Contracting with Risk Aversion and Subjective Beliefs Under Costly State Verification

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Summary: We study the loan contracting model of Gale and Hellwig (1985) under general assumptions of risk aversion of the borrower and diverse subjective beliefs of the borrower and lender about the outcome of the investment. We continue to assume the lender must incur cost in order to observe the outcome of the project, while the borrower can (ex-post) observe the state at no cost. We claim that once we introduce differences in probability beliefs into the contracting environment the complex relationship between two trading parties becomes focal. Contract terms reflect return on capital, insurance and risk sharing arising from the motive to trade on the differences in probabilities. This trading is desired by the parties since there are no financial markets where agents could purchase insurance for state contingencies hence private contracting replaces markets for contingent claims. Under such conditions verification states are not necessarily interpreted as "default" states. We characterize the optimal contract and show that (i) the contractual payoff in verification states varies by states in accord with risk aversion and probability belief of the borrower, and (ii) the verification region may consist of many intervals. We provide conditions and examples to show that when the borrower is more optimistic than the bank, there may be fewer verification regions and the terms of an optimal contract may be simplified.

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1.   Introduction

The seminal work of Gale and Hellwig (1985) extended Townsend's (1978) contracting

approach to incentive compatible debt contracts and has had a significant impact on the way markets

for loans are being studied. Their model is commonly explored in textbooks on banking theory (e.g.

Xavier and Rochet (1997) Chapter 4) and is often used as a benchmark in applications (e.g. Boyd

and Smith (1994), Khalil and Parigi (1998), Mookherjee and Png (1989)). We briefly review the

basic set-up of the model. An entrepreneur has equity capital N used as collateral when seeking to

borrow the amount B from the bank. He aims to invest the amount $N + B$ in a risky project whose

return depends on a stochastic state $s \in S$, and consume his payoff. The standard assumptions are: (i)

consumption is non-negative, (ii) both the borrower and the bank are risk-neutral, (iii) the borrower

and the bank hold the same probability beliefs about the distribution of s. The optimal contract

stipulates the payment by the project as a function of the state but the state is ex-post asymmetrically

observed. The entrepreneur can observe the realized state at no cost while the bank needs to incur

an audit cost of $c(s, B + N) \geq 0$ in order to verify the state. Hence, the contract specifies the

payment as a function of the announced state. A bank's decision to verify the state is *commonly*

interpreted as a default by the borrower who announces that he cannot meet the contractual

payment. The conclusion is a *standard loan contract* partitioning the set of states S into two

intervals. An audit interval S^* of states of the lowest investment returns where the bank spends the

observation cost c to verify the state and an interval $S\backslash S^*$ of high investment returns where no state

verification takes place. If $s \in S^*$ the borrower is in default and consumes nothing while the bank

receives all available resources. For $s \in S\backslash S^*$ the payment to the bank is a constant independent of

the state, contracted to cover the loan B plus interest, and the borrower receives the balance.

Gale and Hellwig briefly considered how the contract might change if the borrower is risk

averse. Garino and Simmons (2001) study in further detail the optimal contract when the borrower

is risk averse but both papers assume that the borrower and the bank hold the same probability belief

about the state. Assuming the verification cost is independent of the state, they show that when the

borrower is risk-averse he consumes *a constant positive quantity* when the state is verified in S^* and

he is in default, but S^* is still an interval of low return states. If there are any non-verification states,

then due to the bank's verification cost, there will normally be a discontinuity in the payoff schedule

of the borrower at the point of transition from verification to non-verification interval. With a view

to remove such discontinuity, and inspired by the Innes' (1990) model of unobserved effort level,

Garino and Simmons (2001) explore the idea that the borrower can destroy output. Under such

indirect assumption of endogenous effort, they show that the consumption level of the borrower

would be monotonic with no discontinuities in the state contingent schedule.

There is no reason to believe an optimizing entrepreneur is less risk averse than any other

economic agent, and hence Gale and Hellwig's (1985) and Garino and Simmons's (2001) discussion

of risk aversion reflects a desire to adapt the model to more realistic conditions. The assumption that

the bank is risk neutral is realistic due to the bank's diversified loan portfolio, but this fact brings out

a *fundamental* aspect of the contract. Consider the realistic case when the borrower is risk averse.

The bank's ability to diversify enables it to become an *insurance agent* for the borrower. The need

for such insurance arises from the fact that markets for contingent claims do not exist due to asymmetric information, hence the borrower cannot purchase as much insurance as he desires. This means that private contracting between the borrower and the bank becomes an institution used to improve risk sharing arrangements that should ideally be performed via financial markets. But then the simple borrower - lender relationship no longer characterizes the problem. We are now engaged in the analysis of the trading pattern between two partners in a multifaceted economic relationship. Nevertheless, under the assumptions of Gale and Hellwig (1985) and Garino and Simmons (2001), the effect of risk aversion is minimal. The bank provides the borrower with default insurance in the form of *constant* utility (instead of zero payoff) over the verification set S^* of low project returns, but the simple *formal* structure of a standard loan contract remains intact.

Our interest in this paper is to extend the analysis of Gale and Hellwig (1985) to an environment where the borrower is risk averse but where he also *has a probability belief about the distribution of the state s which is different from the bank.* But once we introduce differences in probability beliefs, the full force of the complex relationship between two trading parties becomes focal. If probabilities matter then we deal with investment opportunities of an entrepreneur who seeks an investment of capital from a financial institution with a diversification capability. The relationship between these parties may involve complex terms of return on capital, insurance and risk sharing which arise from the motive to trade on the differences in probabilities. This means that in some states the agreed compensation may not be due to the real outcome of the project but rather, it may be due to the difference in probabilities on which the parties trade. With this in mind, it may not be useful to think of S^* a a set of "default" states. Instead, S^* is the set of states in which the contract stipulates a payoff to the borrower that exceeds the project's resources minus the fixed

obligation to the bank. The insured agent then makes a demand for payment by the lender to cover

his compensation. More generally, the constraints do not prohibit the borrower from receiving a

compensation which is even larger than the *total resources* of the entire project. To explain the issue

further, consider the case when S^* is an interval of lowest returns. Due to risk aversion and

probability differences the lender proposes to insure the borrower by a specified function on S^*. In

addition, suppose there is a region S^{**} in the middle of S (unconnected with S^{**}) on whose

realization the lender puts very small probability while the borrower places very high probability.

The borrower may ask to receive a large compensation on S^{**} and, in addition, that on S^{**} the

project should have no obligation to the bank. The lending bank may be willing to guarantee the

borrower this compensation package *simply because he does not believe that these outcomes are*

likely. If the lender can lower the amounts he proposed to the borrower at S^* (a region of high

probability for him) in exchange for higher promises on the set S^{**} (a region of low probability for

him) he may consider doing so. Such a trade would keep the expected return of the lender at the

same level and may improve the utility of the borrower. So the reappearance of sets like S^{**} away

from S^* is simply due to the differences in probabilities: the contract environment provides

opportunities for utility improving risk sharing. One may argue that risk sharing which leads to

trades on probability differences puts the bank in a position of "speculating" on probabilities and this

is not what the function of a lending bank is. This is exactly the point at which the bank's motive to

maximize expected returns alters its simple role of a loan provider and turns it into a trading partner

in an environment of incomplete markets.

