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A Median Voter Theorem for Postelection Politics

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Abstract

We analyze a model of 'postelection politics', in which (unlike in the more common Downsian models of 'preelection politics') politicians cannot make binding commitments prior to elections. The game begins with an incumbent politician in office, and voters adopt reelection strategies that are contingent on the policies implemented by the incumbent. We generalize previous models of this type by introducing heterogeneity in voters' ideological preferences, and analyze how voters' reelection strategies constrain the policies chosen by a rent-maximizing incumbent. We first show that virtually any policy (and any feasible level of rent for the incumbent) can be sustained in a Nash equilibrium. Then, we derive a 'median voter theorem': the ideal point of the median voter, and the minimum feasible level of rent, are the unique outcomes in any strong Nash equilibrium. We then introduce alternative refinements that are less restrictive. In particular, Ideologically Loyal Coalition-proof equilibrium also leads uniquely to the median outcome.

1　Introduction

Following the approach of Downs (1957), most political economy models assume that electoral promises made by political candidates are binding and enforceable (either directly, or implicitly through reputational or other mechanisms). Persson and Tabellini (2000) label this approach the study of "preelection politics". In these models, the primary role of elections is to choose policies *via* the choice of candidates who are irrevocably committed to a known policy stance.[1] The aim is to predict the policy that will be implemented (or, equivalently, the candidate who will be elected), given that voters have heterogeneous preferences over policy. The most famous result in this vein is of course the median voter theorem. As is well known, this depends crucially on the assumption that electoral promises are binding (e.g. Alesina, 1988). This naturally raises the question of how politics can be modeled in contexts where precommitment is unavailable.

There exists an alternative tradition in political economy of what Persson and Tabellini (2000) call "postelection politics". This approach assumes that electoral promises are not enforceable, and analyzes how voters' backward-looking behavior in deciding whether to reelect incumbent politicians can constrain the set of policies chosen by those incumbents. Barro (1973) and Ferejohn (1986) develop models in this tradition that involve moral hazard - the incumbent's policy choice is observed with noise by voters, who adopt retrospective voting rules that seek to discipline the incumbent. Banks and Sundaram (1993, 1998) generalize this framework to consider politicians of different types, and so incorporate adverse selection as well as moral hazard. More recently, Persson, Roland and Tabellini (2000; hereafter PRT) use a symmetric information version of this framework as the basis of a comparative analysis of political institutions. The incompleteness of political contracts gives incumbents some discretionary power while in office. Incumbents are thus able to extract a positive level of rent in equilibrium, even in the absence of the informational rents available in the asymmetric information models.

These models of postelection politics all assume that voters are homogeneous in their policy preferences (or at least that voters coordinate their strategies perfectly). The aim of this paper is to generalize the postelection politics framework to incorporate voter heterogeneity in policy preferences (along some dimensional that is orthogonal to rents). Like PRT, we examine the symmetric information case, and

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[1]In addition to the Downsian framework, models of probabilistic political competition (e.g. Lindbeck and Weibull 1987) can also be regarded as falling within the category of preelection politics.

thus ignore issues of moral hazard and adverse selection. Our framework involves an infinite sequence of periods, within each of which the following game is played between an incumbent officeholder and a finite set of voters. Voters noncooperatively choose their (retrospective) reelection strategies (specifying a minimum level of utility that is required for them to vote for the incumbent's reelection). The incumbent chooses a policy, which involves both a level of rent to extract, and a point along a one-dimensional policy space (similar to that of Downsian models). Then, elections are held, pitting the incumbent against an exogenously chosen challenger. Politicians are assumed to maximize the present value of rents they obtain from office, while voters all wish to minimize rents, but have heterogeneous preferences over the other policy dimension (which we label "ideology").

