Complexity and Efficiency in Repeated Games with Negotiation

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Abstract

This paper considers the "negotiation game" (Busch and Wen [4]) which combines the features of two-person alternating offers bargaining and repeated games. Despite the forces of bargaining, the negotiation game in general admits a large number of equilibria some of which involve delay in agreement and inefficiency. In order to isolate equilibria in this game, we explicitly consider the complexity of implementing a strategy, introduced in the literature on repeated games played by automata. It turns out that when the players have a preference for less complex strategies (even at the margin) only *efficient* equilibria survive. Thus, complexity and bargaining in tandem may offer an explanation for co-operation in repeated games.

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1 Introduction

Busch and Wen [4], henceforth referred to as BW, analyze the following game. In each period, two players bargain - in Rubinstein's alternating-offers protocol - over the distri-

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bution of a fixed and commonly known periodic surplus. If an offer is accepted, the game ends and each player gets his share of the surplus according to the agreement at every period thereafter. After any rejection, but before the game moves to the next period, the players engage in a normal form game to determine their payoffs for the period. The Pareto frontier of the disagreement game is contained in the bargaining frontier. We shall refer to this game as the "negotiation game".

The negotiation game generally admits a large number (continuum) of subgame-perfect equilibria, as summarized by BW in a result that has a same flavor as the Folk theorem in repeated games. BW show that only the structure of the disagreement game, and not the gains from the agreement, determines multiplicity and the range of payoffs that can be sustained as credible equilibria. Moreover, they show that, provided the players are sufficiently patient, the negotiation game in general has a continuum of equilibria which will involve delay in agreement (even perpetual disagreement) and inefficiency.

The negotiation game and its equilibria can be interpreted from two alternative viewpoints. Naturally, we can think of the game as a standard alternating-offers bargaining game with endogenous disagreement payoffs.[1] In fact, Fernandez and Glazer [7] (and also Haller and Holden [11]) derive much of the insights in a well-known application of the game along this bargaining interpretation. They consider the standoff between a union and a firm. During a contract renewal process, a union and a firm renegotiate over the distribution of a periodic revenue, but a disagreement puts them in a strategic situation. After rejecting the firm's wage offer or having their own offer rejected by the firm, the union can forego the status quo wage for one period and strike before a counter-offer is made next period. (The firm is inactive in the disagreement game.) Fernandez and Glazer's characterization of subgame-perfect equilibria in this specific setting contains many of the salient features of the equilibria in the general game, and thus, offers an explanation as to why such socially wasteful activities as strikes may take place even in a situation where the agents are completely rational and have complete information.

The alternative viewpoint focuses on the repeated game aspect of the negotiation game (and this is the interpretation we want to emphasize in the paper). Real world repeated interactions are often accompanied by negotiations which can lead to mutual agreement. While equilibria in standard repeated games are usually given the interpretation of implicit, self-enforcing agreements, the situations depicted by the negotiation game are associated with the possibility of explicit contracts that can bind the players to a particular set of outcomes. For example, we observe firms engaged in a repeated horizontal or vertical relationship negotiating over a long-term contract, or even a merger. Similarly, countries involved in international trade often attempt to settle an agreement

[1] The issue of endogenous disagreement payoffs in a bargaining situation goes back at least to Nash [14] who considers the problem in a co-operative framework.

that enforces fixed quotas and tariffs.

The Folk theorem gives economic theorists little hope of making any predictions in repeated interactions. However, as the aforementioned examples suggest, it seems that negotiation is often a salient feature of real world repeated interactions, presumably to enforce co-operation and efficient outcomes. Can bargaining be used to isolate equilibria in repeated games? Unfortunately, the contributions of BW and others demonstrate that Folk theorem type results with a large number of equilibria (involving delay and inefficiency) may persist even when the players are endowed with an opportunity at the beginning of each period to settle on an efficient outcome once and for all.

In order to enrich this line of enquiry, on the issue of how bargaining can be used to select (efficient) equilibria in repeated games, this paper departs from the standard rationality paradigm and introduces the notion of *complexity* into the negotiation game. Our central message is that the equilibrium strategies supporting inefficient outcomes in this game are unnecessarily too complex to implement. Bargaining combined with the players' preference for less complex strategies (even at the margin) select only *efficient* outcomes in the repeated game (at least if the players are sufficiently patient).

There are many different ways of defining the complexity of a strategy. In the literature on repeated games played by automata the number of states of the machine is often used as a measure of complexity (Rubinstein [19], Abreu and Rubinstein [1], Piccione [17] and Piccione and Rubinstein [18]). This is because the set of states of the machine can be regarded as a partition of possible histories. In particular, Kalai and Stanford [13] show that the counting-states measure of complexity, henceforth referred to as *state complexity*, is equivalent to looking at at the number of *continuation strategies* that the strategy induces at different histories of the game. We extend this notion of strategic complexity to the negotiation game, and facilitate the analysis by considering an equivalent "machine game".

The alternating-offers bargaining imposes an asymmetric structure on the negotiation game which is stationary only every two periods (henceforth we shall refer to every two periods as a "stage"). To account for such structural asymmetry of the game, we shall adopt machine specifications that formally distinguish between the different *roles* played by each player in a given stage. A player can be either proposer or responder. In the main specification used in the analysis, a machine consists of two "sub-machines", each playing a role (of a proposer or a responder) with distinct states, output and transition functions. Transition occurs at the end of each period, from a state belonging to one sub-machine to a state belonging to the other sub-machine as roles are reversed.

We first demonstrate that the result of Kalai and Stanford [13] holds for our specification of machines. The total number of states used by each sub-machine under this specification is equivalent to measuring the total number of continuation strategies that the implemented strategy induces at the beginning of each period.

The concept of Nash equilibrium is then refined to incorporate the players' preference for less complex strategies. In our choice of equilibrium notions, complexity enters a player's preferences, together with the payoffs in the underlying game, either lexicographically or as a positive fixed cost c. The larger this cost is, the more is required of complexity. We can thus interpret it as a measure of the players' "bounded rationality". We will refer to a Nash equilibrium (of the machine game) with fixed complexity cost c by NEMc and adopt the convention of using $c = 0$ (and thus NEM0) to refer to the lexicographic case. We also invoke the notion of subgame-perfection and consider the set of NEMc that are subgame-perfect, referred to as SPEMc.

The selection result is as follows. We first show that, independently of the degree of complexity cost and discount factor, if an agreement occurs in some finite period as the outcome of some NEMc then it must occur within the very first stage of the game, and moreover, the players' equilibrium strategies are stationary (history-independent). Thus, in this case any NEMc outcome is efficient in the limit as the discount factor goes to one.

We then establish the following results on the set of SPEMc profiles when the players are sufficiently patient:

1. For $c = 0$ (lexicographic preferences),

   (i) every SPEM0 of the negotiation game that induces an agreement is stationary and hence (almost) efficient;

   (ii) every SPEM0 of the negotiation game that induces perpetual disagreement is at least long-run (almost) efficient ; that is, the players must reach a finite period in which the continuation game then on is (almost) efficient.

2. For any $c > 0$, every SPEMc of the negotiation game is stationary and hence (almost) efficient.

This implies that we can draw a yet stronger set of conclusions under certain disagreement game structures (given sufficiently high discounting). For example, if every disagreement game outcome is dominated by an agreement, it is not possible to have a SPEMc (for any $c \geq 0$) involving perpetual disagreement. Thus, every equilibrium outcome in this case must reach an agreement in the first stage of the negotiation game, and hence, is stationary and (almost) efficient.

We also explore an alternative machine specification that employs more frequent transitions and hence account for finer partitions of histories and continuation strategies. This machine consists of four sub-machines; while keeping the role distinction, transition occurs twice in each period at the end of bargaining and at the end of the disagreement game. Using this machine specification, we derive a set of SPEMc results that contain much the same flavor as the corresponding results under the two sub-machine specification, but are sharper; the discount factor is now immaterial. Specifically, the results in 1 and 2 above hold for any discount factor and the efficiency property therein is no longer restricted to the limit case.[2]

Our contribution thus takes the study of complexity in repeated games a step further from the aforementioned literature in which complexity has yielded only a limited selective power. (See also Bloise [3] who shows robust examples of two-player repeated games in which the set of Nash equilibria with complexity costs coincides with the set of individually rational payoffs.) This paper demonstrates that complexity and bargaining in tandem may offer an explanation for co-operation and efficiency in repeated games.

There have been extensive and wide-ranging approaches at restricting the unwieldily large set of equilibria resulting from the Folk theorem. Among these attempts, one literature motivates the notion of bargaining and negotiation by invoking the idea that punishments that are inefficient may be vulnerable to renegotiation and hence not credible. This literature suggests a solution concept based on *renegotiation-proofness*.[3]. There are two differences between this line of research and the negotiation game. First, the former takes a "black box" approach to renegotiation. Unlike in the negotiation game, the process of (re)negotiation is not explicitly modelled; rather, the renegotiation arguments are embedded in the additional restrictions imposed on an equilibrium. Secondly, the renegotiation literature does not allow for binding agreements.

We also want to mention several recent papers that have rekindled the issue of complexity in equilibrium selection, and in particular, demonstrated that complexity drives efficient outcomes in some specific games. Chatterjee and Sabourian [5][6] consider the multi-person Rubinstein bargaining game, and Sabourian [20], Gale and Sabourian [9][10] consider market games with matching and bargaining. (These papers are also interested in other issues such as the uniqueness of the equilibrium set and the competitive nature of equilibria in the case of the market games.) In contrast to the present paper, however, these papers build upon a different notion of strategic complexity. They consider the complexity of response rules *within* a period. A simple response rule according to their notion of *response complexity* uses only the information available in the current period and not the history of play up to the period. Introducing this (together with state complexity in Sabourian [20]) delivers the efficiency results in those games.

The paper is organized as follows. In the following section, we describe the negotiation game and BW's main results. We then introduce the notion of complexity in terms of strategies and machines. The machine game will be described. Section 4 presents the main analysis and results. We then run the analogous results with an alternative, more elaborate machine specification in Section 5. We finally conclude. The appendices contain some relegated proofs and also explains that the equilibrium concept we use closely parallels that of Abreu and Rubinstein [1].

[2]These results are also associated with a slightly different (stricter) notion of a stationary strategy.

[3]See Pearce [16] and Chapter 5.4 of Fudenberg and Tirole [8] for a survey.

2 The Negotiation Game

Let us formally describe the negotiation game, as defined by BW. There are two players indexed by $i = 1, 2$. In the alternating-offers protocol, each player in turn proposes a partition of a *periodic* surplus whose value is normalized to one. If the offer is accepted, the game ends and the players share the surplus accordingly at every period indefinitely thereafter. If the offer is rejected, the players engage in a one-shot (normal form) game, called the "disagreement game", before moving onto the next period in which the rejecting player makes a counter-offer.

We index the (potentially infinite) time periods by $t = 1, 2, \ldots$ and adopt the convention that player 1 makes offers in odd periods and player 2 makes offers in even periods. Let $\triangle^2 \equiv \{x = (x_1, x_2) \mid \sum_i x_i = 1\}$ be a partition of the unit periodic surplus. A period then refers to a single offer $x \in \triangle^2$ by one player, a response made by the other player - acceptance "Y" or rejection "N" - and the play of the disagreement game if the response is rejection. The common discount factor is $\delta \in (0, 1)$.

The disagreement game is a two-player normal form game, defined as

$$G = \{A_1, A_2, u_1(\cdot), u_2(\cdot)\}$$

where A_i is the set of player i's actions and $u_i(\cdot) : A_1 \times A_2 \to R$ is his payoff function in the disagreement game. We shall denote the set of action profiles in G by $A = A_1 \times A_2$ with its element indexed by a.[4] Let $u(\cdot) = (u_1(\cdot), u_2(\cdot))$ be the vector of payoff functions, and assume that it is bounded. Each player's minmax payoff in G is normalized to zero. Also, we assume that for any $a \in A$

$$u_1(a) + u_2(a) \leq 1 \ .$$

Therefore, any agreement weakly dominates disagreement in terms of total payoffs. Thus, the bargaining offers the players an opportunity to settle on an efficient outcome once and for all.

Two types of outcome paths are possible in the negotiation game; one in which an agreement occurs in a finite time, and one in which disagreement continues perpetually. Let T denote the end of the negotiation game and $a^t \in A$ the disagreement game outcome (action profile) in period $t < T$. If $T = \infty$, we mean an outcome path in which agreement is never reached. Player i's (discounted) *average* payoff in this case is equal to

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(a^t) \ .$$

[4] The normal form may involve sequential moves. In this case, A_i will represent player i's set of strategies, rather than actions, in the disagreement game.

If $T < \infty$, denote the agreed partition in T by $z = (z_1, z_2) \in \triangle^2$. Player i's payoff from such an outcome path amounts to

$$(1-\delta) \sum_{t=1}^{T-1} \delta^{t-1} u_i \left(a^t\right) + \delta^{T-1} z_i \ .$$

The negotiation game is stationary only every two periods (beginning with an odd one) or "stage". In specifying the players' strategies (and later machines), we shall formally distinguish between the different *roles* played by each player in each stage game. He can be either the proposer (p) or the responder (r) in a given period. We shall index a player's role by k. The role distinction provides a natural framework to capture the structural asymmetry that the alternating offers bargaining imposes on the repeated (disagreement) game.

