Players with Fixed Resources in Elimination Tournaments

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Abstract. We consider two-round elimination tournaments where players have fixed resources instead of cost functions. Two approaches are suggested. If the players have the same resources and a success function is stochastic, then players *always* spend more resources in the first than in the second round in a symmetric equilibrium. Equal resource allocation between two rounds takes place only in the winner-take-all case. However, if the players have independent private resources and the success function is deterministic, then every player spends at least one third of his resources in the first round. The players spend exactly one third of their resources in the winner-take-all case. Applications for career paths, elections, and sports are discussed.

1. Introduction

The aim of this paper is to analyze two-round elimination tournaments in which players have fixed resources instead of cost functions. In the first round players are matched in two pairs for contests and the winner of each contest proceeds to the next round. All losers receive the prize, W_0, and are eliminated from the tournament. There is a trade off here. On the one hand, the more resources a player spends in the first round, the higher her chance to win the current contest and play in the final. On the other hand, the more resources a player spends in the first round, the less her chance to win the final. Each player has to allocate optimally her overall resources between two rounds. This strategic problem is different from the problem analyzed in the contest literature, where players must decide how much effort to spend to win the prize(s) in one contest; see, for example, Dixit (1987, 1999), Baik and Shogren (1992), Baye and Shin (1999), and Moldovanu and Sela (2001).

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Several real-world phenomena have the structure of such tournaments. A participant in a two-stage election campaign, who has a fixed total budget, plays an elimination tournament with fixed resources. A new worker (an assistant professor) at the beginning of his/her career, when he/she has a limited number of working hours for the whole career, has to distribute optimally those hours. Needless to say, that many sportsmen, for example, tennis and soccer players, have to allocate optimally his/her overall energy across all rounds in elimination tournaments, or chess players have to decide in which rounds he/she should use his/her novelties in elimination FIDE World Championship.

The tournament literature has focused theoretically, see, for example, Lazear and Rosen (1981), Rosen (1981, 1986), and empirically, Ehrenberg and Bognanno (1990), Knoeber and Thurman (1994), on players' incentives in tournaments, when players have some costs for exerting effort. Classical papers Lazear and Rosen (1981) and Rosen (1981) analyze a stochastic success functions and show that high differences in prizes in the last round(s) must provide enough incentives for players to insert the same effort in all rounds. We show that the equal resource allocation between two rounds takes place only in the winner-take-all case, if the success function is stochastic. Moreover, in the symmetric equilibrium every player will spend more resources in the first round that in the final round, if the prize scheme is different from the winner-take-all. The intuition is straightforward: if a player keeps her resources until the last round to get higher prizes, then she will be eliminated in the first round.

However, if the success function is deterministic - if the player spends more in the current round than her opponent, then she wins the current round for sure, then the prediction is different. Every player spends at least one third of her resources in the first round, but she can spend more resources in the final now. The intuition is that the players are going to face a strong opponent in the final now, because the opponent is the best in the other pair of the players from the first round. Moreover, in the symmetric monotone equilibrium every player will spend exactly one third of her resources in the first round, if the prize scheme is the winner-take-all. It contrasts with Krishna and Morgan (1998) and Moldovanu and Sela (2001), where the authors show that the winner-take-all prize scheme is often the optimal for the principal, who wants to maximize joint effort of the players.

The rest of this paper is organized as follows. We consider the stochastic model in Section 2 and the deterministic model in Section 3. Section 4 provides a discussion.

