Evolutionarily Stable Correlation

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Abstract

Most existing results of evolutionary games restrict only to the Nash equilibrium. This paper introduces the analogue of evolutionarily stable strategy (ESS) for correlated equilibria. We introduce a new notion of evolutionarily stable correlation (ESC) and prove that it generalizes ESS. We also study analogues of perfection (cf. Dhillon and Mertens (1994)), properness, and replicator dynamics for the correlation equilibrium and discuss their relationships with ESC.

Keywords: Correlated Equilibrium, Evolutionarily Stable Correlation, Evolutionarily Stable State, Random Device.

JEL Classifications: C70, C72

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1 Introduction

The Nash equilibrium — the fundamental concept in the theory of non-cooperative normal form games — deals only with the simple situation where players choose actions independently. Aumann (1974) extends it to the notion of a correlated equilibrium, which deals with the more general situation where players may be independent or may have some correlating signals in choosing their actions. That the correlated equilibrium has natural interpretations in terms of rationality,[1] is a very practically relevant solution,[2] and has numerous applications [3] has been well accepted.

For sharper predictions of the Nash equilibrium, numerous refinements have been studied and become standard tools, e.g., the evolutionary stability (Maynard Smith (1982)), the perfect equilibrium (Selten (1975)), and the proper equilibrium (Myerson (1978)). The analogue of perfection for the correlated equilibrium was pioneered by Myerson (1986) and Dhillon and Mertens (1994). Other refinement analogues, particularly the evolutionary stability, however, have not been well explored.

In this paper, we will study an analogue of evolutionary stability (Maynard Smith (1982)) for the correlated equilibrium. Our central notion is an evolutionarily stable correlation (ESC) that generalizes the traditional (Maynard Smith) notion of evolutionarily stable strategy Nash equilibrium (ESS). We also discuss perfection, properness, and a replicator dynamic process for correlation and relate them to ESC.

We adopt a large population model with a uniform random matching process. We assume that a given (conventional) random device will recommend actions to matched players. When players take the "obedient" strategy, the device will generate a probability distribution on joint actions played — the conventional correlation. We investigate the stability of this correlation with respect to mutations on players' assignment strategies defined as mappings from recommended actions into actions actually played. Suppose a group of mutants appears in the population and they all use the same assignment strategy that is different from the obedient strategy. Although the (conventional) random device remains the same, the resulting correlation of joint actions played will be different from the conventional correlation. We say the conventional correlation is evolutionarily stable (ESC) when an incum-

[1] Such as a Bayesian rationality. See Aumann (1974, 1987) for more details.

[2] As pointed out by Hart and Mas-Colell (2000, p.1128, lines 2-4), "it is hard to exclude a priori the possibility that correlating signals are amply available to the players."

[3] E.g. sunspots and games with communication. See Forges and Peck (1995) and Myerson (1994).

bent, using the obedient strategy, performs better than a mutant, using the non-obedient strategy.

We show that an ESC is a correlated equilibrium, but not vice versa, and characterizes ESC (Proposition 1 and Example 2). We prove that this notion of ESC is a generalization of the traditional notion of ESS (Proposition 4) in the same manner as the correlated equilibrium generalizes the Nash equilibrium. In particular, if a correlation ζ is generated by a mixed action x in the independent manner (i.e. ζ is the product measure of x times x), then ζ is ESC if and only if x is ESS.

The ESS refines the Nash equilibrium and deals with evolutionary stability under the restriction of independent plays. In contrast, the ESC refines the correlated equilibrium and deals with evolutionary stability without the restriction of independent plays. In many cases, the set of ESC's is strictly larger than the set of ESS's. While the set of ESS's is always finite, the set of ESC's need not be always finite. For example, in a standard coordination game (Example 2), the only ESS's are the two pure Nash equilibria but many ESC's are "mixed" correlations. The differences between the sets of ESC's and ESS's sheds some light on clarifying the differences between the implications of the restriction of the evolutionary stability alone and that of the evolutionary stability together with independent plays.

We characterize ESC with regard to its local superiority. We prove that a correlation is ESC if and only if it is locally superior, i.e. incumbents using obedient strategies earn higher payoffs against all nearby mutants than they earn against themselves (Proposition 5).

We introduce the notion of a perfect correlated equilibrium, the notion of a proper correlated equilibrium, and a replicator dynamic process for correlation. We prove that a perfect correlated equilibrium is a generalization of a perfect Nash equilibrium and a proper correlated equilibrium is a generalization of a proper Nash equilibrium (Proposition 7). Also, we show that an ESC is a proper correlated equilibrium and a proper correlated equilibrium is a perfect correlation (Propositions 7 through 11). However, none of the inverse is true. We also study a dynamic stability of ESC and show that an ESC is asymptotically stable with respect to the replicator dynamics (Proposition 12).

This paper also can be applied to the study of evolutionary ecology. It has been accepted (cf. Riechert and Hammerstein (1983)) that only pure symmetrical equilibria in symmetric games are relevant to the biological conflicts of animals and plans, but many interesting games do not have such equilibria (e.g. a hawk-dove game). The results of this paper enable us to interpret ESC as a solution to such conflicts, with the underlying assumption that Nature supplies the random device as a phenotype conditional device in the manner very similar to previous contributions of Selten (1980), Shmida and Peleg (1997), and Cripps (1991). Moreover, our approach and results are more general than these works. For example, our model is simpler in that restrictions of role-asymmetry on the pairwise matching are not necessary, but such restrictions are crucial in their models. Also, our stability notion is less restrictive than their stability notion; indeed our ESC generalizes the classical notion of ESS, but theirs do not. More specifically, their notions are shown to be equivalent to a strict correlated equilibrium, but our notion allows non-strict correlated equilibrium.

We relate our work with several existing works on perfection refinements (Myerson (1986), Dhillon and Mertens (1994)) and learning (Ianni (2001), Foster and Vohra (1997), Fudenburg and Levine (1999), Hart and Mas-Colell (2000) for correlated equilibrium. The work of Dhillon and Mertens (1994) is most closely related to ours; in particular, stability about a correlated equilibrium by dealing with assignment functions (obedient vs non-obedient). Dhillon and Mertens study only the perfection refinement for correlated equilibrium but we cover evolutionary stability and other issues for correlated equilibrium. In our model with symmetric two-player game, our notion of a perfect correlation is the same as Dhillon and Mertens' notion of a perfect direct correlated equilibrium.[4] Similar to a result of Dhillon and Mertens, our perfect correlated equilibrium is, in general, stronger than Myerson's acceptable correlated equilibrium (Example 5).

Foster and Vohra (1997), Fudenburg and Levine (1999), Hart and Mas-Colell (2000), and Ianni (2001) study dynamic simple procedures for playing games that lead to a correlated equilibrium, but not study refinements of the correlated equilibrium.[5] We are interested in refining the correlated equilibrium, from evolutionary and other stability viewpoints. Also, our study of dynamics (replicator) is different from their dynamics: our process

[4] Dhillon and Mertens studied both direct mechanism and indirect mechanism but our approach is concerned only with the direct one.

[5] Foster and Vohra (1997), Fudenburg and Levine (1999), and Hart and Mas-Colell (2000) seek simple procedures for playing a normal game that generate paths of plays whose empirical distributions of joint actions played converge to the set of correlated equilibria of the game. Ianni (2001) studies convergence to correlated equilibrium in an indirect manner: First, she considers a population game (with an underlying finite normal form game, a population of players, and an exogenous random matching technology for players) and shows that a Nash equilibrium in a population game that corresponds to a correlated equilibrium for the underlying normal game (cf. Mailath, Samuelson, and Shaked (1997)). Then, she finds a class of population games and a class of simple procedures generating convergence to a Nash equilibrium, leading to a correlated equilibrium.

is taken in the space of assignment functions (not action space) and our approach is evolutionary (not learning). Also our main interest in studying this dynamics is to relate its stable state to our concept of evolutionary stability.

2 Evoluationarily Stable Correlation

Consider a finite and symmetric two-player normal form game $G = \{S_1, S_2; u_1, u_2\}$, where the finite set $S = \{s_1, s_2, \cdots, s_m\}$ of (pure) actions and payoff function $u : S \times S \to \mathbb{R}$ such that $S_1 = S_2 = S$, and $u_1(s_i, s_j) = u(s_i, s_j) = u_2(s_j, s_i)$ for all $s_i, s_j \in S$. A *strategy* (or *mixed action*) is an $x \in \triangle(S)$.[6] A *correlation* is an $\zeta \in \triangle(S \times S)$.[7] A correlation ζ is *symmetric* if $\zeta(s_i, s_j) = \zeta(s_j, s_i)$ for all $(s_i, s_j \in S)$.