In order to define a strategy, we first need to introduce some further notations. We shall use the following notational convention. Whenever superscripts/subscripts i and j both appear in the same exposition, we mean $i, j = 1, 2$ and $i \neq j$. Similarly, whenever we use superscripts/subscripts k and l together, we mean $k, l = p, r$ and $k \neq l$.

We shall denote by e a history of outcomes in a period of the negotiation game, and this belongs to the set

$$E = \{(x^i, Y), (x^i, N, a)\}_{x^i \in \triangle^2, a \in A, i=1,2}$$

where the superscript i represents the identity of the proposer in the period. Let e^t be the outcome of the period t.

We also need notation to represent information available to a player *within* a period when it is his turn to take an action given his role. To this end, we define a "partial history" (information within a period) d as an element in the following set

$$D = \{\emptyset, (x^i), (x^i, N)\}_{x^i \in \triangle^2, i=1,2}$$

For example, the null set $\emptyset$ here refers to the beginning of a period at which the proposer has to make an offer; (x^i, N) represents a partial history of an offer x^i by player i followed by the other player's rejection.

Also, let us define

$$D_{ik} \equiv \{d \in D \mid \text{it is } i\text{'s turn to play in role } k \text{ after } d \text{ in the period}\}$$

Thus, we have

$$D_{ip} = \{\emptyset, (x^i, N)\}_{x^i \in \triangle^2}$$

and

$$D_{ir} = \{(x^j), (x^j, N)\}_{x^j \in \triangle^2} \ .$$

We denote the set of actions available to player i in the negotiation game by

$$C_i \equiv \triangle^2 \cup Y \cup N \cup A_i \ .$$

Let us denote by $C_{ik}(d)$ the set of actions available to player i given his role k and a corresponding partial history $d \in D_{ik}$. Thus, we have

$$C_{ip}(d) = \begin{cases} \Delta^2 & \text{if } d = \emptyset \\ A_i & \text{if } d = (x^i, N) \end{cases}$$

and

$$C_{ir}(d) = \begin{cases} \{Y, N\} & \text{if } d = x^j \\ A_i & \text{if } d = (x^j, N) \end{cases}$$

Let

$$H^t = \underbrace{E \times \cdots \times E}_{t \text{ times}}$$

be the set of all possible histories of outcomes over t periods in the negotiation game, excluding those that have resulted in an agreement. The initial history is empty (trivial) and denoted by $H^1 = \emptyset$. Let $H^\infty \equiv \cup_{t=1}^\infty H^t$ denote the set of all possible finite period histories.

For the analysis, we shall divide H^∞ into two smaller subsets according to the different roles that the players play in each stage. Let H^t_{ik} be the set of all possible histories over t periods after which player i's role is k. Notice that $H^t_{ik} = H^t_{jl}$. Also, let $H^\infty_{ik} = \cup_{t=1}^\infty H^t_{ik}$. Thus, $H^\infty = H^\infty_{ip} \cup H^\infty_{ir}$ $(i = 1, 2)$.

A strategy for player i is then a function

$$f_i : (H^\infty_{ip} \times D_{ip}) \cup (H^\infty_{ir} \times D_{ir}) \to C_i$$

such that for any $(h, d) \in H^\infty_{ik} \times D_{ik}$ we have $f_i(h, d) \in C_{ik}(d)$. The set of all strategies for player i is denoted by F_i. Also, we shall denote by F^t_i the set of player i's strategies in the negotiation game starting with role distribution given in period t. Thus, if t is odd, $F^t_i = F_i$.

We can define a *stationary* (or history-independent) strategy in the following way.

Definition 1 *A strategy f_i is stationary if and only if $f_i(h, d) = f_i(h', d)$ $\forall h, h' \in H^\infty_{ik}$ and $\forall d \in D_{ik}$ for $k = p, r$. A strategy profile $f = (f_i, f_{-i})$ is stationary if f_i is stationary for all i.*

The behavior induced by such a strategy may depend on the partial history within the current period but not on the history of the game up to the period. For instance, if a strategy f_i is stationary, then it must be such that

$$f_i(h, x, N) = f_i(h', x, N) \ \ \forall h, h' \in H^\infty \ \forall x \in \triangle^2$$

but we may have

$$f_i(h,x,N) \neq f_i(h,x',N) \ \text{ for } x \neq x' \ .$$

Notice also that a stationary strategy profile always induces the same outcome in each stage of the game.

In the spirit of the Folk theorem, BW characterize the set of all subgame-perfect equilibrium (SPE) payoffs of the negotiation game. BW, to this end, compute the lower bound of each player's SPE payoff in the negotiation game with discount factor δ.

Define

$$w_j = \max_{a \in A} \left\{ u_j(a) - \left[\max_{a_i' \in A_i} u_i(a_i', a_j) - u_i(a) \right] \right\}$$

which BW assume to be well-defined. Note also that $w_i \leq 1$ given the assumption that $u_i(a) \leq 1 \ \forall a \in A$, and $w_i \geq 0$ if G has at least one Nash equilibrium.

Then, the infimum of player i's SPE payoffs in the negotiation game beginning with his offer (given δ) is not less than

$$\underline{v}_i(\delta) = \frac{1 - w_j}{1 + \delta}$$

while the infimum of the other player's SPE payoffs in the same game is not less than

$$\underline{v}_j(\delta) = \frac{\delta(1 - w_i)}{1 + \delta} \ .$$

BW show that, provided the players are sufficiently patient, there exists a SPE of the negotiation game (beginning with i's offer) in which the players obtain these lower bounds.

Define the limit of these infima as δ goes to unity by

$$\underline{v}_i = \frac{1 - w_j}{2} \ \text{ and } \ \underline{v}_j = \frac{1 - w_i}{2} \ .$$

We can now formally state the BW's main Theorem.

BW Theorem *For any payoff vector (v_1, v_2) of the negotiation game such that $v_1 > \underline{v}_1$ and $v_2 > \underline{v}_2$, $\exists \ \bar{\delta} \in (0,1)$ such that $\forall \delta \in (\bar{\delta}, 1)$, (v_1, v_2) is a SPE payoff vector of the negotiation game with discount factor δ.*

The forces of bargaining thus restrict the set of feasible equilibrium payoffs in the negotiation game compared to the set of individually rational payoffs in the disagreement (repeated) game. But, if $\underline{v}_1 + \underline{v}_2 < 1$, the negotiation game has many inefficient subgame-perfect equilibria much in the way the Folk theorem characterizes the repeated game (even when the disagreement game payoffs are always uniformly small relative to agreement). The negotiation game has a unique (efficient) SPE payoff if $\underline{v}_1 + \underline{v}_2 = 1$ or $w_1 = w_2 = 0$ which implies that any Nash equilibrium payoff vector of the disagreement game has to coincide with its minmax point.

3 Complexity, Machines, and Equilibrium

There are many alternative ways to think of the "complexity" of a strategy in dynamic games. One natural and intuitive way to measure strategic complexity, which we shall adopt in the paper, is to consider the total number of distinct *continuation strategies* that the strategy induces at different histories (Kalai and Stanford [13]).

In a repeated game, it is natural to take the measure over all its possible subgames. In the negotiation game, each stage game consists of an extensive form game and this means that several different definitions are possible.

In the main analysis, we shall consider the set of all continuation strategies at the beginning of each period of the negotiation game. Formally, let $f_i|h$ be the continuation strategy at history $h \in H^\infty$ induced by $f_i \in F_i$. Thus,

$$f_i|h(h', d) = f_i(h, h', d) \text{ for any } (h, h', d) \in H^\infty_{ik} \times D_{ik} \text{ for any } k \ .$$

Also, let us define the set of all such continuation strategies by $F_i(f_i) = \{f_i|h : h \in H^\infty\}$. Then the cardinality of this set provides a measure of strategic complexity. Let us call it $comp(f_i)$.

The set of continuation strategies can also be divided into smaller sets according to the role specification. Define $F_{ik}(f_i) = \{f_i|h : h \in H^\infty_{ik}\}$ such that we have $F_i(f_i) = \cup_k F_{ik}(f_i)$. Complexity can then be equivalently measured by $comp(f_i) = \sum_k |F_{ik}(f_i)|$.

We can also measure complexity over finer partitions of histories and corresponding continuation strategies. We shall later show that the exact definition of complexity is going to play some role in shaping the precise details of the results.

In dynamic games any strategy can be *implemented* by an automaton or a "machine" (we shall clarify this statement below in our negotiation game context). Moreover, Kalai and Stanford [13] show that in repeated games the above notion of complexity of a strategy (the number of continuation strategies) is equivalent to counting the number of states of the (smallest) automaton that implements the strategy. Thus, one could equivalently describe any result either in terms of underlying strategies and their complexity ($comp(\cdot)$) or in terms of machines and their number of states.

We shall establish below that this equivalence between the two representations of strategic complexity also holds in the negotiation game. Our approach to complexity will then be facilitated in machine terms as this will provide a more economical platform to present the analysis of complexity. Each player's strategy space in the negotiation game will then be taken as the set of all machines and the players simultaneously and independently choose a single machine at the beginning of the negotiation game. This is the "machine game", a term which we shall interchangeably use with the negotiation game.

Here again, the extensive form of the stage game allows for many different machine specifications to equivalently represent a strategy. (The same is also the case in other sequential dynamic games; see Piccione and Rubinstein [18], Chatterjee and Sabourian [5][6] and Sabourian [20]). The fact that the stage game is also asymmetric across its two periods - a player switches his role in the bargaining process - adds to this issue of multiple possible machine specifications.

In this paper, we present two particular machine specifications. We choose to run the analysis first with the simpler of the two. The results are in fact sharper under the other specification, but our chosen order of analysis will serve to strengthen the expositional flow. As we shall see later, counting the number of states for these machines corresponds precisely to the manner in which we divide the histories and accordingly define the notion of complexity in terms of (continuation) strategies.

The following defines a machine that employs two “sub-machines”.

Definition 2 (Two sub-machine (2SM) specification) *For each player i, a machine (automaton), $M_i = \{M_{ip}, M_{ir}\}$, consists of two sub-machines $M_{ip} = (Q_{ip}, q^1_{ip}, \lambda_{ip}, \mu_{ip})$ and $M_{ir} = (Q_{ir}, q^1_{ir}, \lambda_{ir}, \mu_{ir})$ where for any $k, l = p, r$*

$$\begin{aligned}
&Q_{ik} \textit{ is the set of states};\\
&q^1_{ik} \textit{ is the initial state belonging to } Q_{ik};\\
&\lambda_{ik} : Q_{ik} \times D_{ik} \to C_i \textit{ is the output function such that}\\
&\qquad \lambda_{ik}(q_{ik}, d) \in C_{ik}(d),\ \forall q_{ik} \in Q_{ik} \textit{ and } \forall d \in D_{ik};\ \textit{and}\\
&\mu_{ik} : Q_{ik} \times E \to Q_{il} \textit{ is the transition function.}
\end{aligned}$$

Let Φ_i denote the set of player i's machines in the machine game. We also let Φ^t_i denote the set of player i's machines in the machine game starting with role distribution given in period t. Thus, if t is odd, $\Phi^t_i = \Phi_i$.

Each sub-machine in the above definition of a machine consists of a set of *distinct* states, an initial state and an output function enabling a player to play a given role. Transitions take place at the end of each period from a state in one sub-machine to a state in the other sub-machine as roles are reversed each period.

We shall assume that each sub-machine has to have at least one state.[5] But notice that we do not assume finiteness of a machine; each sub-machine may have any arbitrary (possibly infinite) number of states. This is in contrast to Abreu and Rubinstein [1] and others who consider finite automata. Assuming that machines can only have a finite number of states is itself a restriction on the players' choice of strategies.

Notice also that the initial state of the sub-machine that operates in the second period is in fact redundant because the first state used by this sub-machine depends on

[5] We could also define a distinct terminal state for each sub-machine. This is immaterial. We are assuming that if an offer is accepted by the responder, M_i enters the terminal state of the relevant sub-machine and shuts off.

the transition taking place between the first two periods of the game. Nevertheless, we endow both sub-machines with an initial state for expositional ease.

Let us now formally state what we mean by a machine implementing a strategy in the negotiation game. Consider a machine $M_i = \{M_{ip}, M_{ir}\} \in \Phi_i$ where, for $k = p, r$, $M_{ik} = (Q_{ik}, q^1_{ik}, \lambda_{ik}, \mu_{ik})$. For every $k = p, r$ and for any $h \in H^\infty_{ik}$, denote the state at history h by $q_i(h) \in Q_{ik}$. Formally if $h = (e^1, \ldots, e^{t-1})$ then $q_i(h) = q^t_i$ where for any $0 < \tau \leq t$, q^τ_i is defined inductively by

$$q^1_i = \begin{cases} q^1_{ik} & \text{if } i \text{ is in role } k \text{ initially at } t = 1 \\ q^1_{il} & \text{if } i \text{ is in role } l \text{ initially at } t = 1 \end{cases}$$

and for $\tau > 0$

$$q^\tau_i \equiv \begin{cases} \mu_{il}(q^{\tau-1}_i, e^{\tau-1}) & \text{if } i \text{ is in role } k \text{ at } \tau \\ \mu_{ik}(q^{\tau-1}_i, e^{\tau-1}) & \text{if } i \text{ is in role } l \text{ at } \tau \end{cases}$$

Definition 3 *M_i implements f_i if $\forall k$, $\forall h \in H^\infty_{ik}$ and $d \in D_{ik}$ we have*

$$\lambda_{ik}(q_i(h), d) = f_i(h, d)$$

where $q_i(h)$ is defined inductively as above.

