A Unified Approach to Information, Knowledge, and Stability*

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Abstract

Within the context of strategic interaction, we provide a unified framework for analyzing information, knowledge, and the "stable" pattern of behavior. We first study the related interactive epistemology and, in particular, show an equivalence theorem between a strictly dominated strategy and a never-best reply in terms of epistemic states. We then explore epistemic foundations behind the fascinating idea of stability due to J. von Neumann and O. Morgenstern. The major features of our approach are: (i) unlike the *ad hoc* semantic model of knowledge, the state space is constructed by Harsanyi's types that are explicitly formulated by Epstein and Wang (*Econometrica* **64**, 1996, 1343-1373); (ii) players may have general preferences, including subjective expected utility and non-expected utility; and (iii) players may be boundedly rational and have non-partitional information structures. *JEL Classification:* C70, C72, D81.

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1 Introduction

In their classics, von Neumann and Morgenstern (1944) enthusiastically advocated the idea of "stability" by introducing a fascinating solution concept of the vN-M stable set. Ever since then the criterion of stability has been widely applied in economics and other social sciences; see, for instance, Lucas (1994) and Shubik (1982) for surveys. Greenberg (1990) took this line of approach one step further by providing an integrated approach to the study of formal models in the social and behavioral sciences. Chwe (1994), Greenberg *et al.* (1996, 2002), Luo (2001), Nakanishi (1999), and Xue (1998) are some examples of recent applications in game theory and economic theory.

While most applications have concentrated on cooperative environments, von Neumann and Morgenstern (1944, Sections 4.6, 4.7, and 65.1) also referred to the idea of stability as the "accepted standard of behavior" in a fairly wide range of social organizations. Recall that a *vN-M (abstract) stable set* is defined as a subset $\mathcal{K}$ of ordered outcomes satisfying the following two conditions:

(1) [internal stability] no y in $\mathcal{K}$ is dominated by an x in $\mathcal{K}$;

(2) [external stability] every y not in $\mathcal{K}$ is dominated by some x in $\mathcal{K}$.

Accordingly, the stability criterion is fully characterized by a pair of principles: internal stability and external stability.

Within the conventional semantic framework, Luo (2002) first explored microfoundations behind the "stable" pattern of behavior. The rationale behind stability was found to be surprisingly abundant by establishing the formal epistemic linkage between stability and Bayesian rationality. Among others, Luo (2002) proved that (i) common knowledge of rationality (c.k.r.) implies an externally stable set that, in turn, is contained in an internally stable set; and (ii) whenever choice sets are mutually known, rationality alone implies a stable set. A major objective of this paper is to further extend this line of research by allowing for rather general information structures and diverse preferences. In particular, we are interested here in seeking epistemic conditions on the set-valued solution concept in extremely general situations.

In this paper we adopt a decision-theoretic approach to game theory, as suggested first by Harsanyi (1967-1968) and further developed later by Tan and Werlang (1988), in which each player's problem of choosing a strategy is cast as a single agent decision problem under uncertainty. By employing Epstein and Wang's (1996) general construction of Harsanyi's types (cf. also Mertens and Zamir (1985) and Brandenburger and Dekel (1993) for constructions of types in the Bayesian framework), we provide a unified framework for analyzing information, knowledge, and the "stable" pattern of behavior. More specifically, we provide an analytical framework in which the state space represents the exhaustive uncertainty facing each player in a strategic setting — i.e., the primitive uncertainty about the choices of strategy by *all* players, as well as the uncertainty about *all* players' types (each type is homeomorphic to an infinite regress of a hierarchy of "preferences over preferences"). This paper is thereby closely related to Epstein's (1997) work on the study of rationalizability and equilibrium by considerably relaxing the definition of rationality. As emphasized above, this paper focuses on the epistemic analysis of the solution concept of the stable set.

There are three primary reasons for pursuing the study of this paper. Firstly, experimental evidence such as the Ellsberg Paradox contradicts some of the tenets in the Savage model; for example, the Sure-Thing Principle. In particular, decision makers usually display an aversion to uncertainty or ambiguity (see Epstein (2000) for a market counterpart of the Ellsberg Paradox). Under the presumption that uncertainty is important in strategic settings, concern with descriptive accuracy, it is hence a significant research subject to study games where players might have general preferences, including subjective expected utility and non-expected utility; see, e.g., Dow and Werlang (1994), Epstein (1997), Ghirardato and Le Breton (2000), Klibanoff (1993, 1996), Lo (1996, 1999), Luo and Ma (2001), and Marinacci (2000). In the same vein, this paper investigates epistemic foundations behind the set-valued solution concept of the stable set in games where players might exhibit general preferences.

Secondly, much of the work on the epistemic foundations of game-theoretic solution concepts has been done within the *ad hoc* semantic framework (see, e.g., Aumann (1976, 1987, 1995, 1999), Aumann and Brandenburger (1995), and Brandenburger and Dekel (1987)), in which the information structure is assumed to be partitional. However, weakening the assumptions on information is clearly appealing since the assumption of a partitional information structure is rather restrictive in many economic applications (see, e.g., Bacharach (1985), Dekel and Gul (1997), Geanakoplos (1989, 1994), Luo and Ma (2003), Morris (1996), Rubinstein (1998), Samet (1990), and Shin (1993) for discussions; in particular, Brandenburger *at al.'s* (1992) work on correlated equilibrium with generalized information structures). By making use of Epstein and Wang's (1996) general construction of types, the proposed framework in this paper allows players to be boundedly rational and have a non-partitional information structure (see Rubinstein 1998, Chapter 3); for example, players may be "unaware of awareness," "ignoring ignorance," or even convinced of something objectively incorrect — i.e., they might fail to satisfy the basic axioms of knowledge: *the axiom of knowledge, the axiom of transparency,* and *the axiom of wisdom.*

Thirdly, there is a well-known philosophical difficulty with the conventional semantic framework used in game theory. The difficulty is that the notion of a *state of the world,* or simply a *state,* may be self-referential since it consists of a specification of information, knowledge, and strategy.[1] Within the proposed framework in this paper, as a type associated with a state is explicitly constructed from hierarchies of preferences over the constructed state space, the comprehensive representation of a state allows for eliciting, as not being *ad hoc,* all aspects of the *full* description of the world, including information, knowledge, preferences, and the choice of strategy. Our approach, equipped with a rich state space, therefore is immunized from this self-referential criticism. At a conceptual level, our approach is also significant, because within the framework in this paper, the assumption that the model of knowledge is commonly

[1]See, e.g., Aumann (1999, p. 264) and Osborne and Rubinstein (1994, p. 77). As Fagin *et al.* (1999, p. 332) articulated: "The problem is that it is not *a priori* clear what the relation is between a state in an Aumann structure — which is, after all, just an element of a set — and the rather complicated reality that this state is trying to model. ...... This seems to lead to circularity, since the partitions are defined over the states, but the states contain a description of the partitions. One particularly troubling issue, already mentioned in Aumann's original paper, is how the states can be used to capture knowledge about the model itself, such as the fact that the partitions are common knowledge."

known can be stated formally (see 4.1), whereas this sort of assumption must be understood informally in a meta-sense. Moreover, the proposed framework offers a thorough set-up for thinking about the set-valued game-theoretic solution concepts, like the vN-M stable set; it suggests a novel interpretation for the "choice set" associated with a state (see 4.2).

To conclude this introduction, we provide a perspective on the main results of this paper. As a state constructed by Harsanyi's notion of a type can be viewed as the counterpart in the conventional semantic framework, we start (in Subsection 2.3) by investigating the relationship between two different definitions of knowledge. To be sure, while Aumann (1976) and much work that followed defined "knowledge" in terms of an *exogenous* information structure, the notion of knowledge used in this paper is defined *endogenously* as a property of preferences over acts — roughly speaking, a known event's complement is required to be null in the sense of Savage. It is shown that the notion of knowledge elicited from (state-type-)preferences is consistent with the one defined in a semantic fashion (see $K\mathbf{5}$ in Lemma 1). This relationship allows us to exploit the relative familiar and simple semantic way of analysis whenever doing so is more convenient. It is noteworthy to mention that, along the line of Savage's (1954) choice-theoretic approach, Morris (1996, 1997) offered a "similar" framework in which the notions of information and knowledge can be deduced from *exogenously* specified preferences at a state. As a by-product, we extend Morris' (1996) results on properties of knowledge to an *infinite* state space (see K**1-5** in Lemma 1).

We next relate the notion of "payoff dominance" in games to the notion of "never-best response" in terms of epistemic states by establishing an equivalence theorem (Theorem 1). As the former one is used to define the stable set (see Definition 1) on the one hand and the latter one is referred to epistemic rationality on the other hand, the equivalence theorem will play a "bridging" role to link stability with rationality. The main argument for the equivalence theorem is using Glicksberg's Fixed Point Theorem.

We lastly turn to the study of the epistemic foundation for a stable set within the proposed framework in this paper. In order to deal with a set-valued solution concept, we define the notion of "rationality" by requiring that, in face of epistemic uncertainty, the set of plausible strategy choices consist of all the best replies (see Subsection 2.4). Under rather mild conditions, it is shown that rationality as well as c.k.r. prescribe the "stable" pattern of strategy behavior that coincides with iterated strict dominance (Theorem 2). If, in particular, the set of strategy choices is publicly known as a "social norm," rationality coincides with stability (see (3.2) in Theorem 3). The basic logic for these results runs as follows: As is well-known, c.k.r. leads to a set with the "best response" property (see footnote 4) and through our equivalence theorem gives rise to the principle of internal stability. The argument for the principle of external stability relies heavily on Zorn's Lemma (see Lemma 5 in Appendix II). As Epstein and Wang's (1996) general construction of types accommodate, e.g., the ordinal expected utility (Borgers 1993), Choquet expected utility (Schmeidler 1989), and probabilistically sophisticated preferences (Machina and Schmeidler 1992), this paper thus extends Luo's (2002) work to very general preferences (Theorem 3). We would like to point out that there is a discrepancy between (3.1) in Theorem 3 and the aforementioned result (i) in Luo (2002). The discrepancy can be attributed mainly to the completeness of a state space (cf. Lemma 7 in Appendix II).

The sequel of this paper is organized as follows. Section 2 offers a unified framework for analyzing information, knowledge, and the "stable" pattern of behavior. We set up the framework by resorting to some familiar apparatus in the traditional game-theoretic literature so that it is easily accessible to game theorists. Subsection 2.1 introduces the notion of stability; Subsection 2.2 models games in terms of epistemic states; Subsection 2.3 investigates the related interactive epistemology; and Subsection 2.4 establishes a fundamental equivalence theorem between a strictly dominated strategy and a never-best reply in terms of epistemic states. Section 3 studies epistemic foundations for stability. Subsection 3.1 introduces the notion of rationality; and Subsection 3.2 presents the main results about information, knowledge, and stability. Section 4 is devoted to discussions. For convenience, the precise definitions of "regular preferences" and "marginal consistency" are summarized in Appendix I. To facilitate reading, all the proofs are relegated to Appendix II.

2 The Analytic Framework

In this section we provide a unified framework for analyzing strategic behavior as well as its related interactive epistemology. Throughout this paper, we consider an *n-person strategic game* $\mathcal{G} \equiv (N, \{X_i\}, \{\zeta_i\})$, where X_i, for each $i \in N$, is a compact convex metric space of player i's strategies, and $\zeta_i : X \rightarrow [0,1]$ (where $X \equiv \times_{i \in N} X_i$) is a continuous payoff function that assigns each strategy profile $x \in X$ to a number in $[0,1]$. For any subset $Y \subseteq X$, a strategy y_i is *strictly dominated given* Y if there exists $x_i \in X_i$ such that $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$, where $Y_{-i} \equiv \{y_{-i} | \ (x_i, y_{-i}) \in Y$ for some $x_i \in X_i\}$.