Definition 1 (cf. Aumann (1974, 1987)) A *correlated equilibrium* is a correlation ζ such that for all $s_i \in S$,

$$\sum_{s_j \in S} u(s_i, s_j)\zeta(s_i, s_j) \geq \sum_{s_j \in S} u(s_{i'}, s_j)\zeta(s_i, s_j) \qquad \text{for all } s_{i'} \in S.$$

It is well-known that a correlated equilibrium can be viewed as the outcome of Bayesian maximization of players with respect to a random device.

Recall that a random device is a lottery mechanism selecting a private message for each agent (cf. Aumman (1974,1987)). In this paper, we will restrict our attention to the case of direct mechanisms, i.e. the message space $M_i = S$ for each agent $i = 1, 2$. Formally, a *random device* is a pair $F = (S, \zeta)$, where ζ is a correlation, a probability distribution over joint message space $M = M_1 \times M_2 = S \times S$. When an agent receives a message $s_i \in S = M_i$, we interpret that the device recommends the agent to play action s_i.

Thus, a correlated equilibrium is a correlation ζ such that the corresponding random device $F = (S, \zeta)$ is incentive compatible, i.e. being obedient (playing recommended actions) is a best response for an agent if the opponent is also obedient.

Of course, given a random device, an agent may not be obedient, and may choose (pure or mixed) actions different from the recommended actions.

[6] For any finite set X, we use $\triangle(X)$ to denote the set of probability measures over X and $\text{int}(\triangle(X)) = \{\mu \in \triangle(X) : \mu(x) > 0 \ \forall x \in X\}$.

[7] Correlations play two roles in this paper: correlation of messages for recommended actions (see our random-device discussion) or correlation of actions played (Definition 3).

Definition 2 An *assignment function* is a function $\delta_i : S \to \triangle(S)$ for $i = 1, 2$. An *assignment pair* is $\delta = (\delta_1, \delta_2)$ where each δ_i is an assignment function.

Thus, when an agent chooses an assignment function δ and when he receives the signal s_i from the random device, he will choose the mixed action $\delta(s_i)$. We use $\delta(s_j|s_i)$ to denote the probability assigned on s_j under this mixed action $\delta(s_i)$.

We denote the set of pure assignments by $T = \{\delta|\delta : S \to S\}$, the set of assignments by $Q = \{\delta|\delta : S \to \triangle(S)\}$, and the set of probability measures over T by $\triangle(T)$. As a standard, we can map T onto Q by a natural linear function, so we can view the set T as the set Q (Remark A in Appendix).

Suppose there is a random device $F = (S, \zeta)$ and that each player is using the obedient assignment (identity assignment function) δ^{id} where

$$\begin{aligned} \delta^{id}(s_j|s_i) &= 1 \qquad \text{for } j = i \\ &= 0 \qquad \text{for } j \neq i \ . \end{aligned}$$

Then, the resulting correlation (probability distribution) of joint actions actually played will be ζ, the same as the correlation of recommended actions in the device. But when players use other assignments, a different correlation of played joint actions may result.

Definition 3 We say that a correlation μ is *generated by* a random device $F = (S, \zeta)$ and an assignment pair $\delta = (\delta_1, \delta_2)$, denoted by $\mu = K(F, \delta)$, if for all $(s_i, s_j) \in S \times S$,

$$\mu(s_i, s_j) = \sum_{(s'_i, s'_j) \in S \times S} \delta_1(s_i|s'_i)\delta_2(s_j|s'_j)\zeta(s'_i, s'_j).$$

It is natural to interpret a correlation μ (actions actually played) as a coordination mechanism among players. Different assignment functions allow different μ's on $\triangle(S \times S)$, although they need not generate all elements in $\triangle(S \times S)$. In this sense, we endogenize a coordination mechanism μ.

Now, given a random device $F = (S, \zeta)$ and an assignment pair δ, a correlation μ (for joint actions played) is generated, which induces the expected payoffs for the players.

Definition 4 Given $\zeta \in \triangle(S \times S)$ for any assignment pair (δ_1, δ_2), the payoffs for the two players are $U_1 = U(\delta_1, \delta_2)$ and $U_2 = U(\delta_2, \delta_1)$ where:

$$\begin{aligned} U(\delta_1, \delta_2) &= \sum_{(s'_i, s'_j) \in S \times S} \sum_{(s_i, s_j) \in S \times S} \delta_1(s_i|s'_i)\delta_2(s_j|s'_j)u(s_i, s_j)\zeta(s'_i, s'_j) \\ &= \sum_{(s_i, s_j) \in S \times S} u(s_i, s_j)\mu(s_i, s_j) \\ &\qquad \text{where } \mu = K(F, (\delta_1, \delta_2)) \text{ and } F = (S, \zeta) \ . \end{aligned}$$

Our main idea of defining evolutionary stability relies on relative performances of different assignment functions.

It is worthwhile to point out that in our definition of evolutionary stability, we will focus on a symmetric random device, i.e. a device $F = (S, \zeta)$ where ζ is symmetric. This is natural because the underlying normal form game is symmetric and players have symmetric roles.

Example 1 shows that two different assignment functions can generate the same correlation μ (for joint actions played), hence the same payoff (even when the device is symmetric).

Example 1 Let $S = \{X, Y\}$, and $\zeta = (1/4, 1/4, 1/4, 1/4)$ on (XX, XY, YX, YY), and $F = (S, \zeta)$. Let $\delta' = \delta^{id}$. Let δ'' be such that $\delta''(X) = (1/2)X + (1/2)Y$, and $\delta''(Y) = (1/2)X + (1/2)Y$. Then, $K(F, (\delta', \delta')) = K(F, (\delta', \delta'')) = K(F, (\delta'', \delta')) = K(F, (\delta'', \delta'')) = \zeta$.

If two different assignment functions generate the same correlation μ (for joint actions played), it is difficult to say that two assignment functions are really different. For this reason, we give the following definition.

Definition 5 For a given random device $F = (S, \zeta)$, we say that two assignment functions δ', δ'' are *equivalent,* denoted by $\delta' =^* \delta''$, if $\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4$, where, $\zeta_1 = K(F, (\delta', \delta'))$, $\zeta_2 = K(F, (\delta', \delta''))$, $\zeta_3 = K(F, (\delta'', \delta'))$, and $\zeta_4 = K(F, (\delta'', \delta''))$.

Lemma 1 and its immediate Corollary 1 characterize the equivalence relation between two assignment functions.

Lemma 1 *Consider a given random device $F = (S, \zeta)$ where $\zeta \in \triangle(S \times S)$ is symmetric. Then, for all $\delta', \delta'' \in Q$,*

$$[K(F, (\delta', \delta^{id})) = \zeta] \ \Rightarrow \ [K(F, (\delta', \delta'')) = K(F, (\delta^{id}, \delta''))].$$

Proof: See Appendix.

Corollary 1 *Consider a given random device $F = (S, \zeta)$ where $\zeta \in \triangle(S \times S)$ is symmetric. Then:*

a) *for any assignment functions δ', δ'':*

i) $\delta' =^* \delta^{id}$ if and only if $K(F, (\delta^{id}, \delta')) = \zeta$;

ii) if $\delta' =^* \delta^{id}$ and $\delta'' =^* \delta^{id}$, then $\delta' =^* \delta''$ and $K(F, (\delta', \delta'')) = \zeta$.

b) *the set $\{\delta' \in Q : \delta' =^* \delta^{id}\}$ is convex.*

The following definition of stability is the central concept of this paper. A correlation is evolutionarily stable if incumbents with obedient strategies perform better than mutants with non-obedient strategies. We need to be careful in defining mutants since not all non-obedient strategies should be considered mutations (Example 1). For this reason, mutational strategies are only those that are not equivalent to the identical assignment function.

Definition 6 An *evolutionarily stable correlation (ESC)* is a symmetric $\zeta \in \triangle(S \times S)$ such that the identity assignment δ^{id} is evolutionarily stable with respect to $F = (S, \zeta)$, i.e. for every assignment function $\delta' \neq^* \delta^{id}$, there is a $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$:

$$(1-\epsilon)U(\delta^{id}, \delta^{id}) + \epsilon U(\delta^{id}, \delta') \;>\; (1-\epsilon)U(\delta', \delta^{id}) + \epsilon U(\delta', \delta').$$

Suppose there is a large population of individuals whose habit of coordination is the correlation ζ. With respect to the physical random device $F = (S, \zeta)$, regarded as the conventional device, all individuals are programmed to play the obedient assignment δ^{id}. Now, suppose a small group of mutants appears. These mutants use a mutational assignment function $\delta' \neq^* \delta^{id}$, but they cannot change the conventional device. With the physical device ζ, ϵ portion of the population are mutants who use δ', and $1-\epsilon$ portion of the population are incumbents who use δ^{id}. Pairs of individuals in this bimorphic population are repeatedly drawn to play the game with a uniform matching probability. In playing the game, the pair of players faces the conventional random device $F(S, \zeta)$. Then, if ζ is ESC, the incumbents perform better than the mutants; therefore, the evolutionary force drives out the mutants and the incumbents and the conventional random device $F(S, \zeta)$ will last long.