2. Stochastic Model: The Same Resources

We begin the formal analysis by considering equilibrium behavior in two-round, four risk-neutral players, elimination tournaments. In round 1 all players are matched in pairs for fights/contests, where only two winners of the first round continue to fight for higher payoffs in the final. All losers get payoff W_0, each of them wins 0 contest, and are eliminated from the tournament. In the final, round 2, the winners of the first round, 2 players, are matched in pair for new fight/contest. The winner of the final gets payoff W_2 and the loser receives payoff W_1. We make the standard assumption that prizes increase from round to round

$$A1.\ 0 \leq W_0 \leq W_1 < W_2. \tag{1}$$

Each player i has an initial fixed resource E, and must decide how to allocate this resource between two rounds. Denote the spent part of player i's resource in the first round by x_1^i and in the second round by x_2^i. If player i chooses to use a part $x_k^i \in [0, E]$ of her resource in k round, $k = 1, 2$, when her opponent in k round, player j, spends a part $x_k^j \in [0, E]$, then player i wins this fight with probability

$$\frac{g\left(x_k^i\right)}{g\left(x_k^i\right) + g\left(x_k^j\right)}, \tag{2}$$

where $g\left(x\right)$ is a positive, twice differentiable, and increasing function:

$$A2.\ g\left(x\right) > 0,\ g'\left(x\right) > 0,\ \text{and } g''\left(x\right) \leq 0 \text{ on the interval } \left[0, E\right]. \tag{3}$$

We will call the function (2) *success* function. A pure strategy for player i is a rule (x_1^i, x_2^i), which assigns a part of her resources for every round in the tournament, such that $x_1^i + x_2^i = E$, $x_k^i \geq 0$ for any $i \in \{1, ..., 4\}$ and $k \in \{1, 2\}$.[1]

We will call the following prize structure

$$0 \leq W_0 = W_1 < W_2$$

winner-take-all. The main results of the Stochastic Model can be stated now.

2.1. Existence of a symmetric equilibrium. We show first that there exists a symmetric equilibrium in pure strategies. The properties of the symmetric equilibrium are analyzed after that.

Proposition 1. *Suppose that assumptions* (1) *and* (3) *hold. Then there exists a symmetric equilibrium in pure strategies.*

[1]Note that the strategy of player i is completely determined by her choice in the first round, but for the convenience of exposition we will write the strategy of player i as her choices in two rounds.

Proof: Note first that if every player but player 1 plays strategy (y_1, y_2), then player 1 faces one and the same opponent in every round. Given the opponents' resource allocation (y_1, y_2), the player 1's resource allocation decision x_1 in the 1-st round of the tournament is determined by the solution of:

$$\max_{x_1 \in [0,E]} G(x_1, y_1), \tag{4}$$

where the payoff function

$$G(x_1, y_1) = \frac{g(y_1)}{g(x_1) + g(y_1)} W_0 +$$

$$+ \frac{g(x_1)}{g(x_1) + g(y_1)} \left(\frac{g(E - x_1)}{g(E - x_1) + g(E - y_1)} W_2 + \frac{g(E - y_1)}{g(E - x_1) + g(E - y_1)} W_1 \right)$$

or

$$G(x_1, y_1) = \frac{g(x_1)}{g(x_1) + g(y_1)} \left(\frac{g(E - x_1)}{g(E - x_1) + g(E - y_1)} [W_2 - W_1] + [W_1 - W_0] \right) + W_0.$$

Note that

$$\frac{\partial G(x_1, y_1)}{\partial x_1} = \frac{g'(x_1)\, g(y_1)}{[g(x_1) + g(y_1)]^2} \left(\frac{g(E - x_1)}{g(E - x_1) + g(E - y_1)} [W_2 - W_1] + [W_1 - W_0] \right) +$$

$$\frac{g(x_1)}{g(x_1) + g(y_1)} \times \frac{-g'(E - x_1)\, g(E - y_1)}{[g(E - x_1) + g(E - y_1)]^2} [W_2 - W_1]$$

and

$$\frac{\partial^2 G(x_1, y_1)}{\partial x_1^2} = -\frac{2g'(x_1)\, g(y_1)}{[g(x_1) + g(y_1)]^2} \times \frac{g'(E - x_1)\, g(E - y_1)}{[g(E - x_1) + g(E - y_1)]^2} [W_2 - W_1] +$$

$$\frac{g''(x_1) [g(x_1) + g(y_1)]^2 - 2 [g'(x_1)]^2 [g(x_1) + g(y_1)]}{[g(x_1) + g(y_1)]^4} \times$$