While extremely simple, this framework allows us to pose the question of what (ideological) policy can be expected to prevail in equilibrium, in addition to determining the equilibrium level of rent. In contrast, the previous literature has focused only on the latter issue. We first characterize the set of Nash equilibrium outcomes, showing that virtually any point in the policy space can be sustained in equilibrium. This is in sharp contrast to the prediction of the Downsian model of preelection politics that the median voter's preferred outcome will be implemented. Moreover, any feasible level of rent can be extracted by the incumbent in equilibrium. In the existing literature on postelection politics (e.g. PRT), it is assumed that coordination among voters will force the incumbent down to the lowest level of rent consistent with the incumbent's discretionary authority. In contrast, our results show that once we introduce heterogeneity among voters, elections do not necessarily place any constraints on incumbents' rents, even when all voters have identical preferences over the level of rents.

We then refine the set of Nash equilibria, and show that there exists a unique policy that can be supported as a strong Nash equilibrium (in the sense of Aumann (1959)). This policy combines the median voter's preferred ideological outcome and the minimum level of rent that is consistent with the incumbent's short-term discretionary authority. This constitutes our "median voter theorem"; it can be viewed as a counterpart to the analogous result in the theory of preelection politics. However, it relies on a more restrictive solution concept, in particular, on deviations that may not be self-enforcing, and hence on binding commitments (or at least implicit conventions) among voters.

To address this issue, we analyze the coalition-proof Nash equilibria (Bernheim, Peleg, and Whinston, 1987) of the postelection politics game (thus considering only

self-enforcing deviations). While there exist some nonmedian policies that are supported by coalition-proof equilibria, we introduce a refinement, based on "ideological loyalty" in voting strategies, and show that the median voter outcome is the unique policy supported by a coalition-proof equilibrium in ideologically loyal strategies.

The main contribution of this paper is to present a median voter result for postelection politics that can serve as a counterpart to analogous results in the study of preelection politics. There are a number of significant lessons to be drawn from this analysis. Most fundamentally, the game of postelection politics that we analyze yields outcomes that converge on the center of the distribution of voters' preferences; moreover, the rents that incumbent politicians can extract while in office are restricted to the minimal level consistent with the contractual incompleteness of political constitutions. While these results are encouraging, it should be emphasized that they rely on stringent requirements in terms of the enforceability of agreements and the existence of communication or coordination though implicit conventions among voters. A Nash equilibrium outcome alone will not, in general, yield ideological policies that reflect the median voter's preferences, nor will it constrain the rent-seeking of incumbent politicians. Our results thus provide some grounds for pessimism about the ability of electoral politics to constrain politicians' opportunistic behavior in circumstances where policy platforms are not binding.

This paper is related to a number of strands within the political economy literature, in addition to those already mentioned. Our framework, with a large number of voters whose strategic choices affect the incentives facing the incumbent, is analogous in some respects to common agency models of lobbying (e.g. Grossman and Helpman, 1994; Dixit, Grossman and Helpman, 1997; Dharmapala, 1999). In these models, there are many principals (lobby groups) with conflicting interests concerning a policy chosen by the agent (the government). Using the menu auction approach of Bernheim and Whinston (1986), common agency lobbying models allow lobby groups to express their willingness-to-pay for different policies by offering the government payment schedules, conditioned on the implemented policy. We do not adopt this type of framework here, because voters (unlike lobby groups) can only make a binary choice, namely whether or not to reelect the incumbent. Thus, while we also address a situation of common agency, the strategies available to the principals are very different.

In our model, there is a unidimensional policy issue ("ideology") as well as a second dimension - rents - over which voters have common preferences. There has been some analysis in the political science literature of Downsian models with this

type of policy space; the dimension along which voters agree is termed a "valence" issue (e.g. Groseclose, 2001). Such models, however, have been developed within the tradition of preelection politics, assuming binding promises by candidates.

The rest of the paper proceeds as follows. The basic model is presented in Section 2; then, the Nash equilibria are characterized in Section 3. The median voter result is derived in Section 4, and alternative equilibrium refinements are analyzed in Section 5. Section 6 discusses the implications, and concludes.