Clearly, any strategy f_i can be implemented by a machine. For example, consider a machine $M_i = \{M_{ip}, M_{ir}\}$ (where, for $k = p, r$, $M_{ik} = (Q_{ik}, q^1_{ik}, \lambda_{ik}, \mu_{ik})$) which is such that $\forall k$, $\forall h \in H^\infty_{ik}$, $\forall d \in D_{ik}$ and $\forall e \in E$

$$Q_{ik} = H^\infty_{ik},\ \lambda_{ik}(h, d) = f_i(h, d),\ \mu_{ik}(h, e) = (h, e),$$

and $q^1_{ik'} = \emptyset$ where k' is i's role in period $t = 1$. Evidently, this machine implements f_i.

The following defines a *minimal* machine.

Definition 4 *A machine is minimal if and only if each of its sub-machines has exactly one state.*

A minimal machine implements the same actions in every period regardless of the history of the preceding periods, provided that the partial history within the current period (given a role) is the same. Thus, it corresponds to a stationary strategy as in Definition 1. We shall henceforth refer to a minimal machine (profile) interchangeably as a stationary machine (profile).

We have thus far established that machines and strategies are equivalent in the negotiation game. Now let us formally show that *comp*(f_i) is equivalent to counting the total number of states of the machine that implements the strategy f_i. It must be stressed here that the exact specification of a machine is important in qualifying this statement. In fact, it is precisely to establish this equivalence that we have chosen the above machine specification.[6]

Let $\|M_i\| = \sum_k |Q_{ik}|$ be the total number of states (or size) of machine M_i. We now establish that the cardinality of the set of continuation strategies that a strategy induces at the beginning of each period of the negotiation game corresponds to the size of the smallest machine implementing the strategy.

Proposition 1 *For every $f_i \in F_i$ let $\mathcal{M}_i(f_i) \subseteq \Phi_i$ be the set of machines that implement f_i. Also, let $\bar{M}_i = \{\bar{M}_{ip}, \bar{M}_{ir}\}$ be a smallest machine that implements f_i; that is*

$$\|\bar{M}_i\| \in \{M_i \in \mathcal{M}_i(f_i) \mid \|M_i\| \leq \|M_i' \mid \forall M_i' \in \mathcal{M}_i(f_i)\} \ .$$

Then, we have $|F_{ik}(f_i)| = \|\bar{M}_{ik}\|$ for any $k = p, r$ and thus $\|\bar{M}_i\| = comp(f_i)$.

Proof. The proof is a direct application of the proof of Theorem 1 in Kalai and Stanford [13]. For ease of exposition, it is relegated to Appendix A. $\|$

Given this result, we now formally define the notion of complexity in terms of machines, as adopted in the literature on repeated games played by automata *à la* Rubinstein [19] and Abreu and Rubinstein [1].[7]

Definition 5 (State complexity) *A machine M_i' is more complex than another machine M_i if $\|M_i'\| > \|M_i\|$.*

To wrap up the description of the machine game, let us fix some more notational conventions. Let $M = (M_1, M_2)$ be a machine profile. Then, if M is the chosen machine profile, $T(M)$ refers to the end of the negotiation game; $z(M) \in \triangle^2$ is the agreement offer if $T(M) < \infty$; $a^t(M) \in A$ is the disagreement game outcome in period $t < T(M)$; and $q_i^t(M)$ is the state of player i's machine appearing in period $t \leq T(M)$ (the state of the active sub-machine in period t).

[6] Since in defining $comp(f_i)$ we consider continuation strategies *at the beginning of each period*, we need transitions between the states of a machine to take place *between periods* in accordance with the continuation points chosen. It is also important that each sub-machine uses its own distinct set of states.

[7] We also draw attention to the work of Binmore, Piccione, and Samuelson [2] who propose another notion of complexity similar to state complexity considered in this paper and others. According to their "collapsing state condition", an automaton M^1 is less complex than another automaton M^2 if the same implementation can be obtained by consolidating a collection of states belonging to M^2 into a single state in M^1. It will not be difficult to see that our results will also hold under this notion of complexity.

Similarly, we denote by $\pi_i^t(M)$ player i's (discounted) average *continuation payoff* at period $t \leq T(M)$ when the machine profile M is chosen. Thus,

$$\pi_i^t(M) = \begin{cases} (1-\delta)\sum_{\tau=t}^{\infty} \delta^{\tau-t} u_i(a^\tau(M)) & \text{if } T(M) = \infty \\ \\ (1-\delta)\sum_{\tau=t}^{T-1} \delta^{\tau-t} u_i\left(a^\tau(M)\right) + \delta^{T-t} z_i(M) & \text{if } t < T(M) < \infty \\ \\ z_i(M) & \text{if } t = T(M) < \infty \end{cases}$$

We shall use the abbreviation $\pi_i(M) = \pi_i^1(M)$.

For ease of exposition, the argument in M will sometimes be dropped when we refer to one of these variables that depends on the particular machine profile. For example, π_i^t will refer to $\pi_i^t(M)$. Unless otherwise stated, the abbreviated variable will refer to the machine profile in the *claim*.

We now introduce an equilibrium notion that captures the players' preference for less complex machines. (The following definition can be written equivalently in terms of underlying strategies.) There are several ways of refining Nash equilibrium with complexity. We choose an equilibrium notion in which complexity enters a player's preferences *after* the payoffs and with a (non-negative) fixed cost c.[8]

To facilitate this concept, we first define the notion of ϵ-best response.

Definition 6 *For any $\epsilon \geq 0$, a machine M_i is a ϵ-best response to M_{-i} if, $\forall M_i'$,*

$$\pi_i(M_i, M_{-i}) + \epsilon \geq \pi_i(M_i', M_{-i}) \ .$$

If a machine is a 0-best response, then it is a best response in the conventional sense.

Using this, we define a Nash equilibrium of the machine game with complexity cost c.

Definition 7 *A machine profile $M^* = (M_1^*, M_2^*)$ constitutes a Nash equilibrium of the machine game with complexity cost $c \geq 0$ (NEMc) if $\forall i$*

(i) M_i^* is a best response to M_{-i}^*;

(ii) There exists no M_i' such that M_i' is a c-best response to M_{-i}^* and $\|M_i^*\| > \|M_i'\|$.

By definition, the set of NEMc is a subset of the set of Nash equilibria in the negotiation game. The case of zero complexity cost $c = 0$ is closest to the standard equilibrium and corresponds to the case in which complexity enters players' preferences *lexicographically*. Any NEMc with a positive complexity cost $c > 0$ must also be a NEMc with

[8] Sabourian [20] employs this equilibrium notion.

$c = 0$. The magnitude of c therefore can be interpreted as a measure of how much the players care for less complex strategies, or indeed the players' *bounded rationality*.

Abreu and Rubinstein [1], henceforth referred to as AR, propose a general way of describing a player's preference ordering over machine profiles that is increasing in his payoff of the game and decreasing in the complexity of his machine. A Nash equilibrium can then be written in terms of machines that are most preferred against each other. In contrast, our equilibrium concept directly finds a subset of Nash equilibria of the underlying game that fits our complexity cost criterion (at the margin). There is, however, an analytical parallel between our choice of solution concept and that of AR because the latter must also be a Nash equilibrium of the underlying negotiation game (see Appendix B). Our complexity cost criterion can be thought of as an alternative way to embed the trade-off between payoff and complexity that underlies AR's preference ordering.

NEMc strategy profiles are not necessarily credible however. We could introduce credibility, as in Chatterjee and Sabourian [5][6], by introducing trembles into the model and considering the limit of extensive form trembling hand equilibrium (Nash equilibrium with independent trembles at each information set) with complexity cost as the trembles become small. The trembles will ensure that strategies are optimal (allowing for complexity) at every information set that occurs with a positive probability.

A more direct, and simpler, way of introducing credibility would be to consider NEMc strategy profiles that are subgame-perfect equilibria of the negotiation game without complexity cost.

Definition 8 *A machine profile $M^* = (M_1^*, M_2^*)$ constitutes a subgame-perfect equilibrium of the machine game with complexity cost $c \geq 0$ (SPEMc) if M^* is both a NEMc and a subgame-perfect equilibrium (SPE) of the negotiation game.*

Given Proposition 1, we can equivalently define these notions of equilibrium (NEMc and SPEMc) in terms of underlying strategies and the corresponding measure of complexity $comp(\cdot)$. As mentioned earlier, we prefer the machine game analysis for its expositional economy.

4 Analysis: Complexity and Efficiency

4.1 NEMc Results

4.1.1 Some Preliminary Results

We begin by laying out some Lemmas that will pave way for the main results below. These results are derived independently of the magnitudes of complexity cost and discount factor.

We first state an obvious, yet very important, implication of the complexity requirement. Suppose that there exists a state in some player's equilibrium (NEMc) machine that never appears on the equilibrium path. Unless the machine is minimal, however, this cannot be possible because this state can be "dropped" by the player to reduce complexity cost without affecting the outcome and payoff, thereby contradicting the NEMc assumption (that is, there exists another machine identical to the original equilibrium machine except that it has one less state which, given the other player's machine, will generate the same outcome/payoffs). This argument leads to the following Lemma.

Lemma 1 *Assume that $M^* = (M_1^*, M_2^*)$ is a NEMc with $c \geq 0$. Let $M_i^* = \{M_{ip}^*, M_{ir}^*\}$ where, for $k = p, r$, $M_{ik}^* = (Q_{ik}^*, q_{ik}^{1*}, \lambda_{ik}^*, \mu_{ik}^*)$.*[9] *Then, we have the following:*

(i) If $T(M^*) \geq 2$, then $\forall i$, $\forall k$ and $\forall q_i \in Q_{ik}^*$ there exists a period t such that $q_i^t(M^*) = q_i$;

(ii) If $T(M^*) \leq 2$, then $|Q_{ik}| = 1$ $\forall i$ and $\forall k$.

Remark 1 *Note that although a player can choose a machine of any size, it follows from Lemma 1 that for any NEMc profile $M^* = (M_1^*, M_2^*)$ with $c \geq 0$, M_i^* $(i = 1, 2)$ must have a countable number of states.*

Next note that, since any strategy can be implemented by a machine, it follows from its definition that any NEMc profile $M^* = (M_1^*, M_2^*)$ corresponds to a Nash equilibrium of the underlying negotiation game; thus ($\forall c \geq 0$)

$$\pi_i(M_i^*, M_j^*) = \max_{f_i \in F_i} \pi_i(f_i, M_j^*) \quad \forall i, j \tag{1}$$

where, with some abuse of notation, $\pi_i(f_i, M_j^*)$ refers to i's payoff in the game where i and j play according to f_i and M_j^* respectively.

More generally, the equilibrium machines must be best response (in terms of payoffs) *along* the equilibrium path of the negotiation game.

Lemma 2 *Assume that $M^* = (M_1^*, M_2^*)$ is a NEMc with $c \geq 0$. Then, $\forall i, j$ and $\forall \tau \leq T(M^*)$ we have*

$$\pi_i^\tau(M^*) = \max_{f_i \in F_i^\tau} \pi_i(f_i, M_j^*(q_j^\tau))$$

where $q_j^\tau \equiv q_j^\tau(M^*)$, $M_j^*(q_j^\tau) \in \Phi_j^\tau$ is the machine that is identical to M_j^* except that it starts with the sub-machine which operates in period τ with initial state q_j^τ, and again with some abuse of notation, $\pi_i(f_i, M_j^*(q_j^\tau))$ refers to i's payoff in the negotiation game that starts with role distribution given in period τ and is played by i and j according to $f_i \in F_i^\tau$ and $M_j^*(q_j^\tau)$ respectively.

[9] This will henceforth define the equilibrium machines in our claims.

Proof. See Appendix A. ‖

Now it follows that if a state belonging to a player's equilibrium machine appears twice on the outcome path then the continuation payoff of the other player must be identical at both periods.

Lemma 3 *Assume that $M^* = (M_1^*, M_2^*)$ is a NEMc with $c \geq 0$. Then, $\forall i, j$ and $\forall t, t' \leq T(M^*)$ we have the following:*

$$\text{if } \; q_j^t(M^*) = q_j^{t'}(M^*), \; \text{ then } \pi_i^t(M^*) = \pi_i^{t'}(M^*) \; .$$

Proof. This follows from Lemma 2. ‖

Using this information, we can show that if a state belonging to a player's equilibrium machine appears on the outcome path for the first time, then the state of the other player's machine in that period must also be appearing for the first time. This Lemma will provide a critical tool behind the derivation of some of the main results below.

Lemma 4 *Assume that $M^* = (M_1^*, M_2^*)$ is a NEMc with $c \geq 0$. Then, for any i and any $\tau \leq T(M^*)$ we have the following:*

$$\text{if } \; q_i^\tau(M^*) \neq q_i^t(M^*) \; \forall t < \tau, \; \text{ then } q_j^\tau(M^*) \neq q_j^t(M^*) \; \forall t < \tau \; .$$

Proof. Suppose not. So, there exists some i and some $\tau \leq T$ such that $q_i^\tau \neq q_i^t \; \forall t < \tau$ and $q_j^\tau = q_j^{\tau'}$ for some $\tau' < \tau$. Then, by Lemma 3, $\pi_i^\tau = \pi_i^{\tau'}$.