2.1 Stability

For the purpose of this paper, we employ the following natural extension of the notion of a vN-M stable set (see Luo (2001)).

Definition 1. A subset $\mathcal{K} \subseteq X$ is a *(general) stable set* if it is a vN-M stable set with respect to $\succ^{\mathcal{K}}$, where $x \succ^{\mathcal{K}} y$ iff, for some i, x_i strictly dominates y_i given $\mathcal{K}$.

That is, a stable set $\mathcal{K}$ satisfies:

(1) [internal stability] $\forall x \in \mathcal{K}$, $y \not\succ^{\mathcal{K}} x$ for all $y \in \mathcal{K}$, and

(2) [external stability] $\forall x \notin \mathcal{K}$, $y \succ^{\mathcal{K}} x$ for some $y \in \mathcal{K}$.

In other words, $\mathcal{K}$ is free of inner contradictions — i.e., no element in $\mathcal{K}$ can be dominated by an element in $\mathcal{K}$, with respect to the conditional dominance relation $\succ^{\mathcal{K}}$. Furthermore, $\mathcal{K}$ is free of external inconsistencies — i.e., any element outside $\mathcal{K}$ is dominated by some element in $\mathcal{K}$, with respect to the conditional dominance relation $\succ^{\mathcal{K}}$. Clearly, every stable set is in Cartesian-product form. It is worthwhile to point out that, by the equivalence theorem in Subsection 2.4, this notion of a stable set is closely related to Basu and Weibull's (1991) notion of a "tight" curb.

Example 1. Consider a two-person game $\mathcal{G} = (N, \{X_i\}, \{\zeta_i\})$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and for all $x_i, x_j \in [0, 1]$, $i, j = 1, 2$ and $i \neq j$, $\zeta_i(x_i, x_j) = x_i x_j + (1 - x_i)(1 - x_j)$. Let $\mathcal{K}^1 \equiv \{(0, 0)\}$, $\mathcal{K}^2 \equiv \{(0.5, 0.5)\}$, $\mathcal{K}^3 \equiv \{(1, 1)\}$ and $\mathcal{K}^4 \equiv [0, 1] \times [0, 1]$.

Since $\zeta_i(x_i, 0) = 1 - x_i$, 0 strictly dominates any x_i in $(0, 1]$ given $\mathcal{K}^1$. Therefore, $\mathcal{K}^1$ is externally stable. As every singleton is internally stable, $\mathcal{K}^1$ is a stable set. Similarly, $\mathcal{K}^3$ is a stable set.

Since $\zeta_i(x_i, 0.5) = 1$, no x_i in $[0, 1]$ is strictly dominated given $\mathcal{K}^2$. Thus, $\mathcal{K}^2$ violates external stability. As $\mathcal{K}^2 \subset \mathcal{K}^4$, every x_i in $[0, 1]$ is not strictly dominated given $\mathcal{K}^4$. Therefore, $\mathcal{K}^4$ is internally stable. Since the set of all strategy profiles trivially satisfies external stability, $\mathcal{K}^4$ is a stable set.

2.2 Games in terms of epistemic states

In game $\mathcal{G}$, each player (as a decision maker) faces uncertainty not only about the primitive uncertainty corresponding to the strategy choices, but also about players' types in Harsanyi's sense. Accordingly, the state space of states of the world is constructed as: $\Omega \equiv X \times T_1 \times T_2 \times \ldots \times T_n$, where T_i is the space of player i's types. We refer to an element $\omega \in \Omega$ as a *state* and to a (Borel measurable) subset $E \subseteq \Omega$ as an *event*. Denote by t_i^{ω} player i's type projected at ω, and denote by x^{ω} the strategy profile at ω. Thus, a state ω can be written as $(x^{\omega}; t_1^{\omega}, t_2^{\omega}, ..., t_n^{\omega})$.

The objects of each player's choice are *acts*; i.e., Borel measurable functions $f : \Omega \to [0, 1]$. Denote by $\mathcal{F}(\Omega)$ the set of a player's acts and by $\mathcal{P}(\Omega)$ the set of the *preferences over* $\mathcal{F}(\Omega)$. Throughout this paper, we restrict ourselves to the subclass of regular preferences that admit representation by utility functions — i.e., the subclass of regular preferences that satisfy U.1-6 and U.2′ in Appendix I. Based upon Epstein and Wang's (1996) Theorem 6.1, $T_i \sim^{homeomorphic} \mathcal{P}(\Omega)$,[2] and let $\psi : T_i \to \mathcal{P}(\Omega)$ represent such a homeomorphism. Write the utility function associated with t_i^{ω} freely as $\psi \circ t_i^{\omega}$ or u_i^{ω} for convenience.

A strategy $x_i \in X_i$ is referred to as an act $x_i : X \to [0, 1]$, satisfying $x_i(x') = \zeta_i(x_i, x'_{-i})$ for all $x' \in X$. (The strategy x_i is also referred to as an act from

[2] Each player's type space is homogeneous and each player may be ignorant of his own types (cf. 4.3).

Ω to $[0,1]$, satisfying $x_i(\omega) = \zeta_i\left(x_i, \overset{\omega}{x}_{-i}\right)$.) Let $\mathcal{P}_i(X)$ denote the set of the preferences over the set of acts $f : X \to [0,1]$, satisfying $f(x_i, x_{-i}) = f(x_i', x_{-i})$ for all (x_i, x_{-i}) and (x_i', x_{-i}) in X. In what follows, we assume that $\mathcal{P}(E)$ and $\mathcal{P}_i(Y)$ are well defined for any $E \subseteq \Omega$ and $Y \subseteq X$. For the sake of brevity, we use $u_i(x_i)$ to represent the utility of the restriction of x_i to E (or Y) if $u_i \in \mathcal{P}(E)$ (or $u_i \in \mathcal{P}_i(Y)$). Let $X^E \equiv \left\{\overset{\omega}{x} \mid \omega \in E\right\}$. By marginal consistency in Appendix I, $\mathcal{P}(E)$ and $\mathcal{P}_i\left(X^E\right)$ can be treated as the same provided that preferences refer only to player i's strategies.

Given an event E, let $\mathcal{P}(\Omega|E)$ denote the set of i's preferences for which the complement of E is null in the sense of Savage; i.e., any two acts that agree on E are ranked as indifferent. Say i *knows* E *at* ω if there exists a closed subset $\overline{E} \subseteq E$ such that $\psi \circ \overset{\omega}{t_i} \in \mathcal{P}(\Omega|\overline{E})$. (Some reader may prefer the term "believes E" rather than "knows E.") Let K_iE denote the set of all the states where i knows E; i.e.,

$$K_iE \equiv \left\{\omega \in \Omega |\ \psi \circ \overset{\omega}{t_i} \in \mathcal{P}(\Omega|\overline{E}) \text{ for some closed set } \overline{E} \subseteq E\right\}.$$

Thus, for a closed set E, $K_iE = \left\{\omega \in \Omega |\ \psi \circ \overset{\omega}{t_i} \in \mathcal{P}(\Omega|E)\right\}$. Player i's information structure generated by the knowledge operator K_i is the correspondence $P_i : \Omega \rightrightarrows \Omega$, such that for all $\omega \in \Omega$,

$$P_i(\omega) = \bigcap_{\{E \subseteq \Omega |\ K_iE \ni \omega\}} E.$$

The set $P_i(\omega)$ represents *all* aspects of uncertainty on the part of player i — including uncertainty about *all* players' strategic behavior, uncertainty about the uncertainty of *all* players' strategic behavior, and so on *ad infinitum*. It constitutes the standard model for "differential" information.

Example 2. A state ω^* is said to be a *Nash state* in $\mathcal{G}$ if, for all i,

$$\psi \circ \overset{\omega^*}{t_i}\left(\overset{\omega^*}{x_i}\right) \geq \psi \circ \overset{\omega^*}{t_i}(x_i) \text{ for all } x_i \in X_i,$$

where $\overset{\omega^*}{x}_{-i} = \overset{\omega}{x}_{-i}$ for all $\omega \in P_i(\omega^*)$.[3] The profile $\overset{\omega^*}{x}$ is said to be a Nash equilibrium under general preferences.

[3]By marginal consistency, $mrg_{\mathcal{F}_i(X)}\psi \circ \overset{\omega^*}{t_i} \in \mathcal{P}_i\left(X^{P_i(\omega^*)}\right)$. Let $u_i^*\left(x_i, \overset{\omega^*}{x}_{-i}\right) \equiv mrg_{\mathcal{F}_i(X)}\psi \circ \overset{\omega^*}{t_i}(x_i)$. For all i, $u_i^*\left(\overset{\omega^*}{x_i}, \overset{\omega^*}{x}_{-i}\right) \geq u_i^*\left(x_i, \overset{\omega^*}{x}_{-i}\right)$ for all $x_i \in X_i$.

2.3 Interactive epistemology

In this subsection, we start with discussing the related interactive epistemology within the set-up in Subsection 2.2. We first list some important properties satisfied by the knowledge operator K_i and the information correspondence P_i.

Lemma 1. *K_i and P_i satisfy the following properties:*

$K1$: $K_i\emptyset = \emptyset$.

$K2$: $K_i\Omega = \Omega$.

$K3$: $E \subseteq F \Rightarrow K_iE \subseteq K_iF$.

$K4$: $\bigcap_{\lambda\in\Lambda} K_iE^{\lambda} \subseteq K_i(\bigcap_{\lambda\in\Lambda} E^{\lambda})$ *for a family of closed subsets* $\{E^{\lambda}\}_{\lambda\in\Lambda}$.

$K5$: $K_iE = \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$.

$P1$: $P_i(\omega)$ *is nonempty and closed.*

$P2$: $P_i(\omega) = P_i(\omega')$ *whenever* $t_i^{\omega} = t_i^{\omega'}$.

Remark 1. The knowledge operator K_i may fail to satisfy the other three axioms of knowledge: *the axiom of knowledge*, *the axiom of transparency*, and *the axiom of wisdom* (i.e., $K_iE \subseteq E$, $K_iE \subseteq K_i(K_iE)$, and $\Omega \backslash K_iE \subseteq K_i(\Omega \backslash K_iE)$); in particular, the information structure is possibly non-partitional. The property $K5$ gives an alternative semantic definition of knowledge.

We next introduce the notion of "common knowledge." Roughly speaking, an event is *common knowledge* if everyone knows it, and everyone knows that everyone knows it, and everyone knows that everyone knows that everyone knows it, and so on *ad infinitum.* For any $E \subseteq \Omega$, let $KE \equiv \cap_{i\in N} K_iE$ and let $K^lE \equiv K\left(K^{l-1}E\right)$ for all $l \geq 2$ (where $K^1E \equiv KE$). Define

$$CKE \equiv KE \cap K^2E \cap K^3E \cap ....$$

That is, CKE is the event that E is commonly known. Say E is an *self-evident* event if $E \subseteq KE$. Without referring to *any* axiom of knowledge, we have the following useful properties about the common knowledge operator CK.

Lemma 2. *CK satisfies the following properties:*

$CK1$: $CKE = K(E \cap CKE)$.

$CK2$: $\omega \in CKE$ *if* E *is a self-evident event containing* ω.

2.4 Equivalence theorem

In this subsection we formulate the notion of "best response" in terms of epistemic states and then establish a fundamental equivalence theorem between a "strictly dominated strategy" and a "never-best response."

