Competition between market-making Intermediaries

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Abstract

We introduce capacity constrained competition between market-making intermediaries in a model in which agents can choose between trading with intermediaries, joining a search market or remaining inactive. Recently, market-making by a monopolistic intermediary has been analyzed by Rust and Hall (2003) and Gehrig (1993). Market-makers set publicly observable ask and bid prices. Because market-making involves price setting, without further restrictions competition between market-making intermediaries is Bertrand-like and yields the Walrasian outcome, where the ask-bid spread is zero (Rust and Hall 2003, Gehrig 1993). However, positive ask-bid spreads and competition between market-makers can be observed in reality, e.g. in banking and in retailing. Following Kreps and Scheinkman (1983) and Boccard and Wauthy (2000), we therefore introduce physical capacity constraints. This allows for a gradual transition from monopolistic to perfectly competitive intermediation as the number of intermediaries increases. In particular, we show that given Cournot capacities, intermediaries will set Cournot bid and ask prices in the subsequent subgames, so that the equilibrium of the intermediated market coincides with the Walrasian equilibrium as the number of intermediaries becomes large.

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1 Introduction

In this paper, we analyze competition between market-making intermediaries. These intermediaries set bid prices on the input market and ask prices on the output market of intermediation. Our starting point is the model developed by Gehrig (1993). According to this model, individual agents who are buyers or sellers can join a search market, join the monopolistic intermediary or remain inactive. In the search market, agents are randomly matched and the price at which exchange takes place is set bilaterally. Because matching is random, the search market does not exhaust all possible gains from trade. If agents join the intermediary, buyers have to pay an ask price set in advance by the intermediary. Likewise, if they decide to deal through the intermediary, sellers are paid the bid price the monopolistic intermediary previously announced. The intermediary trades simultaneously with both buyers and sellers. Gehrig shows that there is an equilibrium in which the search market and the market of the monopolistic intermediary are simultaneously open and where the intermediary makes positive profits because he trades at a positive ask-bid spread. More precisely, the set of individual agents is tripartite: High valuation buyers and low cost sellers deal through the intermediary. Buyers and sellers with average valuations and costs are active in the search market, and low valuation buyers and high cost sellers remain inactive. This model can also be seen as an instance of competing exchanges. Full efficiency (i.e. the Walrasian or Marshallian market equilibrium) of the more efficient exchange is not established due to limited (i.e. absence of) competition among intermediaries. Two obvious applications are the labor and the housing market, where typically intermediated and search markets co-exist.[1] But we may also think of banks and retailers as providing, among other things, the kinds of services intermediaries achieve in this type of model.

We extend the basic Gehrig model in two ways. (1) We impose a sequential structure by requiring that intermediaries first have to buy the good from the sellers (this is called the input market of intermediation) before it can be sold to buyers on what we call the output market of intermediation. (2) We introduce capacity constraints and competition between capacity constrained intermediaries.

The most important consequence of (1) is that there is a unique subgame

[1]Another application: Foreign exchange market and POW-camp (Radford, 1945).

perfect equilibrium with an active search market, in which the equilibrium analyzed by Gehrig is replicated on the equilibrium path (see Loertscher, 2003). The rationale behind (2) is the following. As is well known, competition among price setting firms is apt to lead to paradoxical results such as the one uncovered by Bertrand, according to which "two is enough for perfect competition." The same is true in our model. However, the problem arises not on the output market but on the input market of intermediation, where without additional restrictions bid price competition is (like in Stahl, 1988) a winner-takes-all competition for the monopoly profit accruing on the output market. The most important contribution to solve the Bertrand paradox has been made by Kreps and Scheinkman (1983, KS hereafter) who analyze a two stage game. In the first stage, two firms set capacities and in the second stage, they compete as price setters on a product market. For our purpose, it is therefore quite natural to follow this approach. The regions of pure strategy equilibria in the bid price setting subgame are the same as in KS and Boccard and Wauthy (2000, BW hereafter) (who extend the KS setting to n firms). If no firm's capacity is strictly larger than its Cournot best response function, given the capacities of all other firms, then there is a pure strategy equilibrium in which all firms set the input market clearing bid price. Therefore, the problem studied by KS and our problem are very similar in that respect.

[to be completed]

Apart from the model developed by Gehrig (1993), our paper is also related to Spulber (1996), Spulber (1999), and to Rust and Hall (2003). The main similarities and differences are best highlighted by briefly commenting on the following quote, taken from Spulber (1996, p.579):

> Intermediation between customers and suppliers often is the primary economic activity of firms, whether they are merchants or manufacturers. The neoclassical model of the firm implicitly recognizes that as intermediaries, firms coordinate input purchases, production, distribution, and output sales. However, since the neoclassical firm takes prices as given, the firm only intermediates on the quantity side by transforming inputs into outputs. In the neoclassical framework, market-making takes place outside the firm though exogenous price adjustment represented by the Walrasian auctioneer. On the other

> hand, models of imperfect competition in the field of industrial organization have brought the price setting role of firms to center stage, but ignore the intermediation role of firms by emphasizing competition in product markets.

We focus on the same problem as Spulber, but in a sense we want to go two steps further by (i) introducing (imperfect) competition between a finite number of market-making intermediaries and (ii) allowing prices to exert more power as they are public rather than private signals.

The paper is structured as follows. Section 2 introduces the model. In section 3, we analyze the equilibrium of the output and the input market, and section 4 contains (preliminary) conclusions. Problems concerning the mixed strategy equilibrium on the input market and some further considerations concerning Cournot competition are in the Appendix.

2 The Model

There is a continuum of buyers willing to buy one unit of an indivisible good of homogenous quality, which is known to every one. Their preferences are described by reservation prices r which are uniformly distributed over the unit interval, $r \sim U[0,1]$. If a buyer with reservation price r buys the product at price p (where the volunteer nature of exchange and individual rationality require $p \leq r$), his utility gain is $r - p$. This generates an aggregate demand schedule $D(p) = 1 - p$, $p \in [0,1]$, which can be interpreted as a (Walrasian) market demand. Analogously, sellers' preferences are described by reservation prices or unit costs of production s which are uniformly distributed on the unit interval $[0,1]$. If a seller with reservation price s sells the product at price p (where volunteer exchange under individual rationality requires $p \geq s$), his utility gain is $p - s$, so that the aggregate (Walrasian) supply function is $S(p) = p$, $p \in [0,1]$. A buyer with reservation price r owns another good that he can exchange for the good in question. This good is called money. We assume that buyers have and sellers accept money in exchange for the good.

Walrasian (or Marshallian) Outcome Given the demand function $D(p) = 1 - p$ and the supply function $S(p) = p$, the Walrasian market outcome is characterized by price $p^W = \frac{1}{2}$ and quantity exchanged $Q^W = \frac{1}{2}$, and buyers with $r \geq \frac{1}{2}$ and sellers with $s \leq \frac{1}{2}$ participate in the market, while the other agents remain inactive.

However, at the core of our model is the assumption that there is no benevolent auctioneer quoting market clearing prices and coordinating trading activities at zero costs. The purpose of our analysis is to study what allocation emerges if agents establish this allocation themselves.

Buyers and sellers can either meet in a decentralized search market where they are randomly matched and where they share the gains from trade evenly. (Alternative bargaining procedures and their consequences are discussed in detail by Loertscher (2003).) Or they can join intermediaries or remain inactive. Intermediaries first set a physical capacity constraint. Then they set a bid price at which they are willing to buy from the sellers, and finally they set an ask price at which they are willing to sell what they have previously bought. In the presence of intermediation, buyers and sellers face thus three decisions. They can either join the intermediary, enter the search market or choose to remain inactive. We denote by I_σ (I_β) the set of all sellers (buyers) who join the intermediation market. The set of sellers (buyers) active in the search market is denoted by S_σ (S_β), and the set of sellers (buyers) who decide not to be active is denoted by Z_σ (Z_β). Finally, we denote by Ω_σ (Ω_β) the set of all sellers (buyers), so that by definition $Z_\sigma \equiv \Omega_\sigma \backslash (I_\sigma \cup S_\sigma)$ and $Z_\beta \equiv \Omega_\beta \backslash (I_\beta \cup S_\beta)$. The (Lebesgue) measure of these sets is denoted by $v(.)$, e.g. $v(I_\sigma)$ is the measure of sellers joining the intermediated market.

2.1 The Dynamic Intermediation Game

We now describe the dynamic intermediation game with capacity constrained intermediaries. We speak interchangeably of intermediaries and firms. There are n profit maximizing intermediaries, indexed by $i = 1, .., n$. Each of them is endowed with a physical capacity constraint $\overline{q}_i$.[2] These capacity constraints are such that intermediary i can trade any quantity $q \leq \overline{q}_i$ at zero marginal costs, whereas trading any quantity greater than $\overline{q}_i$ is prohibitively costly. An example

[2]It would be very desirable if the choice of capacity could be endogenized as e.g. in Kreps and Scheinkman (1983). However, the mixed strategy equilibrium has turned out to be too complicated to allow for this. See Appendix A.

for a physical capacity constraint is the number of counters of an intermediary or his storage capacity for the input purchased.

The capacities of all firms are observed by all other firms and all individual agents. Given these observations, intermediaries then set simultaneously bid prices b_i, which can subsequently not be changed. The legal arrangement is such that each intermediary is obliged to buy any quantity up to the capacity constraint $\overline{q}_i$ sellers are willing to sell to him at bid price b_i. Having observed $\overline{q}_i$ and b_i for all i, sellers decide whether or not to join the intermediated market. The market where sellers interact with intermediaries is called the *input market* (of intermediation). All bid prices are public information. If more than $\overline{q}_i$ sellers want to sell to intermediary i the $\overline{q}_i$ sellers with the lowest cost can sell to i. In other words, we assume that an efficient rationing rule applies. Those sellers who get rationed by intermediary i can then join any other intermediary where again an efficient rationing rule applies. However, joining the intermediation market is an irreversible decision so that sellers who have decided to try to sell to any intermediary but who were rationed cannot subsequently go back to the search market. We assume that agents who cannot expect positive utility gain from joining the intermediated market will not join it. In exchange for the good sellers get money from the intermediary to whom they sell. We assume that all intermediaries have enough money and that sellers are aware of this.[3] When all intermediaries have finished buying, the quantity bought q_i^b by each intermediary i is observed by all agents. From what has just been said we know that $q_i^b \leq \overline{q}_i$. The sets of sellers joining the intermediary is denoted as I_σ and its (Lebesgue) measure is denoted as $\upsilon(I_\sigma)$. For reasons of tractability, we assume also that the set of sellers joining the intermediated market is observed by all agents remaining in the game.

