.

To implement operator B, we need to bound the equilibrium shadow values of investment. For this purpose, it is convenient to use the following dynamic programming argument. We define an auxiliary value function of an individual sequential optimization problem. For a given sequence of factor and bond prices and aggregate capital

$(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = \{r_t(z^t), w_t(z^t), q_t(z^t), K_{t+1}(z^t)\}_{t \geq 0}$, let

$$J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) = \max E \sum_{t=0}^{\infty} \beta^t u^i(c_t(z^t), l_t(z^t), z_t)$$

subject to the sequence of budget constraints (14), as well as constraint (15) for the debt constrained economy, and constraint (16) for the liquidity constrained economy, for given initial conditions k_0^i, b_0^i. That is, $J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is the maximum utility attained for initial k_0^i, b_0^i, over an expected future sequence of equilibrium prices and tax rebates.

For every bounded sequence $(\mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$, the value function $J^i(k_0^i, z_0, b_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is well defined, and continuous. Moreover, mapping $J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is increasing, concave, and differentiable with respect to k_0^i and b_0^i [cf. Rincon-Zapatero and Santos (2009)]. Let $D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ be the partial derivative of function $J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ with respect to (k_0, b_0). Then, $D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ varies continuously with $(k_0^i, b_0^i, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$. The next result readily follows from these regularity properties of the value function.

Proposition 5.3 *For all SCE*
$\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2}, \right)_{t \geq 0}$ *with* $K^{max} \geq \sum_{i=1}^{2} k_0^i(z^0) \geq K^{min}$, *there is a constant vector* $\widehat{\gamma} = (\gamma, \gamma)$ *for* $\gamma > 0$ *such that*
$0 \leq D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \widehat{\gamma}$ *for all* z^t.

Observe that these bounds apply to the following types of utility functions: (i) Both function $u(\cdot, \cdot, z)$ and its derivative are bounded; (ii) function $u(\cdot, \cdot, z)$ is bounded, and its derivative function is unbounded; and (iii) both function $u(\cdot, \cdot, z)$ and its derivative are unbounded. Phelan and Stacchetti (2001) deal with case (i) and Krebs (2004) and Kubler and Schmedders (2003) consider utility functions of type (iii). We provide a uniform

method of proof that covers all three cases, as well as production functions with bounded and unbounded derivatives, and exogenous and endogenous constraints. As a matter of fact, Proposition 5.3 fills an important gap in the literature for production economies with heterogeneous consumers and market frictions, since no general results are available on upper and lower bounds for equilibrium allocations and prices.

For any initial distribution of capital $k_0 = (k_0^1, k_0^2)$, bonds $b_0 = (b_0^1, b_0^2)$ and a given shock z_0, we define the Markov equilibrium correspondence as

$$V^*(k_0, b_0, z_0) = \left\{ \begin{array}{c} \{D_{k,b}J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))\}_{i=1,2} : \\ \text{There is a SCE} \end{array} \right\}. \quad (17)$$

Hence, the set $V^*(k_0, b_0, z_0)$ contains all current equilibrium shadow values of investment returns m_0^i, for every household i.

Corollary 5.4 *Correspondence V^* is non-empty, compact-valued, and upper semicontinuous.*

This corollary is a straightforward consequence of Propositions 5.2 and 5.3. These bounds insure that our operator B maps compact sets into compact sets [cf., Assumption 2.1]. The construction of B follows the same steps of the preceding section.

5.4 Quantitative Experiments

We now explore the quantitative implications of the above two financial scenarios. For comparison purposes we will also report results for the model with complete markets which can be solved using standard dynamic programming techniques.

We assume a one-period utility with stochastic shock $\nu^i(z)$ given by

$$u^i(c, l, z) = \nu^i(z) \frac{\left[c^\eta (1-l)^{1-\eta}\right]^{1-\sigma}}{1-\sigma}, \quad (18)$$

and a Cobb-Douglas production function

$$AF(K, L) = AK^\alpha (L)^{1-\alpha}. \quad (19)$$

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We also consider the following standard parameter values: $\alpha = 0.36, \eta = 0.36$, and $\sigma = 2$. From quarterly data, we let $\beta = 0.99$ and $\delta = 0.025$. We consider a discrete VAR process for the technology shocks with four possible states: $(A^1 = 0.95613, A^2 = 0.95613)$, $(A^1 = 0.95613, A^2 = 1.04480)$, $(A^1 = 1.04480, A^2 = 0.95613)$, $(A^1 = 1.04480, A^2 = 1.04480)$. These states evolve according to the transition matrix

$$\pi = \begin{bmatrix} 0.83022 & 0.07849 & 0.07803 & 0.01326 \\ 0.10821 & 0.77567 & 0.00865 & 0.10747 \\ 0.10971 & 0.00793 & 0.77629 & 0.10607 \\ 0.01354 & 0.07934 & 0.07960 & 0.82752 \end{bmatrix}.$$