2   The Model

2.1   Basic Assumptions

The agents in our model consist of an odd, finite number of voters N, where $N \geq 5$, and a set of politicians. One of the politicians (hereafter referred to as the incumbent) is exogenously chosen to be in office at the beginning of the game (the other politicians are referred to as the challengers). The incumbent's task is to implement a policy, chosen from the set of feasible policies. This set, denoted   , is assumed to be a compact, convex subset of $\mathbb{R}^2$. A typical policy, denoted $(x, r) \in$, consists of two elements:

i) $x \in \mathbb{R}$ is a unidimensional general policy space; this can be viewed as analogous to the policy space in standard Downsian models of preelection politics, and can be interpreted, for example, as "ideology."

ii) $r \in [0, r^{\max}]$ is the "rent" captured by the incumbent while in office; this can be interpreted as the private benefits of officeholding. Rents are nonnegative, and bounded above by some finite amount $r^{\max}$ that represents the maximum amount of rent that an incumbent can extract while in office.

For a given general policy x, and strictly positive r, it is assumed that it is always possible to implement any lower level of r (that is, the set of feasible policies does not constrain reductions in r, unless $r = 0$):

$$(x, r) \in \quad \text{and} \quad 0 \leq r' < r \quad \Rightarrow \quad (x, r') \in \quad . \tag{1}$$

In the basic Downsian model of preelection politics, candidates for political office are assumed to care simply about winning office.[2] Similarly, existing models of postelection politics (e.g. PRT) assume that politicians (including the incumbent) are simply interested in maximizing rents. In essence, we adopt this assumption:

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[2] There have, however, been extensions that analyze the behavior of policy-motivated politicians (e.g. Wittman, 1977; Calvert, 1985) - see Section 6 below.

thus, the objective of the incumbent is to maximize the present value of rents from office. Moreover, all politicians have this aim, conditional on being in office - a nonincumbent politician who is elected to office will also seek to maximize the present value of rents from office. Hence, an incumbent's per-period payoff is simply r. It is assumed that the default payoff for a politician who is not in office is 0. Politicians are assumed to have a discount rate of $\delta \in (0, 1)$.

In a model such as that of PRT, the assumption of rent-maximization leads incumbents to well-defined preferences over the feasible policies. Here, we introduce an additional policy dimension x, that is of no direct concern to politicians. Hence, the choice of x by a purely rent-maximizing politician will not be well-defined (as she would be indifferent among all policies in the x-dimension for given r). In other words, the incumbent's best response may be a correspondence and not a function of voters' reelection strategies. To avoid such mathematical complexities, we introduce the following tie-breaking assumption: if the incumbent is indifferent between (x_1, r) and (x_2, r) with $x_1 \neq x_2$, there exists some exogenously given selection criterion that determines an unique choice of x. Moreover, this selection criterion is identical for all politicians (i.e. for the incumbent and her potential challengers alike). This assumption does not amount to assuming policy-motivated politicians (e.g. Wittman, 1977); rather, it resembles lexicographic preferences. The incumbent prefers the policy with the highest r, but for given r has some means of choosing among the feasible values of x. Hence, politicians' preferences over policy define a complete order over   (rather than simply a partial order); that is, the incumbent is never indifferent between two distinct policies. This assumption greatly improves the tractability of the model, but does not fundamentally affect the results. It should be remembered that we show below that virtually any policy can be sustained in a Nash equilibrium; thus, allowing the incumbent to be indifferent among different x's would only reinforce this.

Voter $i, i \in \{1, ..., N\}$, is assumed to have preferences over the 2-dimensional policy space described by $u_i(x, r) = v_i(x) - r$, where functions $v_i(.)$ (and therefore functions $u_i(., .)$) are continuous and strictly quasi-concave.[3] For some fixed level of rent, an ideal point for voter i along the x-dimension is defined by:

$$\max_{(x,r) \in} \quad u_i(x, r) \qquad s.t. \ r \geq \bar{r} \tag{2}$$

(i.e. $x_i^* = \arg\max_x u_i(x, r)$, subject to $r \geq \bar{r}$). Since u_i is continuous and   is

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[3]Most of results still hold without additively separable utility functions for voters. This assumption has only been made for ease of presentation.