In the second stage, each intermediary i sets the publicly observable ask price a_i and buyers decide whether to join the intermediated market. As with sellers, buyers who have joined the intermediated market cannot go back to the search market in case they are rationed. In case rationing occurs, an efficient rationing

[3]The quantity of money an intermediary holds can be regarded as his short-term capital, while his long-term or fixed capital is embedded in his physical capacity constraint $\overline{q}_i$. In a richer model, one could endogenize the money or short-term capital an intermediary holds. Thus, short-term capital would be a strategic variable in addition to the capacity constraint and might be used a signalling device to attract customers. Section **??** provides a motivation why such an extension might be enriching.

rule applies. The market where buyers interact with intermediaries is called the *output market* (of intermediation). The legal arrangement is such that at the ask price a_i intermediary i is obliged to sell any quantity $q \leq q_i^b$ buyers are willing to buy from him. If an intermediary cannot sell its whole stock q_i^b he can dispose of the excess quantity for free. However, as on the input market intermediaries are committed to the prices they set. We assume that the set I_β is observable. Like sellers, intermediaries accept money in exchange for the good they sell.

In the third and last stage, sellers and buyers who have not joined the intermediated market decide simultaneously whether to join the search market. We assume that agents join the search market only if their expected utility from doing so is positive. This prevents the search market from being overcrowded with agents who never engage in trade. The set of sellers (buyers) joining the search market is denoted by $S_\sigma(S_\beta)$, and their (Lebesgue) measure is denoted by $v(S_\sigma)$ $(v(S_\beta))$. The matching technology is such that if the number of buyers and sellers is the same, each buyer (seller) is matched with probability λ to a seller (buyer), where $\lambda \in [0,1]$. If the number (or measure) of, say, sellers active in the search market is larger than that of buyers, the probability of being matched to a buyer is correspondingly adjusted downwards, while the probability of a match for buyers is still λ. That is, the traders on the long side of the search market are matched with probability $\gamma_i\lambda$, where $\gamma_i = \frac{v(I_j)}{v(I_i)} < 1$ with $i = \sigma, \beta, j \neq i$. There is no further possibility to trade after a match has been established. For those who are not matched, the game is over. As observed by Spulber (1999, p. 561), the search market is static in the sense that search market participants are randomly and pairwise matched at most once. If a buyer with reservation price r and a seller with cost s are matched they share the gain from trade evenly if $r - s > 0$. That is, they agree on the price $p = \frac{r-s}{2}$. After that, the game is over. If $r - s \leq 0$, the game is over for these agents without trade taking place. Finally, the sets of inactive sellers (buyers) is denoted as $Z_\sigma(Z_\beta)$.

Let us summarize the time structure of the dynamic intermediation game. This structure departs from the one in previous versions of the model (Gehrig, 1993; Freixas and Rochet, 1997; Spulber, 1999), where the game is played in simultaneous moves, but it is the same as in Loertscher (2003). The dynamic intermediation game has three stages.

1. **Input Market**: There are n intermediaries indexed by $i = 1, ..n$. They are

endowed with capacity constraints $\overline{q}_i$. Before setting bid prices b_i on the input market, $\overline{q}_i$ is observed by all i and by all sellers and buyers. Up to $\overline{q_i}$, intermediary i is obliged to buy any quantity sellers want to sell to him at bid price b_i. After observing b_i (and $\overline{q_i}$) sellers decide simultaneously whether to join the intermediated market. For all those sellers who join the intermediated market, the game is over, regardless of whether they can actually sell or not. The quantity bought by each intermediary, q_i^b is public information, and the set of sellers joining the intermediary is public information, too.

2. **Output Market**: On the output market, intermediary i sets an ask price a_i at which he has to sell any quantity buyers want to buy up to his whole stock q_i^b. In case there is rationing, an efficient rationing rule applies. For buyers who decide to join the intermediary the game is over, regardless of whether they can buy or get rationed. The set of buyers who have joined the intermediary, I_β is observed by all players remaining in the game.

3. **Search Market**: Sellers and buyers who have not joined the intermediary may join the search market. Those who participate in the search market are randomly matched. The matching technology is such that all traders in the search market are matched with probability $\lambda \in [0, 1]$ if the set of sellers and buyers active in the search market have the same measure. A buyer r and a seller s who are successfully matched share the gains from trade evenly by agreeing on the price $\frac{r-s}{2}$ if $r - s > 0$. If $r - s \leq 0$, they do not exchange the good. After that, the game is over.

2.2 Strategies

There are three types of agents, sellers s, buyers r and intermediaries $i = 1, .., n$. Let $\overline{\mathbf{q}}$ denote the n-tuple of capacities $(\overline{q}_1, ..., \overline{q}_n)$, let $\mathbf{q^b}$ be the n-tuple of quantities bought $(q_1^b, ..., q_n^b)$, and let $\mathbf{b} = (b_1, ..., b_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ be the n-tuples of bid and ask prices, respectively. Finally, as with quantities and capacities, we use the subscripts i and $-i$ to indicate the variable chosen by i and all firms other than i, respectively. For example, b_i is intermediary i's bid price and b_{-i} are the bid prices of all firms.[4] Given these conventions, a strategy for a seller s is

$$\tau_s = (I_s(\mathbf{b}, \overline{\mathbf{q}}); S_s(\mathbf{a}, \mathbf{b}, \overline{\mathbf{q}}, I_\sigma, I_\beta))\,. \tag{1}$$

Similarly, for a buyer a strategy is

$$\rho_r = (I_r(\mathbf{a}, \mathbf{b}, \overline{\mathbf{q}}, I_\sigma,), S_r(\mathbf{a}, \mathbf{b}, \overline{\mathbf{q}}, I_\sigma, I_\beta)) \tag{2}$$

where the functions $I_k(.)$ and $S_k(.)$ specify the conditions under which agent k joins the intermediary or the search market, respectively, $k = s, r$. Note that both for sellers and buyers, we do not have to specify the decision to be inactive, because it is contained in the case where an agent decides to join neither the intermediary nor the search market. Finally, for intermediary i, a (pure) strategy is

$$\varphi_i = \big(b_i(\overline{\mathbf{q}}); a_i(\mathbf{q}^{\mathbf{b}}, \mathbf{b}, I_\sigma)\big)\,, \tag{3}$$

where the $\overline{q}_i$'s are a real positive numbers and $b_i(\overline{\mathbf{q}})$ and $a_i(\mathbf{q}^{\mathbf{b}}, I_\sigma)$ are real valued functions. In general, strategies for this game are cumbersome expressions because there are so many states of the world for which each agent must have a complete contingent plan. For example, every small change in the set of sellers deciding to join the intermediary will require a different optimal response by all other players in subsequent periods. Since there is an infinity of such contingencies, it would not be possible to write down these strategies in closed forms in general. However, as we show next, the space over which these strategies have to defined can be reduced considerably.

2.2.1 Partitioning of Buyers and Sellers

As it turns out, in any equilibrium with an active search market, the set of individual agents is tripartite: High valuation buyers and low cost sellers deal through an intermediary. Buyers and sellers with average valuations and costs are active in the search market, and low valuation buyers and high cost sellers remain inactive.[5] This result is due to Gehrig (1993) and stated formally in the following Proposition. It is important because it allows us to consider only strategies that are defined for such tripartite sets.

[4]Note that in contrast to the case of quantities or capacities, where $\overline{q}_{-i} \equiv \sum_{j\neq i}^{n} \overline{q}_j$, b_{-i} is not a sum but a $(n-1)$-tuple.

[5]Spulber (1996) and Rust and Hall (2003) report the same result in similar contexts.

Proposition 1 (Gehrig (1993), Proposition 1) *In any equilibrium with an active search market,*[6] *there are critical reservation values* $\underline{r}$ *and* $\overline{r}$*, such that the set of buyers can be partitioned into three subsets . If* $r \in [0, \underline{r})$*, then* $r \in Z_\beta$*; if* $r \in [\underline{r}, \overline{r}]$*, then* $r \in S_\beta$ *and if* $r \in (\overline{r}, 1]$*, then* $r \in I_\beta$*. In any equilibrium with an active search market, there are critical unit costs* $\underline{s}$ *and* $\overline{s}$*, such that the set of sellers can be partitioned into three subsets. If* $s \in [0, \underline{s})$*, then* $s \in I_\sigma$*; if* $s \in [\underline{s}, \overline{s}]$*, then* $s \in S_\sigma$ *and if* $s \in (\overline{s}, 1]$*, then* $s \in Z_\sigma$*.*

The Proposition is proved with the help of the following three Lemmas.

Lemma 1 (Gehrig (1993), Lemma 1) *For any positive ask bid spread* $a - b > 0$*, some traders will be active in the search market.*

Proof: Buyers with $r < a$ and sellers with $s > b$ can expect positive utility gains from search market participation.■

Lemma 2 (Gehrig (1993), Lemma 2) *In equilibrium, the sets of inactive buyers and sellers,* Z_β *and* Z_σ*, are closed and convex sets such that* $0 \in Z_\beta$ *and* $1 \in Z_\sigma$*.*

Proof: Let buyer r be inactive and suppose $\tilde{r} < r$ is active. Then r could imitate $\tilde{r}$ and get at least his payoff, whereas his payoff when inactive is zero. Completely symmetric reasoning applies for sellers. Finally, buyer 0 and seller 1 remain inactive because they never expect a positive gain from trade.■

Lemma 3 (Modification of Lemma 3, Gehrig (1993)) *In any equilibrium with an active search market (i.e.* $S_\sigma \neq \emptyset, S_\beta \neq \emptyset$*),*
(i) $r_0 \in S_\beta \Rightarrow r \notin I_\beta$ *for* $r < r_0$ *and*
(ii) $s_0 \in S_\sigma \Rightarrow s \notin I_\sigma$ *for* $s > s_0$*.*

Proof: Parts of the proof very closely mimic the one by Gehrig (1993) and Loertscher (2003). We denote by $\gamma_i, i = \sigma, \beta$ a seller's and a buyer's probability of being successfully matched in the search market with probability λ. Thus, for

[6] We have added this phrase because there is also an equilibrium where no one joins the search market. If no one goes to the search market, unilateral deviation to join the search market does obviously not pay. However, as long as there is no fix cost of joining the search market, in this equilibrium, two continua of agents play weakly dominated strategies.

example a seller is matched with probability $\lambda\gamma_\sigma = \lambda \min[\frac{\upsilon(S_\beta)}{\upsilon(S_\sigma)}, 1]$. Since each agent has measure zero, γ_i for $i = \sigma, \beta$ can be taken as given by every individual agent.

