PRODUCTS OF REPRESENTATIONS CHARACTERIZE THE PRODUCTS OF DISPERSIONS AND THE CONSISTENCY OF BELIEFS

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Abstract. A "dispersion" specifies the relative probability between any two elements of a finite domain. It thereby partitions the domain into equivalence classes separated by infinite relative probability. The paper's novelty is to numerically represent not only the order of the equivalence classes, but also the "magnitude" of the gaps between them. The paper's main theorem is that the many products of two dispersions are characterized algebraically by varying the magnitudes of the gaps between each factor's equivalence classes. An immediate corollary is that the many beliefs consistent with two strategies are characterized by varying each player's "steadiness" in avoiding various zero-probability options.

1. Introduction

1.1. An Example

Theorem 3.4 is this paper's only theorem, and it is best motivated by applying it to games like Figure 1.1. There, the outcome $r2$ results from the sequential equilibrium consisting of the strategy profile $(p_\ell, p_r) = (0,1)$, $(p_1, p_2) = (0,1)$, $(p_\alpha, p_\beta) = (0,1)$ and the belief $(p_{\ell 2}, p_{r1}) = (0,1)$. This equilibrium outcome would vanish if Helen believed $\ell 2$ were more likely than $r1$: she would choose α over β and thereby induce Yolanda to choose 1 over 2. And yet, the equilibrium itself admits no chance that Helen will actually be called upon to make a decision. The consistency of beliefs is thus an important and subtle matter.

As explained in Section 2.1, Xavier's strategy corresponds to a "dispersion," that is, a system of relative probabilities, over his strategy set $X = \{\ell, r\}$. That dispersion conveys the fact that he is infinitely more likely to play r than ℓ. Similarly, Yolanda's strategy corresponds to a dispersion over $Y = \{1, 2\}$ which states that she is infinitely more

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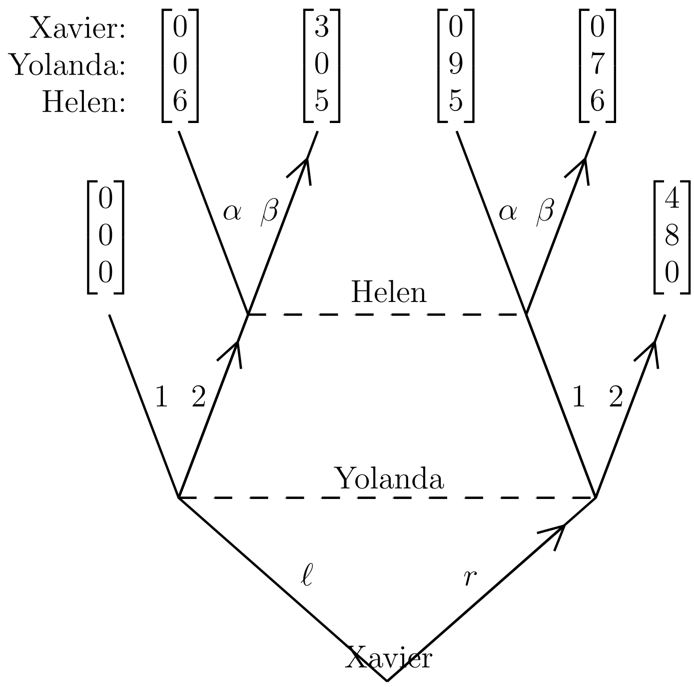

FIGURE 1.1.

likely to play 2 than 1. Consistency requires that Helen's beliefs over $\{\ell 2, r1\}$ accord with a "product" of Xavier's dispersion over X and Yolanda's dispersion over Y. Section 3.2 defines such products as the dispersions over $X{\times}Y$ that not only accord with Xavier's and Yolanda's dispersions, but also satisfy a natural collection of cancellation laws. Theorem 3.4's purpose is to make such products tractable by means of numerical representation.

In particular, a vector of monomials $[c_z n^{e_z}]$ having positive coefficients will be said to "represent" the dispersion in which the probability of z relative to z' is

$$\lim_{n\to\infty} \frac{c_z n^{e_z}}{c_{z'} n^{e_{z'}}} = \begin{pmatrix} \infty & \text{if } e_z > e_{z'} \\ c_z/c_{z'} & \text{if } e_z = e_{z'} \\ 0 & \text{if } e_z < e_{z'} \end{pmatrix}.$$

For example, $(c_\ell n^{e_\ell}, c_r n^{e_r})$ represents Xavier's dispersion when $e_r > e_\ell$, and $(c_1 n^{e_1}, c_2 n^{e_2})$ represents Yolanda's dispersion when $e_2 > e_1$.

Furthermore, the product of these two representations is

$$\begin{array}{c|cc|} \cline{2-3} 2 & c_\ell c_2 n^{e_\ell+e_2} & c_r c_2 n^{e_r+e_2} \\ 1 & c_\ell c_1 n^{e_\ell+e_1} & c_r c_1 n^{e_r+e_1} \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\ell} & \multicolumn{1}{c}{r} \end{array}$$

and these four monomials then represent a dispersion over $X{\times}Y = \{\ell 1, \ell 2, r1, r2\}$. The inequalities $e_r > e_\ell$ and $e_2 > e_1$ imply that $r2$ is infinitely more likely than any other element of $X{\times}Y$ and that $\ell 1$ is infinitely less likely than any other element of $X{\times}Y$. Also note that the probability of $r1$ relative to $\ell 2$ is

$$\text{(1)} \qquad \lim_{n\to\infty} \frac{c_r c_1 n^{e_r+e_1}}{c_\ell c_2 n^{e_\ell+e_2}} = \left(\begin{array}{cl} \infty & \text{if } \; e_r{-}e_\ell > e_2{-}e_1 \\ \left(\frac{c_r/c_\ell}{c_2/c_1}\right) & \text{if } \; e_r{-}e_\ell = e_2{-}e_1 \\ 0 & \text{if } \; e_r{-}e_\ell < e_2{-}e_1 \end{array} \right),$$

which can assume any value in $[0,\infty]$ without violating the inequalities $e_r > e_\ell$ and $e_2 > e_1$.

Theorem 3.4 shows that the products of two dispersions are characterized by the products of their representations. Thus, the preceding paragraph shows that the set of all products of Xavier's dispersion with Yolanda's dispersion are the dispersions over $X{\times}Y$ for which $r2$ is infinitely more likely than any other element of $X{\times}Y$ and for which $\ell 1$ is infinitely less likely than any other element of $X{\times}Y$. Note that there are many such products because the probability of $r1$ relative to $\ell 2$ can vary from one product to the next.

Accordingly, any conceivable belief on Helen's information set $\{\ell 2, r1\}$ is consistent with Xavier's and Yolanda's dispersions. In particular, the belief $(p_{\ell 2}, p_{r1}) = (0,1)$ can be derived from (1) by setting $e_r{-}e_\ell > e_2{-}e_1$, as in

$$\begin{array}{c|cc|} \cline{2-3} 2 & n & n^3 \\ 1 & 1 & n^2 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\ell} & \multicolumn{1}{c}{r} \end{array}\,.$$

Intuitively, Helen believes that $r1$ is infinitely more likely than $\ell 2$ because she believes Xavier's "steadiness" in choosing r over ℓ is infinitely greater than Yolanda's "steadiness" in choosing 2 over 1.

More generally, Section 4 considers any collection of information sets which might follow an arbitrary pair of simultaneous moves. Corollary 4.1 uses product representations to characterize consistency in such a context, and Section 4.3 employs this characterization to resolve a relatively complicated example that had been left unresolved elsewhere. Section 5 summarizes the paper in light of this example.

1.2. Literature

This paper resembles McLennan (1989a,b), Blume, Brandenburger, and Dekel (1991a,b) and Kohlberg and Reny (1997) in the sense that it studies the product of dispersions (i.e., the product of systems of relative probabilities) in order to better understand consistency. Yet synthesizing this paper with the literature is nontrivial because it ventures to rearrange some important concepts and results. I must humbly ask my readers for their patience, and my insightful predecessors for their indulgence.

The concluding paragraphs of Sections 2.1 and 2.2 explain how previous papers have used slightly different terms while studying not only dispersions but also the representation of dispersions by monomials having exponents that are consecutive integers. These papers include Myerson (1986), McLennan (1989a,b), Blume, Brandenburger, and Dekel (1991a), Monderer, Samet, and Shapley (1992), Hammond (1994), and Kohlberg and Reny (1997). In light of these well-known contributions, it would be routine to represent both Xavier's and Yolanda's dispersions by $(n^{-1}, 1)$.

The novelty of this paper is its use of arbitrary exponents. Although the flexibility of arbitrary exponents is superfluous when representing a single dispersion, it becomes significant when two representations are multiplied together. As before, the exponents in each factor order the factor's domain (in the sense that $e_z > e_{z'}$ iff z is infinitely more likely than z'). In addition, they now specify the magnitude of the gaps between that ordering's equivalence classes (in the sense that $|e_z - e_{z'}|$ is the magnitude of the gap between the equivalence class containing z and the equivalence class containing z'). In light of Theorem 3.4, two dispersions have many products when the magnitudes of one factor's gaps can be varied in relation to the magnitudes of the other factor's gaps, as in (1).

In a nutshell, this paper contributes the concept of representation by monomials with arbitrary exponents, Theorem 3.4's characterization of producthood by such representations, and Corollary 4.1's application of this theorem to consistency. Since this paper defines producthood in terms of cancellation laws (Section 3.2), Theorem 3.4 can be tersely summarized as the equivalence, with regard to producthood, of cancellation and representation.

Cancellation and representation are but two ways of understanding producthood. A third and fourth have been the focus of the literature.

McLennan (1989b) and Kohlberg and Reny (1997) use the third alternative: they define their concept of producthood through approximation with a sequence of full-support product distributions, $[\pi_x^n \pi_y^n]$. (Such sequences are also used by Kreps and Wilson (1982) to define consistency.) Notice that the cancellation laws imply a representation $[c_x c_y n^{e_x+e_y}]$ by Theorem 3.4, that such a representation corresponds to the approximation

$$[\pi_x^n \pi_y^n] = \left[\frac{c_x n^{e_x}}{\Sigma_{x'} c_{x'} n^{e_{x'}}} \cdot \frac{c_y n^{e_y}}{\Sigma_{y'} c_{y'} n^{e_{y'}}} \right] ,$$

and that (almost obviously) any such approximation implies the cancellation laws (details at (27)). Thus, Theorem 3.4 implies the equivalence, with regard to producthood, of cancellation, representation, and approximation. The gist of the matter is that a representation corresponds to a particularly pleasant approximation.

Kohlberg and Reny (1997) introduced cancellation and showed the equivalence, with regard to producthood, of cancellation and approximation. While their paper viewed approximation as the definition of producthood, this paper develops cancellation as the definition of producthood and views approximation and representation as two alternative characterizations.

Blume, Brandenburger, and Dekel (1991a) and Hammond (1994) use a fourth way of understanding producthood: they define a product of two dispersions by the product of two nonstandard probability distributions, $[a_x a_y]$. Notice that the cancellation laws imply a representation $[c_x c_y n^{e_x+e_y}]$ by Theorem 3.4, that such a representation corresponds, for any infinitesimal ε, to the nonstandard product

$$[a_x a_y] = [c_x \varepsilon^{-e_x} \cdot c_y \varepsilon^{-e_y}] \ ,$$

and that (almost obviously) any such nonstandard product satisfies the cancellation laws (details at (31)). Thus, Theorem 3.4 implies the equivalence, with regard to producthood, of cancellation, representation, and nonstandard probability. The gist of the matter is that a representation corresponds to a particularly pleasant nonstandard probability distribution.

Since these third and fourth alternatives play no role in this paper, there is a nontrivial gap between this paper and the rest of the literature. Appendix B is dedicated to bridging this gap and to explaining the last three paragraphs. Its eight pages are wholly tangential to the remainder of the paper, and are summarized by Section B.3.

Theorem 3.4 is proven in Appendix A, and a portion of that proof depends upon a theorem of Scott (1964). Krantz, Luce, Suppes, and Tversky (1971) extend Scott's theorem from two coordinates to a finite number of coordinates, and accordingly, it appears quite possible to extend Theorem 3.4 to finite products and to extend Corollary 4.1 to finite games.