Definition 2. A strategy x_i is a *best response given* $E \subseteq \Omega$ if, for some $\omega \in K_i E$, $u_i^{\omega}(x_i) \geq u_i^{\omega}(y_i)$ for all $y_i \in X_i$. That is, a strategy y_i is a *never-best response given* $E \subseteq \Omega$ if, for every $\omega \in K_i E$, $u_i^{\omega}(x_i) > u_i^{\omega}(y_i)$ for some $x_i \in X_i$.

Theorem 1. *Let $E \subseteq \Omega$ be nonempty and compact. Then, a strategy y_i is a never-best response given E if, and only if, it is strictly dominated given X^E.*

Similarly, a strategy y_i is said to be a *never-best response given* $Y \subseteq X$ if, for every $u_i \in \mathcal{P}_i(Y)$, $u_i(x_i) > u_i(y_i)$ for some $x_i \in X_i$. An immediate implication of Theorem 1 is the following.

Corollary 1. *Let $Y \subseteq X$ be nonempty and compact. Then, a strategy y_i is a never-best response given Y if, and only if, it is strictly dominated given Y.*

Remark 2. In the case of finite games with expected utility, Pearce (1984) first proved that a strategy is a never-best response if and only if it is strictly dominated; see also Luo's (2002) Lemma 1. Corollary 1 thereby extends the result to infinite games with general preferences.

We end this section by providing an example to illustrate that, without the compactness assumption, the equivalence theorem could not be true.

Example 3. Consider a two-person game $\mathcal{G} = (N, \{X_i\}, \{\zeta_i\})$, where $N = \{1, 2\}$, $X_1 = X_2 = [0, 1]$, and for all $x_i, x_j \in [0, 1]$, $i, j = 1, 2$ and $i \neq j$, $\zeta_i(x_i, x_j) = 1 - (x_i - x_j)^2$. Let us consider a noncompact event $E = (0, 1] \times (0, 1] \times \{(t_1, t_2)\}$ where $t_i \in T_i$.

Let $\omega \in K_i E$. Then, $u_i^{\omega} \in \mathcal{P}(\Omega | \overline{E})$ for some closed $\overline{E} \subseteq E$. Define $r \equiv \inf_{x_j \in X_j^{\overline{E}}} x_j$. Clearly, $r > 0$ and $\zeta_i(r, x_j) > \zeta_i(0, x_j)$ for all $x_j \in X_j^{\overline{E}}$. By strong monotonicity in Appendix I and Epstein and Wang's (1996) Theorem 4.3, $u_i^{\omega}(r) > u_i^{\omega}(0)$. Therefore, 0 is a never-best response given E. However, since for all $x_i \in (0, 1]$, $\zeta_i(0, x_j) > \zeta_i(x_i, x_j)$ if $x_j < x_i/2$, 0 is not strictly dominated given X^E.

3 The Epistemic Foundation of Stability

3.1 Rationality

From an epistemic perspective, at a state ω, player i knows only the set $P_i(\omega)$. That is, he considers it possible that the true state could be any state in $P_i(\omega)$, but not any state outside $P_i(\omega)$. In particular, at that state player i can conclude only that all his plausible choices of strategy are within the scope of $X_i^{P_i(\omega)}$. We refer to $X_i^{P_i(\omega)}$ as i's *choice set.* To do an epistemic analysis of the set-valued solution concept, we therefore define the notion of "rationality" by requiring that $X_i^{P_i(\omega)}$ consist of all the best replies in face of epistemic uncertainty $P_i(\omega)$. Formally, let

$$BR_i(\omega) \equiv \{x_i \in X_i| \ x_i \text{ is a best response given } P_i(\omega)\}.$$

Define i *is rational at* ω if $X_i^{P_i(\omega)} = BR_i(\omega)$. Let $R_i \equiv \{\omega \in \Omega| \ i \text{ is rational at } \omega\}$. Let $R \equiv \cap_{i \in N} R_i$ denote the event that "everyone is rational."

3.2 Foundation of stability

Up until now, we have imposed no essential condition on regular preferences and hence have allowed for a rather arbitrary knowledge and information structure. In conducting an epistemic analysis of a game-theoretic solution concept, throughout this subsection we impose a *weak* axiom of knowledge (for each player i) — i.e., $X_i^{K_i E} \subseteq X_i^{E}$ whenever $E \subseteq R \cap CKR$. In other words, whenever a player knows an event of "rationality" and "common knowledge of rationality," then this would not be false in terms of his strategy dimension. We are now in a position to present a central result of this section.

Theorem 2. $X^{R \cap CKR}$ *is the largest (w.r.t. set inclusion) stable set and, moreover, yields iterated strict dominance.*

An immediate implication of Theorem 2 is the following.

Corollary 2. $X^{R \cap CKR}$ *is the set of all rationalizable strategy profiles.*[4]

[4] We here use the correlated version of Bernheim (1984) and Pearce's (1984) rationalizability concept. Formally, a subset $Y \subseteq X$ is said to have the "best response property" if, for every $y \in Y$, every player i's strategy y_i is a best response given Y. The set of *rationalizable strategy profiles* is defined as the largest set with the best response property (cf. Epstein's (1997) Definition 3.1).

Next, we extend Luo's (2002) main results to the general framework of this paper. For this purpose, we assume the following two conditions on each player i's information structure:

A1. $X_j^{P_j(\omega)} \subseteq X_j^{P_i(\omega)}$ for all j.

A2. $\times_{j \in N} X_j^{P_i(\omega)} \subseteq X^{P_i(\omega)}$.

That is, A1 states that each player has better information regarding his own choice(s) than an opponent does; A2 states that in purely noncooperative situations, each player would be aware of the "independence" of players' choices. Let $\Psi(\omega) \equiv \times_{i \in N} X_i^{P_i(\omega)}$ denote the Cartesian product of players' choice sets.

Theorem 3. *(3.1) Suppose $\omega \in R$. Then, $\Psi(\omega)$ is an externally stable set and, moreover, there is a stable set $\mathcal{K} \supseteq \Psi(\omega)$ whenever $\omega \in CKR$. (3.2) Suppose that $X^{P_i(\omega)} \subseteq \Psi(\omega)$ for all i — i.e., every player knows "choice sets". Then, $\omega \in R$ iff $\Psi(\omega)$ is a stable set. (3.3) Suppose that $\boxed{\mathcal{K}}$ is a self-evident event satisfying $\boxed{\mathcal{K}} = \left\{\omega \in \Omega \mid X^{P_i(\omega)} \subseteq \Psi(\omega) = \mathcal{K}\right\}$. Then, $\boxed{\mathcal{K}} \subseteq R \cap CKR$ whenever $\mathcal{K}$ is a stable set. (3.4) For any compact stable set $\mathcal{K}$, there is $\omega \in R \cap CKR$ such that $\Psi(\omega) = \mathcal{K}$.*

Remark 3. Following J. von Neumann and O. Morgenstern, a stable set is viewed as a prevailing social norm in a society. Accordingly, a social norm is "well known to the community" (see Shubik 1982, p. 261). Under this sort of assumption of social knowledge, (3.2) states that the "stable" pattern of behavior is sustained by rational players and moreover, the "stable" pattern of behavior is attributed only to rational players. The following two examples illustrate that the conditions in (3.2) are indispensable.

Example 3 Continued. In the game $\mathcal{G}$ of Example 3, let us consider two cases:

Case 1. Let $\omega \in \Omega$ satisfy $P_1(\omega) = [2/3, 1] \times [2/3, 1] \times \{t^\omega\}$ and $P_2(\omega) = [0, 1] \times [0, 1] \times \{t^\omega\}$.[5] Then, $\Psi(\omega) = [2/3, 1] \times [0, 1]$. In this case, 2 does not know $\Psi(\omega)$ since $X^{P_2(\omega)} \nsubseteq \Psi(\omega)$. Clearly, $\omega \in R$. However, since for any $x_1 \in [2/3, 1]$,

$$\zeta_2(x_1, 1) \geq 8/9 > 5/9 \geq \zeta_2(x_1, 0),$$

1 strictly dominates 0 given $\Psi(\omega)$. Therefore, $\Psi(\omega)$ violates internal stability.

[5] See Lemma 7 in Appendix II for existence of such a state.

Case 2. Let $\omega \in \Omega$ satisfy $P_1(\omega) = [0,1] \times [0,1] \times \{t^{\omega}\}$ and $P_2(\omega) = [2/3,1] \times [0,1] \times \{t^{\omega}\}$. Then, $\Psi(\omega) = [0,1] \times [0,1]$. In this case, A1 is violated since $X_1^{P_1(\omega)} \nsubseteq X_1^{P_2(\omega)}$. Clearly, $\Psi(\omega)$ is a stable set. However, since for each $x_2 \in [0, 2/3)$,

$$\zeta_2(2/3, x_1) > \zeta_2(x_2, x_1) \text{ for all } x_1 \in [2/3, 1],$$

by U.2′, $u_2(2/3) > u_2(x_2)$ for all $u_2 \in \mathcal{P}(\Omega|P_2(\omega))$. That is, player 2 is not rational at ω.

Example 4. Consider a three-person game $\mathcal{G} = (N, \{X_i\}, \{\zeta_i\})$, where $N = \{1,2,3\}$, $X_i = [0,1]$, and for all $x_i, x_j, x_k \in [0,1]$, $i,j,k = 1,2,3$, $i \neq j$, $i \neq k$ and $j < k$, $\zeta_i(x_i, x_j, x_k) = 1 - [x_i - (2x_j - x_k)]^2$. Let $\omega \in \Omega$ satisfy

$$P_1(\omega) = [0,1/2] \times \{(x_2,x_3)|\ x_2 = x_3,\ x_2 \in [0,1/2]\} \times \{t^{\omega}\},$$

$$P_2(\omega) = [0,1/2] \times \{(x_1,x_3)|\ x_1 = x_3,\ x_3 \in [0,1/2]\} \times \{t^{\omega}\}, \text{ and}$$

$$P_3(\omega) = [0,1/2] \times \{(x_1,x_2)|\ x_1 = x_2,\ x_1 \in [0,1/2]\} \times \{t^{\omega}\}.$$

Then, $\Psi(\omega) = [0,1/2] \times [0,1/2] \times [0,1/2]$. In this case, A2 is violated since $(0,1/2,0) \notin X^{P_1(\omega)}$, for example. Clearly, $\omega \in R$. However, since for $x_2 = 1/2$ and $x_3 = 0$,

$$\zeta_1(1,1/2,0) = 1 > 3/4 \geq \zeta_1(x_1,1/2,0) \text{ for all } x_1 \in [0,1/2],$$

every $x_1 \in \Psi_1(\omega)$ does not strictly dominate 1 given $\Psi(\omega)$. As $1 \notin \Psi_1(\omega)$, $\Psi(\omega)$ violates external stability.

We end this section by providing an example of a noncompact stable set, which illustrates that the compactness condition in (3.4) is indispensable.

Example 5. Consider a two-person game $\mathcal{G} = (N, \{X_i\}, \{\zeta_i\})$, where $N = \{1,2\}$, $X_1 = X_2 = [0,1]$, and for all $x_i, x_j \in [0,1]$, $i,j = 1,2$ and $i \neq j$, $\zeta_i(x_i,x_j) = \min\{x_i, x_j\}$. Let $\mathcal{K} \equiv (0,1] \times (0,1]$. Since if $x_i \neq 0$ and $x_j \neq 0$, then $\zeta_i(x_i,x_j) > 0$, 0 is strictly dominated given $\mathcal{K}$. Therefore, $\mathcal{K}$ is externally stable. Moreover, any $x_i \in (0,1]$ is not strictly dominated given $\mathcal{K}$ since for $x_j = x_i$, $\zeta_i(x_i,x_j) \geq \zeta_i(x_i', x_j)$ for all $x_i' \in [0,1]$. Therefore, $\mathcal{K}$ is internally stable. Thus, $\mathcal{K}$ is a noncompact stable set.