Consider player i using another machine $M_i' = \{M_{ip}', M_{ir}'\}$ where, for $k = p, r$, $M_{ik}' = (Q_{ik}', q_{ik}^{1\prime}, \lambda_{ik}', \mu_{ik}')$. This machine is identical to M_i^* (as defined before) except that:

- q_i^τ is dropped
- the transition function is such that $\mu_{ik'}'(q_i^{\tau-1}, e^{\tau-1}) = q_i^{\tau'}$ where k' is i's role in period $\tau - 1$.

To be precise, M_i' is such that

- $Q_{ik}' = Q_{ik}^* \backslash q_i^\tau$ and $Q_{il}' = Q_{il}^*$
- $q_{ik}^{1\prime} = q_{ik}^{1*}$ and $q_{il}^{1\prime} = q_{il}^{1*}$
- for every $k = p, r$, every $q_i \in Q_{ik}'$ and every $d \in D_{ik}$,

$$\lambda_{ik}'(q_i, d) = \lambda_{ik}^*(q_i, d)$$

- for every $k \neq k'$, every $q_i \in Q'_{ik}$ and every $e \in E$,
$$\mu'_{ik}(q_i, e) = \mu^*_{ik}(q_i, e)$$
and for $k = k'$, every $q_i \in Q'_{ik}$ and every $e \in E$,
$$\mu'_{ik}(q_i, e) = \begin{cases} q_i^{\tau'} & \text{if } q_i = q_i^{\tau-1} \text{ and } e = e^{\tau-1} \\ \mu^*_{ik}(q_i, e) & \text{otherwise .} \end{cases}$$

Since q_i^τ appears for the first time in period τ on the original equilibrium path, we cannot have $q_i^{\tau-1}$ and $e^{\tau-1}$ appearing together before $\tau - 1$. Thus, playing M'_i against M^*_j does not alter the outcome path up to τ.

But, since $q_j^\tau = q_j^{\tau'}$ and by the definition of M'_i, it follows that, from τ onwards, the outcome path between τ' and $\tau - 1$ will repeat itself *ad infinitum*.

Now we show that this does not change i's payoff from the machine game (given M^*_j). First, we know that

$$\begin{aligned}
\pi_i^{\tau'}(M^*_i, M^*_j) &= \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) + \delta^{\tau-\tau'} \pi_i^{\tau}(M^*_i, M^*_j) \\
&= \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) + \delta^{\tau-\tau'} \pi_i^{\tau'}(M^*_i, M^*_j) \\
&= \frac{1}{1-\delta^{\tau-\tau'}} \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) \qquad (2)
\end{aligned}$$

where the second equality follows from Lemma 3. The new machine also yields the same payoff because

$$\begin{aligned}
\pi_i^{\tau'}(M'_i, M^*_j) &= \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) + \delta^{\tau-\tau'} \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) + \ldots \\
&= \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t)(1 + \delta^{\tau-\tau'} + \delta^{2(\tau-\tau')} + \ldots) \\
&= \frac{1}{1-\delta^{\tau-\tau'}} \sum_{t=\tau'}^{\tau-1} \delta^{t-\tau'} u_i(a^t) \; . \qquad (3)
\end{aligned}$$

Since (M'_i, M^*_j) and (M^*_i, M^*_j) induce the same outcome before τ', it follows that $\pi_i(M'_i, M^*_j) = \pi_i(M^*_i, M^*_j)$. But then, since q_i^τ is dropped, $\|M^*_i\| > \|M'_i\|$. Thus, we have contradiction against NEMc.[10] $\|$

[10]Notice that this result turns on the assumption that each sub-machine uses a distinct set of states. If the sub-machines shared the states, we could not simply "drop" q_i^τ since it could be used for the other sub-machine (playing a different role) before τ'.

4.1.2 Agreement

In this sub-section, we shall show that, independently of c and δ, if an agreement occurs at some finite period as a NEMc outcome, then it must occur within the very first stage (two periods) of the negotiation game, and thus, the associated equilibrium machines (strategies) must be minimal (stationary).

To do so, we first establish that if a NEMc induces an agreement in a finite period beyond the first stage, it must be that the pair of states appearing in the final period are distinct.

Lemma 5 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc with* $c \geq 0$ *and* $T(M^*) < \infty$*. Then,* $q_i^t(M^*) \neq q_i^T(M^*)$ $\forall t < T(M^*)$ *and* $\forall i$*.*

Proof. Suppose not. So, suppose that $q_i^t = q_i^T$ for some i and some $t < T$. Let $z = (z_1, z_2) \in \triangle^2$ be the agreement at T. There are two possible cases to consider.

Case A: Player i is the proposer at T.

Define $\tau = \min\{t | q_i^t = q_i^T\}$. Then, i is the proposer in period τ offering z. By Lemma 3, $\pi_j^\tau = \pi_j^T$. Since there is an agreement on z at T, we have $\pi_j^\tau = z_j$.

Now consider player j using another machine $M_j' = \{M_{jp}', M_{jr}'\}$ where, for $k = p, r$, $M_{jk}' = (Q_{jk}', q_{jk}^{1\prime}, \lambda_{jk}', \mu_{jk}')$. This machine is identical to M_j^* except that:

- q_j^τ is dropped (i.e. $Q_{jr}' = Q_{jr}^* \backslash q_j^\tau$)
- the transition function is such that $\mu_{jp}'(q_j^{\tau-1}, e^{\tau-1}) = q_j^T$.

Since, by Lemma 4, q_j^τ (as does q_i^τ by definition) appears for the first time at τ on the original equilibrium path, this new machine (given M_i^*) generates an identical outcome path as the original machine M_j^* up to τ and then induces the agreement z at τ.

Thus, $\pi_j^\tau(M_i^*, M_j') = z_j$, and hence, $\pi_j(M_i^*, M_j') = \pi_j(M_i^*, M_j^*)$. But since q_j^τ is dropped, $\|M_j^*\| > \|M_j'\|$. This contradicts NEMc.

Case B: Player i is the responder at T.

We can show contradiction similarly to Case A above. $\|$

We are now ready to present our first major result. For any value of complexity cost, any NEMc outcome that reaches an agreement must do so in the very first stage of the negotiation game and hence the associated strategies must be stationary. The intuition is as follows. The state of each player's machine occurring in the last period must be distinct. This implies that, if the last period occurs beyond the first stage of the game, one of the players must be able to drop it and instead use another state in his (sub-)machine to condition his behavior in that period without affecting the outcome of the game. This reduces complexity cost.

Proposition 2 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc with* $c \geq 0$ *and* $T(M^*) < \infty$*. Then, (i)* $T(M^*) \leq 2$*, and (ii)* M_1^* *and* M_2^* *are minimal and hence* M^* *is stationary.*

Proof. If part (i) of the claim is true, part (ii) must be true because of Lemma 1. Let us consider part (i).

Suppose not. So, suppose that an agreement $z \in \triangle^2$ occurs at some $T \in (2, \infty)$. We know from Lemma 5 that q_1^T and q_2^T are both distinct. Now suppose that player i is the proposer at T and consider two possible cases.

Case A: $x^\tau = z$ at some $\tau < T$ where i proposes.

Consider another machine $M_i' = \{M_{ip}', M_{ir}'\}$ where, for $k = p, r$, $M_{ik}' = (Q_{ik}', q_{ik}^{1\prime}, \lambda_{ik}', \mu_{ik}')$ which is identical to M_i^* except that:

- q_i^T is dropped (i.e. $Q_{ip}' = Q_{ip}^* \backslash q_i^T$)
- the transition function is such that $\mu_{ir}'(q_i^{T-1}, e^{T-1}) = q_i^\tau$. (Note that $q_i^\tau \neq q_i^T$ since we have $T > 2$ and q_i^T is distinct by Lemma 5.)

Since $\lambda_{ip}'(q_i^\tau, \emptyset) = z$ and q_i^T appears for the first time at T on the original equilibrium path, this new machine (given M_j^*) generates an identical outcome path and payoff as the original machine M_i^*. But since q_i^T is dropped, we have $\|M_i^*\| > \|M_i'\|$. This contradicts NEMc.

Case B: $x^\tau \neq z$ $\forall \tau < T$ where i proposes.

Consider another machine $M_j' = \{M_{jp}', M_{jr}'\}$ where, for $k = p, r$, $M_{jk}' = (Q_{jk}', q_{jk}^{1\prime}, \lambda_{jk}', \mu_{jk}')$ which is identical to M_j^* except that:

- q_j^T is dropped (i.e. $Q_{jr}' = Q_{jr}^* \backslash q_j^T$)
- the transition function is such that $\mu_{jp}'(q_j^{T-1}, e^{T-1}) = q_j \neq q_j^T$ for some arbitrary but fixed $q_j \in Q_{jr}'$. (Such q_j exists since we have $T > 2$ and q_j^T is distinct by Lemma 5.)
- the output function is such that $\lambda_{jr}'(q_j, z) = Y$.

Since the offer z does not appear anywhere before T on the original equilibrium path when i proposes, the new machine (given M_i^*) does not affect the outcome and payoff. But then, q_j^T is dropped and therefore we have $\|M_j^*\| > \|M_j'\|$. This contradicts NEMc. $\|$

It immediately follows from Proposition 2 that any NEMc involving an agreement must be efficient in the limit as the discount factor goes to one.

Corollary 1 *For any* $\epsilon \in (0, 1)$, $\exists\ \bar{\delta} < 1$ *such that for any* $\delta \in (\bar{\delta}, 1)$ *and any* $c \geq 0$, *any NEMc profile* M^* *of the negotiation game with discount factor* δ *that involves an agreement must be such that* $\sum_i \pi_i(M^*) > 1 - \epsilon$.

4.2 SPEMc Results

4.2.1 Stationary Subgame-perfect Equilibria

We begin the SPEMc characterization of the negotiation game by considering *stationary* subgame-perfect equilibria. Since our notion of a stationary strategy (Definition 1) allows for actions conditional on partial history within a period, a stationary SPE here does not precisely correspond to BW's characterization (see their Proposition 1 and Corollary 1).

Of course, it must be that for a pair of stationary strategies to constitute a SPE of the negotiation game, only a Nash equilibrium of the disagreement game can be played after a rejection (on- or off-the-equilibrium path); otherwise, there will be a profitable deviation for some player as continuation payoffs are history-independent at the beginning of next period.

But, a stationary SPE here is not necessarily efficient. Delay in agreement (either over one period or indefinite) and inefficiency can be sustained in equilibrium because a player who makes a deviating offer can be credibly punished in the disagreement game of the same period if the disagreement game has multiple Nash equilibria.[11]

The next result provides a characterization of stationary subgame-perfect equilibria.

Proposition 3 *Let A^* be the set of Nash equilibria of G. We have the following:*

(i) The negotiation game has a stationary SPE if and only if A^* is non-empty;

[11]For instance, suppose that G has three Nash equilibria $a = (a_1, a_2)$, $a^1 = (a_1^1, a_2^1)$ and $a^2 = (a_1^2, a_2^2)$ such that

$$1 > \sum_i u_i(a) \geq 1 - (1-\delta)\max_i(u_i(a^i) - u_i(a)) \tag{4}$$

and

$$u_i(a^i) > u_i(a) \;\; \forall i \; . \tag{5}$$

Then, there exists a stationary SPE in which the players disagree indefinitely and play a in every period after rejection. We can easily check that given (4) and (5) the following stationary strategy profile $f = (f_1, f_2)$ constitutes a SPE. For each i, f_i is such that $\forall h \in H_{ip}^{\infty}$

$$\begin{aligned} f_i(h, \emptyset) &= x^i = (x_1^i, x_2^i) \in \triangle^2 \text{ such that } x_j^i < u_j(a) \\ f_i(h, x, N) &= \begin{cases} a_i & \text{if } x = x^i \\ a_i^j & \text{if } x \neq x^i \end{cases} \end{aligned}$$

and $\forall h \in H_{ir}^{\infty}$

$$\begin{aligned} f_i(h, x) = Y &\;\; \text{if and only if } x_i > (1-\delta)u_i(a^i) + \delta u_i(a) \\ f_i(h, x, N) &= \begin{cases} a_i & \text{if } x = x^j \\ a_i^i & \text{if } x \neq x^j \; . \end{cases} \end{aligned}$$

(ii) If $f = (f_1, f_2)$ is a stationary SPE of the negotiation game, then we have

$$\sum_i \pi_i(f) \geq 1 - (1-\delta)b$$

where $b = \max_i \sup_{a,a' \in A^*}[u_i(a) - u_i(a')]$.[12]

Proof. (i) Necessity follows from the property that in any stationary SPE the players must play a Nash equilibrium of G after any rejection. Sufficiency follows from Corollary 1 of BW.

(ii) Suppose not. So, suppose that $f = (f_1, f_2)$ is a stationary SPE of the negotiation game and

$$\sum_i \pi_i(f) + \epsilon < 1 - (1-\delta)b \tag{6}$$

for some $\epsilon > 0$. Since this equilibrium is inefficient, agreement cannot happen in the first period.