$$\left(\frac{g(E - x_1)}{g(E - x_1) + g(E - y_1)} [W_2 - W_1] + [W_1 - W_0] \right) g(y_1) +$$

$$\frac{g''(E - x_1) [g(E - x_1) + g(E - y_1)]^2 - 2 [g'(E - x_1)]^2 [g(E - x_1) + g(E - y_1)]}{[g(E - x_1) + g(E - y_1)]^4} \times$$

$$\frac{g(x_1)\, g(E - y_1)}{g(x_1) + g(y_1)} [W_2 - W_1].$$

Since $g'' \leq 0$, by assumption (3), then $\frac{\partial^2 G(x_1, y_1)}{\partial x_1^2} \leq 0$ and the continuous payoff function G is quasi-concave in x_1. It means that we can apply Kakutani's fixed-point theorem to the player 1 best reply correspondence. The fixed point is a symmetric equilibrium in pure strategies. **End of proof.**

2.2. Properties of the symmetric equilibrium. Since Proposition 1 establishes the existence of a symmetric equilibrium in pure strategies, we can analyze properties of this equilibrium.

Proposition 2. *Suppose that assumptions* (1) *and* (3) *hold. Then, in symmetric equilibrium,* (x_1^*, x_2^*), *it must be* $x_1^* \geq x_2^*$, *for any prize structure* (W_0, W_1, W_2). *Equal resource allocation between two rounds,* $x_1^* = x_2^*$, *takes place only in the winner-take-all case.*

Proof: Suppose that all players, but player 1, allocate resource E in the same way (y_1, y_2). Given the opponents' resource allocation (y_1, y_2), the player 1's resource allocation decision x_1 in the $1-st$ round of the tournament is determined by the solution of (4). The first order condition for the problem (4) is

$$\frac{\partial G(x_1, y_1)}{\partial x_1} = \frac{g'(x_1)\, g(y_1)}{[g(x_1) + g(y_1)]^2} \left(\frac{g(E - x_1)}{g(E - x_1) + g(E - y_1)} [W_2 - W_1] + [W_1 - W_0] \right) -$$

$$- \frac{g(x_1)}{g(x_1) + g(y_1)} \frac{g'(E - x_1)\, g(E - y_1)}{[g(E - x_1) + g(E - y_1)]^2} [W_2 - W_1] = 0$$

In symmetric equilibrium, $x_2 = y_2 = x_2^*$, $x_1 = y_1 = x_1^*$ and we have

$$\frac{g'(x_1^*)}{g(x_1^*)} \left([W_2 - W_1] + 2[W_1 - W_0]\right) = \frac{g'(E - x_1^*)}{g(E - x_1^*)} [W_2 - W_1]. \tag{5}$$

Assumption (3) guarantees that the left-hand side (LHS) in equation (5) is a strictly decreasing function of x_1^* on the interval $[0, E]$, and the right-hand side (RHS) in the same equation is a strictly increasing function of x_1^* on the interval $[0, E]$. It follows from the fact that g'/g is a strictly decreasing function since $g''g - [g']^2 < 0$, which is a corollary of the assumption (3).

The existence of a symmetric equilibrium in pure strategies follows from Proposition 1. Hence, equation (5) either has no solution and $x_1^* = E$ is a unique pure strategy symmetric equilibrium or has a unique solution x_1^* inside of the interval $(0, E)$, since it defines the intersection of a decreasing and an increasing continuous functions. Moreover, in the last case, if $x_1^* = E/2$, then $LHS\left(\frac{E}{2}\right) \geq RHS\left(\frac{E}{2}\right)$, because of the assumption (1), with equality if and only if $W_0 = W_1$. It means that a unique solution of the equation (5) must be $x_1^* \geq E/2$. **End of proof.**

Corollary 1. *If the difference in prizes* $(W_1 - W_0)$ *is positive, then the players spend more resources in the first round.*