It is worth noting that in recent years many economists have adopted the view that

incomplete markets should not be characterized _only_ by an absence of financial markets to trade risk

but *also* by the emergence of voluntary private institutions to supplement trading opportunities available in the market. Indeed, some theorists consider contracting to be the correct way to model incomplete markets (e.g. Atkeson and Lucas (1992), Cole and Kocherlakota (2001)) . Without addressing this deeper question of selecting optimal private institutions to trade risk, we stress that a crucial feature of contracting with risk aversion and diverse probability beliefs is the fact that it offers a substitute for financial markets and the multiple regions of verification reflects the complex nature of trading it implies. Moreover, in a more general setting (e.g. to include hidden actions, adverse selection, among others) these regions would reflect the differential demands for state contingent claims of the two parties involved.   But then, are we not moving further and further away from the initial borrower-lender relationship? Bernanke, Gertler and Gilchrist (1999) use the contracting model to develop a financial accelerator and in applying the model to the U.S. economy take an extreme position on this issue. They argue that in advanced economies there exists intermediation between the ownership of capital and the management of capital, hence *all public capital ownership is covered by the state verification model!* They thus do not distinguish between equity and debt investments since they are both subject to state verification. But in this case the Gale and Hellwig (1985) assumptions adopted by Bernanke, Gertler and Gilchrist (1999) are entirely inappropriate since it is a fact that a typical capital investment involves very complex set of contingent obligations and payments, all absent from the *standard loan contract* which they adopt.

We suggest that there are many sound reasons to focus on the case of differential probability beliefs of the two parties.   First, it is a realistic assumption since there is a vast empirical evidence that agents exhibit diverse probability beliefs in financial markets (see, for example, Frankel and Froot (1987), (1990), Frankel and Rose (1995), Kandel and Pearson (1995), Kurz (2001) and

Takagi (1991)). Second, if potential entrepreneurs have diverse beliefs about available projects, it is easy to demonstrate simple selection mechanisms which result in realizations in which investors who seek a bank loan are more optimistic than the bank about the project of their choice. Investors who are pessimistic about the outcomes of potential projects would generally choose not to invest at all. For this reason we are particularly interested in the case when an entrepreneur who seeks a loan from the bank is more optimistic than the bank about the prospects of his project. Third, the case of difference in probability beliefs is a relatively simple setting of trading where we could hope to derive the optimal contract and even be able to solve it numerically. Other settings may be too complex to enable a full characterization of the optimal contract. And, indeed, our conclusions are relatively simple and can be summarized as follows:

(A) The contractual payments to the borrower in verification states are not constant: they vary by states in accord with risk aversion and probability belief of the borrower. There is substantial empirical evidence to support this result since even in states of "default" the consumption level of a borrower is not constant. There are items (e.g. home) not subject to seizure by a lender; in some states the borrower may default only on some loans which are not backed by collateral; there are often items on the balance sheet of a borrower which are not even part of the collateral. Finally, depending on the severity of the situation (i.e. depending on s), if the borrower is unable to pay all his debts then under Chapter 11 he may be able to renegotiate the loan contracts and emerge with positive consumption.

(B) The observation region may consist of *many intervals and the structure of these intervals is sensitive to the difference in probability beliefs.* This is the result which reflect the trading opportunities which private contracting makes possible.

2.   A Solution of the Contracting Problem

We consider the general contracting case where the payoff to a risk averse borrower is not required to be monotonic.[1] To present our formulation we start with the following notation:

N - equity of the borrower;

B - amount borrowed and $C = B + N$.;

$S = \mathbb{R}_+$ the state space;

$F : S \times \mathbb{R} \to \mathbb{R}$: the production function, as function of the state s and the input of capital,

$P : S \to [0,1]$ - the distribution of s according to the bank;

$P_B : S \to [0,1]$ - distribution of s according to the borrower;

u - utility function of the borrower;

c(s , C) - verification cost depending on the state and input of capital;

$S^* \subset S$: verification set: set of states at which observation takes place;

$v(s)$ - state contingent payment to borrower;

R - payment to the bank (i.e. the loan B plus contractual interest) when state is unobserved;

r   - riskless interest rate.

We start with only three assumptions but add assumptions later to clarify the development.

Assumptions:

A.1   u is defined on $\mathbb{R}_+$, C^2 and strictly concave, $u' > 0$, $u'' < 0$;

A.2   The bank is risk neutral.

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[1]Garino and Simmons (2001) explore that implication of permiting the borrowing entrepreneur to destroy revenue and this results in a monotonic payoff function to the borrower.

A.3   The probability measures induced by P and P_B are equivalent.

2.1   A Simple Tool for Characterizing the Solution of the Contracting Problem

We start by a formal statement of the *Contracting Problem*, which is

Select S* measurable, $v(\cdot)$ defined on $S^*, v(s) \geq 0$ and measurable to solve

(1a)
$$\max_{(v,S^*,R,B)} \int_{S^*} u[v(s)]P_B(ds) + \int_{S\backslash S^*} u[F(s,B+N)-R]P_B(ds)$$

such that

(1b) $$\int_{S^*} [F(s,B+N) - v(s) - c(s,B+N)]P(ds) + \int_{S\backslash S^*} R\,P(ds) \geq B(1+r)$$

(1c) $$v(s) \geq F(s,N+B) - R, \ \forall\, s \in S^*$$

(1d) $$F(s,N+B) - R \geq 0, \ \forall\, s \in S\backslash S^*$$

(1b) is a participation constraint of the bank, ensuring the bank a normal return; (1c) is the incentive

compatibility constraints making it unprofitable for the borrower to misrepresent the truth in his

report of the state in S*; (1d) is a feasibility condition on the contractual value of R. As a matter of

notation, we denote by $\tilde{v}, \tilde{S}^*, \tilde{R}, \tilde{B}$ a solution to this problem.

Our strategy for characterizing the solution of problem (1) is to define simpler problems with

the same solution and then develop the first order conditions for the optimization. In this spirit we

denote by $\hat{v}_C(\cdot, k)$, for $k \in \mathbb{R}$ and for a given B, the solution of the following problem (2):

Select $v(\cdot) \geq 0$ and measurable which solves

(2a) $$\underset{v(\cdot)}{\text{Max}} \int_0^\infty u[v(s)]P_B(ds)$$

such that

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(2b)
$$\int_0^\infty [F(s,C) - v(s)] P(ds) \geq k.$$

Problem (2) is much simpler than problem (1) and we derive the first order conditions for it. Let C and k be given such that (2) has a solution. Let $S_{0\delta} = \{s \in S : \hat{v}_C(s,k) > \delta\}$, then have the following first order condition. For any A, B $\subset$ S and ϵ such that P(B) > 0 and $P((A \cup B) \cap S_{o|\epsilon|}) = P(A \cup B)$ we have that

(3) $\int_A (u[\hat{v}_C(s,k) + \epsilon] - u[\hat{v}_C(s,k)]) P_B(ds) + \int_B (u[\hat{v}_C(s,k) - \epsilon \dfrac{P(A)}{P(B)}] - u[\hat{v}_C(s,k)]) P_B(ds) \leq 0.$

Dividing by ϵ and taking limits we conclude that if $P(A \cap S_{o\delta}) = 0$, $\forall \delta > 0$ and $P(B \cap S_{o\delta}) > 0$ for some δ, we can write the first order condition in the more useful form

(4) $\int_A u'(\hat{v}_C(s,k)) P_B(ds) - \int_B u'(\hat{v}_C(s,k)) \dfrac{P(A)}{P(B)} P_B(ds) \leq 0 .$

If, however, P(B) > 0 and $(A \cup B) = P((A \cup B) \cap S_{o\delta})$ then we have equality in (4). Condition (4) will be used later in the development.