compact, ideal points exist. Moreover, the strict quasi-concavity of $u_i(.)$, together with the convexity of , implies the uniqueness of the ideal points. Equivalently, a voter i's ideal point along the x-dimension solves:

$$\max_{x \in} \quad v_i(x) \tag{3}$$

so it is independent of r. Thus, we can identify a voter i simply by her ideal point along the x-dimension, i.e. x_i^*. By duality:

$$(\bar{x}, \bar{r}) \text{ is an ideal point for voter } i \quad \Leftrightarrow \quad \max_{(x,r) \in} r \quad s.t : u_i(x, r) \geq u_i(\bar{x}, \bar{r}) \tag{4}$$

Suppose (without loss of generality) that we order voters according to their ideal points (in ascending order), so that $i < j \implies x_i^* \leq x_j^*$. Hence, x_m^* is the median voter's ideal point in the x-dimension, where $m = \frac{N+1}{2}$. The strict quasi-concavity of $u_i(.,.)$ ensures the single-peakedness of individual preferences (as in Black's (1948) original proof of the median voter theorem). We also impose on voter preferences a condition based on the single crossing property (SCP) of Gans and Smart (1996) (as well as some additional restrictions on individual preferences). Note that the SCP has to be modified in our context because we have more than one policy dimension. Initially, suppose we ignore the r-dimension and assume that the SCP holds for the x-dimension. Let $v_i(x)$ represent voter i's preferences over x. Then, the SCP entails that the following condition is satisfied:

$$\text{If} \quad i < j \quad \text{and} \quad x_1 \geq x_2 \tag{5}$$
$$v_i(x_1) \geq v_i(x_2) \quad \Rightarrow \quad v_j(x_1) > v_j(x_2)$$

Under the separability assumptions made above, it is possible to extend the SCP to the following 2-dimensional version:

$$\text{If} \quad i < j \quad \text{and} \quad x_1 \geq x_2 \tag{6}$$
$$u_i(x_1, r_1) \geq u_i(x_2, r_2) \quad \Rightarrow \quad u_j(x_1, r_1) > u_j(x_2, r_2)$$
$$u_j(x_2, r_2) \geq u_j(x_1, r_1) \quad \Rightarrow \quad u_i(x_2, r_2) > u_i(x_1, r_1)$$

If $i < j$, and $x_1 \geq x_2$, i is more likely to prefer (x_2, r_2) to (x_1, r_1) whereas j is more likely to prefer (x_1, r_1) to (x_2, r_2). However, if i prefers (x_1, r_1) to (x_2, r_2) (for instance, because r_1 is sufficiently smaller than r_2) the SCP condition guarantees that *a fortiori* j also prefers (x_1, r_1) to (x_2, r_2). However, if j prefers (x_2, r_2) to (x_1, r_1), then *a fortiori* i also prefers (x_2, r_2) to (x_1, r_1).

Finally, we assume that politicians are "nonextreme" in the following sense. Consider two policies (x_1, r) and (x_2, r) with the same level of rent r; if x_1 is "extreme" (i.e. $x_1 < x_2^*$ or $x_1 > x_{N-1}^*$), while x_2 is not, then politicians prefer (x_2, r) to (x_1, r). This assumption slightly extends the lexicographic preferences of politicians discussed above, but is a very mild condition, as it only rules out the most extreme ideologies.

2.2 The Game

As in PRT, the political game involves an infinite sequence of periods. At the beginning of period t, one of the politicians (the incumbent) is exogenously in office. Voters choose their reelection strategies (specifying a cutoff level of utility, denoted b_i, below which they will vote against the incumbent's reelection). Then, the incumbent implements a policy $(x, r) \in$, after which an election is held. In the election, each voter votes, either for the incumbent or for a challenger (chosen from the set of nonincumbent politicians, all of whom are identical *ex ante*). If the incumbent is relected, then a new period $(t + 1)$ starts with the same incumbent in office; if the incumbent is not reelected, then a new period starts with the challenger in office (it is assumed that a defeated incumbent can never return to office). This process is repeated infinitely, with the continuation value for an incumbent of reelection being determined endogenously. In such an analysis, attention is typically restricted to stationary equilibria of the infinitely repeated game (i.e. equilibria in strategies that are independent of t). Thus, there is no significant loss of generality in focusing on only one stage of this infinitely repeated game, and assuming an exogenous continuation value (denoted W) from reelection.