Note that the possible correlations (for joint actions played) generated are restricted to be local in the sense that they are generated restrictively by perturbed mutations through assignments with a fixed direct random device $F = (S, \zeta)$. Such mutations need not generate all possible measures on $\triangle(S \times S)$.[8]

This formulation also captures the idea of mutation arising from misperceptions by agents. For example, suppose the conventional random device remains unchanged but some mutants mistakenly perceive a change in the device. With respect to the perceived device, the mutants will change their assignment functions to accommodate the changes. Our notion of evolutionary stability ensures that these mutants will eventually disappear if their mutational assignment function is different from the conventional obedient one.

The ESC is the analogue of the notion of Maynard Smith's evolutionarily stable (Nash) strategy for correlated equilibria.

Definition 7 (Maynard Smith (1982)) An *evolutionarily stable strategy (ESS)* is an $x \in \triangle(S)$ such that for every $y \in \triangle(S)$ with $x \neq y$, there is a $\bar{\epsilon} > 0$ such that for all $\epsilon \in (0, \bar{\epsilon})$:

$$(1-\epsilon)V(x,x) + \epsilon V(x,y) \; > \; (1-\epsilon)V(y,x) + \epsilon V(y,y).$$

($V(\cdot,\cdot)$ denotes the (expected) payoff for the first player, so for $x, y \in \triangle(S)$, we have $V(x,y) = \sum_{(s_i,s_j) \in S \times S} x(s_i) y(s_j) u(s_i, s_j)$.)

The following fact is well-known.

Fact 1 (Weibull (1995), p.37, Proposition 2.1) *A $x \in \triangle(S)$ is ESS if and only if it satisfies:*

$$\begin{array}{l} a)\ V(y,x) \leq V(x,x) \quad \forall y \in \triangle(S); \\ b)\ V(y,x) = V(x,x) \;\Rightarrow\; V(y,y) < V(x,y) \quad \forall y \in \triangle(S) \text{ with } y \neq x. \end{array} \tag{1}$$

We obtain a similar characterization for our ESC. By the standard arguments for the proof of Fact 1 (cf. Weibull (1995), p. 37), Proposition 1 follows immediately.

[8] This contrasts with the fact that it is possible to find an indirect random machine (i.e. whose message space for each agent need not be S) that can generate the whole set of $\triangle(S \times S)$ through varying assignment pairs. Constructing such a random machine is not difficult. We omit it for brevity.

Proposition 1 (Characterization of ESC) *A $\zeta \in \triangle(S \times S)$ is ESC if and only if it satisfies:*

$$\begin{array}{ll} a)\ U(\delta', \delta^{id}) \leq U(\delta^{id}, \delta^{id}) & \text{for every assignment function } \delta'; \\ b)\ U(\delta', \delta^{id}) = U(\delta^{id}, \delta^{id}) \ \Rightarrow \ U(\delta', \delta') < U(\delta^{id}, \delta') & \\ \qquad\qquad\qquad\qquad \text{for every assignment function } \delta' \neq^* \delta^{id}. & \end{array} \quad (2)$$

Proposition 2 is immediate from Proposition 1.

Proposition 2 . *If $\zeta \in \triangle(S \times S)$ is an ESC, then ζ is a correlated equilibrium.*

Example 2 shows that the converse of Proposition 2 is not true and thus ESC gives us a natural selection of correlated equilibria.

Example 2 This example will show that not every correlated equilibrium is ESC. Consider the following coordination game where $S = \{X, Y\}$ and the payoffs are:

	X	Y
X	$(1,1)$	$(0,0)$
Y	$(0,0)$	$(1,1)$

It is easy to verify that a symmetric $\zeta \in \triangle(S \times S)$ is a correlated equilibrium if and only if it is in the form of $\zeta = (a, b, c, c)$ on (XX, YY, XY, YX) with $a \geq c$ and $b \geq c$. We can show that ζ is an ESC if and only if $a > c$ and $b > c$. It is clear that if $a > c$ and $b > c$, then ζ is a strict equilibrium, hence an ESC.[9] Conversely, suppose $a \geq c$ and $b = c$, then it is easy to check that for the mutational pure assignment function δ' with $\delta'(X) = Y$ and $\delta'(Y) = Y$, we have a violation of condition (b) in Proposition 1, hence ζ cannot be an ESC. Similarly, when $a = c$ and $b \geq c$, ζ cannot be an ESC.

Example 2 is interesting because it shows how ESC compares with ESS. It contrasts sharply with the fact that for any symmetric game there are at most finitely many ESS's: in Example 2 there are only two ESS's, but infinitely many ESC's. More importantly, ESS does not allow any of mixed strategy profiles as an ESS but ESC does. This observation was also made by Theorem 4 in Kandori, Mailath and Rob (1993), which shows that the limit distribution places equal probability on each of two pure Nash equilibria. They also argue that this limit equilibrium has the flavor of a correlated

[9]The set of correlated equilibrium is convex. It is not difficult to show that the set of ESC is also convex.

equilibrium. Thus, Example 2 rigorously confirms their observations in the context of evolutionary approach to correlated equilibria.

It is well-known that the existence of ESS is not guaranteed. Neither is ESC.

Example 3 This example shows the possibility of having no ESC. Consider the following game where $S = \{X, Y, Z\}$ and the payoffs are:

	X	Y	Z
X	$(1,1)$	$(2,-2)$	$(-2,2)$
Y	$(-2,2)$	$(1,1)$	$(2,-2)$
Z	$(2,-2)$	$(-2,2)$	$(1,1)$

If $\zeta \in \triangle(S \times S)$ is symmetric, then it is in the form of $aXX + bYY + cZZ + dXY + dYX + eXZ + eZX + fYZ + fZY$. It is easily verified that if ζ is a correlated equilibrium, then we must have $a = b = c = d = e = f = 1/9$, which is the product measure of the Nash equilibrium mixed action $(1/3)X + (1/3)Y + (1/3)Z$. This Nash equilibrium is not ESS, because any mutation of pure action will perform equally as well as that mixed action. Consequently, this game has no ESC.

Proposition 3 is also immediate from Proposition 1.

Proposition 3 *Suppose a symmetric $\zeta \in \triangle(S \times S)$ is a strict correlated equilibrium, i.e. for all $s_i \in S$, if $\zeta(s_i) = \sum_{s_j \in S} \zeta(s_i, s_j) > 0$, then:*

$$\sum_{s_j \in S} u(s_i, s_j)\zeta(s_i, s_j) > \sum_{s_j \in S} u(s_{i'}, s_j)\zeta(s_i, s_j) \quad \textit{for all } s_{i'} \in S \textit{ with } s'_i \neq s_i.$$

Then ζ is ESC.

However, an ESC is not necessarily a strict correlated equilibrium, shown in Example 4.

Example 4 This example shows that an ESC need not be a strict correlated equilibrium. Consider the following game where $S = \{X, Y\}$ and the payoffs are:

	X	Y
X	$(1,1)$	$(1,1)$
Y	$(1,1)$	$(0,0)$

$\zeta = 1$ on XX is ESS, hence also ESC. Clearly, this ESC is not a strict correlated equilibrium.

Proposition 4 shows the relationship between ESS and ESC.

Proposition 4

a) If $x \in \triangle(S)$ is ESS, then $\zeta = x \times x$ is ESC.

b) If $\zeta \in \triangle(S \times S)$ is ESC and $x \in \triangle(S)$ with $\zeta = x \times x$, then x is ESS.

Proof: See Appendix.

The product measure (a random device) such as $\zeta = x \times x$ does not generate differential information for different recommendations. In particular, if a player is recommended to play s_i or to play s_j, and if the opponents are assumed to be obedient, he will make the same prediction about the opponent's mixed action, namely x. Therefore, his best response to obedience will be x. Under the product measure $\zeta = x \times x$, the (constant) assignment δ', that for each recommendation s_i plays the same mixed action x, satisfies $\delta' =^* \delta^{id}$.

This idea allows us to pass a Nash equilibrium in the standard context of uncoordinated play to the present context of coordinated play and vice versa. These two ideas will be used in our proofs for carrying over other standard refinements of the Nash equilibrium to a correlated equilibrium.

We will now study the structures of the set of ESC and its connections with other known refinements of correlated equilibrium.

3 Evolutionary Stability and Local Superiority

It is well-known that any ESS earns a higher payoff against all nearby mutants than these earned against themselves. This characteristic of ESS is called the local superiority. We will obtain similar characteristics for ESC.