4 Discussions

4.1 Epistemic games. Note that a strategic game $\mathcal{G} \equiv (N, \{X_i\}, \{\zeta_i\})$ does not specify players' preferences in the face of uncertainty; it specifies only players' payoff functions ζ_i. From an epistemic perspective, a complete outcome of the game $\mathcal{G}$ is summarized by a state. A "transparent" game associated with $\mathcal{G}$ is determined by epistemic types. Formally, a "transparent" game at type profile t is defined as:

$$(\mathcal{G}, \psi \circ t) \equiv \left\{\omega \in \Omega | \ \left(\overset{\omega}{t_1}, \overset{\omega}{t_2}, \ldots, \overset{\omega}{t_n}\right) = t\right\},$$

where $\psi \circ t = (\psi \circ t_1, \psi \circ t_2, \ldots, \psi \circ t_n)$. The whole state space Ω can be viewed as an "opaque" game in terms of

$$\Omega = \bigcup_{t \in T_1 \times T_2 \times \ldots \times T_n} (\mathcal{G}, \psi \circ t).$$

Moreover, the game associated with a collection of preference models $\mathcal{P}^* (\Omega) \equiv (\mathcal{P}_1^* (\Omega), \mathcal{P}_2^* (\Omega), \ldots, \mathcal{P}_n^* (\Omega))$ in the sense of Epstein (1997) is given by

$$(\mathcal{G}, \mathcal{P}^* (\Omega)) = \bigcup_{(u_1, u_2, \ldots, u_n) \in \mathcal{P}_1^* (\Omega) \times \mathcal{P}_2^* (\Omega) \times \ldots \times \mathcal{P}_n^* (\Omega)} (\mathcal{G}, (u_1, u_2, \ldots, u_n)).$$

Within our framework in this paper, the statement "a game is common knowledge" is a formal statement rather than an informal "meta-sense": A game is common knowledge if, and only if, the game, as a subset of states, is commonly known (see also Zamir and Vassilakis 1993, pp. 496-497). For example, the "opaque" game is commonly known.

4.2 The rationale for associating a set with a state. To do an epistemic analysis of the set-valued solution concept of the stable set, it is easy to see that we have to associate a set with a state. Within the conventional semantic framework, Luo (2002) studied epistemic foundations behind the criterion of stability. In particular, at a state ω, player i is *exogenously* associated with a nonempty subset of strategies $\Psi_i (\omega)$. In our framework in this paper, this set should be viewed as *endogenous* since it is deduced from the information structure $P_i (\omega)$, i.e., $\Psi_i (\omega) = X_i^{P_i(\omega)}$.

While in Savage's framework of a single-person's decision making, the decision maker would be well aware of his choice that affects no states, this is not appropriate here. In the context of strategic interaction, each player's choice of strategy should be included in the description of a state since each player must take into account the choices of the other players. For example, the choice of strategy by i should depend on the choice of strategy by j that, in turn, should depend on the choice of strategy by i.[6] Indeed, a player may not know his own choice of strategy in games with imperfect recall (cf. Rubinstein 1998, Chapter 4).

Of course, a player can do whatever he wants, but he might *not* know what it is he wants, because what a player wants to do often depends on what others want to do (see also 4.3). Consequently, if a player unconsciously makes a choice, then he certainly does not know his own choice; if a player consciously makes a choice, then he perhaps does not know his own choice, because the player might not know what it is he wants. Although a state of the world does specify a strategy for a player, the player simply may not know his own strategy in the face of epistemic uncertainty. What he knows is only the scope of strategies. The correlation of strategy allowed in our framework could be another origin for the ignorance of one's own strategy choice.

It is easy to see that i knows his strategy x_i^ω at ω if, and only if, $X_i^{P_i(\omega)} = \{x_i^\omega\}$. From an epistemic viewpoint, the requirement that a player knows his own using strategy seems to be rather a restrictive assumption in strategic settings. The following example demonstrates this point.

Example 6. Consider a two-person game $\mathcal{G}$. For simplicity, we consider only the probabilistic notion of knowledge — i.e., "belief with probability 1." Consider four states as follows:

[6] J. von Neumann and O. Morgenstern offered a *defensive and concealment* rationale for mixing play in zero-sum games:

> Thus one important consideration for a player in such a game is to protect himself against having his intentions found out by his opponent. Playing several such strategies at random, so that only their probabilities are determined is a very effective way to achieve a degree of such protection: By this device the opponent cannot possibly find out what the player's strategy is going to be, since the player does not know it himself (von Neumann and Morgenstern 1944, p. 146).

Therefore, this classical rationale posits that a player may show a tendency to *consciously choose not to know his choice.* See also Reny and Robson (2002).

$$\begin{cases} \omega_1 = (x_1, x_2; t_1, t_2) \\ \omega_2 = (x_1', x_2; t_1, t_2) \\ \omega_3 = (x_1, x_2; t_1', t_2) \\ \omega_4 = (x_1', x_2; t_1', t_2) \end{cases}.$$

Since $T_i \sim^{homeomorphic} \Delta(X \times T_i \times T_j)$, we let $\mu_{t_1} = \psi \circ t_1$ and $\mu_{t_1'} = \psi \circ t_1'$ such that

$$\mu_{t_1}(\omega_i) = \begin{cases} 1/2, & \text{if } i = 1, 2 \\ 0, & \text{if } i = 3, 4 \end{cases} \text{ and } \mu_{t_1'}(\omega_i) = 1/4 \text{ for } i = 1, 2, 3, 4.$$

Thus, we have

$$P_1(\omega) = \begin{cases} \{\omega_1, \omega_2\}, & \text{if } \omega = \omega_1, \omega_2 \\ \{\omega_1, \omega_2, \omega_3, \omega_4\}, & \text{if } \omega = \omega_3, \omega_4 \end{cases}.$$

While player 1 knows his own type at ω_1, he does not know his own strategy at that state since $X_1^{P_i(\omega)} = \{x_1, x_1'\}$.

4.3 Ignorance of own type. Note that $T_i \sim^{homeomorphic} \mathcal{P}(\Omega)$. A player with an epistemic type is uncertain not only about the strategy profiles, but also about the type profiles.[7] In particular, the player is uncertain about his own types or own preferences (see also Heifetz and Samet's (1998, p. 330) Remark). In Example 6, at ω_3 player 1 does not know whether his type is t_1 or t_1'.

In the case of a single-person decision making, this viewpoint relates to the decision maker's introspection — i.e., he is uncertain not only about the true *state of nature*, but also about his preferences about this uncertainty, his preferences about his preferences about this uncertainty, and so on. As Epstein and Wang (1996, p. 1352) wrote, "... it seems natural given an agent who does not perfectly understand the nature of the primitive state space ... and who reflects on the nature and degree of his misunderstanding. ... uncertainty about own preferences has been shown to be useful also in modeling preference for flexibility (Kreps (1979)) and behavior given unforeseen contingencies (Kreps (1992))." The viewpoint of the ignorance of one's own type puts forward a novel interpretation for using the notion of choice sets in orthodox choice theory.

[7] To expound his theory of games with incomplete information, Harsanyi (1967, p. 171) articulated that: "Each player is assumed to know his own actual type" (cf. also Harsanyi 1995, p. 296). To make sense of the notion of a Bayesian equilibrium, each player should also be aware of his own using strategy.

4.4 The definition of rationality. The set-up used in the paper is a thorough framework for analyzing a set-valued solution concept. Specifically, player i's choice set associated with a state ω is given by $\Psi_i(\omega) = X_i^{P_i(\omega)}$. From an epistemic perspective, at a state ω, i knows only the set $P_i(\omega)$ — i.e., he considers it possible that the true state could be any state in $P_i(\omega)$, but not any state outside $P_i(\omega)$. In particular, at that state player i can conclude only that all his plausible choices of strategy are within the scope of $\Psi_i(\omega)$. We define the notion of "rationality" by requiring that the choice set $\Psi_i(\omega)$ consists of all the best replies in face of epistemic uncertainty $P_i(\omega)$.[8] Accordingly, rationality requires not only that every plausible choice of strategy in $\Psi_i(\omega)$ can be "justifiable," but also that any choice of strategy outside $\Psi_i(\omega)$ cannot be "justified." The requirement for rationality reflects, at an individual level, von Neumann and Morgenstern's (1944, p. 41) philosophy of interpretation of stability as stable "standard of behavior."

The notion of "rationality" used in this paper is based upon the epistemic aspects. To see this, let t_i' and t_i'' be two plausible types that a *rational* type t_i cannot exclude. Suppose that x_i' and x_i'' are best responses with respect to $\psi \circ t_i'$ and $\psi \circ t_i''$, respectively. The rational type t_i would not preclude x_i' and x_i'' from t_i's disposal choices, and should preclude all the strategies that are not a best response to any of his types that he cannot exclude.[9] In contrast, Epstein (1997) defined "player i is rational at ω" as: $u_i^{\omega}\left(x_i^{\omega}\right) \geq u_i^{\omega}(y_i)$ for all $y_i \in X_i$. To make sense of this sort of definition, a player would be aware of his own *true* type and of his own *using* strategy. By $P2$ in Lemma 1, the information structure is thereby partitional in the type dimension. Subsequently, this definition of rationality arises the question about its applicability in general cases where players are boundedly rational with non-partitional information structures. From a different perspective, Morris (1996) also pointed out that there is an intrinsic inconsistency between the non-partitional information structures and Bayesian rationality, because using Bayes rule entails information structures to be partitional; cf. also Epstein and Le Breton (1993).

[8] This definition of "rationality" is the same as Luo's (2002). Samuelson (1992) also used a similar notion to discuss the "common knowledge of admissibility."

[9] The true preferences are irrelevant to evaluating optimal choices. Only the perceivable and conscious preferences matter for this evaluation. See also Harsanyi's (1997) discussion on "actual" vs. "informed" preferences.

We would like to point out that by Theorem 2 and Corollary 2, Epstein's (1997) Theorem 3.2 can be improved by replacing the relation of "never-best response" with the more conventional "payoff dominance" relation. Consequently, rationality as well as c.k.r. in Epstein's sense give the same prediction — iterated strict dominance. As emphasized above, Theorem 2 does not rely on the strong epistemic assumption that a player be aware of his own *true* type and of his own *using* strategy, however.[10] In fact, Theorem 2 still holds true for a weaker version of rationality: $X_i^{P_i(\omega)} \subseteq BR_i(\omega)$.

4.5 The exogenous vs. endogenous models of knowledge. In the semantic framework, the information structure is exogenously given. In contrast, the information structure is endogenously determined in the framework of this paper. The distinction between exogenous and endogenous models of knowledge gives rise to some different ways of approaching epistemology. For example, within the semantic framework it is interesting to ask a question: Is there a model of knowledge in which a game-theoretic solution is sustained under some appealing epistemic conditions? The question is no longer an issue within the framework of this paper.

4.6 The completeness of a state space. In this paper a state is viewed as an endogenous variable since a state is constructed by strategies and Harsanyi's types. A state specifies what every player does, and what every player thinks about what every player does, and so on; it specifies every player's preferences, and every player's preferences about every player's preferences, and so on; it specifies what every player knows, and what every player knows about what every player knows, and so on. The state space includes all possible states and is intrinsically infinite. The completeness of a state space is crucial for our main results in this paper.

Finally, while throughout this paper we restrict attention to a subclass of regular preferences, all results here are not confined with the restriction. As pointed out by Epstein (1997), our analysis can be applied to other specific models of preferences; for example, the subjective expected utility model, the ordinal expected utility model, the probabilistic sophistication model, the Choquet expected utility model, and so on.