The game (which will be denoted by Γ) can thus be summarized as follows:

1. Each voter $i, i \in \{1, ..., N\}$, noncooperatively chooses a reservation utility $b_i \in \mathbb{R}$

2. The incumbent chooses a policy $\left(x^P, r^P\right) \in$ to implement

3. Elections are held. Each voter $i, i \in \{1, ..., N\}$, votes for the reelection of the incumbent iff $u_i \left(x^P, r^P\right) \geq b_i$; otherwise, i votes for the challenger

4. Payoffs are realized

The action space and strategy space for each voter i is simply the choice of a reservation utility $b_i \in \mathbb{R}$. These choices define an $N-$dimensional vector $\mathbf{b}$ of

reservation utilities for all voters. For the incumbent, the action space is simply a policy $(x, r) \in$. However, the strategy space is a *mapping* from **b** to a policy, and can be expressed as follows:

$$((x, r)|\mathbf{b}) : \mathbb{R}^N \mapsto \qquad (7)$$

(the implemented policy will be denoted by $\left(x^P, r^P\right)$ and the optimal strategy for the incumbent, given a vector **b**, by $(x^*, r^*|\mathbf{b})$). Note that when we consider a deviation by one or more voters from the strategy embodied in **b**, the actions of the incumbent cannot be held fixed, but rather must be reoptimized with respect to the new **b**, in accordance with the given strategy $(x, r \mid \mathbf{b})$. Note also that politicians other than the incumbent (the potential challengers) are not strategic actors in this model.

The voters' choices of reservation utilities, **b**, together with the incumbent's choice of strategy $(x, r \mid \mathbf{b})$, define a voting pattern in stage 3 of the game. This can be formally represented by the indicator variable $\mathbb{I}_i$, defined as follows:

$$\mathbb{I}_i = \begin{cases} 1 & \text{if } u_i\left(x^P, r^P\right) \geq b_i \\ 0 & \text{otherwise} \end{cases} \qquad (8)$$

i.e. $\mathbb{I}_i = 1$ if voter i votes for the incumbent's reelection, and is 0 otherwise. The pattern of voting is summarized by the $N-$dimensional vector **I**. It is assumed that voters' payoffs depend only on policy outcomes, and not directly on how they vote. Thus, voter i receives payoff $u_i\left(x^P, r^P\right)$ in stage 4 of the game. The incumbent receives a payoff denoted by U_I, where:

$$U_I = \begin{cases} r^P + \delta W & \text{if } \sum_{i=1}^N \mathbb{I}_i \geq \frac{N+1}{2} \\ r^P & \text{otherwise} \end{cases} \qquad (9)$$

(i.e. the rents and the continuation value of being reelected, if a majority voters vote for reelection, or simply the rents if the challenger is elected).

This game closely resembles the electoral process modeled in PRT. The novel elements here are the heterogeneity of voters' preferences over x, and the lack of cooperation among them. It may appear that the game imposes a requirement that voters are committed to apply in stage 3 a reelection rule decided in stage 1. An assumption of precommitment to a reservation utility would be problematic, especially as one of the rationales for analyzing models of postelection politics is that politicians may find it difficult to precommit to policies. It should be emphasized, however, that precommitment does not in fact play a crucial role in our model. Under our assumptions, all politicians are equally opportunistic. Thus, when stage 3 of the game is reached, voters are indifferent with respect to whether the incumbent

or the challenger will be in office for the next period. Hence, for a voter in stage 3, following the reelection rule announced in stage 1 leads to the same payoff as the maximum payoff available by departing from this rule. That is, all of the Nash equilibria derived below are subgame-perfect. Allowing voters to reoptimize their reelection rule in stage 3 (in effect, making $\mathbb{I}_i$ a choice variable for voter i in stage 3) does not fundamentally affect the results. Rather, as argued by PRT (pp. 1132-1133), the assumption that voters adhere to the announcements made in stage 1 serves more as an equilibrium selection criterion. Allowing reoptimization in stage 3 admits a range of additional equilibria, but does not fundamentally alter the conclusions (bearing in mind that we show in the next section that virtually any policy can be supported as a Nash equilibrium).