But then, consider player 1 making a deviating offer in $t = 1$, $z = (z_1, z_2) \in \triangle^2$ such that

$$z_2 = (1-\delta)(u_2(a^1) + b) + \delta\pi_2^2(f) + \epsilon$$

where a^1 is the disagreement game equilibrium outcome in $t = 1$ when f is chosen. Clearly, given subgame-perfectness of the stationary profile f and the definition of b (as in the claim), f_2 will accept such offer.

We thus want such deviation to be unprofitable for player 1; that is, we want $z_1 \leq \pi_1(f)$, or

$$1 - (1-\delta)(u_2(a^1) + b) - \delta\pi_2^2(f) - \epsilon \leq (1-\delta)u_1(a^1) + \delta\pi_1^2(f)\ .$$

This implies that

$$\sum_i \pi_i(f) + \epsilon \geq 1 - (1-\delta)b\ .$$

But this contradicts (6). $\|$

It immediately follows from Proposition 3 that for a sufficiently large discount factor every stationary SPE must be (almost) efficient.

Corollary 2 *For any $\epsilon \in (0,1)$, $\exists\ \bar{\delta} < 1$ such that, for any $\delta \in (\bar{\delta}, 1)$, we have the following: If f is a stationary SPE of the negotiation game with discount factor δ, then $\sum_i \pi_i(f) > 1 - \epsilon$.*

[12] Notice that $b \in [0,1]$.

We can also deduce that in some cases a stationary SPE is efficient independently of the discount factor.

Corollary 3 *For any δ, every stationary SPE of the negotiation game is efficient if either (i) G has a unique Nash equilibrium; or (ii) $\forall a \in A^*$ we have $\sum_i u_i(a) < 1 - b$ (where b is as defined in Proposition 3).*

Proof. (i) If G has a unique Nash equilibrium, $b = 0$. Thus, the claim follows from Proposition 3.

(ii) Let f be a stationary SPE of the negotiation and assume that it is inefficient. There are two possible cases to consider.

Case A: There is one period of delay, followed by an agreement.

We know from Proposition 3 that $\sum_i \pi_i(f) \geq 1 - (1-\delta)b$. Thus, it must be that

$$(1-\delta)\sum_i u_i(a) + \delta \geq 1 - (1-\delta)b$$

where $a \in A^*$ is the disagreement game outcome in the first period. But this is clearly not possible if $\forall a \in A^*$ $\sum_i u_i(a) < 1 - b$ as in the claim.

Case B: There is infinite delay.

Let $a^1 \in A^*$ and $a^2 \in A^*$ be the disagreement outcome in odd and even periods respectively. Then, by Proposition 3, we must have

$$\frac{\sum_i u_i(a^1) + \delta \sum_i u_i(a^2)}{1+\delta} \geq 1 - (1-\delta)b\ .$$

Again, this is not possible if $\sum_i u_i(a) < 1 - b$, $\forall a \in A^*$. $\|$

4.2.2 SPEMc and Perpetual Disagreement

We now consider SPEMc outcomes in which agreement never occurs. Some of the results here are sensitive to whether the complexity cost c is zero (lexicographic preferences), or positive.

We shall denote by $\Omega^\delta(c)$ the set of SPEMc profiles in the negotiation game with common discount factor δ when complexity cost is c.

First, we show that, given any complexity cost and a discount factor arbitrarily close to one, any SPEMc outcome with perpetual disagreement must be at least *long-run* (almost) efficient; that is, the players must eventually reach a finite period at which the sum of their continuation payoffs is approximately equal to one.

The argument behind this statement turns critically on the fact that every state of each player's equilibrium machine must appear on the equilibrium path (Lemma 1). This implies the following. Suppose that a player deviates from a SPEMc of the negotiation game by making a different offer in some period. What can the other player obtain if he rejects this offer? Since the state of each player's (sub-)machine is fixed for each period (not at each decision node), the ensuing disagreement game of the period may see an outcome that never happens on the original equilibrium path; but then, Lemma 1 implies that the subsequent transition must take the players to some point along the original path for next period. Thus, any punishment for a player who deviates from the proposed equilibrium must itself occur on the equilibrium path (except for the play of the disagreement game immediately after the deviating offer), and as a consequence, the set of equilibrium outcomes is severely restricted.

Informally, we consider the period in which a player gets his maximum continuation payoff in the proposer role. Bargaining can then be used by the other player in the *preceding* period to break up the on-going disagreement if there is any (continuation) inefficiency from then on. In such cases, there exists a Pareto-improving deviation offer because the responder in that period, who will be proposing next, cannot obtain more from punishing the deviant than what he is already getting from the original outcome as of next period. We need the discount factor to be sufficiently large so as to eliminate the importance of the current period in which the deviation is followed immediately by an off-the-equilibrium play of the disagreement game.

Proposition 4 *For any $\epsilon \in (0,1)$, $\exists\ \bar{\delta} < 1$ such that, for any $\delta \in (\bar{\delta}, 1)$, any $c \geq 0$, and any $M^* \in \Omega^\delta(c)$ with $T(M^*) = \infty$, $\exists\ \tau < \infty$ such that $\sum_i \pi_i^\tau(M^*) > 1 - \epsilon$.*

Proof: Fix any $\epsilon \in (0,1)$. Define

$$\beta = \max\left\{1, \max_i \sup_{a,a' \in A} [u_i(a) - u_i(a')]\right\} \tag{7}$$

which is bounded since $u(\cdot)$ is. Define also

$$\bar{\delta} = 1 - \frac{\epsilon}{\beta}\ .$$

Given these, consider any $\delta \in (\bar{\delta}, 1)$ (thus $\epsilon > \beta(1-\delta)$), any $c \geq 0$ and any $M^* = (M_1^*, M_2^*) \in \Omega^\delta(c)$ with $T(M^*) = \infty$.

Define η, t_{ik} and τ_η such that

$$0 < \eta < \epsilon - \beta(1-\delta), \tag{8}$$

$$t_{ik} = \{t|\ i \text{ plays role } k\}, \tag{9}$$

and

$$\tau_\eta = \min\{t \in t_{2p} \mid \pi_2^t + \eta > \pi_2^{t'}\ \forall t' \in t_{2p}\} \tag{10}$$

where π_2^t is player 2's continuation payoff at period t if M^* is chosen. Clearly, $\tau_\eta < \infty$.

Now, given M_1^*, consider player 2's continuation payoff after rejecting any offer in any period belonging to t_{2r}. Notice that since

- every state of M_1^* appears on the equilibrium path of M^* (Lemma 1)
- $\pi_2^t = \max_{f_2 \in F_2^t} (f_2, M_1^*(q_1^t))\ \forall t$ (Lemma 2),

player 2's continuation payoff *at the next period* if he rejects any offer (given M_1^*) is at most $\sup_{t \in t_{2p}} \pi_2^t$. We also have $\pi_2^{\tau_\eta} + \eta > \pi_2^t\ \forall t \in t_{2p}$.

The above implies that if player 2's equilibrium machine M_2^* receives an offer

$$z' = (1 - \pi_{2r}^{\max}, \pi_{2r}^{\max}) \in \triangle^2$$

where

$$\pi_{2r}^{\max} = (1-\delta) \sup_{a \in A} u_2(a) + \delta \left(\pi_2^{\tau_\eta} + \eta\right), \tag{11}$$

it must always accept because of the subgame-perfectness of the profile M^*.

Now, consider player 1 using another machine $M_1' = \{M_{1p}', M_{1r}'\}$ where, for $k = p, r$, $M_{1k}' = (Q_{1k}', q_{1k}^{1\prime}, \lambda_{1k}', \mu_{1k}')$. This machine is identical to M_1^* except for the output function which is such that

$$\lambda_{1p}'(q_1^{\tau_\eta - 1}, \emptyset) = z' \ . \tag{12}$$

Define

$$\tau = \min_t \{t |\ q_1^t = q_1^{\tau_\eta - 1}\} \ . \tag{13}$$

Since M_2^* always accepts the offer z' and M_1' differs from M_1^* only in offers conditional on state $q_1^{\tau_\eta - 1}$, it follows that (M_1', M_2^*) results in agreement z' in period τ.

We also know by Lemma 3 that $\pi_2^\tau = \pi_2^{\tau_\eta - 1}$. Thus, we have

$$\pi_2^\tau = (1-\delta) u_2(a^{\tau_\eta - 1}) + \delta \pi_2^{\tau_\eta} \ . \tag{14}$$

Now, since $\sup_{a \in A} u_2(a) - u_2(a^{\tau_\eta - 1}) \leq \beta$ (where β is given by (7)), we have, by (11) and (14),

$$\pi_{2r}^{\max} - \pi_2^\tau \leq (1-\delta)\beta + \delta\eta \ .$$

Using this, we can write

$$1 - \pi_{2r}^{\max} \geq 1 - (\pi_2^\tau + (1-\delta)\beta + \delta\eta) \ . \tag{15}$$

Since M^* is a SPEMc it must be that $\pi_1^\tau \geq 1 - \pi_{2r}^{\max}$; otherwise the deviation to M_1' is profitable. This implies that (given $\delta < 1$)

$$\pi_1^\tau + \pi_2^\tau > 1 - ((1-\delta)\beta + \eta) \ .$$

But, since by (8) we have $\epsilon > (1-\delta)\beta + \eta$, it follows that at period $\tau < \infty$, $\sum_i \pi_i^\tau > 1 - \epsilon$ as in the claim. $\|$

Proposition 4 does not however rule out the possibility that we observe inefficiency (in terms of continuation payoffs) early on in the negotiation game.[13] Given any $\epsilon > 0$ and δ sufficiently close to one, we can write the total equilibrium payoff from the negotiation game as

$$\sum_i \pi_i(M^*) > (1-\delta)\sum_{t=1}^{\tau-1}\delta^{t-1}u^t + \delta^{\tau-1}(1-\epsilon) \tag{16}$$

where M^* is the equilibrium machine profile (with $T(M^*) = \infty$), $u^t = \sum_i u_i\left(a^t\left(M^*\right)\right)$, and τ is the period in which continuation (first) becomes (almost) efficient. The limit of the right-hand side as $\epsilon \to 0$ and $\delta \to 1$ is not necessarily the efficient level (because τ may depend on δ).[14]

In fact, when complexity cost c is strictly positive, it turns out that any SPEMc of the negotiation game must be stationary (minimal) however small that complexity cost is.

Proposition 5 *For any $c > 0$, $\exists\ \bar{\delta} < 1$ such that, for any $\delta \in (\bar{\delta}, 1)$, every $M^* \in \Omega^\delta(c)$ is stationary.*

Proof. We shall prove the claim by way of contradiction.

Fix any $c \in (0,1)$.[15] Define $\bar{\delta} = 1 - \frac{c}{\beta}$ where β is given by (7) above.

Given these, consider any $\delta \in (\bar{\delta}, 1)$, and any $M^* = (M_1^*, M_2^*) \in \Omega^\delta(c)$ that is not stationary.

Then, both equilibrium machines M_1^* and M_2^* are non-minimal. (Trivially, if one was minimal, then by part (i) of Lemma 10 in Appendix B the other must also be minimal.) Also, by Proposition 2, it must be that $T(M^*) = \infty$.

Similarly to the proof of Proposition 4 above, define η such that

$$0 < \eta < c - \beta(1-\delta)\ . \tag{17}$$

Define as before t_{ik}, τ_η, and τ (see (9), (10), and (13)).

We need to consider the following two cases separately.

[13] To be precise, neither does it rule out the possibility that there will be inefficient disagreement game outcomes even after τ. It is just that the continuation game from then on is almost efficient.

[14] If we restrict each player's machine to use only a finite number of states, then any machine profile must generate *cycles*. But this is not enough to guarantee that Proposition 4 implies ex ante efficiency in the limit as δ goes to one. For this, we need for instance to additionally assume that the size of a machine is uniformly bounded (for any δ) so that the first cycle cannot last beyond a fixed period.

[15] The case of $c \geq 1$ is trivial because then complexity cost (weakly) dominates any feasible average payoff for each player in the negotiation game and thus any equilibrium machine must be minimal. We can refer to BW Result 1 for SPEMc characterization in this case.

Case A: $\tau_\eta > 2$

First note that

$$q_1^t \neq q_1^{\tau_\eta} \quad \forall t < \tau_\eta \ . \tag{18}$$

Otherwise $q_1^t = q_1^{\tau_\eta}$ for some $t < \tau_\eta$. But then, we have $\pi_2^t = \pi_2^{\tau_\eta}$ by Lemma 3. This contradicts the definition of τ_η.

Next, consider player 1 using another machine $M_1' = \{M_{1p}', M_{1r}'\}$ where, for $k = p, r$, $M_{1k}' = (Q_{1k}', q_{1k}^{1\prime}, \lambda_{1k}', \mu_{1k}')$. This machine is identical to M_1^* except that:

- (similarly to (12) above)

$$\lambda_{1p}'(q_1^{\tau_\eta - 1}, \emptyset) = (1 - \pi_{2r}^{\max}, \pi_{2r}^{\max}) = z'$$

where $\pi_{2r}^{\max}$ is defined by (11) with η now given by (17);

- $q_1^{\tau_\eta}$ is dropped (i.e. $Q_{1r}' = Q_{1r}^* \backslash q_1^{\tau_\eta}$).