Example 1. Suppose that the resource is equal to ten, $E = 10$; $g(x) = 1 + x$; and the following prize structure, $W_0 = 0$, $W_1 = 0$ and $W_2 = 30$. Then, the payoff function $G(x_1, 5) = 30\frac{1+x}{1+x+6}\frac{11-x}{11-x+6}$ for value of $y_1 = 5$ looks

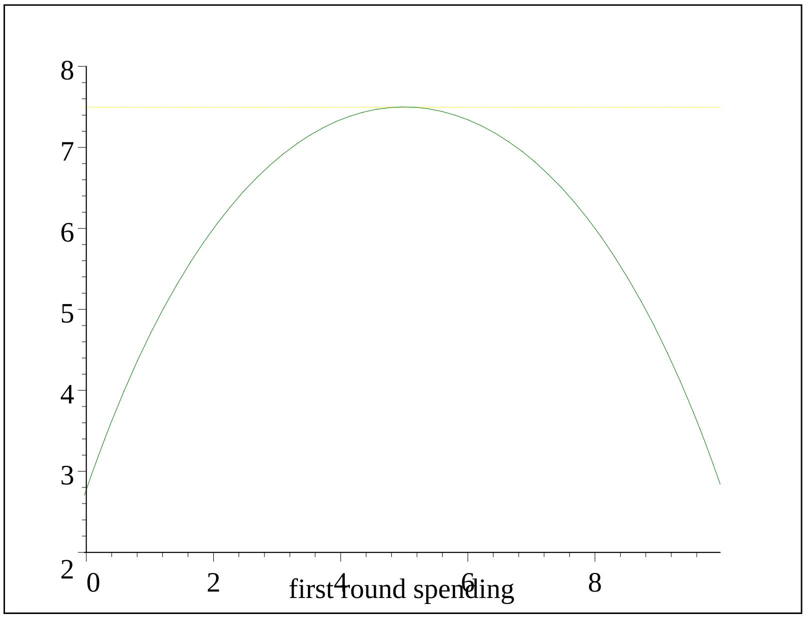

Figure 1

The optimal first round spending is $x_1 = 5$ as we know from Proposition 2. Figure 1 supports our finding.

3. Deterministic Model: Independent Private Resources

We consider elimination tournaments in an environment with independently and identically distributed private values of resources in this section. There are three prizes, $W_2 > W_1 \geq W_0 \geq 0$, in the elimination tournament, where W_2 is the prize for the winner of the tournament (she wins two rounds), W_1 is the prize for the finalist (she wins one round) and W_0 is the prize for the loser of the first round (she wins no rounds). There are four players in the elimination tournament. Each player i assigns a resource of X_i - the maximum amount a player can spend in all rounds of the elimination tournament. A player wins a round if she spends more than her opponent. If both players spend the same amount, then each of them has fifty percent chance to be the winner of the round. Each X_i is independently and identically distributed on some interval $[0, E]$, $E < \infty$, according to the increasing distribution function F. It is assumed that F admits a continuous density $f \equiv F'$ and has full support.

A player i knows the realization x_i of X_i and only that the other players' resources are independently distributed according to F. Players are risk neutral and maximize their expected profits. The distribution function F is common knowledge. A strategy for a player i is a function $\beta_i = (\beta_i^1, \beta_i^2) : [0, E] \to [0, E]^2$, which determines her spending in the first and the second rounds of the tournament. Of course,

$$\beta_i^1 + \beta_i^2 \equiv x_i. \tag{6}$$

We will be interested in the outcomes of a symmetric equilibrium - an equilibrium in which all players follow the same strategy.