3   Characterizing the Set of Nash Equilibria

A Nash equilibrium of the game Γ can be defined as follows:

Definition 1 *A Nash equilibrium of Γ consists of:*
  (i) an $N-$dimensional vector **b***, and*
  (ii) an implemented policy $\left(x^P, r^P\right)$
  (noting that these together define a voting pattern **I***) such that:*
  a) $b_i \in \mathbb{R} \ \forall i = 1, ..., N$
  b) $\left(x^P, r^P\right) \in$
  c) $U_I\left(x^P, r^P | \mathbf{b}\right) \geq U_I\left(x', r'|\mathbf{b}\right)$ for any $(x', r') \neq \left(x^P, r^P\right)$
  d) $u_i\left(x^P, r^P | b_i, b_{-i}\right) \geq u_i\left(x^*, r^*|b_i', b_{-i}\right)$ for any $b_i' \neq b_i$ and for the incumbent's optimal strategy $\left(x^*, r^*|b_i', b_{-i}\right)$, $\forall i = 1, ..., N$ (where b_{-i} denotes the reservation utilities of all voters other than i)

The individual rationality condition for the incumbent ((c) above) can be clarified further by considering the available strategies. Note firstly that the incumbent faces no constraints (other than that of feasibility, $\left(x^P, r^P\right) \in$) while in office. In this sense, models of postelection politics assume that the political contract is incomplete in that it cannot specify politicians' actions while in office (PRT, pp. 1122). The only means of control available to voters is to vote against the incumbent *ex post*. Thus, if the incumbent does not anticipate being reelected, she will always simply choose to extract the maximum possible level of rent $r^{\max}$ - in this case, she in effect does not seek reelection, and receives a lifetime payoff of $r^{\max}$. Alternatively, suppose that she seeks reelection: her problem is then to maximize rents, subject to

the constraint of obtaining a majority of votes. Suppose that r^* is the level of rent that solves this problem, while securing reelection.[4] Then, the incumbent's payoff is $r^* + \delta W$; clearly, unless $r^* + \delta W \geq r^{\max}$, the incumbent will prefer to simply extract $r^{\max}$ and forego the prospect of reelection. Thus, there exists some value of r^* (hereafter denoted by $r^{\min}$) below which seeking reelection is no longer rational for the incumbent; this value is defined by:

$$r^{\min} = r^{\max} - \delta W \qquad (10)$$

In other words, voters must allow the incumbent to secure rents of at least $r^{\min}$, in order to induce the incumbent to not extract $r^{\max}$. Thus, the requirement that

$$r^P \geq r^{\min} \qquad (11)$$

constitutes, in effect, the incumbent's incentive compatibility constraint.

The set of Nash equilibria of Γ is characterized in the following proposition (all proofs are in the Appendix):

Proposition 2 *The set of policies sustained by the Nash equilibria of Γ is:*

$$\left\{ (x, r) \mid x \in \left[x_2^*, x_{N-1}^* \right], r \in [r^{\min}, r^{\max}] \right\} \qquad (12)$$

This implies that virtually any feasible policy can be supported by a Nash equilibrium (and the set would be even larger had we not imposed the assumption of nonextremism, in the interests of tractability). This provides a stark contrast to the Downsian model, where the only policy position supported by a Nash equilibrium among the candidates is the median voter's preferred outcome. Perhaps most significant is the fact that any feasible level of rent (including the maximum possible level $r^{\max}$) can be extracted by the incumbent politician, while securing reelection. In the existing literature on postelection politics (e.g. PRT), it is claimed that coordination among voters will force the incumbent down to the lowest level of rent consistent with the incumbent seeking reelection ($r^{\min}$ in our notation). In contrast, our result shows that once we introduce heterogeneity among voters, elections do not necessarily place any constraints on incumbents' rents, *even when all voters have identical preferences over the level of rents.*