The key tool we use in this paper to solve the contracting problem are the results in Lemmas 1 and 2. They provides a simple characterization of the payoff function in the optimal verification states $\tilde{S}^*$.

Lemma 1: Suppose $k_n \downarrow k$. Then $\hat{v}_C(s, k_n) \uparrow \hat{v}_C(s, k)$ for P-a.a. s.

Proof: We first prove monotonicity by repeated use of the first order condition in (4). Thus suppose

$k > k'$ and define the following sets: $B = \{s \in S : \hat{v}_C(s,k) > 0 , \hat{v}_C(s,k') > 0\}$,

$A = \{s \in S : \hat{v}_C(s,k) > 0 , \hat{v}_C(s,k') = 0\}$, and $A' = \{s \in S : \hat{v}_C(s,k) = 0 , \hat{v}_C(s,k') > 0\}$.

Now suppose $P(A) > 0$. There must be a measurable set $\tilde{B}$ with $P(\tilde{B}) > 0$ on which either

$0 = \hat{v}_C(\cdot,k) < \hat{v}_C(\cdot,k')$ or $0 < \hat{v}_C(\cdot,k) < \hat{v}_C(\cdot,k')$. Hence, by FOC (4) we have

$$(5) \qquad \frac{\int_A u'[\hat{v}_C(s,k')]P_B(ds)}{\int_{\tilde{B}} u'[\hat{v}_C(s,k')]P_B(ds)} \leq \frac{P(A)}{P(\tilde{B})} \leq \frac{\int_A u'[\hat{v}_C(s,k)]P_B(ds)}{\int_{\tilde{B}} u'[\hat{v}_C(s,k)]P_B(ds)}$$

which is a contradiction, since on A, $u'[\hat{v}_C(s,k')] > u'[\hat{v}_C(s,k)]$, while on $\tilde{B}$ we must have

$u'[\hat{v}_C(s,k')] < u'[\hat{v}_C(s,k)]$. We thus conclude that $P(A) = 0$.

Suppose now there is a measurable set $\hat{B} \subset B$ with $P(\hat{B}) > 0$ on which the inequality

$\hat{v}_C(\cdot,k) > \hat{v}_C(\cdot,k')$ holds. Using the FOC (4) we deduce that

$$(6) \qquad \frac{\int_{A'} u'[\hat{v}_C(s,k')]P_B(ds)}{\int_{\hat{B}} u'[\hat{v}_C(s,k')]P_B(ds)} = \frac{P(A')}{P(\hat{B})} \geq \frac{\int_{A'} u'[\hat{v}_C(s,k)]P_B(ds)}{\int_{\hat{B}} u'[\hat{v}_C(s,k)]P_B(ds)}$$

which is a contradiction, since on A', $u'[\hat{v}_C(s,k')] < u'[\hat{v}_C(s,k)]$, while on $\hat{B}$ we must have

$u'[\hat{v}_C(s,k')] > u'[\hat{v}_C(s,k)]$. This proves almost sure monotonicity.

Now let $k_n \downarrow k$. Then there is $\bar{v}$ s.t. for P-a.a. s, $\hat{v}_C(s,k_n) \uparrow \bar{v}(s)$. Suppose that on

some measurable $\tilde{B}$ with $P(\tilde{B}) > 0$, $\bar{v} < \bar{v}_C(\cdot,k)$. Since

$$\int_S [F(s,C) - \bar{v}(s)]P(ds) = k = \int_S [F(s,C) - \hat{v}_C(s,k)]P(ds)$$

there must be some measurable set B' with $P(B') > 0$ s.t. on B', $\bar{v} > \hat{v}_C(\cdot,k)$. But this is in

contradiction with the almost sure monotonicity that was just established, since then there would be a

measurable subset of B' with positive measure on which for large n, $\hat{v}_C(\cdot,k_n) > \hat{v}_C(\cdot,k)$. Hence,

$\hat{v} = \hat{v}_C(\cdot, k)$ a.a. and convergence has been proven. ∎

Lemma 2: Let $\tilde{v}, \tilde{S}^*, \tilde{R}, \tilde{B}$ be a solution to the contracting problem (1). Assume $c(s, C) > 0$ for all s and C. Then there is a $\tilde{k}$ such that $\tilde{v}(s) = \hat{v}_{\tilde{C}}(s, \tilde{k})$ for P - a.a $s \in \tilde{S}^*$ with $\tilde{C} = N + \tilde{B}$.

Proof : Let $\hat{k} = \tilde{B}(1 + r) - P(S\backslash\tilde{S}^*)R + \int_{\tilde{S}^*} c(s, \tilde{B} + N)P(ds)$. Then $\tilde{v}$ solves the problem:

Select $v(\cdot) \geq 0$, measurable and defined on $\tilde{S}^*$ which solves

(7a)
$$\underset{v}{\text{Max}} \int_{\tilde{S}^*} u[v(s)]\, P_B(ds)$$

such that

(7b)
$$\int_{\tilde{S}^*} [F(s, \tilde{B} + N)] - v(s)]P(ds) \geq \hat{k},$$

(7c)
$$v(s) \geq F(s, \tilde{B} + N) - R, \ \forall s \in \tilde{S}^*.$$

Note that $\tilde{v} > F(s, \tilde{B} + N) - R$, P - a.s. To see why note that if $\tilde{v} = F(s, \tilde{B} + N) - R$ on a set G of positive measure, then due to $c(s, C) > 0$ excluding G from the set of observations while leaving the payoff to the borrower unchanged is profitable and would be an improvement. Hence (7c) almost surely will not bind. On the other hand let $\hat{v}$ solve (7) without the constraint (7c). We then have that if $\hat{v} \neq \tilde{v}$ on a set of positive measure, then

$$\int_{S^*} u(\hat{v}(s))P_B(ds) > \int_{S^*} u(\tilde{v}(s))P_B(ds).$$

This follows from strict concavity of u. Suppose then $\hat{v} \neq \tilde{v}$ on a set of positive measure with the purpose of obtaining a contradiction. For any $\epsilon > 0$, let $S_\epsilon \subset \tilde{S}^*$ be a measurable set which fulfills:

$$P(S_\epsilon) < \epsilon$$

$$\exists \delta > 0 \text{ s.t. } \tilde{v}(s) - F(s, \tilde{B} + N) - R < \delta \Rightarrow s \in S_\epsilon.$$

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Now modify $\hat{v}$ as follows:

$$\hat{v}_{\epsilon}(s) \; = \; \max\{F(s\,,\tilde{B}\,+\,N)\,-\,R\,+\,\epsilon\,,\hat{v}(s)\} \quad \text{if} \;\; s \in S_{\epsilon}$$

$$\hat{v}_{\epsilon}(s) \; = \; \hat{v}(s)\,-\,\delta \quad \text{if} \;\; s \in S\backslash S_{\epsilon}\,, \text{ where } \delta \,\geq\, 0 \text{ is chosen such that } \int_{\tilde{S}^{*}} \hat{v}_{\epsilon}(s) \,=\, \hat{k}.$$

Note that for ϵ small, S_{ϵ} may be empty in which case $\delta = 0$ and $\hat{v}_{\epsilon} \,=\, \hat{v}$. Now, as $\epsilon \downarrow 0$, $\hat{v}_{\epsilon} \to \hat{v}$

a.s. so there is an $\epsilon > 0$ for which $\int_{\tilde{S}^{*}} u(\hat{v}_{\epsilon}(s)) P_{B}(ds) > \int_{\tilde{S}} u(\tilde{v}^{*}(s)) P_{B}(ds)$. For that ϵ pick a