[10] Aumann (1987, 1995) so clearly made the assumption that each player knows which strategy he chooses; i.e., the so-called "measurability of strategy with respect to information structure."

Appendix I: Regular Preferences and Marginal Consistency

Let $\mathcal{F}^u(\Omega) = \{f \in \mathcal{F}(\Omega) | \ f(\Omega) \text{ is finite; } f^{-1}([r,1]) \text{ is closed for any } r \in [0,1]\}$. Let $\mathcal{F}^l(\Omega) = \{f \in \mathcal{F}(\Omega) | \ f(\Omega) \text{ is finite; } f^{-1}((r,1]) \text{ is open for any } r \in [0,1]\}$. A preference is said to be *regular* if it has a numerical representation $u : \mathcal{F}(\Omega) \to [0,1]$ satisfying:

U.1. Certainty Equivalence: $u(r) = r$, $\forall r \in [0,1]$.

U.2. Weak Monotonicity: $f' \geq f \Rightarrow u(f') \geq u(f)$, $\forall f, f' \in \mathcal{F}(\Omega)$.

U.3. Inner Regularity: $u(f) = \sup \{u(g) : g \leq f, g \in \mathcal{F}^u(\Omega)\}$, $\forall f \in \mathcal{F}(\Omega)$.

U.4. Outer Regularity: $u(g) = \inf \{u(h) : h \geq g, h \in \mathcal{F}^l(\Omega)\}$, $\forall g \in \mathcal{F}^u(\Omega)$.

For the purpose of this paper, we add the following conditions.[11]

U.5. Uniform Equicontinuity: $\forall \varepsilon > 0$, $\exists \delta$ such that for every $u \in \mathcal{P}(\Omega)$

$$|u(f) - u(f')| < \varepsilon, \text{ whenever } sup_{\omega \in \Omega} |f(\omega) - f'(\omega)| < \delta.$$

U.6. Preference-model Closedness: For any closed subset $E \subseteq \Omega$, $\mathcal{P}(E)$ is closed.

U.2′. Strong Monotonicity: For any subset $E \subseteq \Omega$ and any $u \in \mathcal{P}(E)$, $f' > f \Rightarrow u(f') > u(f)$, $\forall f, f' \in \mathcal{F}(E)$.

Marginal consistency is introduced as a primitive requirement in a case where a player is endowed with an arbitrary set of preferences. For the special case of regular preferences, the "marginal consistency" can be defined as follows. Let $\mathcal{F}_i(X)$ denote the set of acts $f : X \to [0,1]$, satisfying $f(x_i, x_{-i}) = f(x'_i, x_{-i})$ for all (x_i, x_{-i}) and (x'_i, x_{-i}) in X. For any $E \subseteq \Omega$ and $u \in \mathcal{P}(E)$, the "restriction of u to $\mathcal{F}_i(X)$" is referred as a preference in $\mathcal{P}_i(X^E)$, denoted by $mrg_{\mathcal{F}_i(X)} u$. Say u satisfies the *marginal consistency* if, $\forall g \in \mathcal{F}_i(X)$, $\forall f \in \mathcal{F}(E)$, $mrg_{\mathcal{F}_i(X)} u(g) = u(f)$ whenever $g(x^\omega) = f(\omega)$ (in particular, $mrg_{\mathcal{F}_i(X)} u(x_i) = u(x_i)$ $\forall x_i \in X_i$); hence, $\left\{mrg_{\mathcal{F}_i(X)} u | \ u \in \mathcal{P}(E)\right\} = \mathcal{P}_i(X^E)$.

[11] We assume that $\mathcal{F}(\Omega)$ is endowed with sup-norm topology and that $\mathcal{P}(E)$ is endowed with Epstein and Wang's (1996) topology.

Appendix II: Proofs

Lemma 1. *K_i and P_i satisfy the following properties:*

$K1$: $K_i\emptyset = \emptyset$.

$K2$: $K_i\Omega = \Omega$.

$K3$: $E \subseteq F \Rightarrow K_iE \subseteq K_iF$.

$K4$: $\bigcap_{\lambda\in\Lambda} K_iE^{\lambda} \subseteq K_i(\bigcap_{\lambda\in\Lambda} E^{\lambda})$ *for a family of closed subsets* $\{E^{\lambda}\}_{\lambda\in\Lambda}$.

$K5$: $K_iE = \{\omega \in \Omega | \; P_i(\omega) \subseteq E\}$.

$P1$: $P_i(\omega)$ *is nonempty and closed.*

$P2$: $P_i(\omega) = P_i(\omega')$ *whenever* $t_i^{\omega} = t_i^{\omega'}$.

Proof. Clearly, $K1$ holds by U.1; $K2$ and $K3$ hold by the definition of knowledge. To prove $K4$, note that X satisfies the *second axiom of countability* — i.e., the topology on X has a countable basis — since X is a compact metric space (see, e.g., Aliprantis and Border 1999, Chapter 3). We divide this proof into the following three steps.

Step 1. Ω satisfies the second axiom of countability.

By the construction of a type space, $\Omega \subseteq \Omega_0 \times (\times_{k=0}^{\infty}\mathcal{P}^n(\Omega_k))$, where $\Omega_0 = X$ and $\Omega_k = \Omega_{k-1} \times \mathcal{P}^n(\Omega_{k-1})$ for $k \geq 1$. By the fact that the countable Cartesian product of the second countable spaces is the second countable, it suffices to show that $\mathcal{P}(X)$ satisfies the second axiom of countability.

Following Epstein and Wang (1996), consider the topology on $\mathcal{P}(X)$ generated by the subbasis consisting of:

$$\{u : \; u(g) < r, \, g \in \mathcal{F}^u(X), \, r \in [0,1]\} \text{ and } \{u : \; u(h) > r, \, h \in \mathcal{F}^l(X), \, r \in [0,1]\}.$$

Let $\mathcal{B}_\tau$ be a countable basis of this topology on X, and let

$$\mathcal{B} \equiv \left\{B | \; B = \cup_{k=1}^{K} B_k, B_k \in \mathcal{B}_\tau\right\},$$

where K is a positive integer, and let

$$\mathcal{C} \equiv \{C | \; C = X \backslash B, B \in \mathcal{B}\}.$$

Consider the following two classes of functions:

$$\widehat{\mathcal{F}}^u(X) \equiv \left\{ f \in \mathcal{F}^u(X) |\ f = \sum_{k=1}^{K} q_k 1_{C_k};\ q_k \in Q \text{ and } C_k \in \mathcal{C} \right\} \text{ and}$$

$$\widehat{\mathcal{F}}^l(X) \equiv \left\{ f \in \mathcal{F}^l(X) |\ f = \sum_{k=1}^{K} q_k 1_{B_k};\ q_k \in Q \text{ and } B_k \in \mathcal{B} \right\},$$

where Q is the set of all rational numbers in $[0, 1]$. Clearly, $\widehat{\mathcal{F}}^u(X)$ and $\widehat{\mathcal{F}}^l(X)$ are both countable sets. Now consider the following class of sets, denoted by $\mathcal{E}$:

$$\{u : u(g) < q,\ g \in \widehat{\mathcal{F}}^u(X),\ q \in Q\} \text{ and } \{u : u(h) > q,\ h \in \widehat{\mathcal{F}}^l(X),\ q \in Q\}.$$

Note that $h \in \mathcal{F}^l(X)$ can be expressed as $h = \Sigma_{k=1}^{K} r_k 1_{G_k}$, where $r_k \in [0, 1]$ and G_k is open in X (cf. Epstein and Wang 1996, p. 1366). Since $\mathcal{B}_\tau$ is a countable basis, $G_k = \cup_{l=1}^{\infty} B_{l,k}$ (where $B_{l,k} \in \mathcal{B}_\tau$). For r, $r_k \in [0, 1]$, we can find $q_{m,k} \uparrow r_k$ and $q_k \downarrow r$, where $q_{m,k}$, $q_k \in Q$. Define $h_m \equiv \Sigma_{k=1}^{K} q_{m,k} 1_{\cup_{l=1}^{m} B_{l,k}}$. Clearly, $h_m \leq h$ and $h_m(x) \uparrow h(x)$ for each $x \in X$. Now by U.3, for any $\varepsilon > 0$, there exists $g \leq h$, $g \in \mathcal{F}^u(\Omega)$ such that

$$u(h) - \varepsilon < u(g) \leq u(h).$$

By U.5, without loss of generality we may assume $g < h$. Since $g \in \mathcal{F}^u(X)$ can be expressed as $g = \Sigma_{k=1}^{K'} r'_k 1_{F_k}$, where $r'_k \in [0, 1]$ and F_k is closed in X (cf. Epstein and Wang 1996, p. 1366). Therefore, $h_m \geq g$ for sufficiently large m. Thus, $u(h_m) \uparrow u(h)$. Hence,

$$\{u : u(h) > r\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \{u : u(h_m) > q_k\}.$$

Similarly, we have

$$\{u : u(g) < r\} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \{u : u(g_m) < q_k\}.$$

Thus, $\mathcal{E}$ generates the topology on $\mathcal{P}(X)$. Since $\mathcal{E}$ is countable, $\mathcal{P}(X)$ satisfies the second axiom of countability.

Step 2. $K_i E \cap K_i F \subseteq K_i(E \cap F)$ *for any closed sets* $E, F \subseteq \Omega$.

Let $\omega \in K_i E \cap K_i F$. Since E and F are closed, $u_i^{\omega} \in \mathcal{P}(\Omega|E)$ and $u_i^{\omega} \in \mathcal{P}(\Omega|F)$. Therefore,

$$u_i^{\omega}(f) = u_i^{\omega}(1_E f) \text{ and } u_i^{\omega}(f) = u_i^{\omega}(1_F f), \forall f \in \mathcal{F}(\Omega).$$

Thus,

$$u_i^{\omega}(f) = u_i^{\omega}(1_E f) = u_i^{\omega}(1_{E\cap F} f), \forall f \in \mathcal{F}(\Omega).$$

That is, $\omega \in K_i(E \cap F)$.

Step 3. $\bigcap_{\lambda\in\Lambda} K_i E^{\lambda} \subseteq K_i(\bigcap_{\lambda\in\Lambda} E^{\lambda})$.

Let $\omega \in \bigcap_{\lambda\in\Lambda} K_i E^{\lambda}$. Let $\{E^{\lambda}\}_{\lambda\in\Lambda}$ be a family of closed subsets of Ω. Since, by Step 1, Ω satisfies the second axiom of countability, it follows that there exists a countable sequence of open sets $\{\Omega\backslash E^k\}_{k=1}^{\infty}$ such that

$$\bigcup_{\lambda\in\Lambda} \left[\Omega\backslash E^{\lambda}\right] = \bigcup_{k=1}^{\infty} \left[\Omega\backslash E^k\right]$$

or equivalently

$$\bigcap_{\lambda\in\Lambda} E^{\lambda} = \bigcap_{k=1}^{\infty} E^k.$$

Without loss of generality, $\forall k \geqslant 1$, $E^k \supseteq E^{\lambda'}$ for some $\lambda' \in \Lambda$. Since $\omega \in \bigcap_{\lambda\in\Lambda} K_i E^{\lambda}$, $\omega \in K_i E^{\lambda'}$. By $K3$, $\omega \in K_i E^k$, $\forall k \geqslant 1$. Now consider the sequence $\left\{\overline{E}^k\right\}_{k=1}^{\infty}$ such that $\overline{E}^1 = E^1$, $\overline{E}^2 = \overline{E}^1 \cap E^2, ..., \overline{E}^k = \overline{E}^{k-1} \cap E^k, ...$ Clearly, $\overline{E}^k \downarrow \cap_{\lambda\in\Lambda} E^{\lambda}$. By Step 2, $\omega \in K_i \overline{E}^k$, $\forall k \geqslant 1$. The result therefore follows from Epstein and Wang's (1996) Theorem 4.4.