Our characterization of the set of Nash equilibria is intended primarily as a preliminary step in our derivation of the subsequent results. The conclusions are

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[4]Note that the assumption of lexicographic preferences implies that there exists a unique x in the maximization of this problem.

not particularly surprising, and are analogous to results in the existing literature. For instance, Roemer (1998) shows that adding a policy dimension ("religion") that is orthogonal to tax policy can induce a low-income majority of voters to refrain from imposing high taxes on those with high incomes (even though they would do so if taxes were the only policy dimension). Casamatta (2003) introduces an orthogonal dimension into a model of voting over pension policy, and shows that retirees may secure favorable pension policies, even when they are in a minority. It is worth noting, however, that notwithstanding the multiplicity of equilibria, our model implies the following robust result. With heterogeneous voters, equilibrium rents are always (weakly) higher than in the typical postelection politics model with homogeneous voters (e.g. PRT).
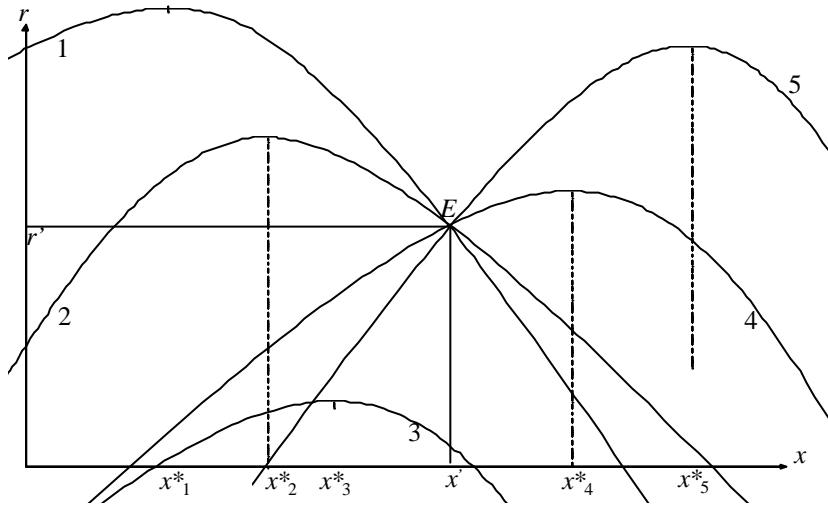

Figure 1: An example of a Nash equilibrium that induces a nonmedian policy and a high level of rents.

Figure 1 illustrates an example of a Nash equilibrium with $N = 5$, $r > r^{\min}$ and $x \neq x_m^*$. Indifference curves for each voter in the policy space are represented by the bold curves. The horizontal axis corresponds to the ideological dimension x, and the vertical axis corresponds to the level of rent r. Hence, the incumbent's policy can be represented as a point in the graph; the higher the point, the better off is the incumbent.

The incumbent can guarantee voter i the same level of utility notwithstanding a higher r, the closer x is to voter i's preferred ideology x_i^*. This, together with the strict quasi-concavity assumption, implies that indifference curves are hump shaped. By the additional separability assumption, each pair of indifference curves of a single voter differs only by a vertical translation. Moving upward in the policy

space corresponds to a lower level of utility. Furthermore, the SCP condition (6) guarantees that each pair of indifference curves of two different voters intersects only once, with that of the voter with the higher index having a larger slope.

For all voters, the choice of a reservation utility b_i is graphically depicted by the selection of one indifference curve in the policy space. Any policy below (above) this indifference curve induces voter i to vote for (against) the incumbent's reelection. Hence, given any profile of reelection strategies (i.e. any profile of indifference curves), an incumbent who seeks reelection selects the highest policy (weakly) below at least three indifference curves. Given the depicted profile of re-election strategies, the incumbent's best response is $E = (x^*, r^*)$. The winning coalition consists of voters $\{1, 2, 4, 5\}$ while 3 votes against reelection.