2. Dispersions

2.1. Definition

Let Z be any finite set, let $q_{z/z'} \in [0,\infty]$ denote the probability of $z \in Z$ relative to $z' \in Z$, and let $Q_Z = [q_{z/z'}] \in [0,\infty]^{Z^2}$ be a *table* listing a relative probability for every pair z/z' taken from Z.

For example, suppose that $Z = \{\ell, m, r\}$ lists the left, middle, and right actions at some node, and that $(\pi_\ell, \pi_m, \pi_r) = (\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$ is the strategy at that node. In this case, the probability of r relative to ℓ is $q_{r/\ell} = \pi_r/\pi_\ell = 3$, and the entire table $Q_Z = [q_{z/z'}] = [\pi_z/\pi_{z'}]$ may be written either as

$$\begin{array}{ccc} q_{\ell/r} = 1/3 & q_{m/r} = 2/3 & q_{r/r} = 1 \\ q_{\ell/m} = 1/2 & q_{m/m} = 1 & q_{r/m} = 3/2 \\ q_{\ell/\ell} = 1 & q_{m/\ell} = 2 & q_{r/\ell} = 3 \end{array}$$

or as

$$\begin{array}{c|cccc} z' & & & & \\ r & 1/3 & 2/3 & 1 & \\ m & 1/2 & 1 & 3/2 & \\ \ell & 1 & 2 & 3 & \\ \hline & \ell & m & r & z \end{array} . \tag{2}$$

A second example corresponds to Xavier's strategy in Figure 1.1. There, the domain Z is Xavier's strategy set $X = \{\ell, r\}$, the probability of his choosing r relative to ℓ is $q_{r/\ell} = \infty$, and the entire table Q_X is

$$\begin{array}{c|ccc} x' & & & \\ r & 0 & 1 & \\ \ell & 1 & \infty & \\ \hline & \ell & r & x \end{array} . \tag{3}$$

A *dispersion* over Z is a table Q_Z such that

$$(\forall z)\ q_{z/z} = 1 \text{ and} \tag{4a}$$

$$(\forall z, z', z'')\ q_{z/z'} \in q_{z/z''} \odot q_{z''/z'} \tag{4b}$$

where the correspondence $\odot$ mapping $[0,\infty]^2$ into subsets of $[0,\infty]$ is defined by

$$s{\odot}t = \begin{pmatrix} [0,\infty] & \text{if } (s,t) \text{ equals } (0,\infty) \text{ or } (\infty,0) \\ \{st\} & \text{otherwise} \end{pmatrix} .$$

Both (2) and (3) are dispersions.

Remark 2.1. *Every dispersion Q_Z satisfies $(\forall z,z')\ q_{z/z'} = 1/q_{z'/z}$.* (Call this property *reciprocity.*)

Proof. (4a) together with (4b) at $(z,z',z'') = (z,z,z')$ yields that $(\forall z,z')\ 1 = q_{z/z} \in q_{z/z'}{\odot}q_{z'/z}$. Thus, it must be the case that either $1 = q_{z/z'}q_{z'/z}$ for some real numbers $q_{z/z'}$ and $q_{z'/z}$ or that one of $q_{z/z'}$ and $q_{z'/z}$ is 0 and the other ∞. □

There are many other ways to specify a system of relative probabilities. [1] A dispersion is equivalent to a system of conditional (as opposed to relative) probabilities as defined in Myerson (1986, page 337) (details in Hammond (1994, Section 4.1, $\mathit{\Delta}_M{\approx}\mathit{\Delta}_C$) and elsewhere). [2] It is equivalent to a conditional system as defined by approximation in McLennan (1989b, page 146) (details in the paragraph containing (28)). [3] It can be specified by a nonstandard probability distribution as in Blume, Brandenburger, and Dekel (1991a) (details in Remark B.8). [4] It is equivalent to a random variable on a relative probability space as in Kohlberg and Reny (1997, pages 282-283) (details in Remark B.5). [5] And finally, the literature's various ways of denoting the equivalence classes of a dispersion can all be regarded as representations of the dispersion (details in the next section).

2.2. Representation

A vector of monomials, $c_Z n^{e_Z} = [c_z n^{e_z}]$, consisting of a coefficient vector $c_Z \in (0,\infty)^Z$ and an exponent vector $e_Z \in \mathbb{R}^Z$, is said to *represent* the table Q_Z defined by

$$(\forall z,z')\ q_{z/z'} = \lim_n \frac{c_z n^{e_z}}{c_{z'} n^{e_{z'}}}\ ,$$

or equivalently, by

$$(\forall z,z')\ q_{z/z'} = \begin{pmatrix} \infty & \text{if} & e_z > e_{z'} \\ c_z/c_{z'} & \text{if} & e_z = e_{z'} \\ 0 & \text{if} & e_z < e_{z'} \end{pmatrix} .$$

Thus the exponents e_Z partition Z into an ordered collection of equivalence classes such that z is in a higher equivalence class than z' iff z is infinitely more likely than z'. The coefficients c_Z specify the nonzero finite relative probabilities within each class. For example, $(1, 5, n) = (1n^0, 5n^0, 1n^1)$ represents

$$
(5) \qquad
\begin{array}{c|ccc}
z' & & & \\
r & 0 & 0 & 1 \\
m & .2 & 1 & \infty \\
\ell & 1 & 5 & \infty \\
\hline
 & \ell & m & r
\end{array}
\quad z \qquad .
$$

It thereby partitions Z into a lower class $\{\ell, m\}$ and an upper class $\{r\}$, and also specifies the nonzero finite relative probabilities within $\{\ell, m\}$.

Rearrangement 2.2. *A table Q_Z is a dispersion iff it is represented by some $c_Z n^{e_Z}$.*

In order to locate this result in the literature, say that a representation $c_Z n^{e_Z}$ is *parsimonious* if the range $E = \{e|(\exists z)e_z{=}e\}$ of the exponent vector e_Z is a set of consecutive nonpositive integers which includes 0, and if the coefficient vector c_Z satisfies $(\forall e{\in}E)\ \Sigma\{c_z|e_z{=}e\} = 1$. Thus a parsimonious representation defines the equivalence classes, the order between the classes, and a full-support probability distribution within each class.

It is well-known that a dispersion is equivalent to a parsimonious representation (details at Hammond (1994, Section 4.1, $\Delta_M{\approx}\Delta_L$) and elsewhere), and that such a parsimonious representation can be denoted in many different ways. For example, dispersion (5) is equivalent to the parsimonious representation $(\frac{1}{6}n^{-1}, \frac{5}{6}n^{-1}, 1)$, which is equivalent to the ordered partition $\{\{\ell, m\}, \{r\}\}$ and the corresponding within-class distributions $\{(\frac{1}{6}, \frac{5}{6}), (1)\}$ of McLennan (1989a, page 127) and Monderer, Samet, and Shapley (1992, page 31), which is equivalent to the lexicographic conditional probability system $\rho = (p_1, p_2)$, $p_1 = (0, 0, 1)$, $p_2 = (\frac{1}{6}, \frac{5}{6}, 0)$ of Blume, Brandenburger, and Dekel (1991a, Definition 5.2).

Rearrangement 2.2's equivalence between dispersionhood and the existence of a representation deviates from the previous paragraph by admitting non-parsimonious representations that appear to be superfluous and by failing to incorporate the fact that each dispersion has exactly one parsimonious representation. Thus, the reader has good reason to suspect that Rearrangement 2.2 is a step in the wrong direction. However, non-parsimonious representations allow one to express the magnitude of the gaps between equivalence classes and these magnitudes afford Theorem 3.4's algebraic characterization of products.

3. Products

3.1. Preproducts

Consider a Cartesian product $X{\times}Y$ and denote one of its elements as xy rather than (x, y). A *preproduct* of two dispersions Q_X and Q_Y is a table over $X{\times}Y$ which satisfies

$$(\forall xy, x'y')\ q_{xy/x'y'} \in q_{x/x'}{\odot}q_{y/y'}\ .$$

The dispersions Q_X and Q_Y are called the *marginals* of the preproduct Q_{XY}. Note that marginals are always dispersions, but that preproducts might not be.

Remark 3.1. *The marginals of a preproduct are unique.*

Proof. Suppose Q_{XY} is a preproduct with marginals Q_X and Q_Y. The unit diagonals of Q_X and Q_Y yield that

$$\text{(6a)}\qquad (\forall y^\circ)(\forall x, x')\ q_{xy^\circ/x'y^\circ} \in q_{x/x'}{\odot}q_{y^\circ/y^\circ} = q_{x/x'}{\odot}1 = \{q_{x/x'}\}$$

$$\text{(6b)}\qquad (\forall x^\circ)(\forall y, y')\ q_{x^\circ y/x^\circ y'} \in q_{x^\circ/x^\circ}{\odot}q_{y/y'} = 1{\odot}q_{y/y'} = \{q_{y/y'}\}\ ,$$

and hence that $(\forall y^\circ)\ [q_{xy^\circ/x'y^\circ}] = Q_X$ and $(\forall x^\circ)\ [q_{x^\circ y/x^\circ y'}] = Q_Y$. This is more than needed: one such y° and one such x° demonstrate that Q_{XY} uniquely determines Q_X and Q_Y. □

Remark 3.2. *A table Q_{XY} is both a dispersion and a preproduct of Q_X and Q_Y iff it is represented by some $[c_{xy}n^{e_{xy}}]$ such that $(\forall y^\circ)$ $[c_{xy^\circ}n^{e_{xy^\circ}}]$ represents Q_X, and $(\forall x^\circ)$ $[c_{x^\circ y}n^{e_{x^\circ y}}]$ represents Q_Y.*

Proof. Dispersionhood yields a representation $[c_{xy}n^{e_{xy}}]$ by Rearrangement 2.2. Preproducthood yields (6). Representation and (6) yield

$$(\forall y^\circ)(\forall x, x')\ \lim_n c_{xy^\circ}n^{e_{xy^\circ}}/c_{x'y^\circ}n^{e_{x'y^\circ}} = q_{xy^\circ/x'y^\circ} = q_{x/x'}$$
$$(\forall x^\circ)(\forall y, y')\ \lim_n c_{x^\circ y}n^{e_{x^\circ y}}/c_{x^\circ y'}n^{e_{x^\circ y'}} = q_{x^\circ y/x^\circ y'} = q_{y/y'}\ .$$

Conversely, the existence of such a representation yields dispersionhood by Rearrangement 2.2 and also yields preproducthood by

$$(\forall xy, x'y')\ \ q_{xy/x'y'} = \lim_n \frac{c_{xy}n^{e_{xy}}}{c_{x'y'}n^{e_{x'y'}}} =$$

$$\lim_n \frac{c_{xy}n^{e_{xy}}}{c_{x'y}n^{e_{x'y}}}\frac{c_{x'y}n^{e_{x'y}}}{c_{x'y'}n^{e_{x'y'}}} \in \lim_n \frac{c_{xy}n^{e_{xy}}}{c_{x'y}n^{e_{x'y}}}{\odot}\lim_n \frac{c_{x'y}n^{e_{x'y}}}{c_{x'y'}n^{e_{x'y'}}} = q_{x/x'}{\odot}q_{y/y'}$$

and (if doubtful of the above inclusion) Lemma A.1. □

For example,

(7)

3	$2n^{-2}$	n^{-1}	1
2	n^{-3}	n^{-2}	$2n^{-1}$
1	n^{-4}	$2n^{-3}$	n^{-2}
	ℓ	m	r

represents a preproduct of the dispersions represented by

$$(a_\ell n^{b_\ell}, a_m n^{b_m}, a_r n^{b_r}) = (n^{-2}, n^{-1}, 1) \text{ and}$$
$$(a_1 n^{b_1}, a_2 n^{b_2}, a_3 n^{b_3}) = (n^{-2}, n^{-1}, 1)\ .$$