By (18), dropping $q_1^{\tau_\eta}$ does not affect the outcome path up to τ. Therefore, by similar arguments as in the proof of Proposition 4, the deviation would induce agreement z' in period τ.

It follows that player 1's deviation payoff here is given also by $1 - \pi_{2r}^{\max}$, and as in (15), we have

$$1 - \pi_{2r}^{\max} \geq 1 - (\pi_2^\tau + (1 - \delta)\beta + \delta\eta) \ .$$

We also have that $1 - \pi_2^\tau \geq \pi_1^\tau$. Thus, player 1's loss from such deviation is

$$\pi_1^\tau - (1 - \pi_{2r}^{\max}) \leq (1 - \delta)\beta + \delta\eta \ . \tag{19}$$

But the new machine M_1' has one less state than M_1^* (since $q_1^{\tau_\eta}$ has been dropped) which means that the deviation also results in a saving on complexity cost by $c > 0$. Since $c > (1 - \delta)\beta + \eta$ by (17), we have

$$\pi_1^\tau - (1 - \pi_{2r}^{\max}) < c$$

implying that the deviation is in fact profitable. (More precisely, this implies that M_1^* is not a *c-best response* to M_2^*.)

This contradicts the proposed SPEMc. Therefore, $T(M^*) < \infty$, and by Proposition 2, M^* must be minimal, and hence, stationary.

Case B: $\tau_\eta = 2$

We know that M_1^* is not minimal. Then, consider player 1 using another minimal machine $M_1' = \{M_{1p}', M_{1r}'\}$ where, for $k = p, r$, $M_{1k}' = (Q_{1k}', q_{1k}^{1\prime}, \lambda_{1k}', \mu_{1k}')$. This machine is constructed such that:

- $Q'_{1p} = \{q_1^1\}$ and $Q'_{1r} = \{q_1^2\}$
- $\lambda'_{1p}(q_1^1, \emptyset) = z'$ (where z' is as appearing in Case A above).

By similar arguments to those behind Case A above, such deviation induces immediate agreement on z' and is (overall) profitable for player 1 since the new machine is minimal and has less states than the original machine. Thus we again have contradiction against SPEMc. The claim then follows as before. $\|$

4.2.3 Summary

Let us summarize our SPEMc results, and in particular, highlight the efficiency property of an equilibrium in the machine game. Putting together Proposition 2, Proposition 4 and Corollary 1, we first characterize the set of all SPEMc in the negotiation game when the players' preferences are lexicographic ($c = 0$). Provided that the players are sufficiently patient, (i) every SPEM0 inducing an agreement must do so in the very first stage of the negotiation game, and thus, be stationary and (almost) efficient; and (ii) every SPEM0 inducing perpetual disagreement must be at least (almost) efficient in the long run.

Theorem 1 (Lexicographic Preferences) *For any $\epsilon \in (0,1)$, $\exists\ \bar{\delta} < 1$ such that for any $\delta \in (\bar{\delta}, 1)$ we have the following:*

(i) If M^* is a NEM0 (and therefore SPEM0) with $T(M^*) < \infty$, then $T(M^*) \leq 2$, M^* is stationary and

$$\sum_i \pi_i(M^*) > 1 - \epsilon \ ;$$

(ii) If M^* is a SPEM0 with $T(M^*) = \infty$, then $\exists\ \tau < \infty$ such that

$$\sum_i \pi_i^\tau(M^*) > 1 - \epsilon \ .$$

If we assume positive complexity cost, the predictions are sharper. Only stationary, and hence, (almost) efficient equilibria are possible under sufficiently patient players. Theorem 2 puts together Proposition 5 and Corollary 2.

Theorem 2 (When Complexity Cost is Positive) *For any $c > 0$ and any $\epsilon \in (0,1)$, $\exists\ \bar{\delta} < 1$ such that, for any $\delta \in (\bar{\delta}, 1)$, every $M^* \in \Omega^\delta(c)$ is stationary and such that*

$$\sum_i \pi_i(M^*) > 1 - \epsilon \ .$$

We can also deduce from Proposition 4 that if the structure of the disagreement game is such that there exists no action profile delivering the efficient surplus (and also the players are sufficiently patient), the players cannot disagree forever for any non-negative complexity cost. Thus, in this case every SPEMc must, induce an agreement in the first stage, be stationary, and be efficient in the limit.

Theorem 3 (When Agreement Dominates Disagreement) *Suppose that we have $\sum_i u_i(a) < 1 \ \forall a \in A$. Then, $\exists \ \bar{\delta} \in (0,1)$ such that, for any $\delta \in (\bar{\delta}, 1)$ and any $c \geq 0$, every $M^* \in \Omega^\delta(c)$ is such that $T(M^*) \leq 2$, and hence, is stationary and efficient in the limit as δ goes to one.*

We can further relate the set of SPEMc in the negotiation game to the structure of the disagreement game G. For instance, Corollary 3 reports some cases where a stationary SPE of the negotiation game is efficient independently of the discount factor. In those cases, we have a stronger set of efficiency results.

Remark 2 *Suppose either G has a unique Nash equilibrium, or $\forall a \in A^*$ we have $\sum_i u_i(a) < 1 - b$ (where b is as defined in Proposition 3). Then, provided that the players are sufficiently patient, for any $c > 0$ every SPEMc is efficient (not just in the limit) as is every SPEM0 involving an agreement.*

5 An Alternative Machine Specification

Since each stage game of the negotiation game has a sequential structure, we can have alternative machine specifications that employ more frequent transitions and hence account for finer partitions of histories and continuation strategies. Let us present a machine which consists of four sub-machines.

The machine under Definition 9 below maintains the role distinction and employs distinct sub-machines to play the bargaining and the disagreement game within each period. Transition thus takes place twice within each period - once after the bargaining and once after the disagreement game.[16]

Definition 9 (Four sub-machine (4SM) specification) *A machine, $M_i = \{M_{ip}, \tilde{M}_{ip}, M_{ir}, \tilde{M}_{ir}\}$, consists of four sub-machines $M_{ik} = (Q_{ik}, q^1_{ik}, \lambda_{ik}, \mu_{ik})$ and $\tilde{M}_{ik} = (\tilde{Q}_{ik}, \tilde{q}^1_{ik}, \tilde{\lambda}_{ik}, \tilde{\mu}_{ik})$ for $k = p, r$. Each sub-machine consists of a set of states,*

[16] We can also construct a machine in which transition occurs at each decision node of the stage game. Six sub-machines will then be required (some of which will in fact serve only to make transition and not output). There are several other ways to divide each stage. But we conjecture that as long as we keep the role distinction for the bargaining part the central results will remain irrespective of the machine specification.

an initial state, an output function and a transition function such that $\forall q_{ik} \in Q_{ik}$*,* $\forall \tilde{q}_{ik} \in \tilde{Q}_{ik}$*,* $\forall x^i, x^j \in \triangle^2$ *and* $\forall a \in A$*,*

$$\begin{aligned} \lambda_{ip}(q_{ip}, \emptyset) &\in \triangle^2; \\ \tilde{\lambda}_{ip}(\tilde{q}_{ip}, \emptyset) &\in A_i; \\ \lambda_{ir}(q_{ir}, x^j) &\in \{Y, N\}; \\ \tilde{\lambda}_{ir}(\tilde{q}_{ir}, \emptyset) &\in A_i; \end{aligned}$$

and

$$\begin{aligned} \mu_{ip}(q_{ip}, x^i, N) &\in \tilde{Q}_{ip}; \\ \tilde{\mu}_{ip}(\tilde{q}_{ip}, a) &\in Q_{ir}; \\ \mu_{ir}(q_{ir}, x^j, N) &\in \tilde{Q}_{ir}; \\ \tilde{\mu}_{ir}(\tilde{q}_{ir}, a) &\in Q_{ip} \ . \end{aligned}$$

As a notational convention, we shall use M_{ik} to denote the sub-machine that plays the bargaining part of the negotiation game to distinguish it from $\tilde{M}_{ik}$, the sub-machine that plays the disagreement game. The components of each sub-machine are denoted with similar conventions. Also, whenever it is necessary to distinguish the new definition from the previously used definition with two sub-machines, we shall refer to them by 4SM and 2SM respectively.

As before $H^t_{ik} (= H^t_{jl})$ refers to the set of t-period histories. Here, we also denote the set of all possible histories at a disagreement game of period t in which i plays role k as $\tilde{H}^t_{ik} = H^t_{ik} \times \{(x, N) \mid \forall x \in \triangle^2\}$. Also, define $\tilde{H}^\infty_{ik} = \cup^\infty_{t=1} \tilde{H}^t_{ik}$. We continue to denote by F_i and Φ_i the set of player i's strategies and machines in the negotiation (machine) game respectively.

A minimal machine in the 4SM specification is again a machine whose sub-machines have only one state each, but it corresponds to an alternative notion of stationarity.[17]

Definition 10 *A strategy* f_i *is Markov-stationary if and only if*

$$\begin{aligned} &f_i(h) = f_i(h') \ \ \forall h, h' \in H^\infty_{ip} \text{ and } \forall h, h' \in \tilde{H}^\infty_{ik} (k = p, r); \\ &f_i(h, x^j) = f_i(h', x^j) \ \ \forall h, h' \in H^\infty_{ir}, \forall x^j \in \triangle^2 \ . \end{aligned}$$

Notice that according to this definition of a stationary strategy the disagreement game actions are independent of partial history within a period. Therefore, this definition captures Markov behavior. Moreover, this definition corresponds precisely to BW's

[17]We shall henceforth refer to a minimal 4SM (profile) interchangeably as a Markov-stationary machine (profile).

notion of stationarity and thus we can appeal to their characterization of stationary subgame-perfect equilibria. Especially, the efficiency of a Markov-stationary SPE does not depend on the discount factor. The following is implied by Proposition 1 and Corollary 1 of BW.

Remark 3 *Let A^* be the set of Nash equilibria of G. We have the following:*

(i) The negotiation game has a Markov-stationary SPE if and only if A^* is non-empty;

(ii) If $f = (f_1, f_2)$ is a Markov-stationary SPE of the negotiation game, then it is efficient; that is $\sum_i \pi_i(f) = 1$;

(iii) Suppose that $\sum_i u_i(a) < 1 \ \forall a \in A^*$. Then, if f is a Markov-stationary SPE of the negotiation game, it induces an immediate agreement.

There may be a Markov-stationary SPE in the negotiation game in which delay takes place, but then, since this equilibrium must be efficient, the disagreement game action profiles (Nash equilibria of G) appearing on the equilibrium path must be efficient.

We define NEMc and SPEMc exactly as before except that the complexity of a machine (4SM) is now given by $\|M_i\| = \sum_k |Q_{ik}| + \sum_k |\tilde{Q}_{ik}|$. We also need to re-define the corresponding definition of complexity in terms of strategies. Let $F_{ik}(f_i) = \{f_i|h : h \in H^\infty_{ik}\}$ be as before and introduce $\tilde{F}_{ik}(f_i) = \{f_i|h : h \in \tilde{H}^\infty_{ik}\}$ to indicate the set of continuation strategies at a disagreement game of period t when i plays role k. It is straightforward to extend Proposition 1 to show that, for any $f_i \in F_i$, $\sum_k |F_{ik}(f_i)| + \sum_k |\tilde{F}_{ik}(f_i)|$ corresponds to the size of a smallest 4SM implementing f_i.

Given this foundation, analyzing the machine game in the 4SM specification is analogous to the previous 2SM case (though a little more cumbersome expositionally). Note here that while the game is being played bargaining alone does not generate any payoffs. Thus, $\pi^t_i(\cdot)$ equally represents i's continuation payoff at every subgame within the period (on the equilibrium path).

We introduce some further notational changes. Let F^t_i and Φ^t_i respectively denote the set of player i's strategies and machines in the negotiation (machine) game starting at the beginning of a period in which the roles are given as in period t. Thus, if t is odd, $F^t_i = F_i$ and $\Phi^t_i = \Phi_i$. Also, let $\tilde{F}^t_i$ and $\tilde{\Phi}^t_i$ respectively denote the set of player i's strategies and machines in the negotiation (machine) game starting at the beginning of the disagreement game of a period in which the roles are given as in period t.

First, it is clear that any NEMc profile in 4SM must be by definition a Nash equilibrium of the underlying negotiation game. It is also true under this alternative machine specification that every state belonging to a (non-minimal) equilibrium machine must appear on the equilibrium path and if the game ends in the first stage the associated equilibrium machines must be minimal (Lemma 1). The following three Lemmas correspond to Lemmas 2, 3, and 4 respectively. (We omit some of the proofs.)