Definition 8 Given a random device $F = (S, \zeta)$, an assignment δ is *locally superior* if there is a neighborhood N of δ in Q such that $U(\delta, \delta') > U(\delta', \delta')$ for all $\delta' \in N$ with $\delta' \neq^* \delta$. We say ζ is *locally superior* if the obedient assignment δ^{id} is locally superior.

Proposition 5 *A symmetric $\zeta \in \triangle(S \times S)$ is ESC if and only if ζ is locally superior.*

Proof: See Appendix.

In Definition 6, the invasion barrier $\bar{\epsilon}$ may depend on the mutant δ'. The following Lemma 2 shows that in the present finite game, the invasion barrier can be taken uniformly for any mutants.

Lemma 2 . *The invasion barrier* $\bar{\epsilon}$ *in Definition 6 can be taken uniformly over all* δ'.

Proof: See Appendix.

Proposition 5 and Lemma 2 are analogues of the similar results in ESS for ESC. (cf. Weibull (1995), Proposition 2.5 and 2.6) We need to make relevant modifications so that we can take care of the $=^*$ relation.

Proposition 6 shows that Definition 8 is a generalization of the standard definition of local superiority for ESS.

Proposition 6 *Let* $x \in \triangle(S)$ *and* $\zeta = x \times x$. *Then,* ζ *is locally superior (in the sense of Definition 8) if and only if* x *is locally superior (in the standard ESS sense), i.e. there is a neighborhood* N *of* x *in* $\triangle(S)$ *such that* $V(x, y) > V(y, y)$ *for all* $y \neq x \in N$.

Proof: It is well-known that an $x \in \triangle(S)$ is ESS if and only if it is locally superior in the standard ESS sense. By Proposition 1 and Proposition 5, x is ESS if and only if ζ is locally superior in the sense of Definition 8. Q.E.D.

4 Asymmetric Animal Conflict

We can apply our model to the study of evolutionary stability on biological conflicts due to Selten (1980), Shmida and Peleg (1997) and Cripps (1991).

It has been noted (cf. Riechert and Hammerstein (1983)) that for applying the ESS notion (Maynard Smith (1982)) to the study of biological conflicts of animals and plants, only pure symmetrical equilibria are relevant, but many games do not admit such equilibria. For example, in a Hawk-Dove game:

	D	H
D	$(2, 2)$	$(2, 4)$
H	$(4, 2)$	$(0, 0)$

the Nash equilibria are: HD, DH, and $(1/2)D + (1/2)H$. To bridge this gap, Selten (1980) introduced a role assignment random process, studied evolutionary stability for behavior strategies ("role conditional strategies"), and proved that any strict (possibly-mixed) Nash equilibrium is the outcome of some process with pure behavior strategy ESS. Shmida and Peleg (1997) and Cripps (1991) extend the work of Selten to cover correlated equilibria (not just Nash equilibria).

For a given symmetric normal form game G with payoff function u and strategy set S as given in the beginning Section 2, the ideas of Shmida and Peleg (1997) and Cripps (1991) can be summarized as follows. [10]

Definition 9 An *asymmetric animal conflict*[11] [12]
is a pair (W, p) such that:

1) $W = W_1 \cup W_2$, where W_1, W_2 are non-empty finite sets that are disjoint,

2) role assignment measure $p \in \triangle(W \times W)$ satisfies $p(w, v) = p(v, w)$ for all $w, v \in W$ and

$$\text{(role asymmetry) } \mathrm{Supp}(p) = W_1 \times W_2 \cup W_2 \times W_1. \tag{3}$$

In an asymmetric animal conflict (W, p), a *behavioral strategy* is a function $b : W \to \triangle(S)$, and we use $b(s|w)$ to denote $(b(w))(s)$. A behavioral strategy b is *pure* if $b(w)$ is a pure (not-mixed) action for every $w \in W$. Given a pair of behavioral strategies b, b', define the expected payoff by:

$$\tilde{U}(b, b') = \sum_{w \in W} \sum_{v \in W} \sum_{s \in S} \sum_{s' \in S} u(s, s') b(s|w) b'(s'|v) p(w, v).$$

As in the standard (Maynard Smith's) sense, a behavioral strategy b is ESS if

$$\begin{array}{ll} \text{a) } \tilde{U}(b', b) \leq \tilde{U}(b, b) & \text{for every behavioral strategy } b'; \\ \text{b) } \tilde{U}(b', b) = \tilde{U}(b, b) \;\Rightarrow & \tilde{U}(b', b') < \tilde{U}(b, b') \\ & \text{for every behavioral strategy } b' \neq b. \end{array} \tag{4}$$

A $\zeta \in \triangle(S)$ is the *outcome* of a behavioral strategy b if

$$\zeta(s, s') = \sum_{w \in W} \sum_{v \in W} b(s|w) b(s'|v) p(w, v) \qquad \text{for all } s, s' \in S.$$

[10] Shmida and Peleg (1997) restricts to such symmetric bimatrix games, Cripps works for general bimatrix games (1991).

[11] The term of asymmetric animal conflict is due to Shmida and Peleg (1997), and it is called a *simple contest* in Cripps (1991).

[12] The role set W discussed below corresponds to our message space S (of recommended actions), and the role assignment function p discussed below corresponds to our correlation ζ, and the behavioral function b discussed below corresponds to our assignment function ζ, and the equation (4) below corresponds to our equation (2).

Definition 10 A $\zeta \in \triangle(S \times S)$ is *SPC* if ζ is the outcome of some pure *ESS* b in some asymmetric animal conflict (W, P).[13]

Then main results of Shmida and Peleg (1997, Theorem 4.1), and Cripps (1991, Theorem) can be stated in this context:

Fact 2 (Shmida and Peleg (1997), and Cripps (1991) *A symmetric* $\zeta \in \triangle(S \times S)$ *is SPC if and only if* ζ *is a strict correlated equilibrium.*

Our work and their work share some similarities in that we commonly relate the notion of correlated equilibrium to the idea of evolutionary stability by using phenotypic conditional behavior. But we are different in several aspects.

First, in motivation, they aim at explaining why a mixed (strict) correlated equilibrium (with asymmetric action pair in its support) is consistent with the idea that only a symmetric pure action equilibrium is relevant in the biological conflict situation. We aim at refining the set of correlated equilibria (allowing non-strict ones) in the spirit of evolutionary stability. Our work admits their interpretation.

Second, in formalizing conditional behavior, they use a role set W, and require the role assignment measure to satisfy the role asymmetry (3), which is essential in their proofs and it makes their space, in general, larger than the action space S. We use the action space as the message space. We also do not need any asymmetry, e.g. we do not require action(or message)-asymmetry, i.e., $\zeta(s, s) = 0$ for all $s \in S$ for a given correlation ζ, although we allow it. (See the fourth comment below).

Third, their notion SPC is too restrictive to be a generalization of the ESS notion. But ESC generalizes ESS. In particular, their SPC is equivalent to the notion of a a strict equilibrium, hence SPC fails to generalize ESS. For example, in the Hawk-Dove game above, the Nash equilibrium $(1/2)H + (1/2)D$ is an ESS, but it is neither a strict Nash equilibrium nor a strict correlated equilibrium, hence it is not SPC. However, it is ESC, as every ESS is ESC.

Fourth, their SPC is in general stronger than our notion of ESC, as a strict correlated equilibrium is also ESC. Their "strict" conclusion is rooted to the requirement of role asymmetry (3) in their formulation. (Cf. Cripps

[13] "SPC" is our own terminology (standing for Shmida-Peleg-Cripps); we use it to avoid confusion. In Shmida and Peleg (1997) (or Cripps (1991)), it is simply called "the outcome distribution of an ESS in an animal conflict (or a simple contest)."

(1991, p. 432, third paragraph.)[14] Interestingly, if a given correlation ζ is action-asymmetric (i.e. $\zeta(s,s) = 0$ for all $s \in S$), then it can easily shown that ζ is ESC if and only if ζ is a strict correlated equilibrium. Thus under action-asymmetry, their SPC and our ESC are equivalent.

5 Evolutionary Stability, Perfection, and Properness

It is well-known that an ESS Nash equilibrium is a perfect equilibrium and even a proper equilibrium. We will give the similar results for correlated equilibria and ESC.

For every number $\epsilon > 0$, we define:

$$Q_\epsilon = \{\delta \in Q : \delta(s_j|s_i) \geq \epsilon \ \forall s_i, s_j \in S\}$$
$$\triangle_\epsilon(S) = \{x \in S : x(s_j) \geq \epsilon \ \forall s_j \in S\} \ .$$

We will first define the standard notions of perfect and proper equilibrium in the context of symmetric games.