We use our method to compute SCE of this two-country model with exogenous and endogenous borrowing constraints. In both scenarios we find that the equilibrium correspondence converges to a function (up to numerical accuracy of 10^{-6}), which indicates that the SCE is unique for given initial conditions. This is the only model of the paper where computational time is a substantial issue. The basic form of our algorithm is fairly easy to implement: It only requires to search for (x, z, m, p) so that the conditions of operator B are to be satisfied. As this process of search is independent across states, it is straightforward to use parallel computing. In terms of running times, as in most algorithms the choice of initial guess matters greatly. The initial guess we employed was the solution of a similar economy but with complete markets and no distortions, which can easily be secured with a standard dynamic programming algorithm. Our grid considers 51 equally spaced points for K and 501 points for m for each country $i = 1, 2$. We ran our C++ MPI code using an IBM Server 1350 Cluster, with 50 Xeon 2.3GHZ processors. The average time per iteration of operator B was 24 minutes. The program took 94 iterations to converge to a desired level of accuracy.

Table 3 reports the simulated moments for the complete markets economy, the liquidity constrained economy, and the debt constrained economy. The resulting simulated sample moments are in line with those reported in Kehoe and Perri (2002) who use a slightly different calibration and a different computational method. Only the debt constrained

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economy offers a chance of generating reasonable correlations. In the first three scenarios, preferences are non stochastic ($\nu(z) = 1$), and there are no taxes ($\tau = 0$). The last column of Table 3 reports a slightly different experiment for the liquidity constrained economy with stochastic preferences and taxes. We assume that $\nu^i = 1.05$ if $A^i > 1$, and $\nu^i = 0.95$ if $A^i \leq 1$. Hence, the representative household is more optimistic (or more willing to consume) in the event of a good productivity shock. Also, $\tau^i = .30$ if $A^i > 1$, and $\tau^i = .25$ if $A^i \leq 1$. Hence, taxes are procyclical. With respect to the debt constrained economy, this last calibration improves the bilateral correlations of investment and labor, but does not do as well for the correlations of consumption c and GDP.

	Data	*complete markets*	*liquidity constrained*	*debt constrained*	*preferences/tax shocks*
Bilateral correlations					
c	0.32	0.8003	-0.8767	0.2264	-0.36
GDP	0.51	-0.5947	-0.7568	0.0170	-0.28
$Investment$	0.29	-0.9117	-0.9953	0.6037	0.41
$labor$	0.43	-0.9341	-0.8714	-0.1062	0.19

Table 3: Statistical properties of the economies with complete markets, and with exogenous or endogenous constraints.

In summary, in this section we apply our reliable algorithm to a two-country general equilibrium model with several real and financial frictions: Incomplete markets, exogenous and endogenous constraints, preference shocks, and taxes. We establish bounds for equilibrium allocations and prices as a key condition for the numerical implementation of our algorithm. Our model simulations broadly confirm the findings of Kehoe and Perri (2002):

Endogenous debt constraints seem instrumental to fix some international business cycles anomalies. We here obtain a related result with pro-cyclical preference shocks and taxation to improve the cross-country correlation of capital and labor. Our computational method can accommodate some other extensions (e.g., time-to-build, adjustment costs), or can be applied to related models of international investment [Bai and Zhang (2010)].

6   Stochastic OLG Economies

OLG models have become quite relevant in the analysis of several macro issues, such as the funding of social security, the optimal profile of savings and investment over the life cycle, the effects of various fiscal and monetary policies, and the evolution of future interest rates and asset prices under current demographic trends.[5] As already stressed, there are no known convergent procedures for the computation of sequential competitive equilibria in OLG models even for frictionless economies with complete financial markets. Our approach delivers a reliable, computable algorithm for the solution of competitive equilibria in a general class of OLG models. As shown below, the application of standard numerical methods that build on the existence of a continuous policy function is not adequate for the computation of these economies since a continuous Markov equilibrium may not exist or there could be a vast multiplicity of equilibria.

6.1   Economic Environment

Time is discrete $t = 0, 1, 2 \cdots$. The exogenous shock z_t follows a Markov process with support $Z = \{z_1, z_2\}$. At each date, a new generation made up of 2 agents appears in the economy. A generation lives for 2 periods. Let (i, z^t) denote an agent of type $i = 1, 2$ born at date-event $z^t = (z_0, z_1, \cdots, z_t)$. At each date, there are 2 perishable commodities available for consumption. Let good 1 be the numeraire commodity, and p the relative price

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[5]For instance, see Conesa and Krueger (1999), Geanakoplos, Magill and Quinzii (2003), Gourinchas and Parker (2002), Imrohoroglu, Imrohoroglu, and Joines (1995), Storesletten, Telmer and Yaron (2004), and Ventura (1999).