Lemma 6 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc in the 4SM specification with* $c \geq 0$. *Then,* $\forall i, j$ *we have:*

$$\begin{array}{ll} (i) & \pi_i^\tau(M^*) = \max_{f_i \in F_i^\tau} \pi_i(f_i, M_j^*(q_j^\tau)) \ \ \forall \tau \leq T(M^*) \\ (ii) & \pi_i^\tau(M^*) = \max_{f_i \in \tilde{F}_i^\tau} \pi_i(f_i, M_j^*(\tilde{q}_j^\tau)) \ \ \forall \tau < T(M^*) \end{array}$$

where $M_j^*(q_j^\tau) \in \Phi_i^\tau$ $(M_j^*(\tilde{q}_j^\tau) \in \tilde{\Phi}_i^\tau)$ *is the machine identical to* M_j^* *except that it starts with the sub-machine operating in the bargaining (disagreement game) part of period* τ *with initial state* q_j^τ $(\tilde{q}_j^\tau)$, *and with some abuse of notation,* $\pi_i(f_i, M_j^*(q_j^\tau))$ $(\pi_i(f_i, M_j^*(\tilde{q}_j^\tau)))$ *refers to i's payoff in the negotiation game that starts with role distribution given in period* τ *and is played by* i *and* j *according to* $f_i \in F_i^\tau$ $(f_i \in \tilde{F}_i^\tau)$ *and* $M_j^*(q_j^\tau)$ $(M_j^*(\tilde{q}_j^\tau))$ *respectively.*

Lemma 7 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc in the 4SM specification with* $c \geq 0$. *Then,* $\forall i, j$ *and* $\forall t, t' \leq T(M^*)$, *we have the following:*

$$\text{if } q_j^t = q_j^{t'} \text{ or } \tilde{q}_j^t = \tilde{q}_j^{t'}, \text{ then } \pi_i^t(M^*) = \pi_i^{t'}(M^*) \ .$$

Lemma 8 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc in the 4SM specification with* $c \geq 0$. *Then, for any* i *and any* $\tau \leq T(M^*)$, *we have the following:*

$$\begin{array}{ll} (i) & \text{if } q_i^\tau(M^*) \neq q_i^t(M^*) \ \forall t < \tau, \text{ then } q_j^\tau(M^*) \neq q_j^t(M^*) \ \forall t < \tau \\ (ii) & \text{if } \tilde{q}_i^\tau(M^*) \neq \tilde{q}_i^t(M^*) \ \forall t < \tau, \text{ then } \tilde{q}_j^\tau(M^*) \neq \tilde{q}_j^t(M^*) \ \forall t < \tau \ . \end{array}$$

Proof. See Appendix A. ‖

Using these Lemmas, it is straightforward to extend the agreement results in Section 4.1.2 to the 4SM case. If a NEMc outcome under this alternative specification ends at some finite period, the pair of states occurring in the last period must be distinct. (Notice that the sub-machines used for playing the disagreement game will not be operating in the final period.)

Lemma 9 *Assume that* $M^* = (M_1^*, M_2^*)$ *is a NEMc in the 4SM specification with* $c \geq 0$ *and* $T(M^*) < \infty$. *Then,* $q_i^t \neq q_i^T$ $\forall t < T$ *and* $\forall i$.

Proof. See Appendix A. ‖

Again, this implies that the agreement must occur within the first stage of the game; otherwise the states in the final period can be "replaced" thereby yielding a saving on complexity cost. (We shall omit the proof of the following result. It is almost identical to that of Proposition 2 and Corollary 1.)

Proposition 6 *If $M^* = (M_1^*, M_2^*)$ is a NEMc in the 4SM specification with $c \geq 0$ and $T(M^*) < \infty$, then (i) $T(M^*) \leq 2$, (ii) M_1^* and M_2^* are minimal and hence M^* is Markov-stationary; and (iii) M^* is efficient in the limit as δ goes to one.*

What we gain from using this alternative machine specification is in the case of SPEMc. Specifically, the SPEMc results in Section 4.2.2 no longer depend on the discount factor. In particular, we derive the following results (corresponding to Propositions 4 and 5). Let $\tilde{\Omega}^\delta(c)$ denote the set of SPEMc in 4SM given discount factor δ and complexity cost c.

Proposition 7 *Fix any $c \geq 0$ and any $\delta \in (0,1)$, and consider any $M^* \in \tilde{\Omega}^\delta(c)$ with $T(M^*) = \infty$. Then for any $\epsilon > 0$, $\exists\ \tau < \infty$ such that $\sum_i \pi_i^\tau(M^*) > 1 - \epsilon$.*

Proof. Fix any ϵ. Also, fix any $c \geq 0$ and any $\delta \in (0,1)$, and consider any $M^* = (M_1^*, M_2^*) \in \tilde{\Omega}^\delta(c)$ with $T(M^*) = \infty$.

Let $M_i^* = \{M_{ip}^*, \tilde{M}_{ip}^*, M_{ir}^*, \tilde{M}_{ir}^*\}$ where, for $k = p, r$, $M_{ik}^* = (Q_{ik}^*, q_{ik}^{1*}, \lambda_{ik}^*, \mu_{ik}^*)$ and $\tilde{M}_{ik}^* = (\tilde{Q}_{ik}, \tilde{q}_{ik}^{1*}, \tilde{\lambda}_{ik}^*, \tilde{\mu}_{ik}^*)$.[18]

Define η such that $0 < \eta < \epsilon$. Define also t_{ik} as before (see (9)) and

$$\tau = \min\{t \in t_{ir} \mid \pi_i^t + \eta > \pi_i^{t'}\ \forall t' \in t_{ir}\}\ . \tag{20}$$

Since

- every state of M_j^* appears on the equilibrium path of M^*
- transition also occurs at the end of bargaining within each period
- $\pi_i^t = \max_{f_i \in \tilde{F}_i^\tau} \pi_i(f_i, M_j^*(\tilde{q}_j^t))\ \forall t$ (Lemma 6),

(given M_j^*) the maximum continuation payoff player i can obtain if he rejects any offer at any $t \in t_{ir}$ is equal to $\sup_{t \in t_{ir}} \pi^t$ which is less than $\pi_i^\tau + \eta$. Then, given the subgame-perfectness of the profile M^*, i's equilibrium machine M_i^* must accept if j offers

$$z'' = (z_i'', z_j'') = (\pi_i^\tau + \eta, 1 - \pi_i^\tau - \eta) \in \triangle^2 \tag{21}$$

at any $t \in t_{ir}$.

Consider now player j using another machine $M_j' = \{M_{jp}', \tilde{M}_{jp}', M_{jr}', \tilde{M}_{jr}'\}$ where, for $p = k, r$, $M_{jk}' = (Q_{jk}', q_{jk}^{1\prime}, \lambda_{jk}', \mu_{jk}')$ and $\tilde{M}_{jk}' = (\tilde{Q}_{jk}', \tilde{q}_{jk}^{1\prime}, \tilde{\lambda}_{jk}', \tilde{\mu}_{jk}')$. This machine is identical to M_j^* except for the output function which is such that

$$\lambda_{jp}'(q_j^\tau, \emptyset) = z''\ . \tag{22}$$

[18] This will henceforth define the equilibrium machines in our claims in this section.

Now, note that $q_j^\tau \neq q_j^t$ $\forall t < \tau$. Otherwise, $\pi_i^t = \pi_i^\tau$ by Lemma 7, which contradicts the definition of τ. Thus, (by similar arguments to those in the proof of Proposition 4 above) (M_i^*, M_j') would induce agreement z'' in period τ.

Since M^* is a SPEMc, it must be that $\pi_j^\tau \geq 1-\pi_i^\tau-\eta$, implying that $\pi_i^\tau+\pi_j^\tau \geq 1-\eta$. But we fixed $\eta < \epsilon$, and thus, at $\tau < \infty$ we have $\sum_i \pi_i^\tau > 1-\epsilon$ as in the claim. $\|$

Proposition 8 *For any $c > 0$ and any $\delta \in (0,1)$, every $M^* \in \tilde{\Omega}^\delta(c)$ is Markov-stationary.*

Proof. Suppose not. Then there exists some $c > 0$, some $\delta \in (0,1)$ and a machine profile $M^* = (M_1^*, M_2^*) \in \tilde{\Omega}^\delta(c)$ that is not minimal (Markov-stationary).

Then, we know that M_1^* and M_2^* are both non-minimal (appealing to Lemma 10 in Appendix B). Also, by Proposition 6, $T(M^*) = \infty$.

Define η such that $0 < \eta < c$. Define also t_{ik} as in (9) and τ as in (20).

Case A: $\tau > 2$

Consider now player j using another machine $M_j' = \{M_{jp}', \tilde{M}_{jp}', M_{jr}', \tilde{M}_{jr}'\}$ where, for $p = k, r$, $M_{jk}' = (Q_{jk}', q_{jk}^{1\prime}, \lambda_{jk}', \mu_{jk}')$ and $\tilde{M}_{jk}' = (\tilde{Q}_{jk}', \tilde{q}_{jk}^{1\prime}, \tilde{\lambda}_{jk}', \tilde{\mu}_{jk}')$.

This machine is identical to M_j^* except that:

- (similarly to (22) above)
$$\lambda_{jp}'(q_j^\tau, \emptyset) = z''$$

where z'' is defined by (21) with η now being such that $0 < \eta < c$;

- $\tilde{q}_j^\tau$ is dropped (i.e. $\tilde{Q}_{jp}' = \tilde{Q}_{jp}^* \backslash \tilde{q}_j^\tau$).

First note that we have $q_j^\tau \neq q_j^t$ and $\tilde{q}_j^\tau \neq \tilde{q}_j^t$ $\forall t < \tau$. Otherwise, $\pi_i^t = \pi_i^\tau$ by Lemma 7, which contradicts the definition of τ. Thus, dropping $\tilde{q}_j^\tau$ would not affect the outcome up to τ.

By similar arguments as in the previous Proposition, (M_i^*, M_j') would induce agreement z'' in period τ. Dropping $\tilde{q}_j^\tau$ is immaterial to this outcome because agreement takes place before the players reach the disagreement game in the period.

Since $\pi_i^\tau \leq 1-\pi_j^\tau$, j's loss from such deviation is η. But the new machine M_j' has one less state than M_j^* and thus there is also a saving in complexity cost by c. We fixed $\eta < c$ and thus the deviation is profitable. This contradicts the proposed SPEMc; therefore, $T(M^*) < \infty$. The claim then follows from Proposition 6.

Case B: $\tau \leq 2$

We can derive contradiction for this case using similar arguments appearing in Case B of the proof of Proposition 5. ‖

We can finally summarize the SPEMc results under the 4SM specification, using Remark 3.

Theorem 4 *For any δ, we have:*

(i) If $M^* \in \tilde{\Omega}^\delta(0)$ is such that $T(M^*) < \infty$, then $T(M^*) \leq 2$, M^* is Markov-stationary and hence

$$\sum_i \pi_i(M^*) = 1 \ ;$$

(ii) If $M^* \in \tilde{\Omega}^\delta(0)$ is such that $T(M^*) = \infty$, then for any $\epsilon > 0 \ \exists \ \tau < \infty$ such that

$$\sum_i \pi_i^\tau(M^*) > 1 - \epsilon \ .$$

Theorem 5 *For any $c > 0$ and any δ, every $M^* \in \tilde{\Omega}^\delta(c)$ is Markov-stationary and hence is efficient.*

Theorem 6 *Suppose that $\sum_i u_i(a) < 1 \ \forall a \in A$. Then, for any $c \geq 0$ and any δ, every $M^* \in \tilde{\Omega}^\delta(c)$ induces an immediate agreement, and hence, is Markov-stationary and efficient.*

6 Conclusion

When players care for complexity of a strategy as well as payoffs, the negotiation game can only display equilibria that are *efficient*. Thus, complexity and bargaining together offer an explanation for co-operation in two-person repeated interactions.

Independently of complexity cost, discount factor and the choice of machine specification, the negotiation game cannot have a NEMc in which an agreement takes place after delay beyond the first stage. If an agreement were to be part of an equilibrium outcome, then it must be so in the very first stage of the game, and the associated strategy profile must be stationary.

The precise efficiency results with perfection, however, depend on whether complexity cost is zero (lexicographic preferences) or is positive, and also, on the chosen machine/complexity specification.

Under the 2SM specification, we find the following set of SPEMc results under sufficiently patient players. If complexity cost is strictly positive, any SPEMc of the negotiation game must be stationary, and hence, (almost) efficient however small that complexity cost is. If complexity cost is zero, in addition to stationary equilibria the negotiation game can also admit a non-stationary equilibrium (SPEM0) in which disagreement persists indefinitely. But this case still has to be (almost) efficient in the long run. It then follows that, regardless of complexity cost, every SPEMc must induce an agreement in the first stage of the negotiation game in cases where disagreement is strictly dominated by agreement.

If we adopt the 4SM specification (thereby accounting for finer partitions of histories and continuation strategies in representing strategic complexity) essentially the same set of insights are derived, but independently of the discount factor.

There are several channels to further generalize the analysis in this paper. Especially, we can reinforce the repeated game flavor of the negotiation game by considering a broader set of payoffs that can be associated with bargaining and agreement. We can, for instance, let the space of offers be some arbitrary (but 'well-behaved') set $P \subset R^2$ such that $u(a) \subseteq P$ for all $a \in A$, thereby allowing an offer to include any disagreement game payoff vector. We conjecture that complexity will still select the efficient outcomes in this case.

7 Appendix A: Relegated Proofs

Proof of Proposition 1. Let $M_i = \{M_{ip}, M_{ir}\}$ be any machine that implements some strategy f_i where $M_{ik} = (Q_{ik}, q_{ik}^1, \lambda_{ik}, \mu_{ik})$ for $k = p, r$.

First, we show that $|Q_{ik}| \geq |F_{ik}(f_i)|\ \forall k$.