3.2. Cancellation Laws

A product will be defined as a table over $X{\times}Y$ that satisfies a large number of cancellation laws. Consider first two full-support distributions π_X and π_Y. Cancelling terms yields results like

$$(\forall x^0y^0, x^1y^1, x^2y^2, x^3y^3)\ \ \frac{\pi_{x^0}\pi_{y^0}}{\pi_{x^1}\pi_{y^2}} = \frac{\pi_{x^0}\pi_{y^3}}{\pi_{x^1}\pi_{y^1}}\frac{\pi_{x^2}\pi_{y^1}}{\pi_{x^2}\pi_{y^2}}\frac{\pi_{x^3}\pi_{y^0}}{\pi_{x^3}\pi_{y^3}}\ .$$

More generally, consider any $m \geq 1$ and any two permutations, σ and τ, of $\{0, 1, 2, ... \, m\}$. Cancelling terms yields

$$(\forall \langle x^iy^i\rangle_{i=0}^m)\ \ \frac{\pi_{x^0}\pi_{y^0}}{\pi_{x^{\sigma(0)}}\pi_{y^{\tau(0)}}} = \Pi_{i=1}^m\ \frac{\pi_{x^{\sigma(i)}}\pi_{y^{\tau(i)}}}{\pi_{x^i}\pi_{y^i}}\ ,$$

and hence, a full-support product distribution $[\pi_{xy}]$ over $X{\times}Y$ must satisfy the cancellation law

$$(\forall \langle x^iy^i\rangle_{i=0}^m)\ \ \frac{\pi_{x^0y^0}}{\pi_{x^{\sigma(0)}y^{\tau(0)}}} = \Pi_{i=1}^m\ \frac{\pi_{x^{\sigma(i)}y^{\tau(i)}}}{\pi_{x^iy^i}}\ .$$

Similarly, a product dispersion $[q_{xy/x'y'}]$ over $X{\times}Y$ will be defined to satisfy the cancellation law

$$(8) \qquad (\forall \langle x^iy^i\rangle_{i=0}^m)\ \ q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} \in \odot_{i=1}^m\ q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

where the product on the right-hand side is defined by

$$\odot_{i=1}^m t_i = \begin{pmatrix} [0,\infty] & \text{if } (\exists i)\, t_i{=}0 \text{ and } (\exists i)\, t_i{=}\infty \\ \{\Pi_{i=1}^m t_i\} & \text{otherwise} \end{pmatrix}$$

for $m \geq 1$, and by $\odot_{i=1}^m t_i = 1$ for $m = 0$. Formally, a *product* is a table over $X{\times}Y$ which satisfies the cancellation law (8) for every $m \geq 0$ and every pair of permutations σ and τ.

Although producthood has not been defined previously in terms of cancellation laws, it is equivalent to concepts in McLennan (1989b), Blume, Brandenburger, and Dekel (1991a), Hammond (1994), and Kohlberg and Reny (1997). These nontrivial equivalences, and the appearance of cancellation laws in Kohlberg and Reny (1997, Theorem 2.10), were introduced in Section 1.2 and are explored fully in Appendix B.

There are a great many cancellation laws. To be precise, there are $((1{+}m)!)^2$ cancellation laws of order m since there are $(1{+}m)!$ permutations of $\{0,1,2,\dots m\}$. The $1{=}(1!)^2$ zero-order law is

$$(\forall xy)\ q_{xy/xy} = 1\ . \tag{9}$$

The $4{=}(2!)^2$ first-order laws are

$$(\forall xy, x'y')\ q_{xy/xy} = q_{x'y'/x'y'} \tag{10a}$$

$$(\forall xy, x'y')\ q_{xy/x'y} = q_{xy'/x'y'} \tag{10b}$$

$$(\forall xy, x'y')\ q_{xy/xy'} = q_{x'y/x'y'} \tag{10c}$$

$$(\forall xy, x'y')\ q_{xy/x'y'} = q_{xy/x'y'}\ , \tag{10d}$$

which are derived from (8) by varying the permutations σ and τ when $x^0y^0{=}xy$ and $x^1y^1{=}x'y'$. One of the $36{=}(3!)^2$ second-order laws coincides with the dispersion criterion (4b) over the domain $X{\times}Y$. That law is

$$(\forall xy, x'y', x''y'')\ q_{xy/x'y'} \in q_{xy/x''y''} \odot q_{x''y''/x'y'}\ , \tag{11}$$

which is derived from (8) and a certain pair of permutations (σ,τ) when $x^0y^0{=}xy$, $x^1y^1{=}x''y''$ and $x^2y^2{=}x'y'$.

Remark 3.3. *A table Q_{XY} satisfies (9), (10), and (11) iff it is both a dispersion and a preproduct. (Hence every product is both a dispersion and a preproduct.)*

Proof. Take any Q_{XY} satisfying (9), (10), and (11). (9) and (11) imply that Q_{XY} is a dispersion. Fix any $x^\star y^\star$. (11) at $x''y'' = x'y$ and (10b&c) yield

$$(\forall xy, x'y')\ q_{xy/x'y'} \in q_{xy/x'y} \odot q_{x'y/x'y'} = q_{xy^\star/x'y^\star} \odot q_{x^\star y/x^\star y'}\ ,$$

and hence that Q_{XY} is a preproduct of $Q_X = [q_{xy^\star/x'y^\star}]$ and $Q_Y = [q_{x^\star y/x^\star y'}]$. Conversely, dispersionhood implies (9) and (11), and preproducthood implies (6a&b) which implies (10b&c). (9) implies (10a), and (10d) is vacuous. □

Another of the $36{=}(3!)^2$ second-order laws is the *cross-cancellation law*

$$(\forall xy, x'y', x''y'')\ q_{xy/x'y'} \in q_{xy''/x''y'} \odot q_{x''y/x'y''}\ , \tag{12}$$

which is derived from (8) and a certain pair of permutations (σ,τ) when $x^0y^0{=}xy$, $x^1y^1{=}x''y'$ and $x^2y^2{=}x'y''$. Example (7) violates this law because

$$q_{r1/\ell 3} = \lim_n \frac{n^{-2}}{2n^{-2}} = 1/2$$

is not the product of

$$q_{r2/m3} = \lim_n \frac{2n^{-1}}{n^{-1}} = 2 \text{ and}$$
$$q_{m1/\ell 2} = \lim_n \frac{2n^{-3}}{n^{-3}} = 2 \ .$$

Hence, a table can be both a dispersion and a preproduct without being a product.

This example is borrowed from Kohlberg and Reny (1997, Figure 1, with their ε set to n^{-1}, their (x,x',x'') set to (r,m,ℓ), and their (y,y',y'') set to $(3,2,1)$). They used the example (their pages 227-228) to draw a similar distinction between "strong independence" (which is equivalent to producthood by Remark B.6(a⇔b^{KR})) and "weak independence" (which is equivalent to the combination of dispersionhood and preproducthood by an omitted argument). Kohlberg and Reny borrowed this example from Blume, Brandenburger, and Dekel (1991a, Figure 7.1 with minor alterations), who in turn credit conversations with Myerson. They used the example to draw a similar distinction between the existence of a "nonstandard product" (which is equivalent to producthood by Remark B.10) and "stochastic independence" (which bears some resemblance to preproducthood). This is an important example, and it seems reasonable to conjecture that there is no simpler example which could draw such distinctions.

3.3. Representation

Here is the paper's only theorem.

Theorem 3.4. *A table Q_{XY} is a product iff it is represented by some $[c_xc_yn^{e_x+e_y}]$. (The product represented by $[c_xc_yn^{e_x+e_y}]$ has its marginals represented by $[c_xn^{e_x}]$ and $[c_yn^{e_y}]$.)*

Appendix A proves the theorem's first sentence. Its second sentence follows from Remark 3.2, the fact that $[c_xn^{e_x}]$ represents the same dispersion as any $[c_xc_{y^\circ}n^{e_x+e_{y^\circ}}]$, and the fact that $[c_yn^{e_y}]$ represents the same dispersion as any $[c_{x^\circ}c_yn^{e_{x^\circ}+e_y}]$.

Remark 3.5. *Any two dispersions have at least one product.*

Proof. Consider any two dispersions Q_X and Q_Y. By Rearrangement 2.2, Q_X has some representation $[c_x n^{e_x}]$ and Q_Y has some representation $[c_y n^{e_y}]$. Hence, by Theorem 3.4, $[c_x c_y n^{e_x+e_y}]$ represents a product of Q_X and Q_Y. □

In particular, consider two dispersions $Q_X = [\pi_x/\pi_{x'}]$ and $Q_Y = [\pi_y/\pi_{y'}]$ which are derived from the full-support probability distributions π_X and π_Y. Since every pair of dispersions has a product (Remark 3.5), since every product is a preproduct (Remark 3.3), and since $[\pi_x\pi_y/\pi_{x'}\pi_{y'}]$ is the only preproduct of $[\pi_x/\pi_{x'}]$ and $[\pi_y/\pi_{y'}]$, it must be that $[\pi_x\pi_y/\pi_{x'}\pi_{y'}]$ is the only product of $[\pi_x/\pi_{x'}]$ and $[\pi_y/\pi_{y'}]$. Thus the product of relative probabilities can be regarded as a natural extension of the product of ordinary probabilities.

Notice that the definition of representation is concerned with ratios. Thus any representation $c_Z n^{e_Z}$ can be simplified by setting some $c_z n^{e_z}$ equal to 1. In other words, one can choose a numeraire. It is particularly useful to establish a numeraire when the domain Z is a product $X \times Y$. For example, Theorem 3.4 implies that products over $X = \{\ell, m, r\}$ and $Y = \{1, 2, 3\}$ are represented by $[c_x c_y n^{e_x+e_y}]$ of the form

(13)

3	$c_\ell c_3 n^{e_\ell+e_3}$	$c_3 n^{e_3}$	$c_r c_3 n^{e_r+e_3}$
2	$c_\ell n^{e_\ell}$	1	$c_r n^{e_r}$
1	$c_\ell c_1 n^{e_\ell+e_1}$	$c_1 n^{e_1}$	$c_r c_1 n^{e_r+e_1}$
	ℓ	m	r

Thus a numeraire like $m2$ in (13) not only eliminates four parameters, but also obviates the need to write the marginal representations $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ separately: these appear in (13) as the second row and second column.

Finally, recall that (7) represents a preproduct which is not a product. This accords with Theorem 3.4 because the $[c_{xy} n^{e_{xy}}]$ of (7) cannot be factored into some $[c_x n^{e_x}]$ and some $[c_y n^{e_y}]$.

4. Consistency

4.1. Definition

This section considers the class of partial games in which an arbitrary pair of simultaneous moves is followed by an arbitrary collection of information sets. Examples include Figures 4.1 and 4.2 (Figure 4.1 is a portion of Figure 1.1). Throughout the class, the two players are Xavier and Yolanda, their action sets are X and Y, and each node in the partial game is identified with the sequence of actions taken to reach it. Accordingly, there is an initial node $\varnothing$ from which Xavier takes an action x, there are $|X|$ nodes of the form x from which Yolanda takes an action y, and there are $|X||Y|$ nodes of the form xy at which the partial game ends. Xavier chooses at the initial information set $\{\varnothing\}$, Yolanda chooses at the information set X, and the partial game ends with a collection $\mathcal{H}$ of disjoint information sets H contained in $X{\times}Y$. For expositional ease, suppose these concluding information sets belong to someone named Helen.

Let p_Z denote a distribution over some finite set Z. If $Z \subseteq \bar{Z}$, a distribution p_Z can be derived from a dispersion $Q_{\bar{Z}}$ by restricting the dispersion $Q_{\bar{Z}}$ to the domain Z, by finding a row of this restricted dispersion that contains no infinite relative probabilities, and by using the finite relative probabilities in that row to determine p_Z. In this fashion, both $(p_\ell, p_r) = (0,1)$ and $(p_\ell, p_m) = (\frac{1}{6}, \frac{5}{6})$ can be derived from the example (5). Formally, p_Z is *induced* by $Q_{\bar{Z}}$ if

$$(\forall z{\in}Z) \;\; p_z = \frac{q_{z/z^\star}}{\Sigma_{z'\in Z}\; q_{z'/z^\star}} \tag{14}$$

for some $z^\star{\in}Z$ such that $(\forall z'{\in}Z)\; q_{z'/z^\star} < \infty$. Note that such a $z^\star$ must exist: if none did, the dispersionhood of Q_Z would be violated by the existence of a sequence $\langle z^n \rangle_{n=1}^{\infty}$ such that $(\forall n)\; q_{z^{n+1}/z^n} = \infty$. Also note that the existence of a second $z^{\star\star}{\in}Z$ satisfying $(\forall z'{\in}Z)\; q_{z'/z^{\star\star}} < \infty$ is inconsequential: the reciprocity of Lemma 2.1 together with $q_{z^{\star\star}/z^\star} < \infty$ and $q_{z^\star/z^{\star\star}} < \infty$ would yield that $q_{z^\star/z^{\star\star}} \in (0,\infty)$, hence that $(\forall z'{\in}Z)\; q_{z'/z^{\star\star}} = q_{z'/z^\star} q_{z^\star/z^{\star\star}}$, and hence that p_Z is invariant to the choice of $z^\star$ or $z^{\star\star}$. Finally, note that the denominator $\Sigma_{z'\in Z}\; q_{z'/z^\star}$ must be positive because $z^\star \in Z$ and $q_{z^\star/z^\star} = 1$.