of good 2. There are two assets in this economy. The first asset is a one-period risk-free bond trading at price $q^b(z^t)$. A Lucas tree is also available trading at price $q^s(z^t)$. The tree generates a random stream of dividends $d(z_t)$. Let $(\theta^{b,i,z^t}, \theta^{s,i,z^t})$ be a pair of bond and share holdings of agent (i, z^t). Shares cannot be sold short: $\theta^{s,i,z^t} \geq 0$. Each individual faces the following budget constraint:

$$p(z^t) \cdot c^{i,z^t}(z^t) + \theta^{b,i,z^t}(z^t)q^b(z^t) + \theta^{s,i,z^t}(z^t)q^s(z^t) \leq p(z^t) \cdot e^{i,z^t}(z_t) \qquad (20)$$

$$p(z^{t+1}) \cdot c^{i,z^t}(z^{t+1}) \leq \theta^{b,i,z^t}(z^t) + \theta^{s,i,z^t}(z^t)[d(z_{t+1}) + q^s(z^{t+1})] + p(z^{t+1}) \cdot e^{i,z^t}(z^{t+1}) \text{ all } z^{t+1}|z^t. \qquad (21)$$

The utility function U^i is separable over consumption of different dates:

$$U^i\left(c^{i,z^t}; z^t, z^{t+1}\right) = u^i\left(c^{i,z^t}\left(z^t\right), z_t\right) + \beta \sum_{z^{t+1} \in \mathbf{Z}} v^i\left(c^{i,z^t}\left(z^{t+1}\right), z_{t+1}\right) \pi\left(z^{t+1}|z^t\right). \qquad (22)$$

Assumption 6.1 *For each $z \in Z$ the one-period utility functions $u^i(\cdot, z), v^i(\cdot, z) : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ are increasing, strictly concave, and continuous. These functions are also continuously differentiable at every interior point $c > 0$.*

6.2 Competitive Equilibrium

As before, a SCE is a collection of vectors $\left\{\left(c^{i,z^t}\left(z^t\right), c^{i,z^t}\left(z^{t+1}\right), \theta^{i,z^t}\left(z^t\right)\right)_{i=1}^I, p(z^t), q\left(z^t\right)\right\}_{t \geq 0}$ such that each consumption-savings plan $\left\{c^{i,z^t}(z^t), c^{i,z^t}(z^{t+1}), \theta^{i,z^t}(z^t)\right\}$ solves the constrained-utility maximization of the agent, and goods and assets markets clear.

Note that in this economy the aggregate commodity endowment is bounded by a portfolio-trading plan [Santos and Woodford (1997)], and hence asset pricing bubbles cannot exist in a SCE. Therefore, equilibrium asset prices must be bounded at each date. It follows that the existence of a SCE can be established by standard methods [e.g., Balasko and Shell (1980), and Schmachtenberg (1988)].

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Then, we define the Markov equilibrium correspondence $V^* : \Theta \times Z \to \mathbb{R}^{JI}_{++}$ as follows:

$$
V^* (\theta_0, z_0) = \left\{ \begin{array}{c} \left(\cdots , (q_0^j (z_0) + d_0^j (z_0)) D_c v^i (c^{i,z^0} (z_0), z_0), \cdots \right) : \\ \left\{ \left(c^{i,z^t} (z^t), c^{i,z^t} (z^{t+1}|z^t), \theta^{i,z^t} (z^{t+1}) \right)_{i=1}^I, p(z^t), q(z^t) \right\}_{t \geq 0} \text{ is a SCE} \end{array} \right\}.
$$

(23)

From the above results on the existence of SCE for OLG economies, we obtain

Proposition 6.1 *Correspondence V^* is nonempty, compact-valued, and upper semicontinuous.*

6.3 Lack of recursive equilibria over the natural state space

We consider first the model specification of Kubler and Polemarchakis (2004) where the real asset is not available. These authors show that no competitive equilibria admits a Markov equilibrium representation over the natural state space. Citanna and Siconolfi (2010) establish *generic* existence of this Markovian property of equilibrium under the additional assumption that the number of agents is sufficiently large. Of course, for computational reasons many economies of practical interest contain a limited number of agents which are given as primitives of the model. As in our previous example the recursive representation of equilibrium in Citanna and Siconolfi (2010) is not necessarily continuous.

The intertemporal objective of agent of type 1 is given by

$$
-\frac{1024}{\left(c_1^{1,z^t} \right)^4} + E_{z_{t+1}|z^t} \left[-\frac{1024}{\left(c_1^{1,z^t} (z_{t+1}) \right)^4} - \frac{1}{\left(c_2^{1,z^t} (z_{t+1}) \right)^4} \right]
$$

while that of agent of type 2 is given by

$$
-\frac{1}{\left(c_1^{2,z^t} \right)^4} + E_{z_{t+1}|z^t} \left[-\frac{1}{\left(c_1^{2,z^t} (z_{t+1}) \right)^4} - \frac{1024}{\left(c_2^{2,z^t} (z_{t+1}) \right)^4} \right].
$$

Each individual receives a random endowment of good 1 in their first period of life. Specifically, $e_1^{1,z^t} (z_t) = 10.4$, $e_1^{2,z^t} (z_t) = 2.6$ if $z_t = z_1$, and $e_1^{1,z^t} (z_t) = 8.6313$, $e_1^{2,z^t} (z_t) = 4.3687$ if

$z_t = z_2$. Endowments during the second period of life are deterministic and include positive amounts of both goods. Namely, $e^{1,z^t}\left(z^{t+1}\right) = (12,1)$ and $e^{2,z^t}\left(z^{t+1}\right) = (1,12)$.