Definition 11 (cf. Selten (1975)) An $x \in \triangle(S)$ is a *perfect Nash equilibrium* if there is a sequence of positive $\epsilon_t \to 0$, and a sequence of $x_t \in \triangle_{\epsilon_t}(S)$ such that $x_t \to x$ and for every t, $V(x_t, x_t) = \max\{V(y, x_t) : y \in \triangle_{\epsilon_t}(S)\}$.

Definition 12 (cf. Myerson (1978)) An $x \in \triangle(S)$ is a *proper Nash equilibrium* if there is a sequence of positive $\epsilon_t \to 0$, and a sequence $x_t \in \text{int}(\triangle(S))$ such that $x_t \to x$ and for every t,

$$V(s_p, x_t) < V(s_q, x_t) \ \Rightarrow \ x_t(s_p) \leq \epsilon_t x(s_q) \ \ \forall s_p, s_q \in S.$$

We now give analogues of these definitions in our coordination context.

Definition 13 A *perfect correlated equilibrium* is a symmetric $\zeta \in \triangle(S \times S)$ such that there is an $\delta \in D = \{\delta' \in Q : \delta' =^* \delta^{id}\}$, a sequence of positive $\epsilon_t \to 0$, and a sequence $\delta_t \in Q_{\epsilon_t}$ such that $\delta_t \to \delta$ and for every t, $U(\delta_t, \delta_t) = \max\{U(\delta', \delta_t) : \delta' \in Q_{\epsilon_t}\}$.

In Definition 14, we will use the following set, $\text{int}(Q) = \{\delta \in Q : \delta(s_j|s_i) > 0 \ \forall s_i, s_j \in S\}$. For every $s_i, s_p \in S$, and $\delta \in Q$, we define:

$$U(s_p, \delta_t|s_i) = \sum_{s_k, s_j \in S} \zeta(s_i, s_j)\delta_t(s_k|s_j)u(s_p, s_k).$$

[14] In short, role asymmetry leads to the situation where mutants may meet, but mutant component (actually mutated action) will never meet itself. Cf. Weibull (1995, p. 66).

Definition 14 A *proper correlated equilibrium* is a symmetric $\zeta \in \triangle(S \times S)$ such that there is an $\delta \in D = \{\delta' \in Q : \delta' =^* \delta^{id}\}$, a sequence of positive $\epsilon_t \to 0$, and a sequence $\delta_t \in \text{int}(Q)$ such that $\delta_t \to \delta$ and for every t and every $s_i \in S$:

$$U(s_p, \delta_t|s_i) < U(s_q, \delta_t|s_i) \;\Rightarrow\; \delta_t(s_p|s_i) \leq \epsilon_t \delta_t(s_q|s_i) \qquad \forall s_p, s_q \in S \; .$$

In Proposition 7, we show that Definitions 11 and 12 are generalizations of perfect Nash and proper Nash equilibria.

Proposition 7 . *Let $x \in \triangle(S)$ and $\zeta \in \triangle(S \times S) = x \times x$. Then:*

a) x is a perfect equilibrium if and only if ζ is a perfect correlated equilibrium;

b) x is a proper equilibrium if and only if ζ is a proper correlated equilibrium.

Proof: See Appendix.

The relationship between a perfect correlated equilibrium and a proper correlated equilibrium is same as the case for the Nash equilibrium.

Proposition 8 *A proper correlated equilibrium is a perfect correlated equilibrium but the converse is not true.*

Proof: The first assertion follows easily along the lines of a standard proof (cf. Myerson (1978)) that a proper Nash equilibrium is a perfect Nash equilibrium, and we omit the details. The second assertion follows from Proposition 7, and the well-known fact that a proper Nash equilibrium is not necessarily a perfect Nash equilibrium (cf. Myerson (1978)). Q.E.D.

We provide two other natural formalizations for perfectness and properness. For a given $\zeta \in \triangle(S \times S)$, we define the game $\Gamma(\zeta) = (T_1, T_2, U_1, U_2)$ where the strategy sets are $T_1 = T_2 = T$ and T is the set of pure assignments (see the paragraph after Definition 2). And for all $x, y \in \triangle(T)$, the payoffs are defined by $U_1(x, y) = U(x, y)$, and $U_2(x, y) = U(y, x)$, where $U(x, y) = \sum_{(\delta,\delta') \in T \times T} x(\delta) y(\delta') U(\delta, \delta')$, and $U(\delta, \delta')$ is defined in Definition 4. In Definitions 13 and 14, we define the perfection and properness for correlated equilibria in the context of the agent normal form for extensive games. Definition 15 defines the perfection for correlated equilibria for normal form games.

Definition 15 A *perfect correlated equilibrium* is a symmetric $\zeta \in \triangle(S \times S)$ such that there is an $\delta \in D$ such that δ is a perfect equilibrium in the game $\Gamma(\zeta)$ where $D = \{\delta \in \triangle(T) : \delta =^* \delta^{id}\}$.

Proposition 9

a) Definition 13 and Definition 15 are equivalent.

b) Definition 15 is equivalent to the following seemingly weakening of Definition 15:

> *a symmetric* $\zeta \in \triangle(S \times S)$ *such that*
>
> δ^{id} *is a perfect equilibrium in the game* $\Gamma(\zeta)$.

Proof: See Appendix.

The condition stated in Proposition 9 is the same as the definition of a perfect direct correlated equilibrium ζ defined by Dhillon and Mertens (1994) in our symmetric context. Therefore, our notion of a perfect correlation is the same as their notion. Many of results in Dhillon and Mertens (1994) are also valid. Their Proposition 3 shows that a perfect direct correlated equilibrium is an acceptable correlated equilibrium (defined by Myerson (1986)). Therefore, our notion of a perfect correlated equilibrium is also an acceptable correlated equilibrium.

However, an acceptable correlated equilibrium is not necessarily a perfect correlated equilibrium. A useful fact by Dhillon and Mertens (1994, Proposition 4) is that in a two-player game, the acceptable correlated equilibria are those correlated equilibria for which only undominated strategies are recommended. Using this fact, the following example shows that an acceptable equilibrium does not necessarily imply a perfect correlated equilibrium. Dhillon and Mertens (1994, Example 2) give such an example with a two-player game that is not symmetric; our Example 5 is a symmetric game.

Example 5 Consider the following symmetric two-player game where $S = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the payoffs are:

	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$(1,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(1,0)$	$(-100,0)$
x_2	$(0,0)$	$(1,1)$	$(0,0)$	$(0,1)$	$(-100,0)$	$(1,0)$
x_3	$(1,0)$	$(0,0)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$
x_4	$(0,0)$	$(1,0)$	$(1,1)$	$(1,1)$	$(0,0)$	$(0,0)$
x_5	$(0,1)$	$(0,-100)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
x_6	$(0,-100)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$

Then, x_1, x_2 are undominated actions. Also, $\zeta = (1/4, 1/4, 1/4, 1/4)$ on $(x_1x_1, x_1x_2, x_2x_1, x_2x_2)$ is a correlated equilibrium (indeed Nash) with only undominated actions, hence it is an acceptable correlated equilibrium.

We will now show that ζ is not an perfect correlated equilibrium and not an ESC. It suffices to show that, given the random device ζ, the obedient assignment δ^{id} is weakly dominated by some δ'. We define $\delta' \in Q$ where $\delta'(x_1) = x_3$ and $\delta'(x_2) = x_4$. We will show that for every $\delta \in \mathrm{int}(Q)$, we have $U(\delta', \delta) > U(\delta^{id}, \delta)$. Then, δ^{id} is weakly dominated by δ', hence δ^{id} cannot be an perfect equilibrium for $\Gamma(\zeta)$, so ζ is not a perfect correlated equilibrium and not an ESC.

Consider any $\delta \in \mathrm{int}(Q)$. We will now show that $U(\delta', \delta) > U(\delta^{id}, \delta)$. We write δ in the forms of:

$$\delta(x_1) = (a, b, c, d, e, f) \text{ on } (x_1, x_2, x_3, x_4, x_5, x_6)$$
$$\delta(x_2) = (a', b', c', d', e', f') \text{ on } (x_1, x_2, x_3, x_4, x_5, x_6)$$

where $a, a', b, b', c, c', d, d', e, e', f, f' > 0$. By a simple calculation, we have:

$$\begin{aligned} U(\delta^{id}, \delta) = & \frac{1}{2}[\frac{a+a'}{2}(1) + \frac{e+e'}{2}(1) + \frac{f+f'}{2}(-100)] \\ & + \frac{1}{2}[\frac{b+b'}{2}(1) + \frac{e+e'}{2}(-100) + \frac{f+f'}{2}(1)] \\ U(\delta', \delta) = & \frac{1}{2}[\frac{a+a'}{2}(1) + \frac{c+c'}{2}(1) + \frac{d+d'}{2}(1)] \\ & + \frac{1}{2}[\frac{b+b'}{2}(1) + \frac{c+c'}{2}(1) + \frac{d+d'}{2}(1)], \end{aligned}$$

so $U(\delta', \delta) > U(\delta^{id}, \delta)$.