We first consider (ii) of Lemma 3. Because there are n intermediaries rationing can occur either at the level of the individual intermediary i and/or at the intermediated market as a whole. At the individual level, rationing occurs whenever the number (measure) of sellers willing to sell to intermediary i at bid price b_i exceeds i's capacity constraint $\overline{q}_i$. Because rationing is assumed to be efficient, the $\overline{q}_i$ sellers with the lowest cost who want to sell to intermediary i can do so in this case. At the market level, rationing occurs if and only if the measure of sellers joining the intermediated market exceeds aggregate capacity, i.e. iff $\upsilon(I_\sigma) > \sum_i^n \overline{q}_i$. Since we must not restrict ourselves to the case where all intermediaries set the same bid prices, rationing at the individual level may always occur. Thus, we are left with the cases with and without rationing at the market level. We first consider the case without. First note that $s_0 \in S_\sigma \Leftrightarrow \gamma_\sigma U_\sigma(s_0) \geq b_i - s_0$, where $U_\sigma(s_0)$ is the expected utility gain of seller s_0 of search market participation for $\upsilon(S_\beta) = \upsilon(S_\sigma)$ and where b_i is the bid price s_0 would get when joining the intermediated market. Note also that due to efficient rationing, $s > s_0$ would get a bid price $b_j \leq b_i$ when joining the intermediated market, implying $b_i - s_0 > b_j - s$. Let $F(r)$ be the cumulative distribution function of buyers active in the search market. Then, we have

$$U_\sigma(s_0) = \lambda \int_{s_0 \leq r} \frac{r - s_0}{2} dF(r), \text{and} \tag{4}$$

$$U_\sigma(s) = \lambda \int_{s \leq r} \frac{r - s}{2} dF(r). \tag{5}$$

Because $s > s_0$, $U_\sigma(s_0) > U_\sigma(s)$. Subtracting (5) from (4) we get

$$\begin{aligned} U_\sigma(s_0) - U_\sigma(s) &= \lambda \int_{s_0 \leq r} \frac{s - s_0}{2} dF(r) - \lambda \int_{s_0 \leq r \leq s} s dF(r), \text{or} \\ U_\sigma(s) &= U_\sigma(s_0) - \lambda \int_{s_0 \leq r} \frac{s - s_0}{2} dF(r) + \lambda \int_{s_0 \leq r \leq s} s dF(r). \end{aligned}$$

Since $s > 0, \lambda \int_{s_0 \leq r \leq s} s dF(r) > 0$, so that

$$U_\sigma(s) > U_\sigma(s_0) - \lambda \int_{s_0 \leq r} \frac{s - s_0}{2} dF(r).$$

Because $\lambda \int_{s_0 \leq r} \frac{s-s_0}{2} dF(r) < s - s_0$,

$$U_\sigma(s) > U_\sigma(s_0) - (s - s_0).$$

Multiplying both sides by γ_σ, $0 < \gamma_\sigma \leq 1$, we get $\gamma_\sigma U_\sigma(s) > \gamma_\sigma U_\sigma(s_0) - \gamma_\sigma(s - s_0)$, so that

$$\gamma_\sigma U_\sigma(s) > \gamma_\sigma U_\sigma(s_0) - (s - s_0).$$

However, since $s_0 \in S_\sigma \Leftrightarrow \gamma_\sigma U_\sigma(s_0) \geq b_i - s_0$,

$$\gamma_\sigma U_\sigma(s) > (b_i - s_0) - (s - s_0) = b_i - s \geq b_j - s,$$

where $b_j - s$ is the utility gain for s of joining the intermediated market. Thus, $s > s_0$ will not join the intermediated market if s_0 joins the search market, which proves part (ii) in the case without rationing (at the market level). For buyers, the case (i) without rationing at the market level is completely analogous and will not be treated here.

Now the case with rationing at the market level can be treated fairly easily. Again, consider (ii) and assume first that s_0 would get get $b_i > s_0$ at the intermediated market. That is, s_0 would not get rationed at the intermediated market. Then $s > s_0$ would get at most $b_j \leq b_i$ at the intermediated market and at worst 0, the worst case occurring when s is one of the sellers who get rationed. Since s gets at most $b_j - s < b_i - s_0$, exactly the same reasoning applies as above. Finally, assume that s_0 would get rationed when joining the intermediated market. Due to efficient rationing, s would then get rationed, too, so that their utility gain from joining the intermediated market is zero. Thus, they will not join it (recall the assumption made in subsection 2.1 that agents who expect zero gain from joining the intermediated market will not do so), and we have $s \notin I_\sigma$. Again, the case for buyers being completely symmetric, it will not be treated here. ■

Proof of Proposition 1: These three Lemmas state that the sets of inactive buyers and sellers and the sets of buyers and sellers active in the search market are convex and directed sets. Therefore, only buyers with high reservation prices and sellers with low costs can potentially gain by trading with the intermediary. ■

2.3 Input Supply and Output Demand Functions

For $a > b$, Lemma 1 implies that all buyers with $r \in [\underline{s}, \overline{r}]$ and all sellers with $s \in [\underline{s}, \overline{r}]$ are active in the search market so that $S_\beta = S_\sigma = [\underline{s}, \overline{r}]$.[7] Therefore, in any equilibrium with $a > b$, $\gamma_\beta = \gamma_\sigma = 1$. Moreover, because reservation prices of all agents are uniformly distributed on the unit interval, we know that for $r \in S_\beta, r \sim U[\underline{s}, \overline{r}]$ and for $s \in S_\sigma, s \sim U[\underline{s}, \overline{r}]$. Therefore, $dF(r) = \frac{1}{\overline{r}-\underline{s}}dr$ and $dG(s) = \frac{1}{\overline{r}-\underline{s}}ds$, where $F(r)$ and $G(s)$ are the cumulative distribution functions of buyers and sellers active in the search market. Since all previous actions are assumed to be observable, $\underline{s}$ and $\overline{r}$ will be known when agents decide whether to join the search market. Therefore, it suffices to condition this decision on $\underline{s}$ and $\overline{r}$, so that a strategy for seller s can be written as $\tau_s = \left(I_s(\mathbf{b}, \overline{\mathbf{q}}); S_s(\mathbf{a}, \mathbf{b}, \mathbf{q^b}, \underline{s}, \overline{r})\right)$. Similarly, for a buyer a strategy can be written as $\rho_r = \left(I_r(\mathbf{a}, \mathbf{b}, \mathbf{q^b}, \underline{s}); S_r(\mathbf{a}, \mathbf{b}, \mathbf{q^b}, \underline{\mathbf{s}}, \overline{r})\right)$, and for an intermediary i, a strategy simplifies to $\varphi_i = \left(b_i(\overline{\mathbf{q}}); a_i(\underline{s}, \mathbf{q^b})\right)$. This allows us to compute explicitly the expected utility gains from search market participation and to characterize completely agents' equilibrium strategies in the game. This is what we do next.

We begin by briefly describing the equilibrium of the bargaining subgame. With even sharing, a buyer r and a seller s who are matched in the search market share the gains from trade $r - s$ equally, provided $r - s > 0$. We will refer to seller $\underline{s}$ and buyer $\overline{r}$ as the critical seller and buyer. The expected utility gain for seller s with $s \in [\underline{s}, \overline{r}]$ from search market participation is then

$$\begin{aligned} U_\sigma(s) &= \lambda \int_s^{\overline{r}} \frac{(r-s)}{2} dF(r) = \frac{\lambda}{2} \frac{1}{\overline{r} - \underline{s}} \int_s^{\overline{r}} (r-s) dr \\ &= \frac{\lambda}{2} \frac{\left[\frac{r^2}{2} - rs\right]_s^{\overline{r}}}{\overline{r} - \underline{s}} = \frac{\lambda}{4} \frac{(\overline{r} - s)^2}{\overline{r} - \underline{s}}, \end{aligned}$$

which is the same as that derived by Gehrig under the alternative bargaining schedule with take-it-or-leave-it offers. Thus, for the critical seller $\underline{s}$ we have

$$U_\sigma(\underline{s}) = \frac{\lambda}{4} (\overline{r} - \underline{s}). \tag{6}$$

[7] More precisely, because only agents who can expect positive utility gain from search market participation are assumed to enter the search market, sellers (buyers) with $s = \overline{r}$ ($r = \underline{s}$) will not participate in the search market, and we should write $S_\beta = (\underline{s}, \overline{r}]$ and $S_\sigma = [\underline{s}, \overline{r})$ or $S_\beta = [\underline{s}^+, \overline{r}]$ and $S_\sigma = [\underline{s}, \overline{r}^-]$, where superscript "+" ("-") means marginally "greater (smaller) than". Nonetheless, the search market would be balanced because $\upsilon(S_\beta) = \upsilon(S_\sigma)$, implying $\gamma_\sigma = \gamma_\beta = 1$.

Likewise, for a buyer with reservation price $r \in [\underline{s}, \overline{r}]$ the expected utility gain from being active in the search market is

$$\begin{aligned} U_\beta(r) &= \lambda \int_{\underline{s}}^{r} \frac{(r-s)}{2} dG(s) = \frac{\lambda}{2} \frac{1}{\overline{r}-\underline{s}} \int_{\underline{s}}^{r} (r-s) ds \\ &= \frac{\lambda}{4} \frac{(r-\underline{s})^2}{\overline{r}-\underline{s}}, \end{aligned}$$

so that for the critical buyer

$$U_\beta(\overline{r}) = \frac{\lambda}{4}(\overline{r}-\underline{s}) = U_\sigma(\underline{s}). \tag{7}$$

Now, the utilities of critical buyers and sellers participating in the search market in equation (7) can be used to derive the reservation prices of these agents for joining the intermediated market.[8] If buyer $\overline{r}$ has to pay the ask price A to get the good from an intermediary with certainty, he is indifferent between joining the intermediated market and the search market if and only if

$$\overline{r} - A = \frac{\lambda}{4}(\overline{r}-\underline{s}). \tag{8}$$

Likewise, if seller $\underline{s}$ is paid the bid price B with certainty when joining the intermediated market, he is indifferent between joining the intermediated and the search market if and only if

$$B - \underline{s} = \frac{\lambda}{4}(\overline{r}-\underline{s}). \tag{9}$$

Solving equations (8) and (9) yields

$$A(\overline{r}, \underline{s}) = \frac{4-\lambda}{4}\overline{r} + \frac{\lambda}{4}\underline{s} \quad \text{and} \quad B(\underline{s}, \overline{r}) = \frac{4-\lambda}{4}\underline{s} + \frac{\lambda}{4}\overline{r}. \tag{10}$$

Thus, $A(\overline{r}, \underline{s})$ and $B(\underline{s}, \overline{r})$ are reservation prices of buyer $\overline{r}$ and seller $\underline{s}$ for joining the intermediated market, given all $s < \underline{s}$ and all $r > \overline{r}$ have joined the intermediated market and provided there is no rationing (at the market level). In general, the ask (or bid) prices set by the intermediaries will not be the same. Therefore, the reservation prices in (10) are to be interpreted as follows. Due to Proposition 1, all agents (including buyer $\overline{r}$) know that if $\overline{r}$ joins the intermediated market, all $r > \overline{r}$ will join the intermediated market, too. Thus, if quantities bought and

[8]Throughout, we assume that all agents - buyers, sellers and intermediaries - are risk neutral.

ask prices are such that $\overline{r}$ can buy the good at the ask price $a_i = A$ and if all buyers with higher reservation prices join the intermediated market, too,[9] buyer $\overline{r}$ would be indifferent between joining the intermediated market and entering the search market. An analogous interpretation applies for the reservation price B.