$\lambda \in (0\,,\,1)$ such that

$$\lambda\,\hat{v}_{\epsilon}(s)\,+\,(1\,-\,\lambda)\,\tilde{v}^{*}(s)\,>\,F(s\,,\tilde{B}\,+\,N)\,-\,R\,,\,\forall\,s \in \tilde{S}^{*}.$$

This is possible because on S_{ϵ}, $\hat{v}_{\epsilon}(s) > F(s\,,\tilde{B}\,+\,N)\,-\,R$ and $\tilde{v}(s) > F(s\,,\tilde{B}\,+\,N)\,-\,R$ in

addition to the fact that on $\tilde{S}^{*}\backslash S_{\epsilon}$, $\tilde{v}(s) > F(s\,,\tilde{B}\,+\,N)\,-\,R\,+\,\delta$ for some $\delta > 0$. By strict

concavity of u we have

$$u[\lambda\,\hat{v}_{\epsilon}(s)\,+\,(1\,-\,\lambda)\,\tilde{v}(s)]\,\geq\,\lambda\,u(\hat{v}_{\epsilon}(s))\,+\,(1\,-\,\lambda)u(\tilde{v}(s))\,,\,\forall\,s \in \tilde{S}^{*}$$

with a strict inequality for a set of s of positive P- measure. Hence we have

$$\int_{\tilde{S}^{*}} u[\lambda\,\hat{v}_{\epsilon}(s)\,+\,(1\,-\,\lambda)\,\tilde{v}(s)]P_{B}(ds)\,>\,\lambda\int_{\tilde{S}^{*}} u[\hat{v}_{\epsilon}(s)]P_{B}(ds)\,+\,(1\,-\,\lambda)\int_{S*} u[\tilde{v}(s)]P_{B}(ds)$$

which implies that

$$\int_{\tilde{S}^{*}} u[\lambda\,\hat{v}_{\epsilon}(s)\,+\,(1\,-\,\lambda)\,\tilde{v}(s)]P_{B}(ds)\,>\,\int_{\tilde{S}^{*}} u[\tilde{v}(s)]P_{B}(ds)\,.$$

However, by construction, $\lambda\,\hat{v}_{\epsilon}\,+\,(1\,-\,\lambda)\,\tilde{v}$ fulfills the two constraints in (7). This contradicts the

fact that $\tilde{v}$ was a solution to (7). Thus it must be the case that $\tilde{v}\,=\,\hat{v}$ a.e. on $\tilde{S}^{*}$. What remains

to be shown is that for some $\tilde{k}$ we have that

(8) $$\tilde{v}(s)\,=\,\hat{v}_{\tilde{C}}(s\,,\tilde{k})\quad,\,\forall\,s \in \tilde{S}^{*}.$$

Clearly, as $k \uparrow \int_{0}^{\infty} F(s\,,\tilde{C})P(ds)$, $\hat{v}_{\tilde{C}}(\cdot\,,k) \downarrow 0$ P - a.s., which means $\int_{\tilde{S}^{*}} \hat{v}_{\tilde{C}}(s\,,k)P(ds) \downarrow 0$.

On the other hand, as $k \downarrow -\infty$, $\hat{v}_{\tilde{C}}(s\,,k) \uparrow \infty$, P - a.s. by the same argument used earlier to prove

monotonicity in Lemma 1. Hence there is a $\tilde{k}$ such that

$$\int_{\tilde{S}^*} \hat{v}_{\tilde{C}}(s, \tilde{k}) P(ds) = \int_{\tilde{S}^*} F(s, \tilde{C}) P(ds) - \tilde{k}$$

and (8) follows. ∎

Lemma 2 has a simple interpretation. Since at $\tilde{S}^*$ the state is known, the problem restricted to $\tilde{S}^*$ is an optimization with full information. Hence the borrower's consumption must be optimal only subject to the participation constraint of the bank and to the condition that $\tilde{k}$ is the exact amount needed to assure the bank's participation. Given that these conditions are satisfied, the optimum condition (4) simply reflects the risk aversion and probability belief of the borrower. We recall that when the agent is risk averse but holds the same probability belief as the bank, Gale and Hellwig (1985) and Garino and Simmons (2001) show the solution is $v(s) = \text{constant for } s \in \tilde{S}^*$. If, the agent is risk neutral and holds the same probability as the bank then we have the *standard loan contract* when $v(s) = 0$, $s \in \tilde{S}^*$. The new dimension here is the diversity of probability beliefs between the bank and the borrower, and Lemma 2 shows that this assumption results in a subtle payoff schedule $\tilde{v}(s)$, $s \in \tilde{S}^*$ which varies with the realized state in the verification set.

Lemma 2 provides a precise characterization of the optimal solution for states in $\tilde{S}^*$. Moreover, the simplicity of problem (2) enables us to show that when sufficient smoothness is available, the solution to problem (1) can be characterized with elementary techniques of calculus.

2.1   *First Order Conditions for the Optimal Solution $\tilde{v}$*

We have already derived in (4) the first order conditions for problem (2). We now combine with the previous lemma to derive the first order conditions for problem (1).

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Necessary Condition for an Optimal Contract $\tilde{v}(s)$ *when* $s \in \tilde{S}^*$

For all $A, B \subset \tilde{S}^*$ s.t. $P(B) > 0$ and $\tilde{v}(s) > 0$ for P – a.a. $s \in \tilde{S}^*$ it is necessary that

$$(9) \qquad \int_A u'(\tilde{v}(s)) P_B(ds) \; - \; \int_B u'(\tilde{v}(s)) \frac{P(A)}{P(B)} P_B(ds) \; = \; 0.$$

If P and P_B are differentiable in s and s', so that they have their densities p and p_B defined in s and s', then (9) can be rewritten in the form

$$(9a) \qquad \frac{u'[\tilde{v}(s)] p_B(s)}{u'[\tilde{v}(s')] p_B(s')} \; = \; \frac{p(s)}{p(s')} \qquad \text{if} \quad \tilde{v}(s') > 0 \;\; , \;\; \tilde{v}(s) > 0.$$

Note, that when $P_B = P$, (9) implies that $\tilde{v}$ is a constant as in Gale and Hellwig (1985). Note also that when p and p_B are continuous so is $\tilde{v}(\cdot)$.

Given the choice of capital $\tilde{C} = \tilde{B} + N$, a contract specifies S^*, k and R with $\tilde{v}(s) = \hat{v}_{\tilde{C}}(s, k)$, for $s \in S^*$ and $\hat{v}(s) = F(s, \hat{C}) - R$ for $s \in S \backslash S^*$. Hence, we can develop necessary conditions for optimality in these three choice variables k, S^* and R. Given the results in (9) and (9a) we would like to sharpen these conditions and hence make *additional assumptions*:

A.4   F is continuous and weakly increasing in s.

A.5   c is continuous in s and $c(\cdot, C) > 0$, P - a.s. for all C.

A.6 P_B and P have densities p_B and p which are continuous.