We assume that the player 1 plays with the player 2 and the player 3 plays with the player 4 in the first round, each player i simultaneously spends amount β_i^1 in the first round, and given $(\beta_1, \beta_2, \beta_3, \beta_4)$, the payoff of the player 1 is

$$\pi_1 = \begin{cases} W_0 & \text{if } \beta_1^1 < \beta_2^1 \\ W_1 & \text{if } \beta_1^1 > \beta_2^1 \text{ and } \beta_1^2 < \beta_{34}^2 \\ W_2 & \text{if } \beta_1^1 > \beta_2^1 \text{ and } \beta_1^2 > \beta_{34}^2 \end{cases},$$

$$\text{where } \beta_{34}^2 = \begin{cases} \beta_3^2 & \text{if } \beta_3^1 > \beta_4^1 \\ \beta_4^2 & \text{if } \beta_3^1 < \beta_4^1 \\ 0.5\left(\beta_3^2 + \beta_4^2\right) & \text{if } \beta_3^1 = \beta_4^1 \end{cases}.$$

Every player faces a simple trade off. An increase in spending in the first round will increase the probability to win the first round and play in the final, at the same time reducing the probability to win the final. To get some idea about how these effects balance off, we begin with a derivation of symmetric equilibrium strategies.

Suppose that players $j \neq 1$ follow the symmetric, increasing and differentiable equilibrium strategy $\beta^* = \beta = (\beta^1, \beta^2)$. Suppose that the player 1 receives a resource, $X_1 = x$, and spends b^1 in the first round and b^2 in the second round. We wish to determine the optimal spending $b = (b^1, b^2)$.

First, notice that it can never be optimal to choose $b^1 > \beta^1(w)$ since in that case, player 1 would win the first round for sure and could do better by reducing her first round spending slightly so that she still wins for sure but increases her chance to win the final. So we need only to consider $b^1 \leq \beta^1(w)$. Second, a player with resource 0 must have $\beta(0) = (\beta^1(0), \beta^2(0)) = (0, 0)$.

The player 1 wins the first round whenever she spends more than the player 2 does, that is, whenever $\beta_2^1(X_2) < b^1$. The player 1 wins the tournament whenever she wins the first round, $\beta_2^1(X_2) < b^1$, and the second round, that is, whenever she spends more in the second round than the winner of the first round pair the player 3 and the player 4, that is, whenever $b^2 > \beta_{34}^2(X_3, X_4)$, where $\beta_{34}^2(X_3, X_4)$ is the second round spending of the winner of the first round pair the player 3 and the player 4. The player 1's expected payoff is therefore

$$F\left(\left[\beta^1\left(b^1\right)\right]^{-1}\right)\left[F^2\left(\left[\beta^2\left(b^2\right)\right]^{-1}\right)W_2 + \left(1 - F^2\left(\left[\beta^2\left(b^2\right)\right]^{-1}\right)\right)W_1\right] +$$

$$+\left(1-F\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)\right)W_{0}, \tag{7}$$

where F^2 is the distribution function of Y_{34}, the highest second round spending of players 3 and 4. Maximizing expression (7) with respect to b^1, given that

$$b^1+b^2\equiv x,$$

yields the first order condition:

$$\frac{f\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}{\beta^{1\prime}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}\left[F^{2}\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)W_{2}+\left(1-F^{2}\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)\right)W_{1}\right]+$$

$$F\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)\left[-2\frac{F\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)f\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)}{\beta^{2\prime}\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)}\left[W_{2}-W_{1}\right]\right]-$$

$$-\frac{f\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}{\beta^{1\prime}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}W_{0}=0. \tag{8}$$

At a symmetric equilibrium

$$b^{1}=\beta^{1}\left(x\right),\ b^{2}=\beta^{2}\left(x\right)\ \text{ and }F\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)\equiv F\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right),$$

and thus we can rewrite (8) as

$$\frac{f\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}{\beta^{1\prime}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}\left[F^{2}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)W_{2}+\left(1-F^{2}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)\right)W_{1}\right]+$$

$$F\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)\left[-2\frac{F\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)f\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}{\beta^{2\prime}\left(\left[\beta^{2}\left(x-b^{1}\right)\right]^{-1}\right)}\left[W_{2}-W_{1}\right]\right]-$$

$$-\frac{f\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}{\beta^{1\prime}\left(\left[\beta^{1}\left(b^{1}\right)\right]^{-1}\right)}W_{0}=0. \tag{9}$$