A strategy is a distribution over an action set, and accordingly, Xavier's strategy is denoted p_X and Yolanda's strategy is denoted p_Y. A belief is a distribution over an information set. Yolanda's belief over her information set X is identical to Xavier's strategy p_X and nothing more about this needs to be said. Meanwhile, Helen's belief profile $\{p_H\}_{H\in\mathcal{H}}$ specifies some distribution p_H at each information set $H{\in}\mathcal{H}$. Such a belief profile $\{p_H\}_{H\in\mathcal{H}}$ is *consistent* with the strategy profile (p_X, p_Y) if there exists a product Q_{XY} with marginals Q_X and Q_Y such that Q_X induces p_X, Q_Y induces p_Y, and Q_{XY} induces each p_H. Remark B.4 and the paragraph following it show that this definition of consistency is equivalent to that of Kreps and Wilson (1982).

4.2. Representation

If $Z \subseteq \bar{Z}$, the distribution p_Z *induced* by the representation $c_{\bar{Z}} n^{e_{\bar{Z}}}$ is defined by

$$(\forall z{\in}Z)\ \ p_z = \frac{c_z\ 1(e_z{=}\max e_Z)}{\Sigma_{z'\in Z}\ c_{z'}\ 1(e_{z'}{=}\max e_Z)}\ , \tag{15}$$

where $1(\cdot)$ is the indicator function assuming a value of 1 when its argument is true and a value of 0 when its argument is false. This formula is simple. It says to ignore $\bar{Z}{\sim}Z$, to use the exponents e_Z to find the highest class in Z, to use the coefficients c_Z to assign positive probabilities within that class, and to assign zero probability elsewhere in Z. For example, $(p_\ell, p_m) = (\frac{1}{6}, \frac{5}{6})$ and $(p_m, p_r) = (0, 1)$ are induced by $(c_\ell n^{e_\ell}, c_m n^{e_m}, c_r n^{e_r}) = (1, 5, n)$.

The following is this paper's contribution to game theory. It is a corollary of Theorem 3.4.

Corollary 4.1. *$\{p_H\}_{H\in\mathcal{H}}$ is consistent with (p_X, p_Y) iff there exists $(c_X n^{e_X}, c_Y n^{e_Y})$ such that $c_X n^{e_X}$ induces p_X, $c_Y n^{e_Y}$ induces p_Y, and $[c_x c_y n^{e_x+e_y}]$ induces every p_H.*

Proof. By definition, consistency is equivalent to the existence of a product Q_{XY} with marginals Q_X and Q_Y such that Q_X induces p_X, Q_Y induces p_Y, and Q_{XY} induces each p_H. By Theorem 3.4, this is equivalent to the existence of a $[c_x c_y n^{e_x+e_y}]$ such that $[c_x n^{e_x}]$ represents Q_X which induces p_X, $[c_y n^{e_y}]$ represents Q_Y which induces p_Y, and $[c_x c_y n^{e_x+e_y}]$ represents Q_{XY} which induces each p_H. This is equivalent to the corollary's conclusion because an inspection of (14) and (15) reveals that a representation induces a distribution iff the dispersion that it represents also induces the distribution. □

A relatively simple example is Figure 4.1. There Corollary 4.1 can be used to show that any belief $p_H = (p_{\ell 2}, p_{r1})$ is consistent with $p_X = (p_\ell, p_r) = (0, 1)$ and $p_Y = (p_1, p_2) = (0, 1)$. To see this, note that $(c_\ell n^{e_\ell}, c_r n^{e_r}) = (1, c_r n^{e_r})$ induces p_X for any c_r and any $e_r > 0$, that $(c_1 n^{e_1}, c_2 n^{e_2}) = (1, n)$ induces p_Y, and that their product

	ℓ	r
2	n	$c_r n^{e_r+1}$
1	1	$c_r n^{e_r}$

induces the distribution over $\{\ell 2, r1\}$ corresponding to the ratio

$$p_{r1}/p_{\ell 2} = \lim_n \frac{c_r n^{e_r}}{n} = \lim_n c_r n^{e_r - 1}\ .$$

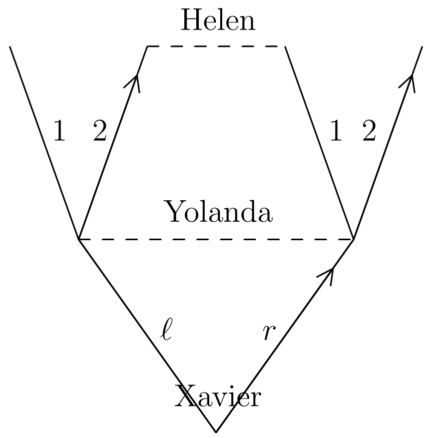

FIGURE 4.1.

In particular, the ratio $p_{r1}/p_{\ell 2} = \infty$ can be obtained by setting $e_r = 2$, the ratio $p_{r1}/p_{\ell 2} = 0$ can be obtained by setting $e_r = 1/2$, and any ratio $p_{r1}/p_{\ell 2} \in (0, \infty)$ can be obtained by setting $e_r = 1$ and $c_r = p_{r1}/p_{\ell 2}$. Hence, by Corollary 4.1, every conceivable belief over $H = \{\ell 2, r1\}$ is consistent with $p_X = (0, 1)$ and $p_Y = (0, 1)$.

The e_r of the previous paragraph can be interpreted as the "steadiness" of Xavier's hand in playing his unit-probability option r as opposed to his zero-probability option ℓ. For example, the ratio $p_{r1}/p_{\ell 2} = \infty$ was obtained by setting $e_r > 1$ so that Xavier's steadiness in playing r was "infinitely greater" than Yolanda's steadiness in playing 2. Similarly, the ratio $p_{r1}/p_{\ell 2} = 0$ was obtained by setting $e_r \in (0, 1)$ so that Xavier's steadiness was "infinitely less" than Yolanda's steadiness. Finally, ratios in $(0, \infty)$ were obtained by setting $e_r = 1$ so that Xavier's steadiness was "finitely comparable" to Yolanda's steadiness.

The vacuousness of consistency in this simple example is not surprising given Kreps and Ramey (1987, Figure 1)'s discussion of a very similar example. What Corollary 4.1 provides is the algebra of product representation and its intuition in terms of steadiness. That algebra and intuition are more prominent in the next example, which has not been fully solved elsewhere.

4.3. A 3×3 EXAMPLE

Formal Discussion

In Figure 4.2, a belief profile $\{p_H\}_{H \in \mathcal{H}}$ is consistent with the strategies $p_X = (p_\ell, p_m, p_r) = (0, 0, 1)$ and $p_Y = (p_1, p_2, p_3) = (0, 0, 1)$ if and

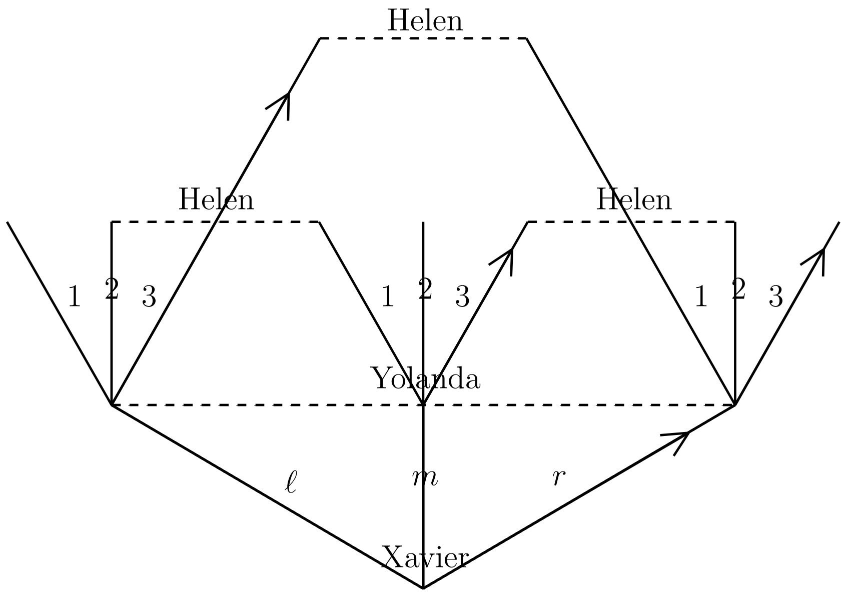

FIGURE 4.2.

only if

$$\frac{p_{r1}}{p_{\ell 3}} \in \frac{p_{r2}}{p_{m3}} \odot \frac{p_{m1}}{p_{\ell 2}} \ . \tag{16}$$

(16) is necessary because the definition of consistency implies the existence of a product Q_{XY} such that $p_{r1}/p_{\ell 3} = q_{r1/\ell 3}$, $p_{r2}/p_{m3} = q_{r2/m3}$, and $p_{m1}/p_{\ell 2} = q_{m1/\ell 2}$, and because the definition of product implies the cross-cancellation $q_{r1/\ell 3} \in q_{r2/m3} \odot q_{m1/\ell 2}$. This half is unsurprising since Kohlberg and Reny (1997, page 297-298) derived the necessity of a condition like (16) in a game like Figure 4.2.

The sufficiency of (16) has not been established elsewhere. This and the next three paragraphs will establish it by applying Corollary 4.1 to 13 cases and subcases. To set the stage, identify a belief profile $\{p_H\}_{H \in \mathcal{H}}$ with the three ratios

$$(\ p_{m1}/p_{\ell 2},\ p_{r1}/p_{\ell 3},\ p_{r2}/p_{m3}\ )\ ,$$

and notice that these three ratios appear in the third, fourth, and fifth columns of Table 4.3. Each row of the table concerns a set of such triples (the symbol + means that the corresponding ratio comes from $(0, \infty)$). For example, the first row concerns the singleton $\{(\infty, \infty, \infty)\}$, the second row concerns the one-dimensional set

$$\{\ (\infty, \infty, p_{r2}/p_{m3}) \mid p_{r2}/p_{m3} \in (0, \infty)\ \}\ ,$$

Case	$c_\ell n^{e_\ell}$	$p_{m1}/p_{\ell 2}$	$p_{r1}/p_{\ell 3}$	p_{r2}/p_{m3}	$c_r n^{e_r}$
1	n^{-2}	∞	∞	∞	n^2
2	n^{-2}	∞	∞	$+$	$(p_{r2}/p_{m3})n$
3a	n^{-2}	∞	∞	0	$n^{1/2}$
3b	$(p_{\ell 3})n^{-3/2}$	∞	$+$	0	$(p_{r1})n^{1/2}$
3c	$n^{-5/4}$	∞	0	0	$n^{1/2}$
4	$(p_{\ell 2}/p_{m1})n^{-1}$	$+$	∞	∞	n^2
5	$(p_{\ell 2}/p_{m1})n^{-1}$	$+$	$\frac{p_{m1}}{p_{\ell 2}}\frac{p_{r2}}{p_{m3}}$	$+$	$(p_{r2}/p_{m3})n$
6	$(p_{\ell 2}/p_{m1})n^{-1}$	$+$	0	0	$n^{1/2}$
7a	$n^{-1/2}$	0	∞	∞	n^2
7b	$(p_{\ell 3})n^{-1/2}$	0	$+$	∞	$(p_{r1})n^{3/2}$
7c	$n^{-1/2}$	0	0	∞	$n^{5/4}$
8	$n^{-1/2}$	0	0	$+$	$(p_{r2}/p_{m3})n$
9	$n^{-1/2}$	0	0	0	$n^{1/2}$

TABLE 4.3.

and the middle row concerns the two-dimensional set

$$\{ \ (\ p_{m1}/p_{\ell 2}, \ \tfrac{p_{m1}}{p_{\ell 2}}\tfrac{p_{r2}}{p_{m3}}, \ p_{r2}/p_{m3} \ ) \mid$$
$$p_{m1}/p_{\ell 2} \in (0,\infty) \text{ and } p_{r2}/p_{m3} \in (0,\infty) \ \} \ .$$

This paragraph notes that every triple which satisfies (16) belongs to a set defined by a row of Table 4.3. Cases 1 through 9 exhaust all possible contingencies for the first ratio $p_{m1}/p_{\ell 2}$ and the third ratio p_{r2}/p_{m3}. (16) implies that the first and third ratios uniquely determine the second ratio in every case but cases 3 and 7. In each of those cases, (16) imposes no restrictions on the second ratio, and hence, cases 3 and 7 each have three subcases.