For a given feasible contract $(\tilde{C}, k, S^*, R)$ let $s_o = \inf \{s : F(s, \tilde{C}) - R = 0\}$. It follows that $[0, s_o) \subset S^*$. We could state the more general condition $P([0, s_o) \cap S^*) = P([0, s_o))$, but such generality does not add much at this point. Also observe that

$$\{\, s \geq s_o : \hat{v}_{\tilde{C}}(s,k) \leq F(s,\tilde{C}) - R \,\} \subset S \backslash S^*,$$

a conclusion which follows from incentive compatibility and the fact that c is non-negative. In the following we want to develop conditions to characterize what members of the set

$$\bar{S} \equiv \{\, s : \hat{v}_{\tilde{C}}(s,\tilde{k}) > F(s,\tilde{C}) - \tilde{R} \geq 0 \,\}$$

belong to S^* and $S \backslash S^*$ respectively.

Proposition 1: Suppose $P(\tilde{S}^*) > 0$ and $P(S \backslash \tilde{S}^*) > 0$. Let $\hat{S}^* \subset \{\, s \in \tilde{S}^* : \hat{v}_{\tilde{C}}(s,\tilde{k}) > 0 \,\}$ be measurable. For P-a.a. $s' \in \tilde{S}^* \cap \bar{S}$ we have:

(10)
$$\frac{u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s',\tilde{C}) - R]}{\hat{v}_{\tilde{C}}(s',k) - [F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})} \frac{p_B(s')}{p(s')} - \frac{\int_{\hat{S}^*} u'[\hat{v}_{\tilde{C}}(s',\tilde{k})] P_B(ds)}{P[\hat{S}^*]} \geq 0.$$

For P-a.a. $s' \in (S \backslash \tilde{S}^*) \cap \bar{S}$ we have

(11)
$$\frac{u[\hat{v}_{\tilde{C}}(s',\tilde{k}) - u[F(s',\tilde{C}) - R]}{\hat{v}_{\tilde{C}}(s',\tilde{k}) - [F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})} \frac{p_B(s')}{p(s')} - \frac{\int_{\hat{S}^*} u'[\hat{v}_{\tilde{C}}(s',\tilde{k})] P_B(ds)}{P[\hat{S}^*]} \leq 0$$

Proof: In view of (2a)-(2b) we consider throughout the proof a continuous version of $\hat{v}_{\tilde{C}}(\cdot,\tilde{k})$. We first prove (10) by contradiction. Thus suppose that $s' \in \text{int } \bar{S}$ and assume (i) $s' \in \tilde{S}^*$ and (ii) $P(\tilde{S}^* \cap [s' - \epsilon, s' + \epsilon]) > 0$, $\forall \epsilon > 0$. Consider the following modification of the contract: change the verification set $\tilde{S}^*$ to $S_\epsilon^* \equiv \tilde{S}^* \backslash [s' - \epsilon, s' + \epsilon]$ and on $\hat{S}^* \backslash [s' - \epsilon, s' + \epsilon]$ change the payment to the entrepreneur to $\hat{v}_{\tilde{C}}(s',\tilde{k}) + \delta$. The change in the profit to the bank is

$$\Delta \Pi(\epsilon,\delta) = \int_{\tilde{S}^* \cap [s' - \epsilon, s' + \epsilon]} \{\hat{v}_{\tilde{C}}(s',\tilde{k}) + c(s,\tilde{C}) - [F(s,\tilde{C}) - \tilde{R}]\} P(ds) - \int_{\hat{S} \backslash [s' - \epsilon, s' + \epsilon]} \delta P(ds).$$

For P -a.a. s', $\Delta\Pi$ is differentiable in $(0,0)$ and

$$\frac{\partial\Delta\Pi}{\partial\epsilon}(0,0) = \hat{v}_{\tilde{C}}(s',\tilde{k}) - [F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})$$

$$\frac{\partial\Delta\Pi}{\partial\delta}(0,0) = - P[\hat{S}^*].$$

However, $\Delta\Pi$ may not be locally C^1 and so the implicit function theorem may not hold. The change

in utility is:

$$\Delta u(\epsilon,\delta) = - \int_{\tilde{S}^*\cap[s'-\epsilon,s'+\epsilon]} \{u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s,\tilde{C}) - \tilde{R}]\}P_B(ds) +$$

$$\int_{\hat{S}^*\backslash[s'-\epsilon,s'+\epsilon]} \{u[\hat{v}_{\tilde{C}}(s',\tilde{k}) + \delta] - u[\hat{v}_{\tilde{C}}(s',\tilde{k})]\}P_B(ds).$$

Since for a generic s' Δu is differentiable at $(0,0)$, (not necessarily C^1) we can deduce

$$\frac{\partial\Delta u}{\partial\epsilon}(0,0) = - \{u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s',\tilde{C}) - \tilde{R}]\}p_B(s')$$

$$\frac{\partial\Delta u}{\partial\delta}(0,0) = \int_{\hat{S}*} u'[\hat{v}_{\tilde{C}}(s,\tilde{k})]P_B(ds).$$

Now suppose that (10) did not hold. For small enough $\epsilon > 0$ define $\delta(\epsilon)$ such that

$$\frac{\int_{\tilde{S}^*\cap[s'-\epsilon,s'+\epsilon]}\{\hat{v}_{\tilde{C}}(s',\tilde{k}) - [F(s',k) - \tilde{R}] + c(s',\tilde{C})\}P(ds)}{\delta(\epsilon)P(\hat{S}^*)} = 1.$$

Then for any $q \in (0,1)$ we have

$$\Delta\Pi(\epsilon,q\delta(\epsilon)) = \int_{\tilde{S}^*\cap[s'-\epsilon,s'+\epsilon]}\{\hat{v}_{\tilde{C}}(s',\tilde{k}) + c(s,\tilde{C}) - [F(s,\tilde{C}) - \tilde{R}]\}P(ds) -$$

$$P(\hat{S}^*)q\delta(\epsilon) + P(\hat{S}^*\cap[s'-\epsilon,s'+\epsilon])q\delta(\epsilon)$$

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which is > 0 for small $\epsilon > 0$. Consequently, if we reduce the area of observation to S_ϵ^* and increase the payment on $\hat{S}^* \setminus [s' - \epsilon, s' + \epsilon]$ by $q\delta(\epsilon)$ the change in utility is

$$\Delta u(\epsilon, q\delta(\epsilon)) = -\{u[\hat{v}_{\tilde{C}}(s', \tilde{k})] - u[F(s, \tilde{C}) - \tilde{R}]\}p_B(s')2\epsilon + o_1(\epsilon) +$$

$$q\delta(\epsilon) \int_{\hat{S}^*} u'[\hat{v}_{\tilde{C}}(s, \tilde{k})]P_B(ds) + o_2(\epsilon)$$

where $\dfrac{o_i(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$. Using the definition of $\delta(\epsilon)$ we get

$$\Delta u(\epsilon, q\delta(\epsilon)) = -\{u[\hat{v}_{\tilde{C}}(s', \tilde{k})] - u[F(s, \tilde{C}) - R]\}p_B(s')2\epsilon +$$

$$(q)\dfrac{\int_{\tilde{S}^* \cap [s' - \epsilon, s' + \epsilon]} \{\hat{v}_{\tilde{C}}(s', \tilde{k}) - [F(s', \tilde{C}) - R] + c(s', \tilde{C})\}P(ds)}{P(\hat{S}^*)} \int_{\hat{S}^*} u'[\hat{v}_{\tilde{C}}(s, \tilde{k})]P_B(ds) + o(\epsilon).$$

By our assumption that (10) does not hold, for q close to 1 (but < 1), this expression is > 0 for small $\epsilon > 0$. But then $\tilde{C}, \tilde{k}, \tilde{S}^*, \tilde{R}$ could not be optimal i.e. we have the desired contradiction.