Kubler and Polemarchakis show that equilibrium bond holdings turn out to be equal to zero in the two states. Hence, to determine consumption when old we must know the realization of the endowment when young.[6] At any state history z^{t-1} with $z_{t-1} = z_1$, and for any possible value of the shock today $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (10.4, 2.6)$, $\left(c_1^{2,z^{t-1}}\left(z^t\right), c_2^{2,z^{t-1}}\left(z^t\right)\right) = (2.6, 10.4)$, and $p = 1$. Likewise, for any state history z^{t-1} with $z_{t-1} = z_2$, ad for any possible value of the shock today $\left(c_1^{1,z^{t-1}}\left(z^t\right), c_2^{1,z^{t-1}}\left(z^t\right)\right) = (8.4, 1.4)$, $\left(c_1^{2,z^{t-1}}\left(z^t\right), c_2^{2,z^{t-1}}\left(z^t\right)\right) = (4.6, 11.6)$, and $p = 7.9$.

Finally, we computed this model using a projection method with collocation and piecewise linear interpolation. This collocation method approximates the Euler equation to search for a continuous equilibrium function over the natural state space – albeit the model does not admit a continuous Markov equilibrium. The computed equilibrium function delivers reasonable Euler equation residuals (i.e., of the order of 10^{-5}). A researcher may again be led to believe that this function is a good approximate solution; however, the computed prices and allocations are quite different from those of the exact equilibrium.

$Statistics$	q	$c_1^{1,z^{t-1}}$	$c_2^{1,z^{t-1}}$	$c_1^{2,z^{t-1}}$	$c_2^{2,z^{t-1}}$
$(\mu_{true}, \mu_{projection})$	(1.0,0.6)	(9.7,9.7)	(2.0,1.7)	(3.6,3.8)	(11.0,11.3)
$(\sigma^2_{true}, \sigma^2_{projection})$	(0.0,0.05)	(1.0,0.2)	(0.36,0.81)	(1.0,0.09)	(0.36,0.08)

Table 1: Statistical properties of the true equilibrium vs. an equilibrium generated by the projection method. Statistics: Mean μ and variance σ^2.

In summary, in equilibrium the relative price of good 2 is a function of the endowment in the previous period. The price is not signaled by the natural state space – current shocks and portfolio holdings – as there is no trade among generations. The equilibrium relative price of good 2 can take on two values and asset holdings take on one single value. This

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[6]Because of an indeterminacy problem of the Euler equation, we can approximate the equilibrium of this more limited economy by letting the stock of trees go to zero.

observation may explain the large differences in Table 1 between the simulated moments generated by the true and computed solutions. Indeed, the computed function by the projection method takes on a single value for the relative price of good 2 midway between the two possible equilibrium values.

6.4 A Monetary Model

Let us now consider a simplified version of the OLG model with money taken from Benhabib and Day (1982) and Grandmont (1985). This model is useful for computation because it can be solved with arbitrary accuracy. Hence, it is possible to compare the true solution of the model with other numerical approximations. Extensions to a stochastic environment are easy to handle with our algorithm but may become problematic for other algorithms.

Each individual receives an endowment e_1 of the perishable good when young and e_2 when old. There is a single asset, money, that pays zero dividends at each given period. The initial old agent is endowed with the existing money supply M. Let P_t be the price level at time t. An agent born in period t chooses consumption c_{1t} when young, c_{2t+1} when old, and money holdings M_t to solve the constrained optimization problem

$$\max u\left(c_{1t}\right) + \beta v\left(c_{2t+1}\right)$$

subject to

$$c_{1t} + \frac{M_t}{P_t} = e_1,$$

$$c_{2t+1} = e_2 + \frac{M_t}{P_{t+1}}.$$

A SEC for this economy is a sequence of prices $(P_t)_{t\geq 0}$, and sequences of consumption and money holdings $\{c_{1t}, c_{2t+1}, M_t\}_{t\geq 0}$ such that an individual solves the budget-constrained utility maximization problem and markets clear. A SEC can be characterized by the following first-order condition:

$$\frac{1}{P_t} u'\left(e_1 - \frac{M}{P_t}\right) = \frac{1}{P_{t+1}}\beta v'\left(e_2 + \frac{M}{P_{t+1}}\right).$$

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Let $b_t = M/P_t$ be real money balances at t. Then,

$$b_t u' (e_1 - b_t) = b_{t+1} \beta v' (e_2 + b_{t+1}).$$

It follows that all competitive equilibria can be generated by an offer curve in the (b_t, b_{t+1}) space. A simple recursive equilibrium would be described by a function $b_{t+1} = g(b_t)$.