$K5$: Let $\omega \in K_i E$. By the definition of $P_i(\omega)$, $P_i(\omega) \subseteq E$. Thus, $K_i E \subseteq \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$. Conversely, suppose $P_i(\omega) \subseteq E$. By the proof of $P1$ in Lemma 1, $K3$ and $K4$ jointly imply that

$$\bigcap_{\left\{\overline{E}\subseteq\Omega |\ K_i\overline{E}\ni\omega \text{ and } \overline{E} \text{ is closed}\right\}} K_i\overline{E} \subseteq K_i E.$$

Therefore, $\omega \in K_i E$. Thus, $K_i E \supseteq \{\omega \in \Omega |\ P_i(\omega) \subseteq E\}$.

$P1$: By the definition of K_iE, it is easy to see that $\omega \in K_iE$ if, and only if, $\omega \in K_i\overline{E}$ for some closed subset $\overline{E} \subseteq E$. It therefore follows that

$$\bigcap_{\{E \subseteq \Omega |\ K_iE \ni \omega\}} E = \bigcap_{\{\overline{E} \subseteq \Omega |\ K_i\overline{E} \ni \omega \text{ and } \overline{E} \text{ is closed}\}} \overline{E}.$$

Hence, $P_i(\omega)$ is closed. Assume, in negation, that $P_i(\omega) = \emptyset$. By $K5$, $\omega \in K_i\emptyset$, which contradicts $K1$.

$P2$: Since $t_i^{\omega} = t_i^{\omega'}$, $\psi \circ t_i^{\omega} = \psi \circ t_i^{\omega}$. Therefore, for any $E \subseteq \Omega$, $\omega \in K_iE$ iff $\omega' \in K_iE$. Hence, $P_i(\omega) = P_i(\omega')$. ■

Lemma 2. *CK satisfies the following properties:*

$CK1$: $CKE = K(E \cap CKE)$.

$CK2$: $\omega \in CKE$ if E is a self-evident event containing ω.

Proof. $CK1$: By $K5$, we have

$$\begin{aligned} K(E \cap CKE) &= KE \cap K(CKE) \\ &= KE \cap K^2E \cap K^3E \cap \ldots \\ &= CKE. \end{aligned}$$

$CK2$: Since E is self-evident, $E \subseteq KE$. By $K3$, $K^{l-1}E \subseteq K^lE$ for all $l \geq 2$. Since $\omega \in E$, $\omega \in K^lE$ for all $l \geq 1$. ■

Theorem 1. *Let $E \subseteq \Omega$ be nonempty and compact. Then, a strategy y_i is a never-best response given E if, and only if, it is strictly dominated given X^E.*

To prove Theorem 1, we need the following two lemmas.

Lemma 3. *$\mathcal{P}(E)$ is convex.*

Proof of Lemma 3. For any $u_1, u_2 \in \mathcal{P}(E)$ and $\alpha \in [0,1]$, we proceed to verify that $\alpha u_1 + (1-\alpha)u_2 \in \mathcal{P}(E)$. Obviously, U.1, U.2, U.2′, U.5, and U.6 hold. Let $f \in \mathcal{F}(E)$. Then,

$$\begin{aligned} & [\alpha u_1 + (1-\alpha)u_2](f) \\ = \; & \alpha u_1(f) + (1-\alpha)u_2(f) \\ = \; & \sup\{\alpha u_1(g) : g \leq f, g \in \mathcal{F}^u(E)\} + \sup\{(1-\alpha)u_2(g) : g \leq f, g \in \mathcal{F}^u(E)\} \\ \geq \; & \sup\{\alpha u_1(g) + (1-\alpha)u_2(g) : g \leq f, g \in \mathcal{F}^u(E)\} \\ = \; & \sup\{[\alpha u_1 + (1-\alpha)u_2](g) : g \leq f, g \in \mathcal{F}^u(E)\}. \end{aligned}$$

Moreover, for sufficiently small $\varepsilon > 0$, there exist $g_1, g_2 \in \mathcal{F}^u(E)$ such that $g_1 \leq f$, $g_2 \leq f$, $u_1(g_1) > u_1(f) - \varepsilon$, and $u_2(g_2) > u_2(f) - \varepsilon$. Define $g'(\omega) \equiv \max[g_1(\omega), g_2(\omega)]$. Clearly, $g' \in \mathcal{F}^u(E)$ and $g' \leq f$. By U.2, it follows that

$$\begin{aligned} & \sup \{[\alpha u_1 + (1-\alpha)u_2](g) : g \leq f, g \in \mathcal{F}^u(E)\} \\ \geq \; & \alpha u_1(g') + (1-\alpha)u_2(g') \\ \geq \; & \alpha u_1(g_1) + (1-\alpha)u_2(g_2) \\ \geq \; & \alpha u_1(f) + (1-\alpha)u_2(f) - \varepsilon \\ = \; & [\alpha u_1 + (1-\alpha)u_2](f) - \varepsilon. \end{aligned}$$

Thus, U.3 holds. Similarly, U.4 holds. ■

Lemma 4. *Let $Y \subseteq X$. Then, $u_i(x_i) > u_i(y_i)$ for all $u_i \in P_i(Y)$ if x_i strictly dominates y_i given Y . Moreover, when Y is closed, x_i strictly dominates y_i given Y if $u_i(x_i) > u_i(y_i)$ for all $u_i \in P_i(Y)$.*

Proof of Lemma 4. Define $E \equiv Y \times \{t\}$ where $t \in T_1 \times T_2 \times \ldots \times T_n$. Suppose that $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$. Since $X_{-i}^E = Y_{-i}$, $x_i(\omega) = \zeta_i(x_i, x_{-i}^\omega) > \zeta_i(y_i, x_{-i}^\omega) = y_i(\omega)$ for all $\omega \in E$. By U.2$'$, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$.

Suppose that $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i(Y)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. For any $\omega \in E$, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E|\{\omega\})$. Since Y is closed, E is compact. By U.1 and Epstein and Wang's (1996) Theorem 4.3, it follows that $x_i(\omega) = u_i(x_i) > u_i(y_i) = y_i(\omega)$ for all $u_i \in \mathcal{P}(E|\{\omega\})$. Therefore, $\zeta_i(x_i, x_{-i}^\omega) > \zeta_i(y_i, x_{-i}^\omega)$ for all $\omega \in E$. Since $X_{-i}^E = Y_{-i}$, $\zeta_i(x_i, y_{-i}) > \zeta_i(y_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$. ■

We now turn to the proof of Theorem 1.

Proof of Theorem 1. *"if part":* Let x_i strictly dominate y_i given X^E. By Lemma 4, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i\left(X^E\right)$. By marginal consistency, $u_i(x_i) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By Epstein and Wang's (1996) Theorem 4.3, $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$. Let $\varphi : \mathcal{P}(E) \to \mathcal{P}(\Omega|E)$ be such a homeomorphism. By the proof of Epstein and Wang's (1996) Theorem 4.3, $\varphi \circ u_i(x_i') = u_i(x_i')$ for all $x_i' \in X_i$. Therefore, $\varphi \circ u_i(x_i) > \varphi \circ u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. Thus, $u_i^\omega(x_i) > u_i^\omega(y_i)$ for all $u_i^\omega \in \mathcal{P}(\Omega|E)$. Since E is compact, it therefore follows that $u_i^\omega(x_i) > u_i^\omega(y_i)$ for all $\omega \in K_i E$.

"only if part": Consider a zero-sum game $\mathcal{G}' \equiv (N', \{X_j'\}, \{\zeta_j'\})$ such that $N' = \{i, -i\}$, $X_i' = X_i$, and $X_{-i}' = \mathcal{P}(E)$. Define the payoff function in $\mathcal{G}'$ as

$$\zeta_i'(x_i, u_i) \equiv u_i^\omega(x_i) - u_i^\omega(y_i), \text{ for all } x_i \in X_i' \text{ and } u_i \in \mathcal{P}(E),$$

where $\overset{\omega}{u_i} \in \mathcal{P}(\Omega|E)$ and $\overset{\omega}{u_i} = \varphi \circ u_i$. By U.6 in Appendix I, $\mathcal{P}(E)$ is compact. By Lemma 3, $\mathcal{P}(E)$ is convex. By Epstein and Wang's (1996) Theorem 3.1, $\mathcal{P}(E)$ is Hausdorff. By continuity of φ and by U.5 in Appendix I, $\zeta_i'(\cdot,\cdot)$ is continuous (see the verification below). Now, by Glicksberg's (1952) Theorem, there exists a Nash equilibrium (x_i^*, u_i^*) in $\mathcal{G}'$. However, since y_i is a never-best response given E, we have

$$\max_{x_i \in X_i'} \zeta_i'(x_i, u_i) = \max_{x_i \in X_i'} \left[\overset{\omega}{u_i}(x_i) - \overset{\omega}{u_i}(y_i)\right] > 0$$

for all $\overset{\omega}{u_i} \in \mathcal{P}(\Omega|E)$. Therefore, for any $u_i \in \mathcal{P}(E)$,

$$\zeta_i'(x_i^*, u_i) \geq \zeta_i'(x_i^*, u_i^*) = \max_{x_i \in X_i'} \zeta_i'(x_i, u_i^*) > 0.$$

Thus, $\overset{\omega}{u_i}(x_i^*) > \overset{\omega}{u_i}(y_i)$ for all $\overset{\omega}{u_i} \in \mathcal{P}(\Omega|E)$. By the proof of Epstein and Wang's (1996) Theorem 4.3, $\overset{\omega}{u_i}(x_i') = \varphi^{-1} \circ \overset{\omega}{u_i}(x_i')$ for all $x_i' \in X_i$. Therefore, $\varphi^{-1} \circ \overset{\omega}{u_i}(x_i^*) > \varphi^{-1} \circ \overset{\omega}{u_i}(y_i)$ for all $\overset{\omega}{u_i} \in \mathcal{P}(\Omega|E)$. Since $\mathcal{P}(E) \sim^{homeomorphic} \mathcal{P}(\Omega|E)$, it follows that $u_i(x_i^*) > u_i(y_i)$ for all $u_i \in \mathcal{P}(E)$. By marginal consistency, $u_i(x_i^*) > u_i(y_i)$ for all $u_i \in \mathcal{P}_i\left(X^E\right)$. By Lemma 4, x_i^* strictly dominate y_i given X^E.

Continuity of $\zeta'(\cdot,\cdot)$: We denote the metric for X_i by d_i and denote the metric for X by $d(x, x') = \sqrt{\sum_{i=1}^n d_i(x_i, x_i')^2}$ for all $x, x' \in X$.

Step 1. If $x_i^m \to x_i$, $sup_{\omega \in \Omega}|x_i^m(\omega) - x_i(\omega)| \to 0$.

Since $\zeta_i(.)$ is continuous and X is compact, $\zeta_i(.)$ is uniformly continuous on X. Hence, for any $\varepsilon > 0$, there exists δ such that whenever $d_i(x_i^m, x_i) < \delta$, we have $|x_i^m(\omega) - x_i(\omega)| = |\zeta_i(x_i^m, \overset{\omega}{x_{-i}}) - \zeta_i(x_i, \overset{\omega}{x_{-i}})| < \varepsilon$ for all ω.

Step 2. For any continuous function $f \in \mathcal{F}(\Omega)$, $u^m(f) \to u(f)$ as $u^m \to u$.