Throughout we use upper case letters A and B to denote the (inverse) demand and supply functions, and lower case letters a_i and b_i to denote the prices set by an individual intermediary i. Similarly, we denote aggregate quantities or aggregate capacities by upper case letters. For example, $\overline{Q} \equiv \sum_{i=1}^{n} \overline{q}_i$ denotes aggregate capacity and $Q^b \equiv \sum_{i=1}^{n} q_i^b$ is aggregate quantity bought. Note that Q^b is the quantity bought *by* intermediaries. We make also use of the notational convention that subscript i denote the variable of intermediary i and subscript $-i$ denote the variable for all intermediaries other than i. Thus e.g. $\overline{q}_i$ is intermediary i's capacity and $\overline{q}_{-i}$ of all intermediaries other than i, so that by definition $\overline{Q} \equiv \overline{q}_i + \overline{q}_{-i}$.

For there to be no rationing on the input market aggregate capacity $\overline{Q}$ has to be at least as great as $\underline{s}$. On the other hand, the quantity intermediaries sell on the output market cannot exceed the quantity bought on the input market, Q^b. Clearly, we thus have $Q^b = \min[\underline{s}, \overline{Q}]$, so that without rationing on either market $\overline{r} \geq 1 - Q^b$. Because there are $1 - \overline{r}$ buyers whose reservation prices are greater than or equal to $\overline{r}$ quantity demanded if all intermediaries set $a = A(\overline{r}, \underline{s})$ is therefore $1 - \overline{r}$. Let $Q^d \equiv 1 - \overline{r}$ denote this quantity and note that this is quantity demanded *at* the intermediaries. For the same reasons as for buyers, there are $\underline{s}$ sellers who are willing to sell at bid price $b(\overline{r}, \underline{s})$, provided the buyer with the highest reservation price in the search market is buyer $\overline{r}$. Therefore, $\underline{s}$ is equal to the quantity the intermediaries can buy at the bid price $b(\overline{r}, \underline{s})$ (if capacities allow them to do so), which is Q^b. If we replace $\overline{r}$ by $1 - Q^d$ and $\underline{s}$ by Q^b in equation (10), we get the inverse output demand and inverse input supply functions

$$A\left(Q^d, Q^b\right) = \frac{4-\lambda}{4} - \frac{4-\lambda}{4} Q^d + \frac{\lambda}{4} Q^b \tag{11}$$

and

$$B\left(Q^d, Q^b\right) = \frac{\lambda}{4} - \frac{\lambda}{4} Q^d + \frac{4-\lambda}{4} Q^b. \tag{12}$$

[9] This is the case if $a_i = \max_j [a_j], j = 1, ..., n$ and if aggregate capacity is equal to or greater than $1 - \overline{r}$, which is the quantity demanded at the intermediated market when $\overline{r}$ is the indifferent buyer.

The output demand and the input supply functions are

$$D\left(a, Q^b\right) = 1 - \frac{4}{4-\lambda}a + \frac{\lambda}{4-\lambda}Q^b, \tag{13}$$

and

$$S\left(b, Q^d\right) = -\frac{\lambda}{4-\lambda} + \frac{4}{4-\lambda}b + \frac{\lambda}{4-\lambda}Q^d, \tag{14}$$

respectively. The ask price elasticity of output demand, given Q^b, is

$$\varepsilon\left(a, Q^b\right) = -\frac{4a}{4-\lambda-4a+\lambda Q^b}. \tag{15}$$

Finally, note also that these functions are valid only under the provision that there is an active search market from which some agents can expect positive utility gains. This requires that $\overline{r} > \underline{s}$. If $\overline{r} \leq \underline{s}$, agents lose the outside option of search market participation. In this case, seller s would join the intermediated market whenever $b > s$ and a buyer r will buy from the intermediaries whenever $a < r$. Graphically, therefore, beyond the point of intersection of the (inverse) output demand function $A\left(Q^d, Q^b\right)$ with the (inverse) Walrasian demand function $1-Q^d$, the willingness to pay for intermediated trade is given by the (inverse) Walrasian demand function. Therefore, the reservation prices of buyers for intermediated trade are actually given by the maximum of these two functions

$$\min\left[A\left(Q^d, Q^b\right), 1-Q^d\right]. \tag{16}$$

It is easy to verify that the intersection of $A\left(Q^d, Q^b\right)$ with $1-Q^d$ is at the point where $1-Q^d = Q^b$. Analogously, the (inverse) input supply function $B\left(Q^d, Q^b\right)$ in equation (12) is valid only to the left of the intersection with Q^b. Beyond that point, expected utility gain from search market participation in not positive, and the reservation prices for trading through the intermediary are given by the (inverse) Walrasian supply function. Hence, the sellers' reservation prices the intermediary faces are given by the maximum of these two functions

$$\max\left[B\left(Q^d, Q^b\right), Q^b\right]. \tag{17}$$

Again, the point of intersection is where $1-Q^d = Q^b$. Finally, when quantity bought equals quantity sold, i.e. $Q^d = Q^b = Q$, we say that trade in the intermediated market is *balanced*. In this case, the inverse demand and supply functions are

$$A(Q) = \frac{4-\lambda}{4} - \frac{2-\lambda}{2}Q \tag{18}$$

$$B(Q) = \frac{\lambda}{4} + \frac{2-\lambda}{2} Q. \tag{19}$$

Below it will be useful to have an expression for the input supply function under balanced trade. This function is

$$S(b) = \frac{4b - \lambda}{2(2-\lambda)}, \tag{20}$$

so that under balanced trade the inverse output demand function can be written as a function of b only

$$A\left(Q^b(b)\right) = 1 - b. \tag{21}$$

Figure **??** depicts the Walrasian demand and supply functions and the search constrained output demand and input supply functions for the intermediaries, under the assumption that intermediated trade is balanced.

[INSERT FIGURE 1 ROUND HERE]

3 Equilibrium

In this section, we show that the dynamic intermediation game with capacity constraints has a subgame perfect equilibrium which replicates the Cournot outcome if firms are given Cournot capacities. We proceed as follows. In section 3.1, we briefly review the basic concepts of Cournot competition and translate their meaning so that they fit to our model. Then in section 3.2, we analyze the output market subgame for any aggregate quantities bought $Q^b \leq \frac{1}{2} = Q^W$, the quantity traded under Walrasian conditions. In the unique equilibrium of this subgame, all firms set the market clearing ask price. In section 3.3 we show that the input market subgame has a unique equilibrium if all firms $i = 1, .., n$ have capacities $\overline{q}_i$ smaller than or equal to the capacities given by the Cournot reaction function. In this equilibrium, all firms set a market clearing bid price. In the Appendix, we show that there is no pure strategy equilibrium if one firm has a larger capacity than given by the Cournot reaction function. A proof for the existence of an equilibrium in the mixed strategy region as well as further considerations concerning Cournot competition are also relegated to the Appendix.

3.1 Preliminary: The Cournot Outcome

Since Cournot competition typically refers to competition on a product market organized by a Walrasian auctioneer we have to make clear what we mean by

Cournot competition and Cournot outcome in the present setting. When speaking of Cournot competition we henceforth mean that the intermediation industry is organized as a Cournot market. Both on the input and on the output market of intermediation a (Walrasian) auctioneer quotes market clearing ask and bid prices, given the quantities intermediaries want to buy and sell and given the inverse supply and demand functions, constrained by the agent' outside option of search market participation. Under Cournot conditions, every intermediary quotes the quantity he wants to trade, and the auctioneer then sets market clearing prices. As a Cournot competitor, each intermediary i thus maximizes his profits by choosing his optimal quantity q_i^*, given the quantities of all other intermediaries, q_{-i} and given the (inverse) supply and demand functions $B(Q)$ and $A(Q)$. Let $\pi_i(q_i, q_{-i})$ denote firm i's profits when setting quantity q_i. The maximization problem for i thus is

$$\begin{aligned}\max_{q_i} \pi_i(q_i, q_{-i}) &= (A(Q) - B(Q))\, q_i \\ &= (A(q_i + q_{-i}) - B(q_i + q_{-i}))\, q_i, \end{aligned} \tag{22}$$

which yields the following first order condition

$$0 = (A'(q_i^* + q_{-i}) - B'(q_i^* + q_{-i}))\, q_i^* + A(q_i^* + q_{-i}) - B(q_i^* + q_{-i}). \tag{23}$$

The solution q_i^* is called i's best response or reaction function and denoted as $r(q_{-i})$. It is implicitly defined as

$$r(q_{-i}) = \frac{A(r(q_{-i}) + q_{-i}) - B(r(q_{-i}) + q_{-i})}{-\left(A'(r(q_{-i}) + q_{-i}) - B'(r(q_{-i}) + q_{-i})\right)}. \tag{24}$$

Because $A(Q)$ has a negative slope and is (weakly) concave and $B(Q)$ is positive sloped and (weakly) convex the maximization problem (22) is a concave problem so that the solution in (24) is the unique (interior) maximum. The corner solution with $r(q_{-i}) = 0$ arises only if q_{-i} is so large that $A(q_{-i}) - B(q_{-i}) \leq 0$.

At this point it is convenient to define the spread function $Z(Q) \equiv A(Q) - B(Q)$. Note that because of the properties of $A(Q)$ and $B(Q)$ just mentioned, $Z(Q)$ is negatively sloped and weakly concave. If we differentiate (23) with respect to q_{-i} and set the result equal to zero, we can solve for $\frac{dr(q_{-i})}{dq_{-i}}$ to get

$$r'(q_{-i}) = \frac{-Z'' - Z'}{rZ'' + 2Z'}, \tag{25}$$

where we have dropped arguments of $r(.)$ and $Z(.)$. The property $r' < 0$ is readily established for any concave function Z. To see this, note that $-Z'' - Z' > 0$ and $rZ'' + Z' < 0$. Moreover, if Z is a linear function, we have $Z'' = 0$, implying that $r' > -1$ and $\frac{d(r(q_{-i})+q_{-i})}{dq_{-i}} > 0$.