Our numerical experiments introduce the following parameterization:

$$u(c) = c^{0.45}, \ v(c) = -\frac{1}{7} c^{-7}, \ \beta = 0.8, \ M = 1, e_1 = 2, e_2 = 2^{6/7} - 2^{1/7}.$$

For this simple example, the offer curve is backward bending (see Figure 2). Hence, the equilibrium correspondence is multi-valued. Then, standard methods – based on the computation of a continuous equilibrium function $b_{t+1} = g(b_t)$ – may portray a partial view of the equilibrium dynamics. There is a unique stationary solution at about $b^* = 0.4181$, which is the point of crossing of the offer curve with the 45-degree line.
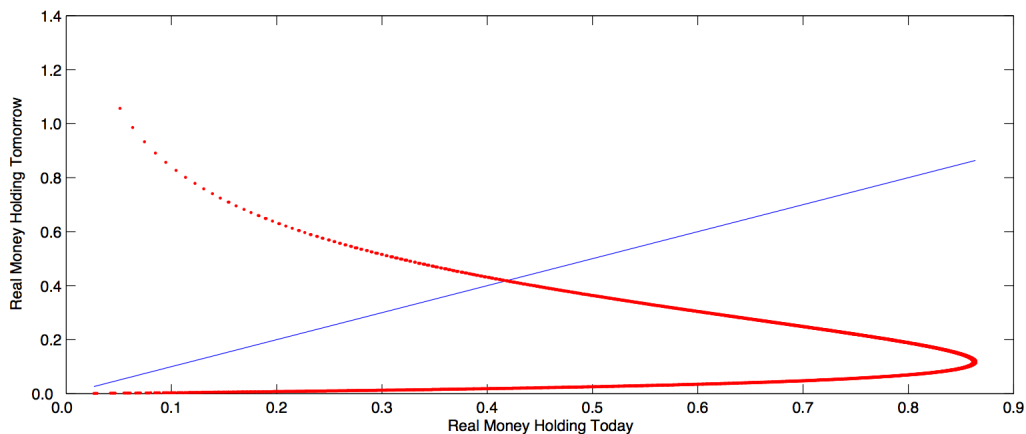

Figure 2: Offer curve.

Implementation of our Algorithm.

Following section 7.2 the implementation our numerical algorithm is fairly straightforward. In fact, since the shadow values of the marginal returns to investment lie in a compact

set, we can follow the same computational steps of previous sections. For this example, we find that the policy correspondence and time series from our method generate an Euler equation residual of order 10^{-6}. Actually, the solution obtained with our algorithm is indistinguishable from the "exact" solution.

Comparison with other Computational Algorithms

A common practice in OLG models is to start the search with an equilibrium guess function $b' = \widehat{g}(b)$, and then iterate over the temporary equilibrium conditions. We applied this procedure to our model. Depending on the initial guess, we find that either the upper or the lower arm of the offer curve would emerge as a fixed point. This strong dependence on initial conditions is a rather undesirable feature of this computational method. In particular, if we only consider the lower arm of the actual equilibrium correspondence then all competitive equilibria converge to autarchy. Indeed, the unique absorbing steady state associated with the lower arm of the equilibrium correspondence involves zero monetary holdings. Hence, even in the deterministic version, we need a global approximation of the equilibrium correspondence to analyze the various predictions of the model. As shown in Figure 3, the approximate equilibrium correspondence has a cyclical equilibrium in which real money holdings oscillate between 0.8529 and 0.0953. It is also known that the model has a three-period cycle. But if we iterate over the upper arm of the offer curve, we find that money holdings converge monotonically to $\frac{\bar{M}}{p} = 0.4181$ (as illustrated by the dashed lines of Figure 3). As a matter of fact, the upper arm is monotonic, and can at most have cycles of period two, whereas the model generates equilibrium cycles of various periodicities.
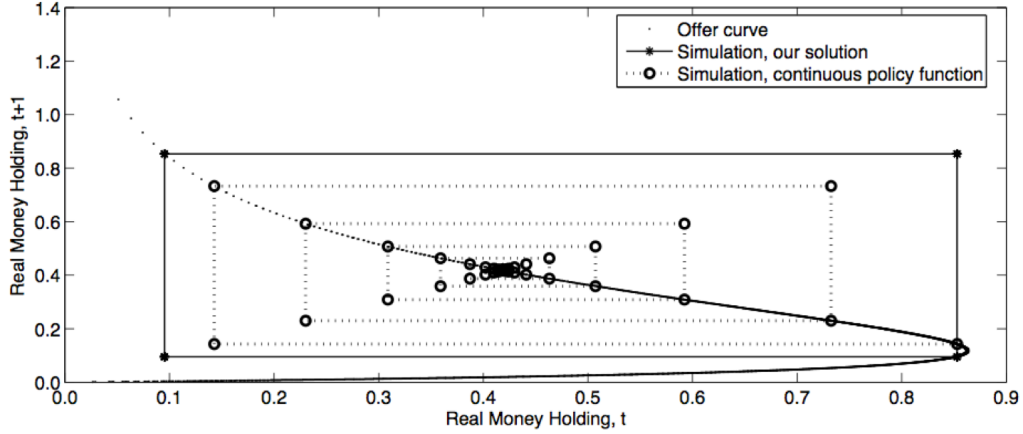
Figure 3: Equilibrium cycles.

In conclusion, for OLG economies, standard computational methods based on iteration of continuous functions do not guarantee convergence to an equilibrium solution, and may miss some important properties of the equilibrium dynamics. In these economies it seems pertinent to compute the set of *all* SCE.