To prove this, it suffices to show that, for all real numbers r, $\{u : u(f) > r\}$ and $\{u : u(f) < r\}$ are open. Since f is continuous, we can find $f_m \in \mathcal{F}^l(\Omega)$ that

$$f_m = \frac{1}{2^m} \sum_{j=1}^{2^m} 1_{G_{mj}}, \text{ where } G_{mj} = \{\omega : f(\omega) > j2^{-m}\}.$$

Clearly, $f_m \uparrow f$ uniformly. By U.5,

$$\{u : u(f) > r\} = \bigcup_{m=1}^{\infty} \{u : u(f_m) > r\}.$$

Thus, $\{u : u(f) > r\}$ is open. Similarly, $\{u : u(f) < r\}$ is open.

Step 3. $\zeta_i'(x_i, u_i)$ is jointly continuous.

Let $(x_i^m, u_i^m) \rightarrow (x_i, u_i)$ be a convergent sequence in $X_i \times \mathcal{P}(E)$. Let $\varepsilon > 0$ be sufficiently small. Then, by Step 1 and U.5, for sufficiently large m, $|u_i'(x_i^m) - u_i'(x_i)| < \varepsilon/3$ for all $u_i' \in \mathcal{P}(E)$. Since the payoff function $\zeta_i(\cdot)$ is continuous, it therefore follows that x_i is a continuous act. By Step 2, for sufficiently large m, $|u_i^m(x_i) - u_i(x_i)| < \varepsilon/3$ and $|u_i^m(y_i) - u_i(y_i)| < \varepsilon/3$. Hence, we have

$$\begin{aligned} |\zeta_i'(x_i^m, u_i^m) - \zeta_i'(x_i, u_i)| &\leq |\overset{\omega m}{u_i}(x_i^m) - \overset{\omega}{u_i}(x_i)| + |\overset{\omega m}{u_i}(y_i) - \overset{\omega}{u_i}(y_i)| \\ &\leq |\overset{\omega m}{u_i}(x_i^m) - \overset{\omega m}{u_i}(x_i)| + |\overset{\omega m}{u_i}(x_i) - \overset{\omega}{u_i}(x_i)| \\ &\quad + |\overset{\omega m}{u_i}(y_i) - \overset{\omega}{u_i}(y_i)| \\ &< \varepsilon. \blacksquare \end{aligned}$$

Corollary 1. *Let $Y \subseteq X$ be nonempty and compact. Then, a strategy y_i is a never-best response given Y if, and only if, it is strictly dominated given Y.*
Proof. Define $E \equiv Y \times \{t\}$ where $t \in T_1 \times T_2 \times \ldots \times T_n$. By the Tychonoff Theorem, E is nonempty and compact. Therefore, $\Big[y_i$ is strictly dominated given $Y\Big]$ $\Longleftrightarrow^{\text{by Theorem 1}}$ $\Big[y_i$ is a never-best response given $E\Big]$ $\Longleftrightarrow^{\text{by Definition 2}}$ $\Big[$for every $\omega \in K_i E$, $\overset{\omega}{u_i}(x_i) > \overset{\omega}{u_i}(y_i)$ for some $x_i \in X_i\Big]$ $\Longleftrightarrow^{\text{by the compactness of } E}$ $\Big[$for every $u_i \in \mathcal{P}(\Omega|E)$, $u_i(x_i) > u_i(y_i)$ for some $x_i \in X_i\Big]$ $\Longleftrightarrow^{\text{by Epstein and Wang's (1996) Theorem 4.3}}$ $\Big[$for every $u_i \in \mathcal{P}(E)$, $u_i(x_i) > u_i(y_i)$ for some $x_i \in X_i\Big]$ $\Longleftrightarrow^{\text{by marginal consistency}}$ $\Big[$for every $u_i \in \mathcal{P}_i(Y)$, $u_i(x_i) > u_i(y_i)$ for some $x_i \in X_i\Big]$ $\Longleftrightarrow$ $\Big[y_i$ is a never-best response given $Y\Big]$. $\blacksquare$

Theorem 2. *$X^{R \cap CKR}$ is the largest (w.r.t. set inclusion) stable set and, moreover, yields iterated strict dominance.*

To prove Theorem 2, we need the following three lemmas.

Lemma 5. *Suppose that $x_i \in X_i$ is strictly dominated given $Y \neq \emptyset$. Then, there exists $x_i^* \in X_i$ such that (a) x_i^* strictly dominates x_i given Y; and (b) x_i^* is not strictly dominated given Y.*

Proof. Consider a partial ordered set $(X_i', \succcurlyeq)$, such that

$$X_i' \equiv \{x_i' \in X_i |\ x_i' \text{ strictly dominates } x_i \text{ given } Y\},$$

and for all $x_i', y_i' \in X_i'$,

(a) $x_i' \succ y_i'$ iff x_i' strictly dominates y_i' given Y;

(b) $x_i' \sim y_i'$ iff $x_i' = y_i'$.

Clearly, any maximal strategy in X_i' is a strictly undominated dominator of x_i. By Zorn's Lemma, it remains to verify that every totally-ordered subset of X_i' has an upper bound in X_i'. Let X_i'' be a totally-ordered subset of X_i'. Since X_i is compact, X_i'' has a convergent subnet $x_i^\lambda \to x_i^*$ in X_i. By continuity of ζ_i, for any $y_{-i} \in Y_{-i}$, $\zeta_i\left(x_i^\lambda, y_{-i}\right)$ increasingly converges to $\zeta_i\left(x_i^*, y_{-i}\right)$. If $\zeta_i\left(x_i^*, y_{-i}\right) > \zeta_i\left(x_i^\lambda, y_{-i}\right)$ for all $y_{-i} \in Y_{-i}$, then $x_i^* \succ x_i^\lambda$; if $\zeta_i\left(x_i^*, y_{-i}\right) = \zeta_i\left(x_i^\lambda, y_{-i}\right)$ for some $y_{-i} \in Y_{-i}$, then for all $x_i^{\lambda'} \succcurlyeq x_i^\lambda$, $\zeta_i\left(x_i^{\lambda'}, y_{-i}\right) = \zeta_i\left(x_i^\lambda, y_{-i}\right)$ and, hence, $x_i^{\lambda'} = x_i^\lambda$. Thus, $x_i^* \sim x_i^\lambda$. Therefore, $x_i^* \succcurlyeq x_i^\lambda$ for all x_i^λ. Since for each $x_i' \in X_i''$, $x_i^\lambda \succcurlyeq x_i'$ for some x_i^λ, $x_i^* \succcurlyeq x_i'$ for all $x_i' \in X_i''$. As $x_i^\lambda \succ x_i$, x_i^* is an upper bound in X_i' for X_i''. ■

Lemma 6. *$\mathcal{K}$ is a stable set iff $\mathcal{K} \neq \emptyset$ and $\mathcal{K} = \left\{x \in X |\ y \not\succ^{\mathcal{K}} x\ \forall y \in X\right\}$.*

Proof. Suppose that $\mathcal{K}$ is a stable set. Then, $\mathcal{K} \neq \emptyset$. As the external stability of $\mathcal{K}$ implies that $\mathcal{K} \supseteq \left\{x \in X |\ y \not\succ^{\mathcal{K}} x\ \forall y \in X\right\}$, it suffices to verify that $\mathcal{K} \subseteq \left\{x \in X |\ y \not\succ^{\mathcal{K}} x\ \forall y \in X\right\}$. Assume, in negation, that there exists $x \in \mathcal{K} \backslash \left\{x \in X |\ y \not\succ^{\mathcal{K}} x\ \forall y \in X\right\}$. Then, for some i, x_i is strictly dominated given $\mathcal{K}$. By Lemma 5, there exists $x_i^* \in X_i$ such that (a) x_i^* strictly dominates y_i given $\mathcal{K}$; and (b) x_i^* is not strictly dominated given $\mathcal{K}$. Define $y \equiv (x_i^*, x_{-i})$. Clearly, $y \in \mathcal{K}$ and $y \succ^{\mathcal{K}} x$, contradicting the internal stability of $\mathcal{K}$.

Suppose that $\mathcal{K} \neq \emptyset$ and $\mathcal{K} = \left\{x \in X |\ y \not\succ^{\mathcal{K}} x\ \forall y \in X\right\}$. Clearly, $\mathcal{K}$ is internally stable. Let $x \notin \mathcal{K}$. Then, for some i, x_i is strictly dominated given $\mathcal{K}$. By Lemma 5, there exists $x_i^* \in X_i$ such that (a) x_i^* strictly dominates x_i given $\mathcal{K}$; and (b) x_i^* is not strictly dominated given $\mathcal{K}$. Since $\mathcal{K} \neq \emptyset$, there is $x' \in \mathcal{K}$. Define $y \equiv \left(x_i^*, x_{-i}'\right)$. Clearly, $y \in \mathcal{K}$ and $y \succ^{\mathcal{K}} x$. Thus, $\mathcal{K}$ is externally stable. ■

Lemma 7. *Suppose that $Y \subseteq X$ is nonempty and closed. Then, there exists ω such that $\omega \in P_i(\omega)$* and *$P_i(\omega) = Y \times \left\{t^\omega\right\}$ for all i.*

Proof. Let $\Omega_0 \equiv X$, and let $\Omega_k \equiv \Omega_{k-1} \times \mathcal{P}^n(\Omega_{k-1})$ for all $k \geq 1$. Since X is a compact metric space, Y has a countable dense subset $\{x^m\}_{m=1}^{\infty}$ (see, e.g., Aliprantis and Border 1999, Chapter 3). Define $u_i^0(f) \equiv \sum_{m=1}^{\infty} 2^{-m} f(x^m)$ for any $f \in \mathcal{F}(X)$. Clearly, $u_i^0 \in \mathcal{P}(X)$. For any $f \in \mathcal{F}(\Omega_k)$, any $k \geq 1$ and $i = 1, ..., n$, define

$$u_i^k(f) \equiv \sum_{m=1}^{\infty} 2^{-m} f(x^m; u_1^0, ..., u_n^0; ...; u_1^{k-1}, ..., u_n^{k-1}).$$

Thus, $u_i^k \in \mathcal{P}(\Omega_k | Y \times \{\times_{l=0}^{k-1} \{(u_1^l, ..., u_n^l)\}\})$ and $mrg_{\mathcal{F}(\Omega_{k-1})} u_i^k = u_i^{k-1}$. Define $t_i \equiv (u_i^0, u_i^1, ...)$, and define $u_i(f) \equiv \sum\limits_{m=1}^{\infty} 2^{-m} f(x^m; t_1, ..., t_n)$ for any $f \in \mathcal{F}(X \times T_1^0 \times \ldots \times T_n^0)$, where $T_j^0 = \times_{k=0}^{\infty} \mathcal{P}(\Omega_k)$ for $j = 1, ..., n$. Clearly, $u_i \in \mathcal{P}(X \times T_1^0 \times \ldots \times T_n^0)$ and $mrg_{\mathcal{F}(\Omega_k)} u_i = u_i^k$ $\forall k$. By Epstein and Wang's Theorem D.2, $\psi \circ t_i = u_i$. Since $u_i \in \mathcal{P}(X \times T_1^0 \times \ldots \times T_n^0 | Y \times \{(t_1, ..., t_n)\})$, $t_i \in T_i$ (see Epstein and Wang's (1996) Section 5). Define $\omega \equiv (y; t_1, ..., t_n)$ where $y \in Y$. As $u_i^{\omega} = \psi \circ t_i$, $u_i^{\omega} \in \mathcal{P}(\Omega | Y \times \{t^{\omega}\})$. Since $Y \times \{t^{\omega}\}$ is closed, $\omega \in K_i(Y \times \{t^{\omega}\})$. Thus, $P_i(\omega) \subseteq Y \times \{t^{\omega}\}$. Since $\{x^m\}_{m=1}^{\infty}$ is dense in Y, for any closed proper subset $Y' \subset Y$, there is some x_m in $(X \backslash Y') \cap Y$. Therefore, $u_i^{\omega} \notin \mathcal{P}(\Omega | Y' \times \{t^{\omega}\})$ and, hence, $\omega \notin K_i(Y' \times \{t^{\omega}\})$. Thus, $P_i(\omega) = Y \times \{t^{\omega}\}$ and, moreover, $\omega \in P_i(\omega)$. ■

We now turn to the proof of Theorem 2.