In our setting, $Z(Q) = \frac{2-\lambda}{2} - (2-\lambda)Q$, so that $r(q_{-i}) = \frac{1}{2} - q_{-i} - r(q_{-i})$. This permits the explicit definition

$$r(q_{-i}) = \frac{1}{4} - \frac{1}{2}q_{-i}. \tag{26}$$

An equilibrium under Cournot competition is reached if and only if all intermediaries $i = 1, .., n$ trade a quantity equal to their best responses, i.e. if

$$q_i = r(q_{-i})\forall i. \tag{27}$$

By symmetry, we have $q_i = q_j = q$ in any equilibrium. Therefore, $q_{-i} = (n-1)q_i = (n-1)q$, so that (27) can be written as

$$q = r((n-1)q)\forall i. \tag{28}$$

Because the left-hand side begins at zero and is increasing in q while the right-hand side begins at $r(0) > 0$ and decreases in q, there is a unique q such that this equality is satisfied. Denote by q^C the value of q such that equality (28) holds. Plugging this into (26) and solving yields $q^C = \frac{1}{2(n+1)}$. Thus, $Q^C = \frac{1}{2}\frac{n}{(n+1)}$, so that $Z^C = \frac{2-\lambda}{2} - \frac{2-\lambda}{2}\frac{n}{(n+1)}$, $A^C = \frac{4-\lambda}{4} - \frac{2-\lambda}{4}\frac{n}{(n+1)}$ and $B^C = \frac{\lambda}{4} + \frac{2-\lambda}{4}\frac{n}{(n+1)}$. Note that as n gets arbitrarily large, $Q^C \overset{-}{\to}= \frac{1}{2} \equiv Q^W$, $Z^C \overset{+}{\to} 0, A^C \overset{+}{\to} \frac{1}{2}$ and $B^C =\overset{-}{\to} \frac{1}{2}$, where "$\overset{-}{\to}$ ($\overset{+}{\to}$)" means "approaches from below (above)". Or put in words: As the number of competing intermediaries gets large, the outcome of Cournot competition converges to the Walrasian market outcome. Finally, note also that for $n = 1$, $Q^C = \frac{1}{4}$, $A^C = \frac{3}{4} - \frac{\lambda}{8}$ and $B^C = \frac{\lambda}{8} + \frac{1}{4}$, which is the equilibrium analyzed by Gehrig (1993).

3.2 The Output Market Subgame

We first prove a Lemma that says that in equilibrium, the quantity traded by intermediaries cannot exceed the Walrasian quantity. This seems very intuitive. The proof, though, is not straightforward. The following Lemma is useful because it allows us to concentrate on aggregate quantity bought $Q^b \leq \frac{1}{2}$.

Lemma 4 *There is no equilibrium in which aggregate quantity bought Q^b exceeds the Walrasian quantity $Q^W = \frac{1}{2}$.*

Proof: Consider a monopolistic intermediary. He would sell his quantity bought Q^b at the ask price a^m, which is defined as the ask price for which the (ask) price elasticity of output demand, $\varepsilon(a, Q^b)$, is minus one if his quantity bought allows him to sell that much (i.e. if Q^b is large enough). Otherwise, he would set the market clearing price for Q^b, which is above a^m. Setting $\varepsilon(a, Q^b)$ in equation (15) equal to minus one and solving for a yields

$$a^m = \frac{1}{2} - \frac{\lambda}{8}(1 - Q^b), \tag{29}$$

which is smaller than $\frac{1}{2}$ for $Q^b < 1$ and $\lambda > 0$. But in order to be able to sell all that is demanded at a^m, Q^b has to be larger than $\frac{1}{2}$. As observed above, for $1 - Q^d < Q^b$, the relevant inverse demand function is $A = 1 - Q^d$ because the search market shuts down. The elasticity of $A = 1 - Q^d$ is -1 at $a = \frac{1}{2}$. Thus, for $Q^b > \frac{1}{2}$, the monopolistic intermediary would set $a^m = \frac{1}{2}$, and the search market shuts down for $Q^b > \frac{1}{2}$. Therefore, the relevant inverse supply function is $B = Q^b$ in this range. Because under efficient rationing aggregate quantity bought is equal to the aggregate supply $S(.)$ at the lowest bid price for which $Q^b(.) > 0$, all firms must therefore pay a bid price greater than $\frac{1}{2}$ in order to buy $Q^b > \frac{1}{2}$.[10] But the aggregate revenue of competing intermediaries whose aggregate quantity bought exceeds $\frac{1}{2}$ will not be larger than the revenue of a monopolistic intermediary with such a large quantity bought. The monopoly's revenue is $\frac{1}{4}(= \frac{1}{2} \times \frac{1}{2})$, while the expenditure needed to acquire Q^b is at least $Q^b \times Q^b > \frac{1}{4}$. Therefore $Q^b > \frac{1}{2}$ implies that the intermediation industry makes negative profits. Because each intermediary has the outside option of making zero profits (e.g. by quoting $b = 0$), this cannot be an equilibrium. ■

Now let us turn to the question what the equilibrium of the ask price setting subgame is, given $Q^b \leq \frac{1}{2}$. But because $Q^d \leq Q^b$, for Q^b the relevant inverse demand function, i.e. $\min[A(Q^d, Q^b), 1 - Q^d]$ as defined in (11), is $A(Q^d, Q^b)$. We first show that ask prices $a_i < A(Q)$ (where $A(Q)$ is as defined in (18)) will not be set in equilibrium. The reason for this is that these prices are strictly

[10] This is true only under the condition that no intermediary sets a bid price greater than the one at which aggregate capacity clears, but in equilibrium this condition is satisfied (see section 3.3).

dominated: At the price a_i firm i sells q_i^b, no matter what ask prices the other firms set, whereas by setting $a^* = A(Q)$ intermediary i would earn $a^* q_i^b > a_i q_i^b$ regardless of the prices the other firms set. Thus, ask prices $a_i < A(Q)$ can be ruled out.

What we have not yet shown is whether higher prices than a^* can occur in equilibrium. That is, whether in equilibrium intermediated trade can be unbalanced with $Q^d < Q^b$. We now show that this is not the case. To see that, suppose that all firms other than i set $a_{-i} = A(Q)$ and consider what the best response of i is. As we have seen in the proof of Lemma 4, the ask price elasticity of output demand is smaller than minus one for $Q^b \leq \frac{1}{2}$. Therefore increasing the ask price by one percent will result in a decrease of quantity demanded by more than one percent. Therefore, increasing a_i will not pay for i. Note that this is the case regardless of whether a proportional or an efficient rationing rule applies.[11] It is also quite intuitive to see that this equilibrium is unique. Suppose that one firm j sets the ask price $a_j > a^*$, where a^* is clearing price, i.e. $a^* = A(Q)$. Then, if $Q^d(a_j^-) \gtrless q_{-j}^b$, the remaining firms' best response will be to set $a_{-j} = A(q_{-j})\left(a_{-j} = a_j^-\right)$. In either case, a_j is not optimal for j. In the former case, j sells nothing, in the latter, he could discontinuously increase his profits by underbidding the competitors' price a_j^- because he would sell (discontinuously) more while the loss due to the lower price is small. Thus, there is no other equilibrium (see also Vives, 1999, ch.5).

The fact that in the unique equilibrium of the output market subgame each intermediary sets the market clearing price a^* is very useful for us because it allows us to treat the output market subgame as a parameterized function that depends only on the (aggregate) quantity bought Q^b.

3.3 The Input Market Subgame

We now turn to the analysis of the bid price setting (or input market) subgame. We show that there is a unique Nash equilibrium in the bid price setting subgame if all intermediaries have capacities no greater than the Cournot capacities. In this equilibrium all intermediaries i play the pure strategy $b_i = b^* \equiv B(\overline{Q})$.

[11] For a brief description and discussion of these rules see e.g. Vives (1999) or the appendix in Loertscher (2003).

3.3.1 Pure Strategy Equilibrium for Cournot capacities

Given the (observed) capacity constraints for all $i = 1, .., n$ firms, on the input market each firm sets a bid price b_i with the aim of maximizing his profits. Recall that on the output market no firm will ever set a price below the one at which market clears (see Subsection 3.2). Very similarly, in the bid price setting subgame, no firm will ever want to set a bid price $b > B(\overline{Q})$. An obvious (and in fact the only)[12] candidate for a pure strategy equilibrium is the market clearing bid price $b^* = B(\overline{Q})$. To see whether $b_i = B(\overline{Q}) \equiv b^*$ for all i is indeed an equilibrium, suppose that all firms other than j set $b_{-j} = B(\overline{Q})$, and consider whether (or when) deviation from b^* pays for j. Bid prices above $B(\overline{Q})$ being strictly dominated, we only have to consider "downward" deviation. As j sets $b_j < b^*$, he faces a residual supply of $\max[S(b_j) - \overline{q}_{-j}, 0]$, which will be his quantity bought q_j^b. Note that for $q_i^b > 0$, aggregate quantity bought will just be $S(b_j)$. This is convenient, because the ask price resulting from behavior on the input market will affect the outcome on the output market. Since the unique equilibrium of the ask price setting game is to set $a_i = A(Q^b)$, the equilibrium price on the output market is a direct function of b_j. If we assume that all other agents do not change their behavior, i.e stick to $b_i = b^*$, it is a function only of j's bid price. If b_j is such that $q_j^b > 0$, then $A(Q^b) = A(S(b_j))$, while for $q_j^b = S(b_j) - \overline{q}_{-j} = 0$, $A(Q^b) = A(\overline{q}_{-j})$. But the latter case will not matter much to j, since with $q_j^b = 0$, his profits are zero independently of $A(.)$. Therefore, assuming b_j is such that $q_j^b > 0$, j's profits when deviating from b^* are given by the following equation:

$$\pi_j(b_j) = [A(S(b_j)) - b_j] \left(S(b_j) - \overline{q}_{-j}\right). \tag{30}$$

Maximizing with respect to b_j yields

$$0 = [A'(S(b_j))S'(b_j) - 1] \left(S(b_j) - \overline{q}_{-j}\right) + \left(A(S(b_j)) - b_j\right) S'(b_j). \tag{31}$$

Dividing by $S'(b_j) \neq 0$ yields

$$0 = \left[A'(S(b_j)) - \frac{1}{S'(b_j)}\right] \left(S(b_j) - \overline{q}_{-j}\right) + A(S(b_j)) - b_j. \tag{32}$$

[12] Vives (1999, p.129) shows this for the case of capacity constrained price setting under a concave demand function.