7   Concluding Remarks

This paper provides a theoretical framework for the computation and simulation of dynamic competitive-markets economies in which the welfare theorems may fail to hold because of market frictions or the existence of an infinite number of generations. We have applied these methods to various macroeconomic models with heterogeneous agents, incomplete financial markets, exogenous and endogenous borrowing constraints, taxes, and money. Our numerical algorithm was especially helpful for the simulation of an international business cycle model and for an OLG economy. These models are not amenable to computation by social planning problems because of the existence of real and financial frictions. They are not amenable to computation by projection methods with continuous equilibrium functions because a continuous recursive representation of equilibrium may not exist. And they are

not amenable to computation by perturbation methods because the ergodic region may be quite large: Agents accumulate assets to accommodate idiosyncratic and aggregate risks. The application of both projection and perturbation methods may be rather cumbersome in problems with market frictions, and exogenous and endogenous constraints.

We employed our algorithm to contrast the quantitative implications of models with exogenous vis-a-vis endogenous borrowing constraints for risk sharing problems. In the asset pricing framework of Kehoe and Levine (2001), the economy with exogenous borrowing limits displayed higher volatility in asset prices and consumption than the economy with endogenous borrowing constraints. The economy with endogenous constraints had enough securities so that markets were otherwise complete, and equilibrium had a simple symmetric form with constant prices. Exogenous constraints did not allow for such simple equilibria, which in turn generated higher volatility of asset prices and almost twice as much volatility of individual consumption. We then considered an expanded version of Kehoe and Perri (2002) with cross-country risk sharing in a full-blown model with capital accumulation and standard production functions. Here, endogenous borrowing constraints improve the predictions of the model relative to the data, in line with the findings of Kehoe and Perri (2002)]. As these authors point out, models with additional frictions may be necessary to make the theory fully compatible with the data. We showed that preference shocks and taxes may improve the cross-country correlation of investment and labor. All of these results add to a large body of literature in which nonoptimal dynamic equilibrium models are critical to improve on the quantitative predictions of representative-agent models. Reliable methods for the numerical approximation of these economies should be instrumental for making further progress. For instance, Feng (2011) generalizes our methods to study the quantitative implications of time inconsistency for optimal taxation problems in realistically calibrated model economies.

Our quantitative analysis ends with the study of two different versions of standard overlapping generation economies, which illustrate some of the pitfalls that may result from the

use of methods without solid foundations. The first test case is the economy of Kubler and Polemarchakis (2004) where competitive equilibria do not admit a recursive representation on the natural state space. We apply a standard projection algorithm and easily find a fixed point with reasonable Euler equation residuals (of order 10^{-5}). Nevertheless, this approximation is shown to yield substantially biased predictions relative to the true equilibrium and dynamic behavior of the model. The second test case is a classical monetary economy with a backward bending offer curve. A standard projection method could only capture one of the equilibrium arms. The selected arm depends on initial conditions. Cycles and some rich dynamic behavior of the model are not generated by this standard computation method. Finally, this example illustrates that randomizations of the equilibrium correspondence may result in unacceptable approximation errors.

For optimal economies, sequential competitive equilibria are generated by a continuous policy function that is the fixed-point solution of a contractive operator. Continuity of the policy function allows for various methods of approximation and functional interpolation, and is essential to validate laws of large numbers for the simulated paths. Differentiability and contractive properties are useful for the derivation of error bounds that can guide the computation process. For economies with distortions or with an infinite number of generations a continuous Markov equilibrium may not exist. We establish a general result on the existence of a Markovian equilibrium solution in a suitably expanded space of state variables, and provide upper and lower bounds for equilibrium allocations and prices. We construct a numerical algorithm that has desirable approximation properties regarding global convergence of the iterative procedure and numerical accuracy. In particular, we show that as we refine our approximations, the numerical solution converges uniformly to the exact solution of the model.

There are three main properties of our algorithm that should be of interest for quantitative work in this area. First, the existence of a Markovian competitive equilibrium is obtained in an enlarged space of state variables with shadow values of investment and

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continuation utility values. Our choice of the shadow values of assets returns is dictated by computational considerations. The continuation utility values allows us to deal with endogenous borrowing constraints. Hence, this set of variables is a minimal addition to the state space to restore existence of a Markovian equilibrium and with the property that the added variables enter linearly into the Euler equation. Second, the algorithm iterates in a space of candidate equilibrium sets – rather than in a space of functions. Iteration over candidate equilibrium sets computes the whole set of competitive equilibria, and insures convergence to the fixed-point solution – even if Markov equilibria are not continuous. For the OLG economy of section 6 it was pertinent to have a global understanding of the equilibrium dynamics. And third, the algorithm provides a reliable method for model simulation. There are many papers following the lead of Kydland and Prescott (1980) and Abreu, Pierce and Stacchetti (1990) on recursive characterizations of equilibria, but none of these contributions is concerned with the numerical implementation. Hence, our results should provide a useful benchmark for the construction of other algorithms.

Of course, our methods must face some computational challenges. Iteration over sets is computationally much more costly than iteration over functions. Therefore, the expansion of the state space along with iteration over sets should certainly be manifested into an additional computational burden. Since the many computational tasks in our algorithm can be decentralized, the development of high-performance, parallel computing will certainly make our methods more attractive.