We also give the definition of properness for correlated equilibria in the context of normal form games.

Definition 16 A *strongly proper correlated equilibrium* is a symmetric $\zeta \in \triangle(S \times S)$ such that there is an $\delta \in D$ such that δ is a proper equilibrium in the game $\Gamma(\zeta)$.

Proposition 10 *If $\zeta \in \triangle(S \times S)$ is a strongly proper correlated equilibrium, then it is a proper correlated equilibrium.*

Proof: See Appendix.

The following Proposition 11 summarizes the relationships among the perfection, the properness, and ESC.

Proposition 11 *If ζ is ESC, then it is a strongly proper correlated equilibrium.*

Proof: See Appendix.

The following Example 6 illustrates Proposition 11

Example 6 Consider the following Rock-Scissor-Paper type game.

	X	Y	Z
X	$(0,0)$	$(2,1)$	$(1,2)$
Y	$(1,2)$	$(0,0)$	$(2,1)$
Z	$(2,1)$	$(1,2)$	$(0,0)$

This game has a strict correlated equilibrium assigning probability 1/6 on XY, XZ, YX, YZ, ZX, ZY, where each player gets a payoff of 3/2. This equilibrium is ESC and strongly proper. This equilibrium does not require the correlated equilibrium to be a product measure and thus is different from the unique mixed strategy Nash equilibrium, where each player randomizes between the three actions with equal probability.

As a consequence of Propositions 10 and 11, if ζ is ESC, then it is a proper correlated equilibrium, hence it is also a perfect correlated equilibrium by Proposition 8. However a proper or perfect correlated equilibrium need not be an ESC. In Example 3, the totally mixed Nash equilibrium $x = (1/3)X + (1/3)Y + (1/3)Z$ is a proper equilibrium, hence it gives a proper correlated equilibrium. However, this is not an ESC.

Whether the notion of strong properness is strictly stronger than that of properness in correlated equilibria remains open. Whether the properness of Nash equilibria implies the strong properness of correlated equilibria also remains open.

6 Evolutionary Stability and Replicator Dynamic Stability

The previous sections study the stability with respect to both a mutation process and mistakes trembling. We now turn to the dynamic stability with respect to replicator dynamics.

We will consider games $\Gamma(\zeta)$. A population state is an element $x \in \triangle(T)$. We will obtain the stability result for the set $D = \{x \in \triangle(T) : x =^* \delta^{id}\}$.

Recall that for a given $\zeta \in \triangle(S \times S)$ and any $x, y \in \triangle(T)$, we define:

$$U(x,y) = \sum_{(\delta,\delta') \in T \times T} x(\delta)y(\delta')U(\delta,\delta')$$

where $U(\delta, \delta')$ is defined in Definition 4.

Definition 17 Given a symmetric $\zeta \in \triangle(S \times S)$, the *replicator dynamic* is the dynamic system on $\triangle(T)$ described by the differential equation:

$$\dot{x}(\delta) = [U(\delta, x) - U(x, x)]x(\delta)$$

where $\dot{x}(\delta)$ denotes $dx(\delta)/dt$.

By the Picard-Lindelöff Theorem, the equation defines a solution mapping $X : \mathbb{R} \times \triangle(T) \to \triangle(T)$, which to each initial state x_0 and time $t \in \mathbb{R}$ assigns the population state $X(t, x_0) \in \triangle(T)$ at time t. Note that our formulation reduces to a standard Nash context so we can apply existing literature of the replicator dynamics. We recall standard stability criteria from the literature.

Definition 18 A closed set $A \subseteq \triangle(T)$ is *Lyapunov stable* if every neighborhood $N \subseteq \triangle(T)$ of A contains a neighborhood $N_0 \subseteq \triangle(T)$ of A such that $X(t, x_0) \in N$ for all $x_0 \in N_0$ and all $t \geq 0$. A closed set $A \subseteq \triangle(T)$ is *asymptotically stable* if it is Lyapunov stable and there exists a neighborhood $N^* \subseteq \triangle(T)$ of A such that:

$$X(t, x_0) \to A \text{ as } t \to \infty \qquad \text{for all } x_0 \in N^*.$$

Proposition 12 *If ζ is ESC, then D is asymptotically stable.*

Proof: See Appendix.

The converse of Proposition 12 is not true. It is well-known that in the standard Nash context, $x \in \triangle(S)$ that is asymptotically stable is not necessarily an ESS. For example, as shown in Weibull (1995, p. 102, Example, 3.9), the following game

	X	Y	Z
X	$(1, 1)$	$(5, 0)$	$(0, 5)$
Y	$(0, 5)$	$(1, 1)$	$(5, 0)$
Z	$(5, 0)$	$(0, 5)$	$(4, 4)$

has a unique Nash equilibrium $x = (3/18, 8/18, 7/18)$ on (X, Y, Z), which is asymptotically stable but not an ESS. Thus, the product measure $\zeta = x \times x$ is not an ESC. Consider any $y \in \triangle(T)$. Because ζ is a product measure, it is easy to verify that y has a unique $\tilde{y} \in \triangle(S)$ such that the assignment pair (y, y) induces the same distribution as $\tilde{y} \times \tilde{y}$. Moreover, the replicator dynamics of y induces a dynamics on $\triangle(S)$, which is the same as the standard replicator dynamics of $\tilde{y}$. Then, for every open set $W \ni x$ in $S(\triangle)$, we define an open set $N = \{y \in \triangle(T) : \tilde{y} \in W\}$ of D in $\triangle(T)$. Because x is Lyapunov stable, D is Lyapunov stable. Also, as x is asymptotically stable, we can find a neighborhood N^* such that for every initial $y^0 \in N^*$, the path $y(t) \in \triangle(T)$ has its counterpart $\tilde{y}(t) \in \triangle(S)$ converging to x. As $\tilde{y}(t)$ converges to x, $y(t)$ must converge to D. Therefore, D is also asymptotically stable.

7 Concluding Remarks

The conventional ESS approach focuses only on the Nash equilibrium, where Nature provides private and independent signals to each player. Even though the ESS gives a good selection of the Nash equilibrium which, all too often are multiple, it is restricted in the sense that people in many cases observe public and correlated signals. The concept of correlated equilibrium is a natural extension of the Nash equilibrium to deal with this case. However, the correlated equilibrium is not free from the multiplicity of equilibria either.

In this paper, we provide an evolutionary approach to the correlated equilibrium as a natural selection criterion. In this regard, this paper fills the gap between ESS and correlated equilibria. Proposition 4 makes clear that the new concept of ESC is a generalization of ESS. We note that our work can be applied to the study of evolutionary ecology (animal conflict). We also provide other refinements for ESC and those have corresponding analogues in the refinements of Nash equilibria. We studied a perfect correlated equilibrium, a proper Nash equilibrium for this purpose and established the relationship between these concepts and corresponding analogues for Nash equilibria. Propositions 7 through 11 show that the concept of ESC gives a good selection of correlated equilibria in the sense that an ESC is proper and perfect but not vice versa.

One interesting topic that we did not investigate in this paper is to extend this approach to study long-run characteristics of the correlated equilibrium in the context of Kandori, Mailath, and Rob (1993) and Young (1993). Our approach and theirs share some interesting observations. In the pure coordination game in Example 2, they predict the society spends equal time in two pure Nash equilibria. A similar observation was made in this paper, even through ESC allows more equilibria. The conventional ESS only predicts two pure Nash equilibria.

Our approach is only concerned with the direct mechanism. One limitation of this restriction is that the direct mechanism with mutations on assignment functions does not generate all the possible measures on $S \times S$. If indirect mechanisms are allowed, it is possible to generate all possible joint distributions simply by varying assignment functions. Thus, an interesting extension would be to generalize the message space of the random device.

8 Appendix

Proof of Lemma 1: Let $S = \{s_1, \cdots, s_m\}$. For notational simplicity, for all $i, j \leq m$, write: $\zeta_{ij} = \zeta(s_i, s_j)$, $\delta'_{kj} = \delta'(s_j|s_k)$, and $\delta''_{kj} = \delta''(s_j|s_k)$. Since $K(F, (\delta,', \delta^{id})) = \zeta$, we have:

$$\zeta_{ij} = \sum_{k=1}^{m} \zeta_{kj}\delta'_{ki} \qquad \text{for all } i, j.$$

Then, for the $\zeta'' = K(F, (\delta', \delta''))$, by definition for all i, j, the number $\zeta''(s_i, s_j)$ satisfies:

$$\begin{aligned} \zeta''(s_i, s_j) &= \sum_{1 \leq p,k \leq m} \delta'_{ki}\zeta_{kp}\delta''_{pj} \\ &= \sum_{p=1}^{m} \delta''_{pj}(\sum_{k=1}^{m} \zeta_{kp}\delta'_{ki}) \\ &= \sum_{p=1}^{m} \delta''_{pj}\zeta_{ip} \end{aligned}$$

Therefore, $\zeta'' = K(F, (\delta^{id}, \delta''))$. Q.E.D.