Define by $x(\overline{q}_{-j})$ the optimal quantity j buys when all other firms set b^* and have capacities $\overline{q}_{-j}$. Obviously, $x(\overline{q}_{-j}) = S(b_j) - \overline{q}_{-j}$ as given in equation (32). Noting that with $x(\overline{q}_{-j})$ thus defined, $b_j = B(x(\overline{q}_{-j}) + \overline{q}_{-j})$, $S'(b_j) = \frac{1}{B'(x(\overline{q}_{-j})+\overline{q}_{-j})}$ and making the appropriate substitutions, we can write (32) as

$$\begin{aligned} 0 \;=\; & \left[A'(x(\overline{q}_{-j}) + \overline{q}_{-j}) - B'(x(\overline{q}_{-j}) + \overline{q}_{-j})\right] x(\overline{q}_{-j}) \\ & + A(x(\overline{q}_{-j}) + \overline{q}_{-j}) - B(x(\overline{q}_{-j}) + \overline{q}_{-j}). \end{aligned} \tag{33}$$

As noted above, the spread, defined as $Z(y) \equiv A(y) - B(y)$, is a decreasing, (weakly) concave function in y. Therefore we can rewrite the above equation to get

$$0 = Z'(x(\overline{q}_{-j}) + \overline{q}_{-j})x(\overline{q}_{-j}) + Z(x(\overline{q}_{-j}) + \overline{q}_{-j}), \tag{34}$$

from where it becomes clear that $x(\overline{q}_{-j})$ is j's Cournot best response function (with no production cost), i.e. $x(\overline{q}_{-j}) \equiv r(q_{-j})$ because $x(.)$ in (34) is defined by exactly the same condition as $r(q_{-j})$ in equation (24) above. That is, $x(\overline{q}_{-j})$ is the best response function if both the input and the output market were organized in Cournot-Walras manner. Put differently, if intermediaries brought binding pledges how much they are willing to buy (and subsequently to sell) to the Walrasian auctioneer and the auctioneer then set market clearing prices, intermediaries' best responses were given by the function $x(.)$ defined above. Therefore, from now on we write $r(.)$ for the Cournot reaction function with zero production cost on the spread $Z(.)$.

Let us now come back and finally answer the question under what conditions it pays firm j to underbid if all other firms set b^*. Then, whenever $B(r(\overline{q}_{-j})+\overline{q}_{-j}) \geq b^* = B(\overline{Q})$, j's best response is to set $b_j = b^*$ since bid prices above b^* are strictly dominated: If firm j could, it would buy $r(\overline{q}_j)$, but because this is more than $\overline{q}_j$, it cannot buy that much. Therefore, it does not pay for j to set a price higher than b^*. Clearly, we therefore have an equilibrium where all firms set b^* if for all firms i, $\overline{q}_i \leq r(\overline{q}_{-i})$.[13] The argument needed to establish uniqueness is analogous to the one of the output market subgame. Bid prices above $b^* = B(\overline{Q})$ being strictly dominated, the only alternative candidates for an equilibrium are bid prices smaller than b^*. However, whenever a firm i sets a bid price $b_i < b^*$, at

[13]This condition is exactly the same that has to hold in Kreps and Scheinkman (1983) for there to be pure strategy equilibrium in region I.

least one other firm, say, j will optimally set a price above b_i (but below b^*) so that firm i's profits would discontinuously increase by setting a slightly higher price than j does. Thus, there is no other equilibrium.

Let $\overline{q}^C \equiv q^C$ denote the Cournot capacity as defined in (28). Then we can neatly summarize our findings as follows:

Proposition 2 *For capacities $\overline{q}_i \leq \overline{q}^C$ for all $i = 1, .., n$, there is a unique Nash equilibrium in the input market subgame, in which all intermediaries set the market clearing bid price $b^* = B(\overline{Q})$.*

Proof: The proof follows directly from the above analysis.■

Equilibrium for the full game A question of great interest is of course whether setting Cournot capacities is an equilibrium (or more precisely, part of a subgame perfect equilibrium strategy profile) for the full game. In order to show this we must investigate whether firm i has an incentive to deviate from setting the capacity constraint $\overline{q}^C$ if all $-i$ set $\overline{q}_{-i} = \overline{q}^C$. It is clear from the results of Cournot competition that deviation to $\overline{q}_i < \overline{q}^C$ will not pay. Why? Recall from the two previous subsections that in this case, equilibrium prices both on the input and on the output market will be market clearing. Since this is the situation prevailing under Cournot competition, the deviation $\overline{q}_i < \overline{q}^C$ will not be profitable. Thus the only candidate deviation we have to consider entail $\overline{q}_i > \overline{q}^C$. In this case, the equilibrium in the bid price setting subgame involves (non-degenerate) mixed strategies. However, as it turns out it is very hard to characterize the equilibrium revenue (and more so to characterize the equilibrium strategies) for a deviating firm, and thus far we have not been able to pin down this revenue (see also Appendix A).

4 Conclusions

In this paper, we have introduced capacity constrained competition between market-making and price setting intermediaries. Capacity constraints prevent competition between price setters to degenerate into Bertrand-style perfect competition. We have shown that intermediaries endowed with Cournot capacities (or with smaller than Cournot capacities) set market clearing bid and ask prices on the input and output market. Therefore, given Cournot capacities, firms set the same prices and trade the same quantities on the subgame perfect equilibrium path of our game as would be set and traded if input and output market were organized by a Walrasian auctioneer. A corollary of this is that the equilibrium outcome of our model coincides with the Walrasian perfect competition outcome when the number of intermediaries with Cournot capacities becomes large.

The fact that the search market only shuts down completely if the number of intermediaries approaches infinity and if there is no cost associated with intermediating may seem somewhat odd. Assuming that these two conditions hold is certainly not less demanding than the the assumptions underlying the Walrasian model. However, this problem can be easily mended by introducing a fix cost to search market participation. Then, the search market shuts down for a quantity traded smaller than the Walrasian one, and firms can make positive profits (or at least set positive ask-bid spreads) in the absence of an active search market.

The paper can also be seen as an attempt to analyze competition between exchange mechanism that differ with respect to their efficiency. In a next step, we consider introducing a minimal size for the search market to be operational.

[To be completed]

Appendix

A Region of Mixed Strategy Equilibria

We first show that for capacities $\overline{q}_i > \overline{q}^C$ and $\overline{q}_{-i} = \overline{q}^C$, there is no equilibrium in pure strategies. The reason is as follows. Recall from subsection 3.3 that given $b_{-i} = b^* = B(\overline{Q})$, intermediary i's best response is to set $b_i = B(r(\overline{q}_{-i}) + \overline{q}_{-i}) < B(\overline{Q})$. Note that $B(r(\overline{q}_{-i}) + \overline{q}_{-i}) > B(\overline{q}_{-i})$ because for $\overline{q}_{-i} = (n-1)\overline{q}^C$, $r(\overline{q}_{-i}) > 0$ and denote the bid price $B(r(\overline{q}_{-i}) + \overline{q}_{-i})$ as $\underline{b}$. Now, given that i sets $\underline{b}$, setting $b_{-i} = b^*$ is not a best response for the other firms since each of them can buy the same quantity $q^b = \overline{q}^C$ by setting a lower price. Because $\underline{b} > B(\overline{q}_{-i})$, the lowest bid price at which this is possible is $\underline{b}^+$ for $b_i = \underline{b}$. But with $b_{-i} = \underline{b}^+$, i's profits increase if he sets a bid price slightly larger than $\underline{b}^+$ since by doing so he can buy a discontinuously larger quantity while the loss due to the higher price is negligible. Because this type of reasoning applies for *any* constellation of bid prices, there is no equilibrium in pure strategies for capacities $\overline{q}_i > \overline{q}^C$ and $\overline{q}_{-i} = \overline{q}^C$.

This raises the question whether there is an equilibrium in mixed strategies. Since there are no equilibria in pure strategies and because strategies are continuous while firms' payoffs are discontinuous, it is not a priori clear that the game has an equilibrium.[14] Speaking somewhat loosely, we may say that Dasgupta and Maskin (1986) (DM hereafter) show that sufficient conditions for the existence of a mixed strategy equilibrium in discontinuous games are that

1. discontinuities arise only at particular strategy combinations (e.g. in the Bertrand model when both firms set the same price)
2. the sum of payoff functions is upper semi-continuous (which is the case e.g. in the Bertrand model, where in the absence of production costs aggregate profits are $p_i D(p_i)$ no matter what price firm j sets, provided only $p_j \geq p_i$)
3. individual payoff function are bounded and weakly lower semi-continuous (as defined by Dasgupta and Maskin (1986, p.7)

It can be shown that these conditions hold in the present model. However, since we wish to compute expected profits in the region with mixed strategy equilibria, it will not do to know that an equilibrium exists. Rather, we would have to determine the equilibrium strategies, or more precisely, the expected equilibrium revenues in this region (which is in principle possible without an explicit characterization of the equilibrium strategies).

A.1 Existence of an Equilibrium

In order to show that the game we consider has an equilibrium, we must show that the conditions of Theorem 5 of DM are satisfied.

[14] Nash (1950, 1951)'s proof guaranteed the existence of a (mixed strategy) equilibrium for finite games, that is for games with a finite number of strategies for each player and finite number of players. Debreu (1952), Fan (1952) and Glicksberg (1952) then proved the existence of an equilibrium for a wider class of games, and some authors, notably the above-mentioned Levitan and Shubik (1972), proved the existence of an equilibrium for particular games. See Dasgupta and Maskin (1986) or Fudenberg and Tirole (1991) for more details. However, general and sufficient conditions for the existence of an equilibrium in discontinuous games had to await the seminal contribution by Dasgupta and Maskin (1986).

We prove existence only for $n = 2$ firms. The proof for any n goes along the same line, but necessitates a some additional notation.

Recall that a pure strategy of player i is choice of $b_i \in [0, \infty)$, so that for $n = 2$, the strategy space is $[0, \infty) \times [0, \infty) \subset R^2$. Translated to our setting, Theorem 5 of DM states that sufficient conditions for our game to have an equilibrium are

1. $\pi_i(b_1, b_2)$ is discontinuous in b_i for $i = 1, 2$ only on a subset of $\{b_1, b_2 \mid b_1 = b_2\}$
2. $\sum_i \pi_i(b_1, b_2)$ is upper semi-continuous (u.s.c.) and $\pi_i(b_1, b_2)$ is bounded and weakly lower semi-continuous (w.l.s.c.).