Our general accuracy results imply uniform convergence to the true solution but lack error bounds. This lack of accuracy should be expected because our models cannot be restated as optimization programs, and miss some common concavity, differentiability, and contractive properties. In terms of numerical implementation, the innovative techniques for error estimation proposed by Judd, Yeltekin, and Conklin (2003) require convexity of the approximations, but convexity cannot be imposed on our algorithm because it may arbitrarily expand the set of equilibria. It is therefore of great interest to extend these

innovative techniques on estimation of error bounds to the present context. Finally, the numerical implementation of our algorithm starts with an initial correspondence of potential equilibrium values. In most numerical work it is imperative to bound the ergodic region in order to minimize computer costs. This task, however, may become much more delicate for non-optimal economies. In our applications above we have developed various procedures to bound equilibrium allocations and prices by ruling asset pricing bubbles and by defining a value function for each household over future equilibrium paths. This value function is convenient because it can embed exogenous and endogenous borrowing constraints, and real and financial frictions. Hence, market imperfections do not have to be dealt with explicitly in establishing bounds for equilibrium allocations and prices. Our techniques should certainly be valuable to establish equilibrium bounds in related models with heterogeneous agents and market distortions.

8 Appendix

In this Appendix we prove some key results formally stated in sections 2 and 3. For convenience, we also offer a proof of existence for the model of section 5, and establish equilibrium bounds. All other claims in the paper may rely on similar arguments.

Proof of Theorem 2.1: Let $\widehat{V}_0 \supset V^*$ and $\widehat{P}_0 \supset P^*$, and $(\widehat{V}_i, \widehat{A}_i) = B(\widehat{V}_{i-1}, \widehat{A}_{i-1})$ for all $i \geq 1$. To insure monotone convergence, let us now redefine these sets as $V_n = \cup_{i=n}^{\infty} \widehat{V}_i$ and $P_n = \cup_{i=n}^{\infty} \widehat{P}_i$ for all $n \geq 0$. Then $(V_n, P_n) = B(V_{n-1}, P_{n-1})$ and $(V_n, P_n) \subset (P_{n-1}, P_{n-1})$ for all $n \geq 1$. It follows that the sequence $\{(V_n, P_n)\}$ must converge to a set (V^U, P^U). Further, $(V^U, P^U) = \cap_{n=1}^{\infty}(V_n, P_n)$. Therefore, $(V^U, P^U) = B(V^U, P^U)$. We next prove that $(V^U, P^U) = (V^*, P^*)$. Indeed, by the monotonicity of operator B we get that $(V^*, P^*) \subset (V^U, P^U)$; also, $(V^U, P^U) \subset (V^*, P^*)$ since every fixed point conforms an equilibrium – given that the transversality conditions are trivially satisfied in this model. To complete the proof of the theorem, just note that $(V^U, P^U) \subset (V^*, P^*) \subset (V_n, P_n)$ for all $n \geq 1$. Since we have already established that $(V_n, P_n) \to (V^U, P^U)$, we get that $V_n \to V^*$ and $P_n \to P^*$. It is clear from these arguments that (V^*, P^*) is the largest fixed-point of operator B.

Proof of Theorems 3.1: (i) Obvious. Operator $B^{h,N}$ is monotone, $(V_0, P_0) \supseteq (V^*, P^*)$ and $B^{h,N}(V^*, P^*) \supset (V^*, P^*)$.

(ii) Proof follows similar arguments as in proof of Theorem 2.1. Actually, $(V_n^{h,N}, P_n^{h,N}) \supset (V^{*,h,N}, P^{*,h,N})$, and our discretized procedure allows for a finite number of set-values. Hence, pointwise convergence implies uniform convergence.

(iii) Note that operator $B^{h,N}$ converges to B as $h \to 0$ and $N \to \infty$. Since $(V^*, P^*) \subset (V^{*,h,N}, P^{*,h,N})$, we get that $(V^{*,h,N}, P^{*,h,N}) \to (V^*, P^*)$ as $h \to 0$ and $N \to \infty$.

Proof of Theorem 3.2: The proof goes by contradiction. Since S is a compact set every sequence must have a convergent subsequence. Hence, if the assertion of Theorem 3.2 is not true there is a converging sequence $\{(x_n^{h,N}, z, y_n^{h,N}, m_n^{h,N}, p_n^{h,N})\} \rightarrow (x, z, y, m, p)$ with $(x_n^{h,N}, z, y_n^{h,N}, m_n^{h,N}, p_n^{h,N}) \in graph(V_n^{h,N}, P_n^{h,N})$ and $d(graph(V_n^{h,N}, P_n^{h,N}), graph(V^*, P^*)) > \epsilon$. As $h \rightarrow 0$, and N and $n \rightarrow \infty$, we must have [cf. Theorem 2.1] that $(x, z, y, m, p) \in graph(V^*, P^*)$. However, this is in contradiction with the previous assertion that $d(graph(V_n^{h,N}, P_n^{h,N}), graph(V^*, P^*)) > \epsilon$ for all N, h, n.