Proof of Proposition 4:
(Part a) Let $x \in \triangle(S)$ be ESS, and $\zeta = x \times x$. Then (x, x) is a Nash equilibrium by Fact 1, so ζ is a correlated equilibrium; therefore, ζ satisfies (2a) in Proposition 1. Thus it suffices to show that ζ satisfies (2b) in Proposition 1. Consider any assignment function $\delta' : S \to \triangle(S)$ with $\delta' \neq^* \delta^{id}$. Define a $y \in \triangle(S)$ by $y(s_j) = \sum_{s_i \in S} x(s_i)\delta'(s_j|s_i)$. The following relations are immediate from Definition 3 and y:

$$\begin{array}{rcl} K(F, (\delta^{id}, \delta^{id})) &=& x \times x \\ K(F, (\delta^{id}, \delta')) &=& x \times y \\ K(F, (\delta', \delta^{id})) &=& y \times x \\ K(F, (\delta', \delta')) &=& y \times y \end{array} \tag{5}$$

Because $\delta' \neq^* \delta^{id}$, we must have $y \neq x$. Now, also note that:

$$\begin{array}{ll} U(\delta^{id},\delta^{id}) = V(x,x) & U(\delta',\delta^{id}) = V(y,x) \\ U(\delta^{id},\delta') = V(x,y) & U(\delta',\delta') = V(y,y) \ . \end{array} \tag{6}$$

Then, (2b) follows from (1b).

(Part b) Let $\zeta \in \triangle(S \times S)$ be an ESC (in our sense), and $x \in \triangle(S)$ with $\zeta = x \times x$. For any $y \in \triangle(S)$ with $y \neq x$, we define an assignment function $\delta' : S \to \triangle(S)$ by $\delta'(s_j|s_i) = y(s_j)$. Then, (5) and (6) hold. Because $y \neq x$ by (5) we have $\delta^{id} \neq^* \delta'$. Therefore, (1a) and (1b) in Fact 1 follows from (2a) and (2b) in Proposition 2. Q.E.D.

Remark A. Consider the linear $h : \triangle(T) \to Q$ defined by:

$$(h(x))(s_j|s_i) = \sum_{\{\delta:\delta\in T\&\delta(s_i)=s_j\}} x(\delta) \ .$$

This maps $\triangle(T)$ onto Q because there is a function $g : Q \to \triangle(T)$ such that $h(g(\delta)) = \delta$ for all $\delta \in Q$. For example, one can choose the function $g : Q \to \triangle(T)$ defined by $g(\delta) = x$ where

$$x(\hat{\delta}) = \Pi_{s_i \in S}\delta(\hat{\delta}(s_i)|s_i) \qquad \text{for all } \hat{\delta} \in T.$$

Therefore, we often identify an element $x \in \triangle(T)$ with the element $h(x) \in Q$. Also, with this h function, we can view the set Q as simplex generated by the extreme points $\delta \in T$.

In fact, for every small $\epsilon > 0$, the function h maps $\triangle_\epsilon(T)$ onto $Q_{\epsilon'}$, where $\epsilon' = \epsilon\#(S)^{\#(S)-1}$. This can be proved by constructing a function $g_\epsilon : Q_{\epsilon'} \to \triangle_\epsilon(T)$ such that $h(g_\epsilon(\delta)) = \delta$ for all $\delta \in Q_{\epsilon'}$.[15]

Proof of Lemma 2. The "if" part is trivial. We only need to prove the "only if" part. By using the linear mapping h as given in Remark A, we identify Q with $\triangle(T)$, the simplex spanned by T. For a given ESC ζ, we

[15] We give such a function g_ϵ. First, for each $\delta \in T$, define x_δ to be an element in $\triangle(T)$ such that $x_\delta(\delta') = \epsilon$ for all $\delta' \in T$ with $\delta' \neq \delta$. Then, $A = \{x_\delta : \delta \in T\}$ is the set of vertices spanning $\triangle_\epsilon(T)$ and $\triangle_\epsilon(T) = \text{con}(A)$. Also, for each $s_i \in S$, define y_{s_i} to be the element in $\triangle(S)$ such that $y_{s_i}(s_j) = \epsilon'$ for all $s_j \in S$ with $s_j \neq s_i$. Then, $B = \{y_{s_i} : s_i \in S\}$ is the set of vertices spanning $\triangle_{\epsilon'}(S)$ and $\triangle_{\epsilon'}(S) = \text{con}(B)$. Note that for all x_δ, $h(x_\delta)$ is a "vertex" of $Q_{\epsilon'}$, in particular, $h(x_\delta) \in Q_{\epsilon'}$ is such that $h(x_\delta)(s_i) = y_{\delta(s_i)}$ for all $s_i \in S$. For each $\delta' \in Q_{\epsilon'}$, there are non-negative scalars $\lambda_{y_{s_j},s_i}$ where $s_j, s_i \in S$ and $\sum_{s_j\in S}\lambda_{y_{s_j},s_i} = 1$ and $\delta'(s_i) = \sum_{s_j\in S}\lambda_{y_{s_j},s_i}y_{s_j}$ for all s_i. Then, we can define $g_\epsilon(\delta') = \sum_{\delta\in T}\mu_\delta x_\delta$ where scalars $\mu_\delta = \Pi_{s_i\in S}\lambda_{h(x_\delta)(s_i),s_i}$. It can be verified that this g_ϵ satisfies the desired property.

define $Z \subseteq \triangle(T) = Q$ to be the union of all boundary faces of $\triangle(T)$ that do not contain any element of the closed convex set $D = \{\delta' \in Q : \delta' =^* \delta^{id}\}$. Then Z is compact and we can take a uniform barrier $\bar{\epsilon}$ for all $\delta' \in Z$. Note that for every $\delta'' \in Q \backslash D$, there is a number $\lambda \in (0,1)$, a $\delta^1 \in Z$ and a $\delta^2 \in D$ such that $\delta'' = \lambda\delta^1 + (1-\lambda)\delta^2$. So, it follows that $(1/\lambda)$ is also a barrier for δ''. Therefore, $\bar{\epsilon}$ is a uniform barrier. Q.E.D.

Proof of Proposition 5. First, we prove the "if" part. Suppose an open set $N \subseteq Q$ is a neighborhood of δ^{id} as given in Definition 8. Note that for every $\delta' \in Q$, there is a small $\epsilon > 0$ such that $w = \epsilon\delta' + (1-\epsilon)\delta^{id} \in N$. If $\delta' \neq^* \delta^{id}$, then we have $w \neq^* \delta^{id}$. Then the local superiority of δ^{id} and bilinearity of u imply $(1-\epsilon)U(\delta^{id}, \delta^{id}) + \epsilon U(\delta^{id}, \delta') \;>\; (1-\epsilon)U(\delta', \delta^{id}) + \epsilon U(\delta', \delta')$. Therefore, ζ is an ESC.

Second, we prove the "only if" part. We choose the sets Z and D as defined in the proof of Lemma 2 and choose the uniform invasion barrier $\bar{\epsilon} \in (0,1)$ as given in Lemma 2. Define the open set N by $N = \{\delta' \in Q : \delta' = \epsilon\delta^1 + (1-\lambda)\delta^2$ for some $\delta^1 \in D$ and $\delta^2 \in Z$ and $\epsilon \in [0, \bar{\epsilon})\}$. It is easy to verify that the set N satisfies the condition as given in Definition 8. Q.E.D.

Proof of Proposition 7. (*Part a*) Let $x \in \triangle(S)$ be a perfect Nash equilibrium and let ϵ_t and x_t be as given in Definition 11. Now, define $\delta \in Q$ and $\delta_t \in Q_{\epsilon_t}$ by:

$$\begin{aligned} \delta(s_j|s_i) &= x(s_j) \qquad \forall s_i, s_j \in S; \\ \delta_t(s_j|s_i) &= x_t(s_j) \qquad \forall s_i, s_j \in S. \end{aligned}$$

Then, $\delta =^* \delta^{id}$ and $\delta_t \to \delta$. Also, each $\delta_t \in Q_{\epsilon_t}$. It is easy to verify that $U(\delta_t, \delta_t) = \max\{U(\delta', \delta_t) : \delta' \in Q_{\epsilon_t}\}$. Therefore, $\zeta = x \times x$ is a perfect correlated equilibrium.