The equation (9) yields the differential equation

$$\frac{f\left(x\right)}{\beta^{1\prime}\left(x\right)}\left[F^{2}\left(x\right)W_{2}+\left(1-F^{2}\left(x\right)\right)W_{1}\right]+$$

$$F(x)\left[-2\frac{F(x)f(x)}{\beta^{2\prime}(x-\beta^1(x))}[W_2-W_1]\right]-\frac{f(x)}{\beta^{1\prime}(x)}W_0=0. \tag{10}$$

Using (6), we obtain

$$\beta^{1\prime}+\beta^{2\prime}\equiv 1. \tag{11}$$

Using (11), the differential equation (10) can be rewritten as

$$\begin{aligned}&\left(1-\beta^{1\prime}(x)\right)\left[F^2(x)W_2+\left(1-F^2(x)\right)W_1\right]=\\&=2F(x)\left[F(x)\left[W_2-W_1\right]\right]\beta^{1\prime}(x)+\left(1-\beta^{1\prime}(x)\right)W_0\end{aligned} \tag{12}$$

or equivalently,

$$\left\{3F^2(x)\left[W_2-W_1\right]+\left[W_1-W_0\right]\right\}\beta^{1\prime}(x)=F^2(x)\left[W_2-W_1\right]+\left[W_1-W_0\right]$$

and

$$\beta^{1\prime}(x)=\frac{F^2(x)\left[W_2-W_1\right]+\left[W_1-W_0\right]}{3F^2(x)\left[W_2-W_1\right]+\left[W_1-W_0\right]}.$$

Since $\beta^1(0)=0$, we have

$$\beta^1(x)=\int_0^x\frac{F^2(s)\left[W_2-W_1\right]+\left[W_1-W_0\right]}{3F^2(s)\left[W_2-W_1\right]+\left[W_1-W_0\right]}ds.$$

Define

$$c=\frac{W_1-W_0}{W_2-W_1}.$$

Then

$$\beta^1(x)=\frac{x}{3}+\frac{2}{3}\int_0^x\frac{c}{3F^2(s)+c}ds \tag{13}$$

and

$$\beta^2(x)=\frac{2}{3}\left\{x-\int_0^x\frac{c}{3F^2(s)+c}ds\right\}. \tag{14}$$

The derivation of the function β is only heuristic because (12) is merely a necessary condition - we have not formally established that if the other three players follow β, then it is indeed optimal for a player with resource x to spend $\beta^1(x)$ in the first round. The following proposition verifies that this is indeed correct.

Definition. Let a set of pairs $\{y,z\mid x\}=\Omega(x)\in\mathbb{R}^2$ be the set of the solutions of the following maximization problem

$$\max_{y,z}F(y)\left[F^2(z)+c\right]$$

$$s.t.\ y+2z+\int_z^y\frac{cds}{3F(s)+c}=3x.$$

Proposition 3. *Suppose that, function F and constant c are such that for all $x \in [0, E]$, $(x, x) \in \Omega(x)$. A symmetric equilibrium of the elimination tournament is given by the function $\beta = (\beta^1, \beta^2)$ defined as* (13) *and* (14).

Proof: Suppose that all but the player 1 follow the strategy $\beta^* \equiv \beta = (\beta^1, \beta^2)$ given in (13) and (14). We will argue that in that case it is optimal for the player 1 to follow β also. First, notice that β is an increasing and continuous function. Thus, in equilibrium the player with the highest resource spends more in both rounds and wins the tournament. Denote $\gamma = \beta^1$ and $\eta = \beta^2$. It is never optimal for the player 1 to spend an amount $b^1 > \gamma(w)$ in the first round. The expected payoff of the player 1 with the resource x if she spends $b^1 \leq \gamma(w)$ in the first round is calculated as follows. Denote by $y = \gamma^{-1}(b^1)$ and $z = \eta^{-1}(b^2) = \eta^{-1}(x - \gamma(y))$ the resources for which b^1 and b^2 are the equilibrium bids in the first and the second rounds, that are, $\gamma(y) = b^1$ and $\eta(z) = b^2$. Then we can write the player 1's expected payoff from spending $\gamma(y)$ in the first round and $\eta(z)$ in the second round when her resource is x as follows:

$$\begin{aligned}\pi(y, z \mid x) &= F(y)\left[F^2(z) W_2 + \left(1 - F^2(z)\right) W_1\right] + (1 - F(y)) W_0 \\ &= F(z) F^2(z) [W_2 - W_1] + F(z) [W_1 - W_0] + W_0,\end{aligned}$$

where

$$\begin{aligned}\gamma(y) &= \frac{y}{3} + \frac{2}{3} \int_0^y \frac{c}{3F^2(s) + c} ds, \\ \eta(z) &= \frac{2}{3}\left\{z - \int_0^z \frac{c}{3F^2(s) + c} ds\right\}\end{aligned}$$

and

$$\gamma(y) + \eta(z) = x. \tag{15}$$

Since, by assumption, $(x, x) \in \Omega(x)$, for all x, then $\pi(x, x \mid x) - \pi(y, z \mid x) \geq 0$ for all y and z such that the equality (15) holds. **End of proof.**

Example 2. Suppose that $c = 0$, or $W_0 = W_1 = 0$, $E = 1$ and $F(x) = x$. Find the set $\Omega(x)$. We have the following maximization problem

$$\max_{y,z} yz^2$$

$$s.t.\ y + 2z = 3x.$$

It follows

$$\max_{z \in [0,E]} (3x - 2z) z^2$$

or

$$z = x \text{ and } y = x \text{ for any } x \in [0, E].$$

It means that all assumptions of the Proposition 3 holds and therefore

$$\beta^1(x) = \frac{x}{3} \text{ and } \beta^2(x) = \frac{2x}{3}.$$

4. Discussion

We consider two-round elimination tournaments, where risk-neutral players have fixed resources. Two approaches are analyzed: stochastic and deterministic. In the stochastic model all players spend more resources in the first than in the second round in the symmetric equilibrium. The intuition is straightforward: the expected payoffs are higher in the first than in the second round of the tournament. The same reasoning is valid for Rosen's (1986) model, where players have costs for exerting effort in every round instead of fixed resources. Rosen (1986) shows that prizes must increase over rounds to provide enough incentives for players to exert the same effort in every round, if players have trade off between costs and expected high future payoffs. In our model, if a principal/designer of the tournament wants players to allocate resources equally in two rounds, then he must implement the winner-take-all prize scheme.

Moldovanu and Sela (2001) analyze a contest with multiple, nonidentical prizes with deterministic relation between effort and output. They show that the winner-take-all is the optimal prize scheme, if cost functions are linear or concave in effort. Deterministic assumption should be contrasted with stochastic result of a contest in every round of the tournament in Rosen (1986) and the stochastic model of this paper. Moreover, our results for the deterministic model are different not only from the results for the stochastic model, but also from Moldovanu and Sela (2001). If the principal wants to maximize joint spending in the deterministic model, then he should make the difference $W_1 - W_0$ as high as possible and never use the winner-take-all scheme.

Although elimination tournaments are usually associated with sports: tennis, soccer and chess, for example, there are many applications for hierarchy in a firm, academic career, and election campaigns as well. This simple model helps to explain why an assistant professor must work harder at the beginning of his/her career, tennis and soccer players have to exert a lot of effort at the beginning of an elimination tournament, and why new workers spend all day long in their offices.

Some work has been done to test prediction of Lazear and Rosen (1981) theory, see for example Ehrenberg and Bognanno (1990) and Knoeber and Thurman (1994). It will be interesting to test the relationship between prizes/relative prizes and allocation of players' resources in experimental and nonexperimental frameworks.

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