Proof of Proposition 5.2: The existence of a SCE can be established by approximating the infinite-horizon economy by a sequence of finite economies. This is the strategy followed by Jones and Manuelli (1999), but their proof does not apply to sequential competitive economies. Of course, the hardest part is to provide upper bounds for equilibrium quantities over all the finite-horizon economies. These bounds follow from Proposition 5.3 below.

Hence, following Jones and Manuelli (1999), we consider the following steps for the proof of a SCE: (i) Existence of an equilibrium for a finite horizon economy. This result is covered by the general proofs of existence of competitive equilibria for economies with taxes, externalities, and incomplete markets [Arrow and Hahn (1971), Levine and Zame (1996), Mantel (1975), and Shafer and Sonneschein (1976)]. (ii) Uniform bounds for equilibrium allocations and prices of finite-horizon economies. As already pointed out, these bounds are established in Proposition 5.3 below. (iii) Existence of SEC as a limit point of finite equilibria. The preceding steps (i) and (ii) guarantee that there is a collection of vectors $\left(\{c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)), K_{t+1}^i(z^t), L_t^i(z^t), r_t^i(z^t), w_t^i(z^t)\}_{i=1,2}, q(z^t, \xi_{t+1}^l(z^t))\right)_{t \geq 0}$ that can be obtained as limits of equilibria of finite economies. It is obvious that for such limiting solution the market clearing conditions must be satisfied at each z^t, and that one period-profits are maximized. Moreover, for each agent i the limiting allocation $(c_t^i(z^t), l_t^i(z^t), k_{t+1}^i(z^t), b^i(z^t, \xi_{t+1}^l(z^t)))$ must satisfy the sequence of budget constraints (14), as well as the exogenous or endogenous constraints. This allocation is optimal since the

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discounted utility function is continuous in the product topology over the set of feasible consumption/leisure plans $\left(c_t^i\left(z^t\right), 1 - l_t^i\left(z^t\right)\right)_{t \geq 0}$ which are preferred to the endowment allocation $\left(e_t^i\left(z_t\right), 1\right)_{t \geq 0}$. This is because feasible consumption plans $\left(c_t^i\left(z^t\right)\right)_{t \geq 0}$ are bounded above, and the endowment process $\left(e_t^i\left(z_t\right)\right)_{t \geq 0}$ is bounded below by a positive quantity and the endowment of leisure is always equal to one.

Proof of Proposition 5.3: We first show that there are positive constants K^{max} and K^{min} such that for every equilibrium sequence of physical capital vectors $\left(k_{t+1}^i(z^t)\right)_{t \geq 0}$ if $K^{max} \geq \sum_{i=1}^2 k_0^i(z^0) \geq K^{min}$ then $K^{max} \geq \sum_{i=1}^2 k_{t+1}^i(z^t) \geq K^{min}$ for all z^t. The existence of K^{max} follows directly from Assumption 5.1, since the marginal productivity of capital converges to zero as K goes to ∞ for every fixed $0 \leq L \leq 1$. Also, it obvious that $K^{min} \geq 0$.

We now claim that there are constants r^{max} and w^{max} such that for every equilibrium sequence of factor prices $\left(r_t^i\left(z^t\right), w_t^i\left(z^t\right)\right)_{t \geq 0}$ we have $0 \leq r_t^i(z^t) \leq r^{max}$ and $0 \leq w_t^i(z^t) \leq w^{max}$ for all z^t. The existence of w^{max} follows from continuity properties of the utility function. The household is endowed with one unit of labor. Hence, if the wage is arbitrarily high it would be optimal to consume a large amount of consumption by giving up a small quantity of leisure. If along an equilibrium path we have that r_t^i is arbitrarily large, then k_t^i must go to zero. From the Euler equation, consumption c_t^i must also go to zero. But this is not possible under either exogenous or endogenous constraints, as $e_t^i > 0$ is bounded below by a positive quantity, and in the debt constrained economy the household can switch to autarky. Moreover, using a simple arbitrage argument, we have that q_t is also bounded. Hence, the value function $J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is well defined. As already pointed out the derivative $D_{k,b}J^i(\cdot, \cdot, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ is continuous in $(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$.[7] Moreover, by a simple notational change it follows from (14) that function J^i can be rewritten as $J^i(a_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{q}(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$

---

[7]Note that if b_0^i is a large negative number then the value function is well defined, but the agent will switch to autarky. In the autarky region the derivative of J^i with respect to b_0^i is zero. Hence, at the point of switching to autarky, the derivative of J^i will not be continuous but the differential is a compact correspondence.

$\mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$, where $a_0^i = e_0^i(z_0) + (1 - \tau)\, r_0 k_0^i$. Then we can conclude that

$0 \leq D_{k,b} J^i(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0)) \leq \widehat{\gamma}$, since $e_0^i(z_0)$ is bounded below by a positive number, and all feasible vectors $(k_0^i, b_0^i, z_0, \mathbf{r}_0(\mathbf{z}_0), \mathbf{w}_0(\mathbf{z}_0), \mathbf{K}(\mathbf{z}_0))$ lie in a compact set.

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