Let $\zeta = x \times x$ be a perfect correlated equilibrium. Let δ, ϵ_t, and δ_t be as given in the Definition 13. Define $x_t \in \triangle_{\epsilon_t}(S)$ by:

$$x_t(s_j) = \sum_{s_i \in S} x(s_i)\delta_t(s_j|s_i) \ .$$

Now that for every $y \in \triangle$, the product measure $y \times x_t$ is equal to $K(\zeta, (\delta_t, \delta'))$, where $\delta' \in Q$ is defined by $\delta'(s_j|s_i) = y(s_j)$ for all $s_i, s_j \in S$. Moreover, $y \in \triangle_{\epsilon_t}(S)$ if and only if $\delta' \in Q_{\epsilon_t}$. Therefore, we have $V(x_t, x_t) = \max\{V(y, x_t) : y \in \triangle_{\epsilon_t}(S)\}$. Thus, x is a perfect Nash equilibrium.

(*Part b*) We will use similar arguments.

Let $x \in \triangle(S)$ be a proper Nash equilibrium and let ϵ_t and x_t be as given in Definition 12. Now define $\delta \in Q$ and $\delta_t \in Q_{\epsilon_t}$ by:

$$\delta(s_j|s_i) = x(s_j) \qquad \forall s_i, s_j \in S;$$
$$\delta_t(s_j|s_i) = x_t(s_j) \qquad \forall s_i, s_j \in S.$$

Then, $\delta =^* \delta^{id}$ and $\delta_t \to \delta$. Also, each $\delta_t \in \text{int}(Q)$. And for each $s_i, s_p, s_q \in S$,

$$U(s_p, \delta_t|s_i) = x(s_i)V(s_p, x_t)$$
$$U(s_q, \delta_t|s_i) = x(s_i)V(s_q, x_t)$$

Therefore, as x_t satisfies the required property in Definition 12, these δ_t also satisfies the required property in Definition 14. Thus, $\zeta = x \times x$ is a proper correlated equilibrium.

Let $\zeta = x \times x$ be a proper correlated equilibrium. Let δ, ϵ_t, and δ_t be as given in the Definition 14. Define $x_t \in \triangle_{\epsilon_t}(S)$ by:

$$x_t(s_j) = \sum_{s_i \in S} x(s_i)\delta_t(s_j|s_i) \ .$$

Now, note that for every t, the product measure $x_t \times x_t \in \triangle(S \times S)$ is equal to $K(\zeta, (\delta_t, \delta_t))$. Moreover, $x_t \in \text{int}(S)$ as $\delta_t \in \text{int}(Q)$. Also, note that because $\zeta = x \times x$ is a product measure, by independence for all s_i, s_p, s_q:

$$U(s_p, \delta_t|s_i) = x(s_i)V(s_p, x_t)$$
$$U(s_q, \delta_t|s_i) = x(s_i)V(s_q, x_t) \ .$$

Therefore, for all $s_p, s_q \in S$:

$$U(s_p, \delta_t|s_i) < U(s_q, \delta_t|s_i) \text{ for some } s_i \in S$$
$$\Leftrightarrow U(s_p, \delta_t|s_i) < U(s_q, \delta_t|s_i) \text{ for all } s_i \in S \text{ with } x(s_i) > 0 \ .$$

Recall that $V(s_p, x_t) = \sum_{s_i \in S} U(s_p, \delta_t|s_i)$ and $V(s_q, x_t) = \sum_{s_i \in S} U(s_q, \delta_t|s_i)$. Therefore, if $V(s_p, x_t) < V(s_q, x_t)$, then:

$$U(s_p, \delta_t|s_i) < U(s_q, \delta_t|s_i) \qquad \text{for all } s_i \in S \text{ with } x(s_i) > 0$$

hence:

$$\delta_t(s_p|s_i) \leq \epsilon_t \delta_t(s_q|s_i) \qquad \text{for all } s_i \in S \text{ with } x(s_i) > 0$$

consequently $x_t(s_p) \leq \epsilon_t x_t(s_q)$. Thus x is a proper equilibrium. Q.E.D.

Proof of Proposition 9: (Part a). Definitions 13 and 15 are equivalent because for every small $\epsilon > 0$, the function h given in Remark A maps $\triangle_\epsilon$ onto the set $Q_{\epsilon'}$, where $\epsilon' = \epsilon\#(S)^{\#(S)-1}$. (Part b) Because the game $\Gamma(\zeta)$ is a symmetric two-player game, an $x \in \triangle(T)$ is a perfect equilibrium if and only if x is not weakly dominated by any strategy. (See van Dame (1984), Theorem 3.22, cf. Weibull (1995), Proposition 1.4). Then, δ^{id} is undominated if and only if some $x \in D$ is undominated if and only if every $x \in D$ is undominated. Q.E.D.

Proof of Proposition 10: It clearly suffices to show that, for any small $\epsilon > 0$, if $x \in \text{int}(\triangle(T))$ satisfies

$$V(\delta', x) < V(\delta'', x) \;\Rightarrow\; x(\delta') \leq \epsilon x(\delta'') \;\; \forall \delta', \delta'' \in T, \tag{7}$$

then $h(x)$ is such that for every $s_i \in S$, and all $s_p, s_q \in S$,

$$U(s_p, h(x)|s_i) < U(s_q, h(x)|s_i) \;\Rightarrow\; (h(x))(s_p|s_i) \leq \epsilon(h(x))(s_q|s_i) \tag{8}$$

where $h(x)$ is defined in Remark A.

First, suppose $U(s_p, h(x)|s_i) < U(s_q, h(x)|s_i)$ and (7) holds. Define T_{-i} to be the set of functions $\delta_{-i} : S\backslash\{s_i\} \rightarrow S$. For every δ_{-i}, define (s_p, δ_{-i}) to be the element in T, which agrees with δ_{-i} over $S\backslash\{s_i\}$ and assigns s_p at s_i, and similarly for (s_q, δ_{-i}). For every $\delta_{-i} \in T_{-i}$, we have:

$$V((s_p, \delta_{-i}), x) - V((s_q, \delta_{-i}), x) = U(s_p, h(x)|s_i) - U(s_q, h(x)|s_i).$$

As $U(s_p, h(x)|s_i) < U(s_q, h(x)|s_i)$, we have $V((s_p, \delta_{-i}), x) < V((s_q, \delta_{-i}), x)$, therefore,

$$x((s_p, \delta_{-i})) \leq \epsilon x((s_q, \delta_{-i})) \qquad \text{for all } \delta_{-i} \in T_{-i}.$$

Then:

$$(h(x))(s_p|s_i) = \sum_{\delta_{-i} \in T_{-i}} x((s_p, \delta_{-i})) \leq \sum_{\delta_{-i} \in T_{-i}} \epsilon x((s_q, \delta_{-i})) = \epsilon(h(x))(s_q|s_i).$$

Thus, (8) holds. Q.E.D.

Proof of Proposition 11: The set $D = \{\delta \in \triangle(T) : \delta =^* \delta^{id}\}$ is a closed and convex set in $\triangle(T)$. Moreover, because ζ is ESC, by definition D is an ESS set (defined by Swinkels (1992), Definition 10) in the normal form game $\Gamma(\zeta)$, i.e. D is closed and contained in the set of Nash equilibria of

$\Gamma(\zeta)$ such that there is an $\epsilon' > 0$ such that for all $\epsilon \in (0, \epsilon')$, all $x \in D$ and all $y \in \triangle(T)$:

$$C(x) \subseteq B((1-\epsilon)x + \epsilon y)) \;\Rightarrow\; (1-\epsilon)x + \epsilon y \in D,$$

where $C(x) = \{\delta \in T : x(\delta) > 0\}$, $B(z) = \{\delta \in T : U(\delta, z) = \max\{U(w, z) : w \in \triangle(T)\}$ for all $z \in \triangle(T)$. Therefore, as proved by Theorem 5 in Swinkels (1992) such a convex and closed ESS set contains a proper equilibrium. This proves Proposition 10. Q.E.D.

Proof of Proposition 12: Because ζ is ESC, it is locally superior. Therefore, we can choose the neighborhood $N \supset D$ as constructed in the proof of Proposition 11. Then for every $x \in D$, the neighborhood $N_x = N$ of x satisfies $U(x, y) > U(y, y)$ for all $y \in N \backslash D$. In other words, the set D is evolutionary stable* in the standard (Nash) context (Weibull 1995, Definition 3.1). Then, by a well-known result of replicator dynamics (cf. Weibull 1995, Proposition 3.13), D is asymptotically stable. Q.E.D.

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