    <u>Remark</u>: Note that if $\hat{v}_{\tilde{C}}(\hat{s}, \tilde{k}) = F(\hat{s}, \tilde{C}) - \tilde{R}$ the inequality in (10) cannot hold in a neighborhood of $\hat{s}$.

We turn next to prove (11). Suppose (i) $s' \in \text{int } S \cap (S \setminus S^*)$ and (ii) $P((S \setminus \tilde{S}^*) \cap [s' - \epsilon, s' + \epsilon]) > 0$ $\forall \epsilon > 0$. By the remark just made, for every N there is a $\delta_N > 0$ such that if $0 < \delta < \delta_N$ then

$$\{s \in \bar{S} : s \le N, \hat{v}_{\tilde{C}}(s, \tilde{k}) - \delta > F(s, \tilde{C}) - \tilde{R}\} = \{s \in \bar{S} : s \le N\}.$$

With this in mind, define $\hat{S}_{+N}^* = \hat{S}^* \cap \{s \in \bar{S} : s < N\}$. Now suppose we had

$$-u[\hat{v}_{\tilde{C}}(s', \tilde{k})] - u[F(s', \tilde{C}) - \tilde{R}]\}p_B(s') +$$

$$\dfrac{\{\hat{v}_{\tilde{C}}(s', \tilde{k}) - [F(s', \tilde{C}) - \tilde{R}] + c(s', \tilde{C})\}p(s')}{P[\hat{S}_{+N}^*]} \int_{\hat{S}_{+N}^*} u'[\hat{v}_{\tilde{C}}(s', \tilde{k})]P_B(ds) < 0.$$

Consider the following contract modification: On $(S \setminus S^*) \cap [s' - \epsilon, s' + \epsilon]$ change the payment to the

entrepreneur from $F(s,\tilde{C}) - \tilde{R}$ to $\hat{v}_{\tilde{C}}(s,\tilde{k}) - q\delta(\epsilon)$, where $q > 1$ and $\delta(\epsilon) > 0$ is small, and add

this set to S^*, the set of observations. On $\hat{S}^*_{+N}$ change the payments to $\hat{v}_{\tilde{C}}(s,k) - q\delta(\epsilon)$.

Everywhere else, leave the contract unchanged. For small enough $\epsilon > 0$, $\delta(\epsilon)$ is defined by

$$\frac{\int_{(S\backslash S^*)\cap[s'-\epsilon,s'+\epsilon]} \{\hat{v}_{\tilde{C}}(s',\tilde{C}) - [F(s',\tilde{C}) - \tilde{R}] + c(s,\tilde{C})\}P(ds)}{\delta(\epsilon)P[\hat{s}^*_{+N}]} = 1$$

The change in profit to the bank is

$$-\int_{(S\backslash S^*)\cap[s'-\epsilon,s'+\epsilon]} \{\hat{v}_{\tilde{C}}(s',\tilde{k}) - [F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})\}P(ds) + q\delta(\epsilon)P(\hat{S}^*_{+N}\cup[s'-\epsilon,s'+\epsilon])$$

and for any $q > 1$ this quantity is positive for sufficiently small $\epsilon > 0$. The change in utility for the

entrepreneur is

$$\int_{(S\backslash\tilde{S}^*)\cap[s'-\epsilon,s'+\epsilon]} \{u[\hat{v}_{\tilde{C}}(s,\tilde{k}) - q\delta(\epsilon)] - u[F(s,\tilde{C}) - \tilde{R}]\}P_B(ds) -$$

$$\int_{\hat{S}^*_{+N}} \{u[\hat{v}_{\tilde{C}}(s,\tilde{k})] - u[\hat{v}_{\tilde{C}}(s,\tilde{k}) - q\delta(\epsilon)]\}P_B(ds) = u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s,\tilde{C}) - \tilde{R}]\}p_B(s')2\epsilon -$$

$$\int_{\hat{S}^*_{+N}} q\delta(\epsilon)u'[\hat{v}_{\tilde{C}}(s,\tilde{k})]\}P_B(ds) - o(\epsilon)$$

(where $\dfrac{o(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$). Using the definition of $\delta(\cdot)$ this expression is in turn equal to

$$u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s,\tilde{C}) - \tilde{R}]\}p_B(s')2\epsilon -$$

$$\int_{\hat{S}^*_{+N}} q\frac{\int_{(S\backslash S^*)\cap[s'-\epsilon,s'+\epsilon]} \{\hat{v}_{\tilde{C}}(s',\tilde{k}) - F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})\}P(ds)}{P[\hat{S}^*_{+N}]} u'[\hat{v}_{\tilde{C}}(s,\tilde{k})]\}P_B(ds) - o(\epsilon).$$

Hence, for q small (but > 1) the quantity above is positive for small $\epsilon > 0$, contradicting the

assumed optimality of the original contract. Hence, for all N we have

$$- \{u[\hat{v}_{\tilde{C}}(s',\tilde{k})] - u[F(s',\tilde{C}) - R]\}\, p_B(s') +$$

$$\frac{\{\hat{v}_{\tilde{C}}(s',\tilde{k}) - [F(s',\tilde{C}) - \tilde{R}] + c(s',\tilde{C})\}\, p(s')}{P[\hat{S}^*_{+N}]} \int_{\hat{S}^*_{+N}} u'[\hat{v}_{\tilde{C}}(s',\tilde{k})]\, P_B(ds) \le 0.$$

Consequently, this equality also holds for $N = \infty$ and we have proved (11) ∎

The interpretation of (10) and (11) are relatively simple and follows from the manner in which we have derived them. The first term in (10) describes the gain in expected utility per unit of expected money spent on including the state s' in the set of observation. The second term quantifies the gain in expected utility per unit of expected money spent on increasing the payment to the entrepreneur marginally on the set of observations. In a state of observation the first minus the second has to be positive. In a state of non-observation, it has to be negative (else state would have to be included in $\tilde{S}^*$).

Under the assumptions stated in the proposition we have two corollaries which specify the optimal contract over the entire state space. We state and prove them next.

<u>Corollary 1</u>: For an optimal contract $(\tilde{C},\tilde{k},\tilde{S}^*,\tilde{R})$ if we let $\tilde{v}: S \to \Re_+$ be the payment to the borrower in any state s (that is both on $\tilde{S}^*$ and on $S\backslash\tilde{S}^*$), then we have P-a.a.:

(i) $\tilde{v}(s) = \hat{v}_{\tilde{C}}(s,\tilde{k})$ for all s satisfying $F(s,\tilde{C}) - \tilde{R} < 0$;

(ii) $\tilde{v}(s) = \hat{v}_{\tilde{C}}(s,\tilde{k})$ for all s satisfying:

(a) $\hat{v}_{\tilde{C}}(s,\tilde{k}) > F(s,\tilde{C}) - \tilde{R} \ge 0$;

(b) $\dfrac{u[\hat{v}_{\tilde{C}}(s,\tilde{k})] - u[F(s,\tilde{C})]}{\hat{v}_{\tilde{C}}(s,\tilde{k}) - [F(s,\tilde{C}) - \tilde{R}] + c(s,\tilde{C})} > u'[\hat{v}_{\tilde{C}}(s,\tilde{k})]$;

(iii) $\tilde{v}(s) = F(s,\tilde{C}) - \tilde{R}$ for all s satisfying:

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(a) $\hat{v}_{\tilde{C}}(s,\tilde{k}) > F(s,\tilde{C}) - \tilde{R} \ge 0$;

$$(b) \quad \frac{u[\hat{v}_{\tilde{C}}(s,\tilde{k})] - u[F(s,\tilde{C})]}{\hat{v}_{\tilde{C}}(s,\tilde{k}) - [F(s,\tilde{C}) - \tilde{R}] + c(s,\tilde{C})} < u'[\hat{v}_{\tilde{C}}(s,\tilde{k})].$$

(iv) $\tilde{v}(s) = F(s,\tilde{C}) - \tilde{R}$ for all s satisfying $\hat{v}_{\tilde{C}}(s,\tilde{k}) \le F(s,\tilde{C}) - \tilde{R}$.