Thus, it remains to be shown that every triple in every row of Table 4.3 is consistent with the strategies $p_X = (p_\ell, p_m, p_r) = (0,0,1)$ and $p_Y = (p_1, p_2, p_3) = (0,0,1)$. By Corollary 4.1, it suffices to show that every such triple is induced by some $[c_x c_y n^{e_x+e_y}]$ such that $c_X n^{e_X}$ induces p_X and $c_Y n^{e_Y}$ induces p_Y.

The second and sixth columns of Table 4.3 show how this can be done. In particular, define $c_X n^{e_X}$ by setting $c_m n^{e_m} = 1$ and by taking $c_\ell n^{e_\ell}$ and $c_r n^{e_r}$ from the second and sixth columns. This $c_X n^{e_X}$ induces $p_X = (0,0,1)$ because $e_\ell < 0 < e_r$ in every case. Next define $c_Y n^{e_Y} =$

$(n^{-1}, 1, n)$ and note that it induces $p_Y = (0, 0, 1)$. The product of $c_X n^{e_X}$ with $c_Y n^{e_Y}$ is

	ℓ	m	r
3	$c_\ell n^{e_\ell+1}$	n	$c_r n^{e_r+1}$
2	$c_\ell n^{e_\ell}$	1	$c_r n^{e_r}$
1	$c_\ell n^{e_\ell-1}$	n^{-1}	$c_r n^{e_r-1}$

which induces the ratios

$$p_{m1}/p_{\ell 2} = \lim_n \frac{n^{-1}}{c_\ell n^{e_\ell}} = \lim_n (1/c_\ell) n^{|e_\ell|-1} \tag{17a}$$

$$p_{r1}/p_{\ell 3} = \lim_n \frac{c_r n^{e_r-1}}{c_\ell n^{e_\ell+1}} = \lim_n (c_r/c_\ell) n^{e_r+|e_\ell|-2} \tag{17b}$$

$$\text{and } p_{r2}/p_{m3} = \lim_n \frac{c_r n^{e_r}}{n} = \lim_n c_r n^{e_r-1} \,. \tag{17c}$$

These formulas generate the third, fourth, and fifth columns in every case (this is easiest to verify by going down the columns rather than across the rows, and by starting with the fifth column).

Informal Discussion

Although the preceding proof of the sufficiency of (16) cannot be replaced with an informal discussion, one can intuitively appreciate how the four parameters $(c_\ell, e_\ell, c_r, e_r)$ span the set of all beliefs satisfying (16). In a nutshell, this example is just slightly more than a two-fold product of Figure 4.1's example with one information set, and consequently, it can be managed with four parameters rather than two.

As in Figure 4.1's example with one information set, the two parameters (c_r, e_r) span all conceivable values of p_{r2}/p_{m3}, and the exponent e_r can be understood as the steadiness of Xavier's hand. Similarly, the two parameters (c_ℓ, e_ℓ) span all conceivable values of $p_{m1}/p_{\ell 2}$, and the absolute value of the exponent $|e_\ell|$ can be understood as the steadiness of Xavier's "other" or "left" hand. In all but cases 3 and 7, the $p_{r1}/p_{\ell 3}$ uniquely determined by (17b) and such a $(c_\ell, e_\ell, c_r, e_r)$ happens to coincide with the $p_{r1}/p_{\ell 3}$ uniquely determined by (16), $p_{m1}/p_{\ell 2}$, and p_{r2}/p_{m3}. In this sense, these seven cases are just a two-fold product of Figure 4.1's example with one information set.

But Case 3 is more interesting. There Xavier is "left-handed" in the sense that his left hand is infinitely more steady than Yolanda's while his right hand is infinitely less steady than Yolanda's (there is no need to distinguish between Yolanda's "two hands"). This leads to three possibilities: Xavier's "total steadiness" ($|e_\ell|+e_r$) is greater than, equal to, or less than Yolanda's "total steadiness" (which is $|-1|+1=2$). Those three possibilities correspond to the three subcases of case 3. Similarly, Xavier is "right-handed" in case 7. In this light, the nine cases might be more intuitively labeled in terms of the steadiness of Xavier's two hands relative to Yolanda's hand: 1. steady, 2. steady-similar, 3. steady-shaky (i.e., left-handed), 4. similar-steady, 5. similar, 6. similar-shaky, 7. shaky-steady (i.e., right-handed), 8. shaky-similar, and 9. shaky.

5. Summary

Consider an arbitrary collection of disjoint information sets which follow a pair of simultaneous moves. It is well-known that many belief profiles can be consistent with a given strategy profile, and that this multiplicity can arise when strategies specify zero probabilities. This paper's Corollary 4.1 has shown that all consistent belief profiles can be found by partitioning the zero-probability options of each player into equivalence classes, by ordering those classes, and by specifying the magnitude of the gaps between them. The contribution of this paper has been to introduce and employ these magnitudes. Each can be interpreted as the steadiness with which the player plays from the higher equivalence class.

This contribution was illustrated by two examples. In Figure 4.1, it happened that any conceivable belief on the example's one information set could be found by varying the steadiness with which Xavier played his unit-probability option (r) rather than his zero-probability option (ℓ). In Figure 4.2, the cross-cancellation law eliminated many belief profiles over the example's three information sets, and any of the remaining belief profiles could be found by varying the steadiness with which Xavier played his unit-probability option (r) rather than one of his zero-probability options (m) and by varying the steadiness with which he played this zero-probability option (m) rather than another zero-probability option (ℓ).

This characterization of consistency (Corollary 4.1) was based upon the introduction of non-parsimonious representations (Section 2.2) and upon a fundamental theorem (Theorem 3.4) which showed that the products of two dispersions are characterized algebraically by the products of their representations. Accordingly, many products arise by varying the representations of each factor, which in turn arise by varying the magnitudes of the gaps between the equivalence classes of each factor.

Appendix A. Proof of Theorem 3.4

A.1. Necessity of a Product Representation

Suppose Q_{XY} is a product of Q_X and Q_Y.

Exponents e_X and e_Y.

Define the binary relation $\succeq$ on $X{\times}Y$ by

$$(\forall xy, x'y')\ xy \succeq x'y' \Leftrightarrow q_{xy/x'y'} > 0\ . \tag{18}$$

Since $q_{xy/x'y'} > 0 \Leftrightarrow q_{x'y'/xy} < \infty$ by the reciprocity of Remark 2.1, (18) is equivalent to

$$(\forall xy, x'y')\ x'y' \preceq xy \Leftrightarrow q_{x'y'/xy} < \infty\ ,$$

which is equivalent to

$$(\forall xy, x'y')\ xy \preceq x'y' \Leftrightarrow q_{xy/x'y'} < \infty\ . \tag{19}$$

This paragraph establishes Scott (1964, page 243, conditions (1_V) and (2_V), at $(A, A^*) = (X, Y)$, $xx^* = xy$, $V = \succeq$, $n = m{+}1$, and $(\pi, \sigma) = (\sigma^{-1}, \tau^{-1})$). The first of these two conditions is the completeness of $\succeq$, which follows from (18) and (19) and the fact that $q_{xy/x'y'} > 0$ or $q_{xy/x'y'} < \infty$ must hold at any $(xy, x'y')$. To prove the second condition, consider any $m \geq 1$, any permutations σ and τ of $\{0, 1, 2, ... m\}$, and any $\langle x^i y^i \rangle_{i=0}^m$ such that

$$(\forall i{\geq}1)\ x^{\sigma(i)} y^{\tau(i)} \succeq x^i y^i\ .$$

Since (18) yields that $(\forall i{\geq}1)\ q_{x^{\sigma(i)} y^{\tau(i)}/x^i y^i} > 0$, it must be that

$$0 \notin \odot_{i=1}^m q_{x^{\sigma(i)} y^{\tau(i)}/x^i y^i}\ .$$

Thus, since the producthood of Q_{XY} implies the cancellation law

$$q_{x^0 y^0/x^{\sigma(0)} y^{\tau(0)}} \in \odot_{i=1}^m\, q_{x^{\sigma(i)} y^{\tau(i)}/x^i y^i}\ ,$$

it must be that $q_{x^0 y^0/x^{\sigma(0)} y^{\tau(0)}} > 0$, and hence, by (18) that

$$x^0 y^0 \succeq x^{\sigma(0)} y^{\tau(0)}\ .$$

The previous paragraph and Scott (1964, Theorem 3.1, with "utility functions" set to e_X and e_Y) yield the existence of $e_X \in \mathbb{R}^X$ and $e_Y \in \mathbb{R}^Y$ such that $xy \succeq x'y'$ iff $e_x{+}e_y \geq e_{x'}{+}e_{y'}$. Thus, by (18) and (19) we arrive at

$$q_{xy/x'y'} = \infty \text{ iff } xy \succ x'y' \text{ iff } e_x{+}e_y > e_{x'}{+}e_{y'} \tag{20a}$$

$$q_{xy/x'y'} \in (0, \infty) \text{ iff } xy \approx x'y' \text{ iff } e_x{+}e_y = e_{x'}{+}e_{y'} \tag{20b}$$

$$q_{xy/x'y'} = 0 \text{ iff } xy \prec x'y' \text{ iff } e_x{+}e_y < e_{x'}{+}e_{y'}\ . \tag{20c}$$

(Now forget $\succeq$.)

Coefficients c_X and c_Y.