Proof of Theorem 2. Let $\mathcal{D}^0 \equiv X$, and for $l \geq 1$ define recursively

$$\mathcal{D}^l \equiv \mathcal{D}^{l-1} \backslash \left\{ y \in \mathcal{D}^{l-1} | \; \exists i \; \exists x \in \mathcal{D}^{l-1} \text{ s.t. } x_i \text{ strictly dominates } y_i \text{ given } \mathcal{D}^{l-1} \right\}.$$

Define $\mathcal{D} \equiv \cap_{l=0}^{\infty} \mathcal{D}^l$. We prove Theorem 2 by the following three steps.

Step 1. $\mathcal{D}$ is a stable set.

(1) [internal stability]. Let $x \in \mathcal{D}$. By Dufwenberg and Stegeman's (2002) Theorem 1, it follows that for all $y \in \mathcal{D}$ and for all i, y_i does not strictly dominate x_i given $\mathcal{D}$. Thus, $y \not\succ^{\mathcal{D}} x$ for all $y \in \mathcal{D}$.

(2) [external stability]. Let $x \notin \mathcal{D}$. Then, $x \notin \mathcal{D}^l$ for some $l \geq 1$. Therefore, for some i, x_i is strictly dominated given $\mathcal{D}^l$. As $\mathcal{D} \subseteq \mathcal{D}^l$, x_i is strictly dominated given $\mathcal{D}$. By Dufwenberg and Stegeman's (2002) Lemma, for some $y \in \mathcal{D}$, y_i strictly dominates x_i given $\mathcal{D}$. Thus, $y \succ^{\mathcal{D}} x$ for some $y \in \mathcal{D}$.

Step 2. $\mathcal{D}$ is the largest stable set.

Let $\mathcal{K}$ be a stable set. By Lemma 6, $\mathcal{K} \subseteq \left\{x \in X | \ y \not\succ^{\mathcal{K}} x \ \forall y \in X\right\}$. Therefore, $\mathcal{K} \subseteq \mathcal{D}^l$ for all $l \geq 0$. Thus, $\mathcal{K} \subseteq \mathcal{D}$.

Step 3. $X^{R\cap CKR} = \mathcal{D}$.

Let $\omega \in R \cap CKR$. Since $\omega \in R_i$, $X_i^{P_i(\omega)} = BR_i(\omega)$ for all i. Since by $K5$ in Lemma 1, $\omega \in K_i P_i(\omega)$, $x_i^\omega \in X_i^{K_i P_i(\omega)}$. Since by $CK1$ in Lemma 2, $P_i(\omega) \subseteq R \cap CKR$, by the weak axiom of knowledge, $x_i^\omega \in X_i^{P_i(\omega)}$. Therefore, x_i^ω is a best response given $P_i(\omega)$. By Theorem 1, x_i^ω is not strictly dominated given $X^{P_i(\omega)}$. Since again by $CK1$, $P_i(\omega) \subseteq R \cap CKR$, for all i, x_i^ω is not strictly dominated given $X^{R\cap CKR}$. Therefore, $x^\omega \in \mathcal{D}^l$ for all $l \geq 0$. Thus, $X^{R\cap CKR} \subseteq \mathcal{D}$.

By Dufwenberg and Stegeman's (2002) Theorem 1, $\mathcal{D}$ is nonempty and compact. By Lemma 7, there exists ω such that $\omega \in P_i(\omega)$ and $P_i(\omega) = \mathcal{D} \times \{t^\omega\}$ $\forall i$. By Step 1 and Lemma 6,

$$\mathcal{D} = \{x \in X | \text{ for all } i, \, x_i \text{ is not strictly dominated given } \mathcal{D}\}.$$

By Theorem 1, $BR_i(\omega) = X_i^{P_i(\omega)}$ for all i. Thus, $\omega \in R$. Since by $P1$, $BR_i(\omega') = BR_i(\omega) = X_i^{P_i(\omega)} = X_i^{P_i(\omega')}$ for all $\omega' \in P_i(\omega)$, $P_i(\omega) \subseteq R$. Let $E \equiv \mathcal{D} \times \{t^\omega\}$. As by $P1$, $P_i(\omega') = E$ for all $\omega' \in E$, E is a self-evident event containing ω. By $CK2$ in Lemma 2, $\omega \in CKE$. Since $E \subseteq R$, $\omega \in R \cap CKR$. Since by $CK1$, $P_i(\omega) \subseteq R \cap CKR$, $\mathcal{D} \subseteq X^{R\cap CKR}$. ■

Corollary 2. $X^{R\cap CKR}$ *is the set of all rationalizable strategy profiles.*
Proof. Define

$$\mathcal{L} \equiv \cup_{\mathcal{K} \subseteq \{x\in X | \ y \not\succ^{\mathcal{K}} x \ \forall y \in X\}} \mathcal{K}.$$

Since, for all $l \geq 0$, $\mathcal{K} \subseteq \mathcal{D}^l$ whenever $\mathcal{K} \subseteq \left\{x \in X | \ y \not\succ^{\mathcal{K}} x \ \forall y \in X\right\}$, $\mathcal{L} \subseteq \mathcal{D}$. By Step 1 and Lemma 6, $\mathcal{D} \subseteq \left\{x \in X | \ y \not\succ^{\mathcal{D}} x \ \forall y \in X\right\}$. Therefore, $\mathcal{L} = \mathcal{D}$ and $\mathcal{L} \subseteq \left\{x \in X | \ y \not\succ^{\mathcal{L}} x \ \forall y \in X\right\}$. By Dufwenberg and Stegeman's (2002) Theorem 1, $\mathcal{D}$ is nonempty and compact. By Corollary 1, $\mathcal{L}$ has the best response property. Since by Lemma 4, every best response is a strictly undominated strategy, it is easy to see that $\mathcal{K} \subseteq \left\{x \in X | \ y \not\succ^{\mathcal{K}} x \ \forall y \in X\right\}$ whenever $\mathcal{K}$ has the best response property. Thus, $\mathcal{L}$ is the largest set with the best response property. By Theorem 1, $X^{R\cap CKR} = \mathcal{L}$. That is, $X^{R\cap CKR}$ is the set of all rationalizable strategy profiles. ■

Theorem 3. *(3.1) Suppose* $\omega \in R$*. Then,* $\Psi(\omega)$ *is an externally stable set and, moreover, there is a stable set* $\mathcal{K} \supseteq \Psi(\omega)$ *whenever* $\omega \in CKR$*. (3.2) Suppose that* $X^{P_i(\omega)} \subseteq \Psi(\omega)$ *for all* i *— i.e., every player knows "choice sets". Then,* $\omega \in R$ *iff* $\Psi(\omega)$ *is a stable set. (3.3) Suppose that* $\boxed{\mathcal{K}}$ *is a self-evident event satisfying* $\boxed{\mathcal{K}} = \left\{\omega \in \Omega |\ X^{P_i(\omega)} \subseteq \Psi(\omega) = \mathcal{K}\right\}$*. Then,* $\boxed{\mathcal{K}} \subseteq R \cap CKR$ *whenever* $\mathcal{K}$ *is a stable set. (3.4) For any compact stable set* $\mathcal{K}$*, there is* $\omega \in R \cap CKR$ *such that* $\Psi(\omega) = \mathcal{K}$*.*

Proof. (3.1) Let $y \in X \backslash \Psi(\omega)$. Since $\omega \in R$, $y_i \notin BR_i(\omega)$ for some i. By $P1$ and Theorem 1, y_i is strictly dominated given $X^{P_i(\omega)}$. By Lemma 5, there exists $x_i^* \in X_i$ such that (a) x_i^* strictly dominates y_i given $X^{P_i(\omega)}$; and (b) x_i^* is not strictly dominated given $X^{P_i(\omega)}$. Since $\omega \in R_i$, by Theorem 1, $x_i^* \in X_i^{P_i(\omega)}$.Thus, $x_i^* \in \Psi_i(\omega)$. Since by $P1$, $\Psi(\omega) \neq \emptyset$, there is x' in $\Psi(\omega)$. Define $x \equiv (x_i^*, x'_{-i})$. Then, $x \in \Psi(\omega)$. Since by A1 and A2, $\times_{j \in N} X_j^{P_j(\omega)} \subseteq X^{P_i(\omega)}$, x_i^* strictly dominates y_i given $\Psi(\omega)$. Therefore, $x \succ^{\Psi(\omega)} y$ for some $x \in \Psi(\omega)$. Thus, $\Psi(\omega)$ is an externally stable set.

Now, let $\mathcal{K} \equiv X^{R \cap CKR}$. By Theorem 2, $\mathcal{K}$ is a stable set. Since by $CK1$, $P_i(\omega) \subseteq R \cap CKR$ for all i, $\Psi(\omega) \subseteq \mathcal{K}$.

(3.2) First of all, since by A1 and A2, $\times_{j \in N} X_j^{P_j(\omega)} \subseteq X^{P_i(\omega)}$, $\Psi(\omega) \subseteq X^{P_i(\omega)}$. As $X^{P_i(\omega)} \subseteq \Psi(\omega)$, $\Psi(\omega) = X^{P_i(\omega)}$ $\forall i$.

"if part": Suppose that $\Psi(\omega)$ is a stable set. By Lemma 6, $x \in \Psi(\omega)$ iff, for all i, x_i is not strictly dominated given $\Psi(\omega)$. Since $\Psi(\omega) = X^{P_i(\omega)}$, by $P1$ and Theorem 1, $x \in \Psi(\omega)$ iff, for all i, x_i is a best response given $P_i(\omega)$. Therefore, $X_i^{P_i(\omega)} = BR_i(\omega)$ for all i. That is, $\omega \in R$.

"only if part": Suppose $\omega \in R$. By (3.1), $\Psi(\omega)$ is externally stable. Therefore, it remains to verify that $\Psi(\omega)$ is internally stable. Assume, in negation, that $y \succ^{\Psi(\omega)} x$ for some $x, y \in \Psi(\omega)$. Then, for some i, y_i strictly dominates x_i given $\Psi(\omega)$. Since $\Psi(\omega) = X^{P_i(\omega)}$, by $P1$ and Theorem 1, x_i is not a best response given $P_i(\omega)$. Since $\omega \in R_i$, $x_i \notin X_i^{P_i(\omega)}$. Thus, $x \notin \Psi(\omega)$, which is a contradiction.

(3.3) Suppose that $\mathcal{K}$ is a stable set. Since $X^{P_i(\omega)} \subseteq \Psi(\omega)$ for all $\omega \in \boxed{\mathcal{K}}$, by (3.2), $\boxed{\mathcal{K}} \subseteq R$. Since $\boxed{\mathcal{K}}$ is self-evident, by $CK2$, $\boxed{\mathcal{K}} \subseteq CKR$. Thus, $\boxed{\mathcal{K}} \subseteq R \cap CKR$.

(3.4) The proof is totally similar to the last part of Step 3 in the proof of Theorem 2. We therefore omit it. ■

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