Proof: For s, s' such that $\hat{v}_{\tilde{C}}(s,\tilde{k}) > 0$, $\hat{v}_{\tilde{C}}(s',\tilde{k}) > 0$ we have

$$u'[\hat{v}_{\tilde{C}}(s,\tilde{k})p_B(s) = p(s)u'[\hat{v}_{\tilde{C}}(s',\tilde{k})]\frac{p_B(s')}{p(s')}$$

Thus

$$\frac{\int_{\hat{S}^*} u'[\hat{v}_{\tilde{C}}(s,\tilde{k})]P_B(ds)}{P[\hat{S}^*]} = u'[\hat{v}_{\tilde{C}}(s',\tilde{k})]\frac{p_B(s')}{p(s')} \qquad \blacksquare$$

Suppose now that the set of s for which both (10) and (11) hold has zero measure under

P. Then the previous corollary provides a complete characterization of the optimal contract and

apart from a possible set of measure zero, we have the following conclusion

Corollary 2: $\tilde{S}^*$ is a at most a countable union of intervals.

Proof: This follows from the inequalities of Proposition 1 and the continuity of all functions

involved in the optimal contract.

$\blacksquare$

2.2 When is S^* an Interval?

We now address the question of when is S^* an interval? To study it we explore a simple

conclusion one can deduce from our previous analysis. Suppose the following conditions hold

(12a) $\hat{v}_C(s, k) > 0$, $\forall C$, $\forall k$, almost all s,

(12b) $c(s, C) > 0$, $\forall C > 0$, $\forall s$.

For any k and C there is only one $s = s^*$ such that

(12c)
$$\frac{u[\hat{v}_C(s,k) - u[F(s,C) - R]}{\hat{v}_C(s,k) - [F(s,\tilde{C}) - R] + c(s,\tilde{C})} = u'[\hat{v}_{\tilde{C}}(s, k)].$$

When (12a) - (12c) hold, then S^* is an interval starting in 0. This fact follows from continuity of the functions involved and the fact that any change from observation to non-observation or from non-observation to observation must take place in the region where

$\hat{v}_{\tilde{C}}(s,k) > F(s,\tilde{C}) - R \geq 0$ and thus (12c) must hold when such a change takes place. We can rewrite (12c) as

(13)
$$\frac{\dfrac{u[\hat{v}_c(s,k)] - u[F(s,C) - R]}{\hat{v}_c(s,k) - [F(s,c) - R]} \dfrac{1}{u'[\hat{v}_C(s,k)]}}{1 + \dfrac{c(s,C)}{\hat{v}_C(s,k) - [F(s,C) - R]}} = 1.$$

The following two sets of assumptions imply that (13) has a single solution for any k, C and R:

(14a) $\hat{v}_c(s, k) - [F(s,C) - R]$ is non-increasing in s;

(14b) $c(s, C)$ is non-decreasing in s;

(14c) $\dfrac{u[\hat{v}_c(s, k)] - u[F(s,C) - R]}{\hat{v}_c(s, k) - [F(s,C) - R]} \dfrac{1}{u'[v_c(s, k)]}$ is *strictly* decreasing in s.

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Or, the second set is

(15a) $\quad\quad\quad \hat{v}_c(s,k) - [F(s,C) - R]$ is non-increasing in s;

(15b) $\quad\quad\quad c(s,C)$ is strictly increasing in s;

(15c) $\quad\quad \dfrac{u[\hat{v}_c(s,k)] - u[F(s,C) - R]}{\hat{v}_c(s,k) - [F(s,C) - R]} \quad \dfrac{1}{u'[v_c(s,k)]}$ is non-increasing in s.

To explain how we derive (14c) or (15c) one may exploit the identity

(16) $\quad \dfrac{d}{dx}\left[\dfrac{u(x) - u(x - \Delta(x))}{\Delta(x)u'(x)}\right] = \dfrac{d\Delta}{dx}\dfrac{\partial}{\partial\Delta}\left[\dfrac{u(x) - u(x-\Delta)}{\Delta u'(x)}\right] + \dfrac{\partial}{\partial x}\left[\dfrac{u(x) - u(x-\Delta)}{\Delta u'(x)}\right].$

By assumption (14a) - (15a) $\quad \dfrac{d\Delta}{dx} \le 0.$ Since u is strictly concave we have

$$\dfrac{\partial}{\partial\Delta}\left[\dfrac{u(x) - u(x-\Delta)}{\Delta u'(x)}\right] = \dfrac{u'(x-\Delta) - \dfrac{u(x) - u(x-\Delta)}{\Delta}}{\Delta} > 0.$$

Thus if the following condition holds

(17) $$\dfrac{\partial}{\partial x}\left[\dfrac{u(x) - u(x-\Delta)}{\Delta u'(x)}\right] < 0$$

then (14c) holds. If (17) holds with a weak inequality, (15c) holds.

Examples of (17)

(i) $\quad$ <u>$u(x) = x^a$, $0 < a < 1$</u>.

Then we have that

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$$[u'(x) - u'(x - \Delta)]u'(x) - u''(x)[u(x) - u(x - \Delta)] =$$

$$[ax^{a-1} - a(x - \Delta)^{a-1}]ax^{a-1} - a(a - 1)x^{a-2}[x^a - (x - \Delta)^a] =$$

$$- a^2[x - \Delta]^{a-1}x^{a-1} + ax^{2(a-1)} + a(a - 1)x^{a-2}(x - \Delta)^a$$

which equals 0 when $\Delta = 0$. Next differentiate this expression w.r.t. Δ to deduce that

$$a^2(a-1)[x - \Delta]^{a-2}x^{a-1} - a^2(a - 1)x^{a-2}(x - \Delta)^{a-1} = x^{a-1}[a^2(a-1)]\{[x - \Delta]^{a-2} - x^{-1}(x - \Delta)^{a-1}\} < 0.$$

Since $a < 1$ and $0 < x - \Delta < x$ it follows that (17) holds.

(ii) $\underline{u(x) = - e^{-ax}, \ a > 0}$.

In this case we have that

$$[u'(x) - u'(x-\Delta)]u'(x) - u''(x)[u(x) - u(x-\Delta)] = [ae^{-ax} - ae^{-a(x-\Delta)}]ae^{-ax} + a^2e^{-ax}[-e^{-ax} + e^{-a(x-\Delta)}] = 0$$

and (17) holds with weak inequality. Hence, if $\hat{v}_c(s,k) - [F(s,C) - R]$ is strictly decreasing, or if it is just non-increasing while $c(s, C)$ is strictly increasing in s, then we can conclude that S^* is an interval.