It remains to find positive numbers (c_X, c_Y) such that

$$(\forall\ (xy, x'y') \in E)\ (c_x c_y)/(c_{x'} c_{y'}) = q_{xy/x'y'}\ ,$$

where

$$E = \{\ (xy, x'y') \mid e_x{+}e_y = e_{x'}{+}e_{y'}\ \}\ .$$

Since (20b) yields that $q_{xy/x'y'} \in (0,\infty)$ for every $(xy, x'y')$ in E, this is equivalent to finding real numbers (d_X, d_Y) such that

$$(\forall\ (xy, x'y') \in E)\ d_x{+}d_y{-}d_{x'}{-}d_{y'} = \ln q_{xy/x'y'}\ .$$

Index E as $\langle(x_j^0 y_j^0, x_j^1 y_j^1)\rangle_{j=1}^{|E|}$ (E is nonempty since it must include diagonal elements). Note that we seek a solution $(d_X, d_Y) \in \mathbb{R}^{X\cup Y}$ to the $|E|$ linear equations

$$(\forall j)\ (\mathbf{1}_{x_j^0}{+}\mathbf{1}_{y_j^0}{-}\mathbf{1}_{x_j^1}{-}\mathbf{1}_{y_j^1}){\cdot}(d_X, d_Y) = \ln q_{x_j^0 y_j^0/x_j^1 y_j^1}$$

($\mathbf{1}_w \in \mathbb{R}^{X\cup Y}$ is the unit vector of $w \in X{\cup}Y$). Since the coefficients on the variables are all rational, this system of linear equations has a solution if (and only if), for all integers $\langle \ell_j\rangle_{j=1}^{|E|}$,

$$\Sigma_j\ \ell_j\ (\mathbf{1}_{x_j^0}{+}\mathbf{1}_{y_j^0}{-}\mathbf{1}_{x_j^1}{-}\mathbf{1}_{y_j^1}) = \mathbf{0}$$
$$\text{implies}\ \ \Sigma_j\ \ell_j \ln q_{x_j^0 y_j^0/x_j^1 y_j^1} = 0\ .$$

Accordingly, consider any integers $\langle \ell_j\rangle_{j=1}^{|E|}$ such that

$$\Sigma_j\ \ell_j\ (\mathbf{1}_{x_j^0}{+}\mathbf{1}_{y_j^0}{-}\mathbf{1}_{x_j^1}{-}\mathbf{1}_{y_j^1}) = \mathbf{0}\ . \tag{21}$$

First, define $\langle(\theta_j, \theta_j')\rangle_{j=1}^{|E|}$ by

$$(\theta_j, \theta_j') = \begin{pmatrix} (0,1) & \text{if } \ell_j \geq 0 \\ (1,0) & \text{if } \ell_j < 0 \end{pmatrix}\ .$$

Note that (21) is equivalent to

$$\Sigma_j\ |\ell_j|\ (\mathbf{1}_{x_j^{\theta_j}}{+}\mathbf{1}_{y_j^{\theta_j}}{-}\mathbf{1}_{x_j^{\theta_j'}}{-}\mathbf{1}_{y_j^{\theta_j'}}) = \mathbf{0}\ , \tag{22}$$

and that the symmetry of E implies

$$(\forall j)\ (x_j^{\theta_j} y_j^{\theta_j}, x_j^{\theta_j'} y_j^{\theta_j'}) \in E\ . \tag{23}$$

Next, define $m^\star = \Sigma_j |\ell_j|$ and define $\langle(x_i y_i, x_i^\star y_i^\star)\rangle_{i=1}^{m^\star}$ by

$$(x_i y_i, x_i^\star y_i^\star) = (x_j^{\theta_j} y_j^{\theta_j}, x_j^{\theta_j'} y_j^{\theta_j'})$$
$$\text{for } i \in \{\ \Sigma_{k=1}^{j-1}|\ell_k|{+}1,\ \Sigma_{k=1}^{j-1}|\ell_k|{+}2, \ldots\ \Sigma_{k=1}^{j-1}|\ell_k|{+}|\ell_j|\ \}$$

(at $j = 1$, $\Sigma_{k=1}^{j-1}|\ell_k| = 0$; at any j, the set is empty if $\ell_j = 0$; and at $j = |E|$, $\Sigma_{k=1}^{j-1}|\ell_k|+|\ell_j| = \Sigma_{j=1}^{|E|}|\ell_j| = \Sigma_j|\ell_j| = m^\star$). Note that (22) is equivalent to

$$\Sigma_{i=1}^{m^\star} (\mathbf{1}_{x_i}+\mathbf{1}_{y_i}-\mathbf{1}_{x_i^\star}-\mathbf{1}_{y_i^\star}) = \mathbf{0} \ , \tag{24}$$

and that (23) yields

$$(\forall i)\ (x_iy_i, x_i^\star y_i^\star) \in E \ . \tag{25}$$

Finally, note that (24) is equivalent to

$$\Sigma_{i=1}^{m^\star} \mathbf{1}_{x_i} = \Sigma_{i=1}^{m^\star} \mathbf{1}_{x_i^\star} \ \text{ and } \ \Sigma_{i=1}^{m^\star} \mathbf{1}_{y_i} = \Sigma_{i=1}^{m^\star} \mathbf{1}_{y_i^\star}$$

($\mathbf{1}_x \in \mathbb{R}^X$ is the unit vector of $x \in X$ and $\mathbf{1}_y \in \mathbb{R}^Y$ is the unit vector of $y \in Y$), which is in turn equivalent to the existence of permutations $\sigma^\star$ and $\tau^\star$ of $\{1, 2, ... m^\star\}$ such that

$$(\forall i)\ x_i = x^\star_{\sigma^\star(i)} \text{ and } y_i = y^\star_{\tau^\star(i)} \ .$$

The producthood of Q_{XY} implies

$$(\forall \langle x^iy^i\rangle_{i=1}^m)\ 1 \in \odot_{i=1}^m q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

for any $m \geq 1$ and any permutations σ and τ of $\{1, 2, ...m\}$ (this follows from (8) by defining $\sigma(0) = 0$ and $\tau(0) = 0$). By applying this at $m^\star$, $\sigma^\star$, $\tau^\star$, and $\langle x_i^\star y_i^\star\rangle_{i=1}^{m^\star}$, one obtains

$$1 \in \odot_{i=1}^{m^\star} q_{x^\star_{\sigma^\star(i)}y^\star_{\tau^\star(i)}/x_i^\star y_i^\star}$$

which by the definition of $\sigma^\star$ and $\tau^\star$ is equivalent to

$$1 \in \odot_{i=1}^{m^\star} q_{x_iy_i/x_i^\star y_i^\star} \ .$$

Since every $q_{x_iy_i/x_i^\star y_i^\star} \in (0, \infty)$ by (20b) and (25), this is equivalent to

$$\Pi_{i=1}^{m^\star} q_{x_iy_i/x_i^\star y_i^\star} \ = \ 1$$

and also to

$$\Sigma_{i=1}^{m^\star} \ln q_{x_iy_i/x_i^\star y_i^\star} \ = \ 0 \ .$$

By the definitions of $m^\star$ and $\langle(x_iy_i, x_i^\star y_i^\star)\rangle_{i=1}^{m^\star}$, this is equivalent to

$$\Sigma_j|\ell_j| \ln q_{x_j^{\theta_j}y_j^{\theta_j}/x_j^{\theta'_j}y_j^{\theta'_j}} = 0 \ ,$$

which is equivalent to

$$\Sigma_{j|\ell_j<0}\ (-\ell_j) \ln q_{x_j^{\theta_j}y_j^{\theta_j}/x_j^{\theta'_j}y_j^{\theta'_j}} + \Sigma_{j|\ell_j\geq 0}\ \ell_j \ln q_{x_j^{\theta_j}y_j^{\theta_j}/x_j^{\theta'_j}y_j^{\theta'_j}} = 0 \ .$$

By the definition of $\langle(\theta_j,\theta'_j)\rangle_{j=1}^{|E|}$, this is equivalent to

$$\Sigma_{j|\ell_j<0}\ (-\ell_j)\ln q_{x_j^1y_j^1/x_j^0y_j^0}+\Sigma_{j|\ell_j\geq 0}\ \ell_j\ln q_{x_j^0y_j^0/x_j^1y_j^1}=0\ ,$$

which by the reciprocity of Remark 2.1 is equivalent to

$$\Sigma_{j|\ell_j<0}\ \ell_j\ln q_{x_j^0y_j^0/x_j^1y_j^1}+\Sigma_{j|\ell_j\geq 0}\ \ell_j\ln q_{x_j^0y_j^0/x_j^1y_j^1}=0$$

and also to

$$\Sigma_j\ \ell_j\ln q_{x_j^0y_j^0/x_j^1y_j^1}=0\ .$$

A.2. Sufficiency of a Product Representation.

This half of the theorem resembles a limiting argument of Kohlberg and Reny (1997, page 305, Proof of Theorem 2.10, first paragraph).

Lemma A.1. *Suppose that* $\{t_j^n\}_j$ *is a finite set of sequences in* $(0,\infty)$*, that each* $\lim_n t_j^n$ *exists in* $[0,\infty]$*, and that* $\lim_n\Pi_j t_j^n$ *exists in* $[0,\infty]$*. Then* $\lim_n\Pi_j t_j^n\in\odot_j\lim_n t_j^n$*.*

Proof. If each $\lim_n t_j^n<\infty$, then $\lim_n\Pi_j t_j^n=\Pi_j\lim_n t_j^n$ and $\odot_j\lim_n t_j^n=\{\Pi_j\lim_n t_j^n\}$. If some $\lim_n t_j^n=\infty$ and every $\lim_n t_j^n>0$, then $\lim_n\Pi_j t_j^n=\infty$ and $\odot_j\lim_n t_j^n=\{\infty\}$. Finally, if some $\lim_n t_j^n=\infty$ and some other $\lim_n t_j^n=0$, the conclusion $\lim_n\Pi_j t_j^n\in\odot_j\lim_n t_j^n$ is vacuous because $\odot_j\lim_n t_j^n=[0,\infty]$. □

If Q_{XY} is represented by some $[c_xc_yn^{e_x+e_y}]$, it must be a product because

$$(\forall xy)\quad q_{xy/xy}=\lim_n\frac{c_xc_yn^{e_x+e_y}}{c_xc_yn^{e_x+e_y}}=1\ ,$$

and because Lemma A.1 yields

$$(\forall\langle x^iy^i\rangle_{i=0}^m)\quad q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}}=\lim_n\ \frac{c_{x^0}c_{y^0}n^{e_{x^0}+e_{y^0}}}{c_{x^{\sigma(0)}}c_{y^{\tau(0)}}n^{e_{x^{\sigma(0)}}+e_{y^{\tau(0)}}}}=$$

$$\lim_n\ \Pi_{i=1}^m\ \frac{c_{x^{\sigma(i)}}c_{y^{\tau(i)}}n^{e_{x^{\sigma(i)}}+e_{y^{\tau(i)}}}}{c_{x^i}c_{y^i}n^{e_{x^i}+e_{y^i}}}\in\odot_{i=1}^m\ \lim_n\ \frac{c_{x^{\sigma(i)}}c_{y^{\tau(i)}}n^{e_{x^{\sigma(i)}}+e_{y^{\tau(i)}}}}{c_{x^i}c_{y^i}n^{e_{x^i}+e_{y^i}}}$$

$$=\odot_{i=1}^m\ q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

for all $m\geq 1$ and all permutations σ and τ.

Appendix B. Synthesis with the Literature

This appendix is tangential to the rest of the paper. It was introduced within Section 1.2 and is summarized by Section B.3.

B.1. Approximation by Full-Support Distributions

Remarks B.1, B.3, and B.4 resemble Rearrangement 2.2, Theorem 3.4, and Corollary 4.1. The latter half of this section will use these remarks to discuss Kreps and Wilson (1982), McLennan (1989b), and Kohlberg and Reny (1997).

A sequence π^n_Z of full-support probability distributions over Z is said to *approximate* the table Q_Z if

$$(\forall z, z')\ q_{z/z'} = \lim_n\ \pi^n_z/\pi^n_{z'}\ .$$

For example,

z'				
r	0	0	1	
m	.2	1	∞	
ℓ	1	5	∞	
	ℓ	m	r	z

is approximated by $(\frac{1}{n+6}, \frac{5}{n+6}, \frac{n}{n+6})$ just as it is represented by $(1, 5, n)$.

Remark B.1. *A table Q_Z is a dispersion iff it is approximated by some π^n_Z.*

Proof. A dispersion has a representation $c_z n^{e_z}$ by Rearrangement 2.2. Set each $\pi^n_z = c_z n^{e_z}/(\Sigma_{z'} c_{z'} n^{e_{z'}})$. Conversely, if Q_Z is approximated by π^n_Z, then (4a) follows from $q_{z/z'} = \lim_n \pi^n_z/\pi^n_z = 1$, and (4b) follows from

$$(\forall z, z')\ q_{z/z'} = \lim_n(\pi^n_z/\pi^n_{z'}) =$$
$$\lim_n(\pi^n_z/\pi^n_{z''})(\pi^n_{z''}/\pi^n_{z'}) \in \lim_n(\pi^n_z/\pi^n_{z''}) \odot \lim(\pi^n_{z''}/\pi^n_{z'}) = q_{z/z''} \odot q_{z''/z'}$$

by Lemma A.1. □

Although Remark B.1 can thus be regarded as a corollary of Rearrangement 2.2, it is close to Myerson (1986, Theorem 1) and is equivalent to McLennan (1989b, Lemma 2.1) (dispersionhood is equivalent to his (2.5) by the text around (28), and approximation is used to define his conditional system).