Condition (1) is easily seen to hold. Consider profits of firm 1. Then, for $b_1 < \min[B(\overline{q}_2), b_2]$, $\pi_1(b_1, b_2) = 0$, which is continuous. For $b_1 \in [B(\overline{q}_2), b_2)$,$\pi_1(b_1, b_2) = (A(b_1)-b_1)(S(b_1)-\overline{q}_2)$, which is continuous because both $A(b_1)-b_1$ and $S(b_1)-\overline{q}_2$ are continuous functions. Finally, for $b_1 > b_2$, profits of firm 1 are $\pi_1(b_1, b_2) = \big(A(\min[b_2, B(\overline{Q})] - b_1)\big) \min[\overline{q}_1, S(b_1)]$, which is a continuous function in b_1.

It takes a bit more to establish that condition (2) is met. However, a sufficient condition for upper semi-continuity of the sum of profits is that the sum of profits is continuous, which is quite easily seen. Since discontinuities of individual profits occur only if both firms set the same bid price, we have to investigate the sum of profits only at points where $b_1 = b_2 = b$.

For simplicity, consider first the case, where b and capacities are such that $\min_i[\overline{q}_i] \geq \frac{S(b)}{2}$ and that $\max_i[\overline{q}_i] < S(b)$. Then, for $b_1 = b$, profits of 1 and 2 are $(A(b) - b)\frac{S(b)}{2}$, so that the sum of profits is $(A(b) - b)S(b)$. Now assume that firm 1 sets $b_1 > b_2$. Then $\pi_1(b_1, b) = (A(b) - b_1)\overline{q}_1$ and $\pi_2(b_1, b) = (A(b) - b)(S(b) - \overline{q}_1)$, so that the sum of profits is $(A(b) - b)S(b) - (b_1 - b)\overline{q}_1$. As b_1 approaches b, this is $(A(b) - b)S(b)$, which is continuous.

Let us now turn to the other cases, where b and $\overline{q}_i$ are not such that $\min_i[\overline{q}_i] \geq \frac{S(b)}{2}$ and/or $\max[\overline{q}_i] < S(b)$.[15] Assume without loss of generality that $\overline{q}_1 \geq \overline{q}_2$. If $\min[\overline{q}_1, 2\overline{q}_2] > S(b)$, then $\sum_i \pi_i(b, b) = (A(b) - b)S(b)$ as above. But $\overline{q}_1 > S(b)$ implies $\pi_2(b_1, b) = 0$ for b_1 only slightly larger than b, so that $\sum_i \pi_i(b_1, b) = \pi_1(b_1, b) = (A(b_1) - b_1)S(b_1)$. Again, therefore, $\lim_{b_1 \to b} \sum_i \pi_i(b_1, b) = (A(b) - b)S(b) = \sum_i \pi_i(b, b)$. Thus, we are left with the case, where $\overline{q}_2 < \frac{S(b)}{2}$. There are two possibilities in this case. Either (a) $\overline{Q} \leq S(b)$ or (b) $\overline{Q} > S(b)$. If (a) is

[15] But for this to be a mixed strategy equilibrium, $\min[\overline{q}_i] < Q^W$ must still hold, of course.

the case, then quantity bought will be $\overline{Q}$ at b and b_1. Thus, inverse demand will be $A(B(\overline{Q}))$ regardless of whether 1 sets b or b_1, and aggregate profits will be $(A(B(\overline{Q}))-b)\overline{Q}$ if both firms set b. For $b_1 > b$, aggregate profits are $(A(B(\overline{Q}))-b_1)\overline{q}_1 + (A(B(\overline{Q}))-b)\overline{q}_2$, so that, again, $\lim_{b_1\to b}\sum_i \pi_i(b_1,b) = \sum_i \pi_i(b,b)$. In case (b), $\sum_i \pi_i(b,b) = (A(b)-b)S(b)$ and $\sum_i \pi_i(b_1,b) = (A(b)-b_1)\overline{q}_1 + (A(b)-b)(S(b)-\overline{q}_1)$, implying as before $\lim_{b_1\to b}\sum_i \pi_i(b_1,b) = \sum_i \pi_i(b,b)$.

To see that $\pi_i(b_1,b_2)$ is bounded, let $\hat{b}$ denote the highest admissible price. Then $\pi_1(b_1,b_2) \geq (A(B(\overline{Q}))-\hat{b})\overline{q}_1$ and $\pi_1(b_1,b_2) \leq (A(B(r(0)))-B(r(0)))r(0)$. Similarly, $\pi_2(b_1,b_2) \geq (A(B(\overline{Q}))-\hat{b})\overline{q}_2$ and $\pi_2(b_1,b_2) \leq (A(B(r(0)))-B(r(0)))r(0)$. Thus, $\pi_i(.)$is bounded for $i = 1,2$. Finally, we turn to weakly lower semi-continuity. In order to do so, we first apply Definition 6 of DM to our problem.

Definition 1 (DM Definition 6) $\pi_1(.)$ *is called weakly lower semi-continuous (w.l.s.c.) in* b_1 *if for all* b_1', b_2' *for which* $\pi_1(.)$ *is discontinuous, there is a* $\lambda \in [0,1]$ *such that*

$$\lambda \lim \inf_{b_1 \xrightarrow{-} b_1'} \pi_1(b_1,b_2') + (1-\lambda)\lim \inf_{b_1 \xrightarrow{+} b_1'} \pi_1(b_1,b_2') \geq \pi_1(b_1',b_2') \text{ [16]}$$

For $\pi_2(.)$, *w.l.s.c. is defined in complete analogy.*

To see that both functions satisfy w.l.s.c., note that whenever $\pi_i(.)$ is discontinuous, i's profits strictly increase by either underbidding or overbidding the other player's bid price. Accordingly, let λ be one in the former and zero in the latter case, and w.l.s.c. is established.

Therefore, Theorem 5 of DM applies, from which we conclude that our game has a (mixed strategy) equilibrium.

A.2 Expected revenue in the mixed strategy equilibrium

Recall (once more) from above that i's best response when the other firms sets $b_{-i} = B(\overline{Q})$ is to set $\underline{b} \equiv B(r(\overline{q}_{-i}) + \overline{q}_{-i})$. Interestingly, setting $b_j = B(r(\overline{q}_{-j}) + \overline{q}_{-j})$ is the optimal bid price for *any* intermediary j who is certain that he sets the lowest bid price, $j = 1,...,n$. That is, if $b_{-j} > b_j$ for all $-j$, then j's best response is to set $B(r(\overline{q}_{-j}) + \overline{q}_{-j})$. To see this, note that when j sets the lowest bid price, aggregate quantity bought will be $S(b_j)$.[17] Because everything that is bought on

[16] As above, $\xrightarrow{-}$ ($\xrightarrow{+}$) means that b_1 approaches b_1' from the left (right).
[17] This is of course true only under the condition that b_j is such j can buy something.

the input market will subsequently be sold on the output market at the market clearing ask price (see section 3.2) the equilibrium ask price intermediary j (and any other intermediary) will get is a function of $S(b_j)$ and thus a function of b_j only. Therefore, j's expected profits in this case are

$$\pi_j(b_j \mid b_j > b_j, \overline{\mathbf{q}}) = (A(S(b_j)) - b_j)\left(S(b_j) - \overline{q}_{-j}\right), \tag{35}$$

where $\left(S(b_j) - \overline{q}_{-j}\right) = q_j^b$. From section 3.3 we know that the solution to this maximization problem is to choose b_j such that $q_j^b = r(\overline{q}_{-j})$, implying that the optimal bid price b_j is equal to $B(r(\overline{q}_{-j}) + \overline{q}_{-j})$. Let us denote this price by $\underline{b}_j$. Because in a mixed strategy equilibrium, agents are indifferent between the pure strategies over which they randomize this firm's expected equilibrium revenue is the revenue accruing when setting this bid price, which is the Stackelberg follower revenue.

The main problem for determining the expected equilibrium revenue of the deviating firm is that it is not easily possible to say whih firm sets $\underline{b}_j$. The reason for this that it is hard to determine the upper bound of prices over which firms randomize in the mixed strategy equilibrium. This is in contrast to the procedure applied by Kreps and Scheinkman (1983) and Deneckere and Kovenock (1996), where both bounds of the support are quite "easy" to identify and where these bounds are used to determine which firm(s) get(s) the Stackelberg follower revenue.

B The Cournot Model

What we call the Cournot model (or Cournot auctioneer model)[18] is the following. There are $n \geq 1$ producers of a homogenous good who seek to maximize their own profits. They know each others' cost function and the downward sloping demand function $P(.)$.[19] These n firms produce simultaneously before bringing their produce to the common market place, where they give it into the hands of a benevolent agent, the so called (Walrasian) auctioneer. The task of this auctioneer consists of organizing the market, that is, he collects the quantities

[18] Of course, the label "Cournot auctioneer" model is ahistorical since the auctioneer due to Walras. Nonetheless, we use this label because we think it is an accurate description.

[19] For technical reasons, and depending on the firms' cost functions, a negative slope is not a sufficient condition, so the function is also often required not to be too convex.

produced by all the n firms and then sets the price at which the market clears. After the market has cleared, it is shut down and further trade is made impossible. This last assumption is often not made explicitly, but it is a necessary one because in Cournot equilibrium with a finite number of firms, the market clears at a price above marginal costs. Since there remain buyers willing to pay a price above marginal costs after market clearing has taken place, firms have incentives to serve these buyers and to increase their profits.

At least some of the assumptions underlying the Cournot auctioneer model are overtly fictitious and have been heavily criticized. Most prominent is the classical criticism by Bertrand (1883) who pointed out that as a matter of fact, firms actually do set prices and not merely quantities. Nonetheless, the Cournot model is intellectually appealing, though not because of its assumptions but for the predictions it makes. This appeal is neatly formulated in the following quote, taken from Mas-Collel et al. (1995, p.394):

> ...[M]any economists have thought that the Cournot model gives the right answer for the wrong reasons.

In our judgement, what makes it particularly valuable is that it contains the whole range of market outcomes. Basically, as the number of firms is one, the market price (and profits) are high. When this number increases, price and profits fall, until in the limit profits are zero and the market outcome is that of perfect competition. The appeal of the Cournot model is not least witnessed in the large and still growing literature (spanning now over more than a century), which makes an effort to find the right reasons for the right predictions, if we want to paraphrase Mas-Collel et al. (1995). Undoubtedly, the most spectacular and influential contribution into this direction has been made by KS, with which we deal extensively below.[20] After this brief introduction to the Cournot (auctioneer) model, we discuss next the basic concepts of this model.