(iii) $\underline{\text{when } u(x) = ax - bx^2, \ \frac{\partial}{\partial x}\left[\frac{u(x) - u(x - \Delta)}{\Delta u'(x)}\right] > 0.}$

In this case we have that

$$[a - 2bx - a + 2b(x - \Delta)][a - 2bx] + 2b[ax - bx^2 - a(x - \Delta) + b(x - \Delta)^2] = 2b^2\Delta^2 > 0.$$

2.3   Conditions on beliefs

In our applications of the results presented here we are interested in the case when the borrower is "optimistic" (or maybe "pessimistic") relative to the bank. We need to define this concept more precisely.

Suppose that P has a density p which is positive on $[0, \bar{s})$ and where $\bar{s} = \infty$ is possible.

We shall explore the following condition

(18)
$$\forall s : \frac{\partial}{\partial s}\left(\frac{p_B(s)}{p(s)}\right) \geq 0.$$

To interpret (18) observe that it typically requires $p_B(s) < p(s)$ for small values of s while for

large values of s it requires $p_B(s) > p(s)$. We thus interpret (18) to express the *relative optimism*

of the borrower. To that end we assumed that the Radon-Nikodym derivative, $\dfrac{p_B(s)}{p(s)}$ is

differentiable. For the more general case an equivalent condition would be:

$$\frac{P_B[s , \bar{s}]}{P[s , \bar{s}]} \quad \text{is non-decreasing in } s.$$

Recalling that we have assumed that F(s , C) is non-decreasing in s, the condition (18) means that

the relative gap between the probability that the borrower attaches to the event $\{F(s , C) \geq \bar{F}\}$

and the probability that the bank attaches to it is non-increasing as $\bar{F}$ increases.

The first order condition which is applicable to $\hat{v}_C(\cdot , k)$ can now be restated as

(19)
$$u'[\hat{v}_C(s , k)] = \frac{p(s)}{p_B(s)} u'[\hat{v}_C(s' , k)] \frac{p_B(s')}{p(s')}.$$

For P - a.a. s , s'. If u is C^2 and $\dfrac{p(s)}{p_B(s)}$ is differentiable, then (18) is equivalent to $\hat{v}_C(s , k)$

being differentiable in s with a derivative $\dfrac{\partial}{\partial s}\hat{v}_C(s , k) > 0$ which is continuous, P- a.a. One

more implication of (18) and (19) is that $\hat{v}_C(s , k)$ is *strictly concave in s* if two conditions are

satisfied: (i) $u''' > 0$, (ii) add to (18) the requirement that $\dfrac{p(s)}{p_B(s)}$ is concave in s.

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<u>Some Further Examples</u>

(i) Suppose $u(x) = - e^{-ax}$ and that both the bank and the borrower use exponential distribution .

We then have that $\dfrac{p_B(s)}{p(s)} = \dfrac{\lambda_B}{\lambda} e^{(\lambda - \lambda_B)s}$ where $\lambda > \lambda_B$ so that indeed $\dfrac{\partial}{\partial s} \dfrac{p_B(s)}{p(s)} > 0$. In this

case $\hat{v}_c(\cdot, k)$ is affine on its positive segment: $\hat{v}_C(s, k) = b + \dfrac{\lambda - \lambda_B}{a} s$. To see this, check

that the first order conditions hold:

$$ a e^{-a[b + \frac{\lambda - \lambda_b}{a} s]} \frac{\lambda_B}{\lambda} e^{(\lambda - \lambda_B)s} = \frac{\lambda_B}{\lambda} a e^{-ab} \ , \ \forall s. $$

If $F(s, C) = f(C)s$ with $f(0) = 0$ and $c(s, k)$ is non-decreasing in s then the set of observations,

S^* will always be an interval if $\tilde{C} > 0$ for an optimal contract and $b > 0$. If $f(\tilde{C}) \geq \dfrac{\lambda - \lambda_B}{a}$

then $\hat{v}_{\tilde{C}}(\cdot, k) - F(s, \tilde{C})$ is decreasing and S^* will be a compact interval. If $f(\tilde{C}) < \dfrac{\lambda - \lambda_B}{a}$ then

$S^* = \mathbb{R}_+$ is possible, in which case all states are observed.

In the next example we develop a density that leads to $\hat{v}$ being increasing and concave.

(ii) Suppose $u(x) = x^a$, $0 < a < 1$ and that $\dfrac{p_B(s)}{p(s)} = D s^b$ on $[0, \bar{s})$, 0 else, with $b > 0$.

So the first order condition for $\hat{v}$ is

$$ a \left[\hat{v}_C(s, k) \right]^{a-1} D s^b = a \left[\hat{v}_C(s', k) \right]^{a-1} D s'^b \ , \ \forall s $$

so, $\hat{v}_C(s, k) = K s^{\frac{b}{1-a}}$

$$ \frac{d}{ds} \hat{v}_C = \frac{b}{1-a} K s^{\frac{b}{1-a} - 1} > 0. $$

Assuming that $\dfrac{b}{1-a} < 1$ we conclude $\dfrac{d^2}{ds^2} \hat{v}_C < 0$ hence we have concavity of $\hat{v}$.

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The densities, p_B and p that lead to the relation $\dfrac{p_B(s)}{p(s)} = Ds^b$ could be as follows:

$$p_B(s) = \frac{1-\beta}{\bar{s}^{(1-\beta)}} s^{-\beta} \quad , \quad p(s) = \frac{1-\alpha}{\bar{s}^{(1-\alpha)}} s^{-\alpha} \text{ and then } b = \alpha - \beta \text{ and } D = \frac{(1-\beta)\bar{s}^{1-\alpha}}{(1-\alpha)\bar{s}^{1-\beta}} .$$

3.   Concluding Remarks

We have studied the form of the optimal financial relationship under costly state verification making the more natural assumptions of a risk averse borrower and diverse subjective expectations. What we have discovered is that in this more general settings the contract tends to become more complex since it has to be sensitive to both preferences and beliefs. When information is asymmetric resulting in incomplete markets, financial contracts act as substitutes for markets and under such conditions a financial contract may have to serve multiple purposes. In our case the contract is not used only for the transfer of money between periods. It also provides an insurance vehicle in addition to the fact that it permits agents with different probability beliefs to trade on their different assessment of the prospects for different future states of the world. Such multi faceted contracts clearly are what we find in today's financial markets. Of course, all the theoretical possibilities allowed in our abstract framework may not have exact real counterparts, since some features of contracts found in real markets depend not only on preferences and beliefs but also on the institutional setting. Moreover, real contracts may also depend upon other types of asymmetric information not considered in our abstract setting.

In the final part of the paper we have provided conditions under which the optimal contract retains one feature which is often found in reality, namely that the region of observations (which can in that case be interpreted as a region of non-performance, or bankruptcy) is a single

interval consisting of all low-outcome states. Such a contract have the important feature that the pay-off to the entrepreneur varies with the observed state. We have explained that this feature is often found in observed contracts. In our setting this conclusion flows directly from the nature of diversity of beliefs.

A natural extension of our work would be to isolate other classes of contracts which are permitted in the model environment and that could be interpreted as having real world counterparts. Such classes could be identified by specific configurations of the model parameters. For this reason we hope to examine further how the optimal contract varies with changes of the parameters in our more general framework. We are particularly interested in studying how the contracting environment affects economic volatility. This question is motivated by the general use of the classical Gale Hellwig model in the literature on the "Financial Accelerator" (e.g. Bernanke, Gertler, and Gilchrist, (1999)). We plan to take up these issues in future research.

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