Lemma B.2. *A table Q_{XY} is both a dispersion and a preproduct of Q_X and Q_Y iff it is approximated by some $[\pi^n_{xy}]$ such that $(\forall y^\circ)$ $[\pi^n_{xy^\circ}]$ approximates Q_X and $(\forall x^\circ)$ $[\pi^n_{x^\circ y}]$ approximates Q_Y.*

Proof. Since a table which is both a dispersion and a preproduct of Q_X and Q_Y has a representation $[c_{xy} n^{e_{xy}}]$ satisfying Remark 3.2's properties, this remark's properties are satisfied by the approximation

$[\pi^n_{xy}] = [c_{xy}n^{e_{xy}}/(\Sigma_{x'y'}c_{x'y'}n^{e_{x'y'}})]$. Conversely, the existence of such an approximation yields dispersionhood by Remark B.1 and yields pre-producthood by

$$(\forall xy, x'y')\ q_{xy/x'y'} = \lim_n \frac{\pi^n_{xy}}{\pi^n_{x'y'}} =$$

$$\lim_n \frac{\pi^n_{xy}}{\pi^n_{x'y}} \frac{\pi^n_{x'y}}{\pi^n_{x'y'}} \in \lim_n \frac{\pi^n_{xy}}{\pi^n_{x'y}} \odot \lim_n \frac{\pi^n_{x'y}}{\pi^n_{x'y'}} = q_{x/x'} \odot q_{y/y'}$$

and Lemma A.1. □

Remark B.3. *A table Q_{XY} is a product iff it is approximated by some $[\pi^n_x \pi^n_y]$. (The product approximated by $[\pi^n_x \pi^n_y]$ has its marginals approximated by $[\pi^n_x]$ and $[\pi^n_y]$.)*

Proof. A product has a representation $[c_x c_y n^{e_x+e_y}]$ by Theorem 3.4. Set each

$$\pi^n_x = \frac{c_x n^{e_x}}{\Sigma_{x'} c_{x'} n^{e_{x'}}} \text{ and } \pi^n_y = \frac{c_y n^{e_y}}{\Sigma_{y'} c_{y'} n^{e_{y'}}} . \tag{26}$$

Conversely, $(\forall xy)\ q_{xy} = \lim_n (\pi^n_x \pi^n_y)/(\pi^n_x \pi^n_y) = 1$ and Lemma A.1 yields

$$(\forall \langle x^i y^i \rangle_{i=0}^m)\ \ q_{x^0 y^0/x^{\sigma(0)} y^{\tau(0)}} = \lim_n \frac{\pi^n_{x^0} \pi^n_{y^0}}{\pi^n_{x^{\sigma(0)}} \pi^n_{y^{\tau(0)}}} =$$

$$\lim_n \Pi_{i=1}^m \frac{\pi^n_{x^{\sigma(i)}} \pi^n_{y^{\tau(i)}}}{\pi^n_{x^i} \pi^n_{y^i}} \in \odot_{i=1}^m \lim_n \frac{\pi^n_{x^{\sigma(i)}} \pi^n_{y^{\tau(i)}}}{\pi^n_{x^i} \pi^n_{y^i}} = \odot_{i=1}^m q_{x^{\sigma(i)} y^{\tau(i)}/x^i y^i} \tag{27}$$

for any $m \geq 1$ and any permutations σ and τ. (The remark's second sentence follows from Lemma B.2, the fact that $[\pi^n_x]$ approximates the same dispersion as any $[\pi^n_x \pi^n_{y^\circ}]$, and the fact that $[\pi^n_y]$ approximates the same dispersion as any $[\pi^n_{x^\circ} \pi^n_y]$.) □

Although Remark B.3 can thus be regarded as a corollary of Theorem 3.4, it is originally due to Kohlberg and Reny (1997, Theorem 2.10) (see the paragraph before Remark B.6).

Remark B.4. *$\{p_H\}_{H \in \mathcal{H}}$ is consistent with (p_X, p_Y) iff there is a pair of sequences (π^n_X, π^n_Y) such that $p_X = \lim_n \pi^n_X$, $p_Y = \lim_n \pi^n_Y$, and*

$$(\forall H)\ p_H = \lim_n \left[\frac{\pi^n_x \pi^n_y}{\Sigma_{x'y' \in H}\ \pi^n_{x'} \pi^n_{y'}} \right]_{xy \in H} .$$

Proof. This paragraph shows that, if $Q_{\bar{Z}}$ is approximated by $\pi^n_{\bar{Z}}$, then p_Z is induced by $Q_{\bar{Z}}$ iff

$$(\forall z \in Z)\ p_z = \lim_n \frac{\pi^n_z}{\Sigma_{z' \in Z} \pi^n_{z'}}$$

(which is equivalent to $p_Z = \lim_n \pi_Z^n$ when $Z = \bar{Z}$). Accordingly, assume that $Q_{\bar{Z}}$ is approximated by $\pi_{\bar{Z}}^n$. Let $z^\star \in Z$ be such that $(\forall z' \in Z)$ $q_{z'/z^\star} < \infty$. Note that

$$(\forall z \in Z) \quad \frac{q_{z/z^\star}}{\Sigma_{z' \in Z} q_{z'/z^\star}} = \frac{\lim_n \pi_z^n / \pi_{z^\star}^n}{\Sigma_{z' \in Z} \lim_n \pi_{z'}^n / \pi_{z^\star}^n} =$$

$$\lim_n \frac{\pi_z^n / \pi_{z^\star}^n}{\Sigma_{z' \in Z} \pi_{z'}^n / \pi_{z^\star}^n} = \lim_n \frac{\pi_z^n}{\Sigma_{z' \in Z} \pi_{z'}^n}$$

because $(\forall z' \in Z)$ $q_{z'/z^\star} < \infty$ implies that all these limits exist in $\mathbb{R}$. Hence, p_Z is induced by $Q_{\bar{Z}}$

$$\text{iff} \quad (\forall z \in Z) \; p_z = \frac{q_{z/z^\star}}{\Sigma_{z' \in Z} q_{z'/z^\star}}$$

$$\text{iff} \quad (\forall z \in Z) \; p_z = \lim_n \frac{\pi_z^n}{\Sigma_{z' \in Z} \pi_{z'}^n} \; .$$

Necessity of (π_X^n, π_Y^n). Since a sequence of full-support product distributions $[\pi_x^n \pi_y^n]$ is equivalent to a pair of sequences (π_X^n, π_Y^n), Remark B.3 shows that the definition of consistency (Section 4.1) implies the existence of a pair of sequences (π_X^n, π_Y^n) such that p_X is induced by the Q_X that is approximated by π_X^n, p_Y is induced by the Q_Y that is approximated by π_Y^n, and each p_H is induced by the Q_{XY} that is approximated by $\pi_X^n \pi_Y^n$. This (π_X^n, π_Y^n) satisfies the remark's properties by the previous paragraph applied at p_X, again at p_Y, and again at each p_H.

Sufficiency of (π_X^n, π_Y^n). Suppose that (π_X^n, π_Y^n) satisfies the remark's properties. Since $[0, \infty]$ is compact, there exists a subsequence (π_X^m, π_Y^m) such that every $\lim_m \pi_x^m / \pi_{x'}^m$, every $\lim_m \pi_y^m / \pi_{y'}^m$, and every $\lim_m (\pi_x^m \pi_y^m) / (\pi_{x'}^m \pi_{y'}^m)$ exists. Hence (π_X^m, π_Y^m) not only satisfies the remark's properties, but also approximates some Q_X, Q_Y, and Q_{XY}. Thus, by several applications of the proof's first paragraph, one finds that p_X is induced by the Q_X that is approximated by π_X^n, that p_Y is induced by the Q_Y that is approximated by π_Y^n, and that each p_H is induced by the Q_{XY} that is approximated by $\pi_X^n \pi_Y^n$. This implies consistency by Remark B.3. □

Remark B.4 is important because it demonstrates the equivalence of this paper's concept of consistency and Kreps and Wilson (1982, page 872)'s concept of consistency (their concept is the same as a (π_X^n, π_Y^n) satisfying the remark's properties).

McLennan (1989b) defines a product concept en route to a beautiful equilibrium existence proof. This paper coincides with his to the extent that the two papers' specifications of relative probability and producthood are equivalent.

Concerning the specification of relative probability, McLennan's conditional system is the logarithm of a dispersion. This equivalence is identical to his Lemma 2.1 because the remainder of this paragraph shows that a table Q_{XY} is a dispersion iff its logarithm satisfies his equation (2.5). In particular, that equation at $\mu = \ln Q_X$ consists of the two statements

$$(\forall x, x')\ \ln q_{x/x'} = -\ln q_{x'/x}\ , \text{ and} \tag{28a}$$

$$(\forall x, x', x'')\ \ln q_{x/x'} = \ln q_{x/x''} + \ln q_{x''/x'} \tag{28b}$$
$$\text{if } (\ln q_{x/x''}, \ln q_{x''/x'}) \text{ is neither } (-\infty, \infty) \text{ nor } (\infty, -\infty)\ .$$

(28a) is equivalent to the reciprocity of Remark 2.1 and (28b) is equivalent to the dispersion criterion (4b). Hence (28) is equivalent to dispersionhood because (28a) implies a unit diagonal (4a) and conversely because dispersionhood implies reciprocity by Remark 2.1.

Concerning the specification of producthood, consider McLennan (1989b, page 170)'s definition of the set Ψ in the special case that his $n = 2$, his $S_1 = X$, and his $S_2 = Y$. Since a conditional system is the logarithm of a dispersion, his Ψ consists of the logarithms of all dispersions that can be approximated by some $[\pi_x^n \pi_y^n]$. Hence, by Remark B.3, his Ψ consists of the logarithms of all products.

Kohlberg and Reny (1997, henceforth "KR") specify a dispersion as a random variable on a relative probability space, define producthood as strong independence, and derive a result equivalent to Remark B.3. Remark B.5 formally states the first of these three assertions, and Remark B.6 states the remaining two.

A *relative probability space* (KR, Definition 3.1) is a finite set Ω together with a function $\rho{:}\mathcal{P}(\Omega)^2 \to [0, \infty]$ satisfying

$$\rho(S, S) = 1\ , \tag{29a}$$

$$\rho(S {\cup} T, U) = \rho(S, U) + \rho(T, U) \tag{29b}$$
$$\text{if } S {\cap} T = \varnothing\ , \text{ and}$$

$$\rho(S, U) = \rho(S, T)\rho(T, U) \tag{29c}$$
$$\text{if } (\rho(S, T), \rho(T, U)) \text{ is neither } (0, \infty) \text{ nor } (\infty, 0)\ ,$$

for all S, T, and U in $\mathcal{P}(\Omega)$ (recall that $\mathcal{P}(\Omega)$ is the collection of all subsets of Ω). A *random variable* on a relative probability space (Ω, ρ) is a surjective function $\boldsymbol{z}$ from Ω onto some set Z. As in KR, use boldface for the random variable, use normal typeface for the values the random variable assumes, and denote a level curve of a random variable by

$$[z] = \{\ \omega{\in}\Omega \mid \boldsymbol{z}(\omega) = z\ \}\ .$$

Remark B.5. *A table Q_Z is a dispersion iff there exists a random variable $\boldsymbol{z}$ on a relative probability space (Ω, ρ) such that*

$$(\forall z, z')\ q_{z/z'} = \rho([z], [z'])\ .$$

(A one-page proof has been omitted.)

On a relative probability space (Ω, ρ), two random variables $\boldsymbol{x}{:}\Omega{\to}X$ and $\boldsymbol{y}{:}\Omega{\to}Y$ are *strongly independent* (KR Definition 2.7) if there exists a sequence of full-support probability measures $\mu^n{:}\mathcal{P}(\Omega){\to}(0,1]$ such that

$$(\forall S, T)\ \rho(S, T) = \lim_n\ \mu^n(S)/\mu^n(T)\ ,$$

and, on each ordinary probability space (Ω, μ^n), the random variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are independent in the ordinary sense. Remark B.6(a⇔b^{KR}) states that strong independence is equivalent to producthood. (An omitted remark states that KR's concept of weak independence is equivalent to the combination of preproducthood and dispersionhood.)

Remark B.6 also shows that KR Theorem 2.10 is equivalent to Remark B.3: (a^{KR}[2]⇔b^{KR}[2]) restates KR Theorem 2.10, and (a⇔b) restates Remark B.3. The substance of both results is that the cancellation laws are equivalent to approximation by a sequence of full-support marginals (insubstantial differences in terminology are translated by (a⇔a^{KR}) and (b⇔b^{KR})). The two papers view this equivalence from different perspectives. While KR defined strong independence in terms of approximation, this paper defined producthood in terms of cancellation laws and avoided approximation until this tangential appendix.