B.1 The reaction function

The most important concept is the (Cournot) reaction or best response function.

[20] A more detailed and accurate description would at this point also refer to the classical treatment of the problem by Edgeworth (1897) and to Levitan and Shubik (1972). Dealing with a simple example with linear demand, the latter authors derived the pure and the mixed strategy equilibria when firms set prices, given capacity constraints.

There are $n \geq 1$ firms indexed as $i = 1, .., n$. We denote by q_i the quantity produced by firm i and by $Q = \sum_j^n q_j$ the aggregate quantity produced. Finally, let q_{-i} denote the aggregate quantity produced by all firms other than i, i.e. $q_{-i} = Q - q_i$. Note that $Q = q_i + q_{-i}$ by definition and that $dq_{-i}/dq_i = -1$.

Under Cournot competition, profits of firm i when the other firm sells quantity q_j and when i's cost of production are zero are $q_i P(Q) = q_i P(q_i + q_{-i})$. The (zero production cost) reaction function $r(q)$ when all other firms produce q is defined

$$r(q) \in \arg\max rP(r+q),$$

implying

$$0 = rP'(r+q) + P(r+q), \tag{36}$$

so that $r(q)$ is implicitly given by

$$r(q) = \frac{P(r(q)+q)}{-P'(r(q)+q)}. \tag{37}$$

Note that because $D(p)$ and $P(Q)$ are (weakly) concave, the second derivative of $rP(r+q)$ with respect to r is always negative (i.e. the function $r(Pr+q)$ is concave, too). Therefore, the set of $\arg\max rP(r+q)$ contains a single element, and thus the solution $r(q)$ in equation (38) is unique.

It turns out to be very useful to know two properties about the slope of $r(q)$. Therefore, let us differentiate equation (36) with respect to q, set this equal to zero and solve for $r'(q)$ to get[21]

$$r' = \frac{-rP'' - P'}{rP'' + 2P'}, \tag{38}$$

where for notational simplicity and because it does not give way to confusion, we have dropped the arguments.

First, we note that $r' < 0$, which follows from the fact that $P(.)$ is (weakly) concave, and therefore the denominator is negative and the nominator positive. Second, $r' > -1$. To see this, multiply $\frac{-rP''-P'}{rP''+2P'} > -1$ by the denominator and cancel terms to get $1 < 2$ (keep in mind that the denominator is negative, so that the sign changes after the multiplication). The fact that $r' > -1$ implies that $r(q) + q$ strictly increases in q, which is a result that will repeatedly be used below.

[21]The idea behind this is that equation (36) holds for any q. Therefore, as q changes, $r(q)$ must change in such a way that equality (36) still holds.

It is also useful to have an expression for the revenue of firm i with zero production costs as a function of the quantity of all firms other than i when i uses its best response $r(q_{-i})$. This revenue is $R(q_{-i}) := r(q_{-i})P(r(q_{-i}) + q_{-i})$. Note that this is firm i's maximal revenue as a function of the quantity set by all other firms. If for example $n = 2$, then firm 1's maximal revenue is $R(q_{-1}) := r(q_{-1})P(r(q_{-1}) + q_{-1}) = r(q_2)P(r(q_2) + q_2)$.

Cournot-Nash Equilibrium An equilibrium is thus defined as a quantity q^* such that $r(q^*) - q^* = 0$.

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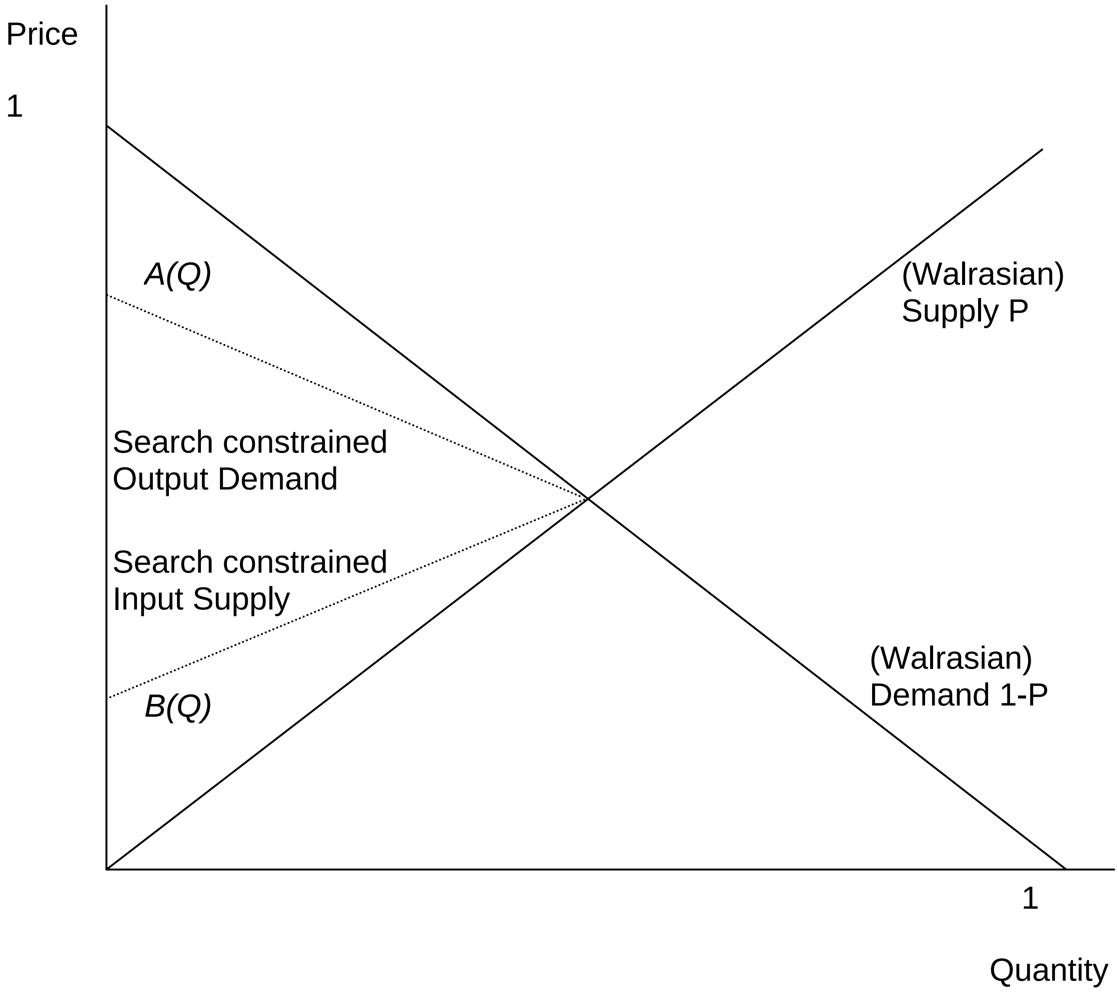

Figure 1:
The search market constrained supply and demand functions. Note that trade is assumed to be balanced.

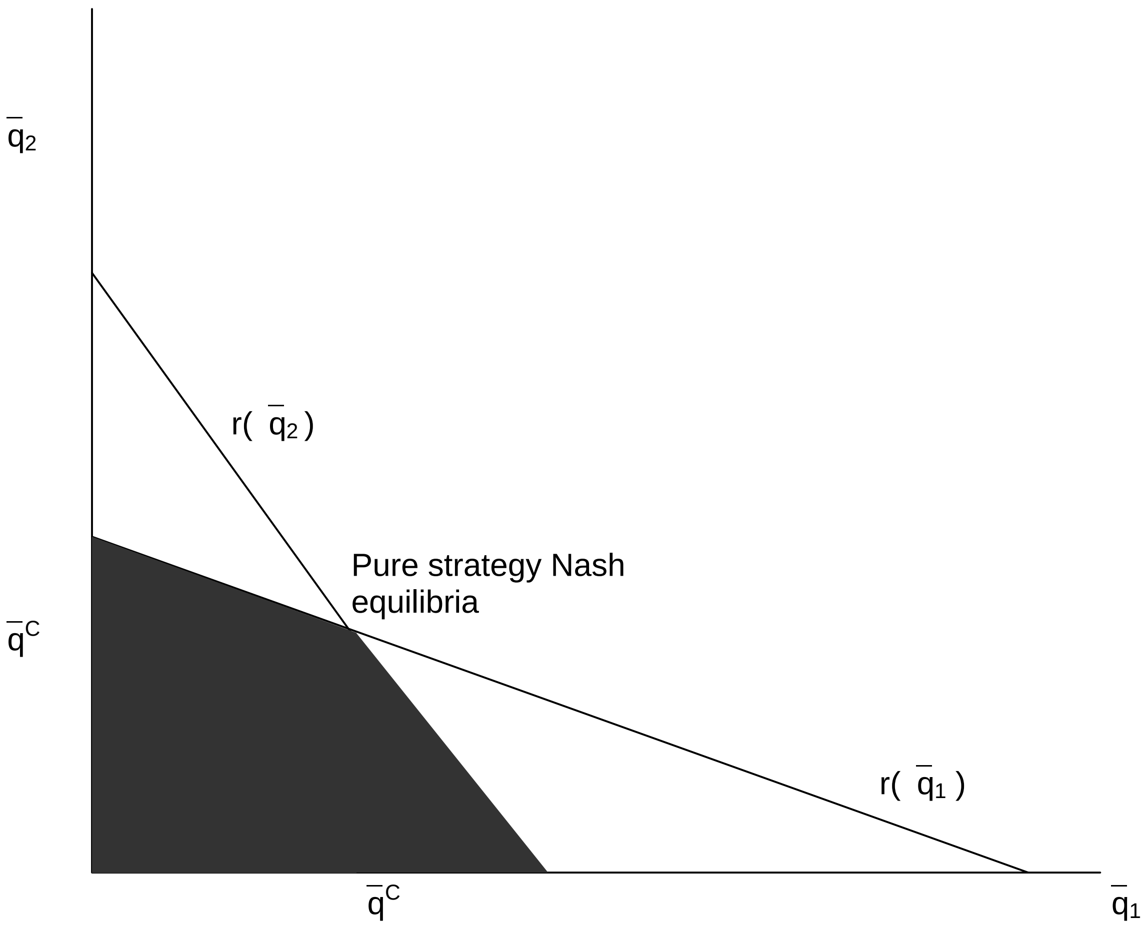

Figure 2: Region of pure strategy Nash equilibria for *n*=2.