For any $q_i \in Q_{ik}$ and $k = p, r$, let $M_i(q_i) = \{M_{ik}(q_i), M_{il}\}$ be the machine that is identical to M_i except that it starts with the sub-machine $M_{ik}(q_i)$ where $M_{ik}(q_i) = (Q_{ik}, q_i, \lambda_{ik}, \mu_{ik})$ is the sub-machine identical to M_{ik} except for the initial state q_i.

Note that for every $\bar{f}_i \in F_{ik}(f_i)$ and $k = p, r$, there exists some $h \in H_{ik}^{\infty}$ such that $\bar{f}_i = f_i|h$. Now define a function $\Gamma_{ik} : Q_{ik} \to F_{ik}(f_i)$ such that $\Gamma_{ik}(\bar{q}_i)$ is the strategy implemented by $M_i(\bar{q}_i)$ for any $\bar{q}_i \in Q_{ik}$. It then follows that for every $\bar{f}_i \in F_{ik}(f_i)$, there must exist a distinct state $\bar{q}_i \in Q_{ik}$ such that $\Gamma_{ik}(\bar{q}_i) = \bar{f}_i$. Simply let $\bar{q}_i = q_i(h)$ (as defined inductively in Section 3) where h is the history such that $\bar{f}_i = f_i|h$.[19]

Second, we show that there exists a machine implementation of f_i which only uses $F_{ip}(f_i)$ and $F_{ir}(f_i)$ as the set of states for its corresponding sub-machines.

Define $\bar{M}_i = \{\bar{M}_{ip}, \bar{M}_{ir}\}$ such that, for $k = p, r$, $\bar{M}_{ik} = (F_{ik}(f_i), f_{ik}^1, \bar{\lambda}_{ik}, \bar{\mu}_{ik})$ where

- $f_{ik}^1 \in F_{ik}(f_i)$ is the initial state and if i plays role k at the initial history then $f_{ik}^1 = f_i$;
- for any $\bar{f}_i \in F_{ik}(f_i)$ and $d \in D_{ik}$, $\bar{\lambda}_{ik}(\bar{f}_i, d) = \bar{f}_i(\emptyset, d)$ where $\emptyset$ is the empty history;

[19]It is critical here that each sub-machine uses its own distinct set of states. Otherwise, a single state can be used to activate two distinct continuation strategies, one in each role.

- $\bar{\mu}_{ik}(\bar{f}_i, e) = \bar{f}_i|h, e$ for any $h \in H_{ik}^{\infty}$ and $e \in E$.

This machine has $\sum_k |F_{ik}(f_i)|$ states (each k sub-machine with $|F_{ik}(f_i)|$ states) and implements f_i. $\|$

Proof of Lemma 2. Suppose not. Then, for some i and $\tau \leq T(M^*)$, there exists another machine $\bar{M}_i = \{\bar{M}_{ip}, \bar{M}_{ir}\} \in \Phi_i^{\tau}$ such that

$$\pi_i^{\tau}(M^*) < \pi_i(\bar{M}_i, M_j^*(q_j^{\tau})) \ .$$

Now, consider player i using at the outset another machine $M_i' = \{M_{ip}', M_{ir}'\}$ where, for $k = p, r$, $M_{ik}' = (Q_{ik}', q_{ik}^{1\prime}, \lambda_{ik}', \mu_{ik}')$. This machine is constructed in the following way.

Let $q_i^t \in Q_{ik}^*$ denote the state of M_i^* appearing in period t (where i is in role k) when M^* is chosen. Also let e^t be the outcome in period t when M^* is chosen.

For every $t < \tau$ and $k = p, r$, there exists a *distinct* state $q_i'(t) \in Q_{ik}'$ such that

$$\lambda_{ik}'(q_i'(t), d) = \lambda_{ik}^*(q_i^t, d) \text{ for all } d \in D_{ik} \ .$$

The transition function of the new machine is such that

$$\mu_{ik}'(q_i'(t), e^t) = \begin{cases} q_i'(t+1) & \forall t < \tau - 1 \\ \bar{q} & \text{for } t = \tau - 1 \end{cases}$$

where $\bar{q} \in Q_{ik}'$ is another distinct state such that $M_i'(\bar{q}) = \bar{M}_i$.

Thus, M_i' played against M_j^* replicates the outcome path up to τ such that

$$\sum_{t=1}^{\tau-1} \delta^{t-1} u_i \left(a^t(M_i', M_j^*)\right) = \sum_{t=1}^{\tau-1} \delta^{t-1} u_i \left(a^t(M_i^*, M_j^*)\right)$$

followed by activation of $\bar{M}_i$ at τ. It then follows that $\pi_i(M_i', M_j^*) > \pi_i(M_i^*, M_j^*)$. But this contradicts (1) above. $\|$

Proof of Lemma 8. (i) Suppose not. So, there exists some $\tau \leq T$ such that $q_i^{\tau} \neq q_i^t$ $\forall t < \tau$ and $q_j^{\tau} = q_j^{\tau'}$ for some $\tau' < \tau$. By Lemma 7, $\pi_i^{\tau} = \pi_i^{\tau'}$.

But then, consider i using another machine $M_i' = \{M_{ip}', \tilde{M}_{ip}', M_{ir}', \tilde{M}_{ir}'\}$ where, for $k = p, r$, $M_{ik}' = (Q_{ik}', q_{ik}^{1\prime}, \lambda_{ik}', \mu_{ik}')$ and $\tilde{M}_{ik}' = (\tilde{Q}_{ik}', \tilde{q}_{ik}^{1\prime}, \tilde{\lambda}_{ik}', \tilde{\mu}_{ik}')$. This machine is identical to M_i^* except that:

- q_i^{τ} is dropped
- the transition function is such that $\tilde{\mu}_{ik}'(\tilde{q}_i^{\tau-1}, a^{\tau-1}) = q_i^{\tau'}$ $(k \in \{p, r\})$.

Since $q_i^\tau \neq q_i^t \; \forall t < \tau$, this preserves the outcome path up to $\tau - 1$ while making the path between τ' and $\tau - 1$ repeat from τ on.

Similarly to the proof of Lemma 4 above, we can show that this will not change i's payoff. But, since q_i^τ has been dropped, $\|M_i^*\| > \|M_i'\|$. We thus have a contradiction against NEMc.

(ii) This part can be proven similarly to (i) above. $\|$

Proof of Lemma 9. Suppose not. So, suppose that $q_i^t = q_i^T$ for some i and some $t < T$. Let $z = (z_1, z_2) \in \triangle^2$ be the agreement at T. There are two possible cases to consider.

Case A: Player i is the proposer at T.

Define $\tau = \min\{t | q_i^t = q_i^T\}$. By Lemma 7, $\pi_j^\tau = \pi_j^T$. Since there is an agreement on z at T, we have $\pi_j^\tau = z_j$.

Now consider player j using another machine $M_j' = \{M_{jp}', \tilde{M}_{jp}', M_{jr}', \tilde{M}_{jr}'\}$ where, for $p = k, r$, $M_{jk}' = (Q_{jk}', q_{jk}^{1\prime}, \lambda_{jk}', \mu_{jk}')$ and $\tilde{M}_{jk}' = (\tilde{Q}_{jk}', \tilde{q}_{jk}^{1\prime}, \tilde{\lambda}_{jk}', \tilde{\mu}_{jk}')$. This machine is identical to M_j^* except that:

- q_j^τ is dropped (i.e. $Q_{jr}' = Q_{jr}^* \backslash q_j^\tau$)
- the transition function is such that $\tilde{\mu}_{jp}'(\tilde{q}_j^{\tau-1}, a^{\tau-1}) = q_j^T$.

Since, by Lemma 8, q_j^τ (as does q_i^τ by definition) appears for the first time at τ on the original equilibrium path, this new machine (given M_i^*) generates an identical outcome path as the original machine M_j^* up to τ and then induces the agreement z at τ. We know $\pi_j^\tau = z_j$, and thus, it follows that $\pi_j(M_i^*, M_j') = \pi_j(M_i^*, M_j^*)$. But since q_j^τ is dropped, $\|M_j^*\| > \|M_j'\|$. This contradicts NEMc.

Case B: Player i is the responder at T.

We can show contradiction similarly to Case A above. $\|$

8 Appendix B: An Alternative Equilibrium Concept

The following defines the general preference ordering over machine profiles proposed by Abreu and Rubinstein [1] (AR).

Definition 11 *Let $\succ_i^s$ (and $\sim_i^s$) denote player i's preference ordering over the set of machines profiles. For any pair of machine profiles $M = (M_i, M_{-i})$ and $M' = (M_i', M_{-i}')$, we have $M \succ_i^s M'$ if one of the following holds:*

$$\begin{array}{ll} (i) & \pi_i(M_i, M_{-i}) > \pi_i(M_i', M_{-i}') \text{ and } ||M_i|| \leq ||M_i'|| \\ (ii) & \pi_i(M_i, M_{-i}) \geq \pi_i(M_i', M_{-i}') \text{ and } ||M_i|| < ||M_i'|| \; . \end{array}$$

A Nash equilibrium can then be written in terms of machines that are most preferred against each other.

Definition 12 *A machine profile $M^* = (M_1^*, M_2^*)$ constitutes a Nash equilibrium of the machine game (NEM) if $\forall i$ there exists no M_i' such that*

$$(M_i', M_{-i}^*) \succ_i^s (M_i^*, M_{-i}^*) \ .$$

The following Lemma extends Lemma 168.2 in Osborne and Rubinstein [15] (also part (a) of AR's Theorem 1) to the negotiation game. Any NEM must be such that each player's machine uses an equal number of states, and consequently, must correspond to a Nash equilibrium of the negotiation game.

Lemma 10 *Suppose either A is compact and $u_i(\cdot)$ is continuous for all i, or A is finite. Then, if $M^* = (M_1^*, M_2^*)$ is a NEM, we have*

$$\begin{array}{ll} (i) & ||M_1^*|| = ||M_2^*||; \\ (ii) & \pi_i(M^*) = \max\limits_{f_i \in F_i} \pi_i(f_i, M_{-i}^*) \ \ \forall i. \end{array}$$

***Proof*.** Consider machines in the 2SM specification. (The 4SM case can be treated similarly.)

(i) Fix player j's machine $M_j = \{M_{jp}, M_{jr}\}$ where, for $l = p, r$, $M_{jl} = (Q_{jl}, q_{jl}^1, \lambda_{jl}, \mu_{jl})$. Then, consider player i solving his dynamic optimization problem for the machine game ignoring complexity.

Let S_{ik} define the set of player i's *one-period strategies* in the extensive form game that he plays in role $k \in \{p, r\}$ every other period of the negotiation game. We denote its element by $s_{ik} \in S_{ik}$. Let $v_i(s_{ik}, s_{jl})$ denote player i's (one-period) payoff given the pair of strategies.

For each $q_j^t \in Q_{jp} \cup Q_{jr}$, let $V_i(q_j^t) = \max_{f_i \in F_i^t} \pi_i(f_i, M_j(q_j^t))$ (where $M_j(q_j^t)$ is j's machine starting with the sub-machine that is active in period t with initial state q_j^t). Also, $\forall q_j^t \in Q_{jp} \cup Q_{jr}$, let $S_i(q_j^t)$ be the solution set to the problem

$$\max_{s_{ik}} \left\{ v_i(s_{ik}, \lambda_{jl}(q_j^t)) + \delta V_i\left(\mu_{jl}(q_j^t, s_{ik})\right) \right\} \tag{23}$$

where k and l are the role of i and j in period t respectively. Then, i's strategy is optimal given M_j if and only if $\forall q_j^t \in Q_{jp} \cup Q_{jr}$ the one-period strategy it plays when j's machine is in state q_j^t belongs to $S_i(q_j^t)$.[20] (Existence is guaranteed if S_{ip} and S_{ir} are compact, which is true if A is compact and $u_i(\cdot)$ is continuous or if A is finite.)

[20]For the case of machines with a finite state space, this statement is established by the Blackwell's theorem. For the general (countable) state space case, which we consider in the paper, see Hinderer [12] and the references therein.

Let $s_i^*(q_j^t) \in S_i(q_j^t)$ denote an optimal one-period strategy for player i given $q_j^t \in Q_{jp} \cup Q_{jr}$. Also, let $s_i^*(q_j^t, d)$ be the action $s_i^*(q_j^t)$ induces at the partial history d.

Now, consider player i's machine $M_i = \{M_{ip}, M_{ir}\}$ defined as follows: $\forall k, l$, we set $Q_{ik} = Q_{jl}$, $q_{ik}^1 = q_{jl}^1$, $\lambda_{ik}(q_j, d) = s_i^*(q_j, d)$ $\forall q_j \in Q_{jl}$ and $\forall d \in D_{ik}$, and $\mu_{ik}(q_j, e) = \mu_{jl}(q_j, e)$ $\forall e \in E$. This machine implements i's best response to M_j using only the states used by j's machine.

Thus, if $M^* = (M_1^*, M_2^*)$ is a NEM profile, then $||M_i^*|| \leq ||M_{-i}^*||$ $\forall i$. It follows that $||M_1^*|| = ||M_2^*||$.

(ii) This follows from part (i). ‖

Lemma 10 connects our notion of NEMc (Definition 7) with AR's equilibrium notion (Definition 12). Effectively, both definitions take the set of Nash equilibria of the negotiation game and select outcomes that capture some measure of "trade-off" between payoffs and complexity. In this sense, the equilibrium notions used in this paper closely parallel those of AR.

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