Remark B.6. *The following are equivalent for any table Q_{XY}.*

(a) Q_{XY} is a product.

(a^{KR}) There exists $(\Omega, \rho, \boldsymbol{x}, \boldsymbol{y})$ such that

[1] $(\forall xy, x'y')\ q_{xy/x'y'} = \rho([x]{\cap}[y], [x']{\cap}[y'])$ *and*
[2] $(\forall m, \sigma, \tau)(\forall \langle x^i y^i \rangle_{i=0}^m)\ 1 \in \odot_{i=0}^m \rho([x^{\sigma(i)}]{\cap}[y^{\tau(i)}], [x^i]{\cap}[y^i])$.

(b^{KR}) There exists $(\Omega, \rho, \boldsymbol{x}, \boldsymbol{y})$ such that

[1] $(\forall xy, x'y')\ q_{xy/x'y'} = \rho([x]{\cap}[y], [x']{\cap}[y'])$ *and*
[2] $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are strongly independent.*

(b) Q_{XY} is approximated by some $[\pi_x^n \pi_y^n]$.

(A two-page proof has been omitted.)

B.2. Nonstandard Probability Distributions

Remarks B.8 and B.10 resemble Rearrangement 2.2 and Theorem 3.4. The section's concluding paragraphs will use them to discuss Blume, Brandenburger, and Dekel (1991a) and Hammond (1994).

Let ${}^*\mathbb{R}$ be a non-standard extension of $\mathbb{R}$, and let ${}^*\mathbb{R}_{++}$ denote its positive elements. Such a ${}^*\mathbb{R}_{++}$ is an attractive place to study probability and relative probability because it contains both infinitesimal and infinite numbers and because its addition, multiplication, and division operators are always well-defined. In particular, the ill-defined expression 0∞ does not arise because neither 0 nor ∞ belong to ${}^*\mathbb{R}_{++}$. However, a related ambiguity arises: if ε is infinitesimal and K is infinite, the product εK could be infinitesimal, or finite but not infinitesimal, or infinite.

Recall that every non-standard number $a \in {}^*\mathbb{R}$ is either infinite or has a standard part, denoted $\mathrm{st}(a) \in \mathbb{R}$, which is the unique standard number which differs from a by an infinitesimal. For convenience, define the standard part of every positive non-standard number $a \in {}^*\mathbb{R}_{++}$ to be

$$\mathrm{sp}(a) = \begin{pmatrix} \mathrm{st}(a) & \text{if} & a \text{ is finite} \\ \infty & \text{if} & a \text{ is infinite} \end{pmatrix} .$$

Lemma B.7. *Suppose that* $\{a_j\}_j$ *is a finite set of numbers from some* ${}^*\mathbb{R}_{++}$. *Then* $\mathrm{sp}(\Pi_j a_j) \in \odot_j \mathrm{sp}(a_j)$.

Proof. If every a_j is finite, $\mathrm{sp}(\Pi_j a_j) = \mathrm{st}(\Pi_j a_j) = \Pi_j \mathrm{st}(a_j)$ and $\odot_j \mathrm{sp}(a_j) = \odot_j \mathrm{st}(a_j) = \{\Pi_j \mathrm{st}(a_j)\}$. If some a_j is infinite and no a_j is infinitesimal, $\mathrm{sp}(\Pi_j a_j) = \infty$ because $\Pi_j a_j$ is infinite, and $\odot_j \mathrm{sp}(a_j) = \{\infty\}$ because some $\mathrm{sp}(a_j) = \infty$ and every $\mathrm{sp}(a_j) > 0$. If some a_j is infinite and some other a_j is infinitesimal, the conclusion $\mathrm{sp}(\Pi_j a_j) \in \odot_j \mathrm{sp}(a_j)$ is vacuous because $\odot_j \mathrm{sp}(a_j) = [0, \infty]$. $\square$

A vector $a_Z \in {}^*\mathbb{R}^Z_{++}$ is said to *express* the table Q_Z if

$$(\forall z, z')\ q_{z/z'} = \mathrm{sp}(a_z / a_{z'}) \ .$$

For example,

z'				
r	0	0	1	
m	.2	1	∞	
ℓ	1	5	∞	
	ℓ	m	r	z

is expressed by $(\varepsilon, 5\varepsilon, 1)$ for any infinitesimal number ε and also by $(1, 5, K)$ for any infinite number K.

REMARK B.8. *A table Q_Z is a dispersion iff it is expressed by some a_Z.*

Proof. By Rearrangement 2.2, a dispersion is represented by some $c_Z n^{e_Z}$. Let ε be an infinitesimal in some ${}^*\mathbb{R}_{++}$, and set each $a_z = c_z \varepsilon^{-e_z}$. Conversely, if Q_Z is expressed by a_Z, then (4a) follows from $q_{z/z} = \text{sp}(a_z/a_z) = 1$, and (4b) follows from

$$(\forall z, z')\ q_{z/z'} = \text{sp}(a_z/a_{z'}) =$$
$$\text{sp}((a_z/a_{z''})(a_{z''}/a_{z'})) \in \text{sp}(a_z/a_{z''}) \odot \text{sp}(a_{z''}/a_{z'}) = q_{z/z''} \odot q_{z''/z'}$$

by Lemma B.7. □

LEMMA B.9. *A table Q_{XY} is a dispersion and a preproduct of Q_X and Q_Y iff it is expressed by some $[a_{xy}]$ such that $(\forall y^\circ)$ $[a_{xy^\circ}]$ expresses Q_X and $(\forall x^\circ)$ $[a_{x^\circ y}]$ expresses Q_Y.*

Proof. Since a table which is both a dispersion and a preproduct of Q_X and Q_Y has a representation $[c_{xy} n^{e_{xy}}]$ satisfying Remark 3.2's properties, this remark's properties are satisfied by $[a_{xy}] = [c_{xy}\varepsilon^{-e_{xy}}]$ where ε is an infinitesimal in some ${}^*\mathbb{R}_{++}$. Conversely, the existence of such an $[a_{xy}]$ yields dispersionhood by Remark B.8 and yields preproducthood by

$$(\forall xy, x'y')\ q_{xy/x'y'} = \text{sp}(a_{xy}/a_{x'y'}) =$$
$$\text{sp}((a_{xy}/a_{x'y})(a_{x'y}/a_{x'y'})) \in \text{sp}(a_{xy}/a_{x'y}) \odot \text{sp}(a_{x'y}/a_{x'y'}) = q_{x/x'} \odot q_{y/y'}$$

and Lemma B.7. □

REMARK B.10. *A table Q_{XY} is a product iff it is expressed by some $[a_x a_y]$. (The product expressed by $[a_x a_y]$ has its marginals expressed by $[a_x]$ and $[a_y]$.)*

Proof. By Theorem 3.4, a product is represented by some $[c_x c_y n^{e_x+e_y}]$. Let ε be an infinitesimal in some ${}^*\mathbb{R}_{++}$, and set

$$[a_x] = [c_x \varepsilon^{-e_x}] \text{ and } [a_y] = [c_y \varepsilon^{-e_y}]\ . \tag{30}$$

Conversely, $(\forall xy)\ q_{xy/xy} = \text{sp}((a_x a_y)/(a_x a_y)) = 1$, and Lemma B.7 yields

$$(\forall \langle x^i y^i \rangle_{i=0}^m)\ q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} = \text{sp}\left(\frac{a_{x^0y^0}}{a_{x^{\sigma(0)}y^{\tau(0)}}}\right) = \tag{31}$$
$$\text{sp}\left(\Pi_{i=1}^m \frac{a_{x^{\sigma(i)}y^{\tau(i)}}}{a_{x^iy^i}}\right) \in \odot_{i=1}^m \text{sp}\left(\frac{a_{x^{\sigma(i)}y^{\tau(i)}}}{a_{x^iy^i}}\right) = \odot_{i=1}^m q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

for any $m \geq 1$ and any permutations σ and τ. (The remark's second sentence follows from Lemma B.9, the fact that $[a_x]$ expresses the same dispersion as any $[a_x a_{y^\circ}]$, and the fact that $[a_y]$ expesses the same dispersion as any $[a_{x^\circ} a_y]$.) □

Recall from Section 2.2 that a dispersion is equivalent to a lexicographic conditional probability system as defined by Blume, Brandenburger, and Dekel (1991a, Definition 5.2). In light of this equivalence, their Theorem 5.3 axiomatizes the preferences over a mixture space that can be characterized by a dispersion over the state space together with a function over the set of consequences (they provide a number of other results, including an axiomatization of more general preferences). If one regards a nonstandard probability distribution as a means of expressing a dispersion (as in their Theorem 5.3 and throughout this section), then Remark B.10 shows that the producthood considered throughout this paper is equivalent to the producthood of their Definition 7.1 (translate their Ω^1 as X, their Ω^2 as Y, and their $p^1(\omega^1){\times}p^2(\omega^2)$ as $a_x a_y$).

Hammond (1994, page 45) suggests that Blume, Brandenburger, and Dekel (1991a) can be further refined by seeking a comparatively simple nonstandard extension $^*\mathbb{R}$. Thus he introduces rational probability functions, which are constructed as ratios of polynomials in a single infinitesimal ε. This paper follows Hammond's lead in seeking comparatively simple nonstandard numbers.

B.3. Summary

Theorem 3.4, Remark B.3, and Remark B.10 together state that the following are equivalent for any table Q_{XY}.

Q_{XY} is a product (that is, obeys the cancellation laws).
Q_{XY} is represented by some $[c_x c_y n^{e_x+e_y}]$.
Q_{XY} is approximated by some $[\pi_x^n \pi_y^n]$.
Q_{XY} is expressed by some $[a_x a_y]$.

The key is the difficult half of Theorem 3.4, namely, that any product is represented by some product of representations, $[c_x c_y n^{e_x+e_y}]$. The remainder is then straightforward. Any product of representations can be regarded as a product of approximations, $[\pi_x^n \pi_y^n]$, by (26), and any of these approximates a table satisfying the cancellation laws by (27). Similarly, any product of representations can be regarded as a product of nonstandard probability distributions, $[a_x a_y]$, by (30), and any of these expresses a table satisfying the cancellation laws by (31).

References

Blume, L., A. Brandenburger, and E. Dekel (1991a): "Lexicographic Probabilities and Choice Under Uncertainty," *Econometrica*, 59, 61–79.

——— (1991b): "Lexicographic Probabilities and Equilibrium Refinements," *Econometrica*, 59, 81–98.

Hammond, P. J. (1994): "Elementary Non-Archimedean Representations of Probability for Decision Theory and Games," in *Patrick Suppes: Scientific Philosopher, Volume 1*, ed. by P. Humphreys, pp. 25–61. Kluwer.

Kohlberg, E., and P. J. Reny (1997): "Independence on Relative Probability Spaces and Consistent Assessments in Game Trees," *Journal of Economic Theory*, 75, 280–313.

Krantz, D. H., R. D. Luce, P. Suppes, and A. Tversky (1971): *Foundations of Measurement, Volume I: Additive and Polynomial Representations*. Academic Press.

Kreps, D. M., and G. Ramey (1987): "Structural Consistency, Consistency, and Sequential Rationality," *Econometrica*, 55, 1331–1348.

Kreps, D. M., and R. Wilson (1982): "Sequential Equilibria," *Econometrica*, 50, 863–894.

McLennan, A. (1989a): "The Space of Conditional Systems is a Ball," *International Journal of Game Theory*, 18, 125–139.

——— (1989b): "Consistent Conditional Systems in Noncooperative Game Theory," *International Journal of Game Theory*, 18, 141–174.

Monderer, D., D. Samet, and L. S. Shapley (1992): "Weighted Values and the Core," *International Journal of Game Theory*, 21, 27–39.

Myerson, R. B. (1986): "Multistage Games with Communication," *Econometrica*, 54, 323–358.

Scott, D. (1964): "Measurement Structures and Linear Inequalities," *Journal of Mathematical Psychology*, 1, 233–247.

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