This profile of reelection strategies is a Nash equilibrium only if each voter is unable to strictly increase her payoff by a unilateral deviation. Voter 3 can only change the implemented policy by decreasing her reservation utility; this, however, implies a decrease in her payoff. For any other voter, increasing her reservation utility removes her from the winning coalition without changing the implemented policy. Finally, decreasing her reservation utility can only change the implemented policy in a way that decreases her payoff.

Thus, none of these voters can unilaterally induce the incumbent to change policy. However, a simultaneous deviation by $\{1, 2\}$ to higher reservation utilities can move policy towards the median outcome. We thus turn in the next section to refining the set of equilibria, using concepts based on multilateral deviations by coalitions of voters.

4 A Median Voter Theorem

A well-known refinement of Nash equilibrium in multiplayer games is the concept of strong Nash equilibrium (Aumann, 1959):

Definition 3 *A Strong Nash Equilibrium (SNE) is a Nash equilibrium at which there does not exist any joint deviation by a subset of voters that makes each deviating voter strictly better off.*

SNE is a very restrictive equilibrium concept, and strong Nash equilibria do not exist in many games. However, we show in this section that there exist strong Nash equilibria of Γ, and that, moreover, there is a unique policy that is supported by any strong Nash equilibrium. This policy involves the implementation of the median

voter's preferred outcome along the x-dimension, together with the lowest level of rent consistent with the incumbent's incentive compatibility constraint (i.e. $r^{\min}$).[5] We label this a "median voter theorem" because, like the most famous result in the theory of preelection politics (Downs, 1957), it establishes conditions under which the median voter's preferred outcome can be expected to prevail as the implemented policy.

Proposition 4 *There exists a unique policy outcome of Γ that can be supported by a Strong Nash Equilibrium profile of voter strategies; this policy is $(x_m^*, r^{\min})$, where x_m^* is the median voter's ideal point.*

The basic intuition for this result is the following. Consider any Nash equilibrium that induces a nonmedian policy (i.e. some policy other than $(x_m, r^{\min})$). Such an equilibrium is always vulnerable to a deviation by a majority of voters. For instance, suppose that the policy involves an x that is strictly to the right of the median. Then, all voters to the left of the median, together with the median voter, will be strictly better off by deviating to a profile of strategies that induces the incumbent to implement $(x_m, r^{\min})$. Moreover, since this subset of deviating voters constitutes a majority, their deviation will be successful in inducing this policy. This is sufficient to establish that an x that is strictly to the right of the median cannot be sustained in a strong Nash equilibrium (analogous arguments are possible for an x that is strictly to the left of the median, and for a policy (x_m, r) where $r > r^{\min}$). When the equilibrium policy is $(x_m, r^{\min})$, however, those voters who wish to change policy in a given direction are in a minority - all voters strictly to the left of the median wish to move policy to the left, while all those strictly to the right of the median wish to move policy to the right, but neither group constitutes a majority.

This median voter theorem establishes that the policy outcomes of postelection politics can be predicted in certain circumstances. It should be noted, however, that this result relies on the use of the strong Nash equilibrium concept, which is more restrictive than that in the previous section (and in the previous literature). It is generally thought to require binding commitments among players, because the joint deviations that are envisaged in the derivation of strong Nash equilibria (and of Majoritarian Nash Equilibria) may not be self-enforcing, in the sense that there may exist further profitable deviations by subsets of voters from these joint deviations.

[5] This result also holds for a slightly less retrictive equilibrium concept that we term Majoritarian Nash equilibrium (MNE), defined as follows: a Majoritarian Nash Equilibrium (MNE) is a Nash equilibrium at which there does not exist any joint deviation by a subset of at least m voters that makes each deviating voter strictly better off (i.e. there does not exist any joint deviation that makes a majority of voters better off). See Appendix 7.2 for details.

However, the existence of a unique policy that can be sustained as a SNE can perhaps be argued to serve as a coordination device for voters. In addition, it could be argued that political parties are likely to develop in this type of setting, and that they can coordinate profitable deviations from Nash behavior (see Carbonell-Nicolau and Klor (2003) for a similar argument justifying the use of SNE in a voting model). Of course, some of these deviations may not be self-enforcing, involving behavior that is not individually rational for each deviator (for instance, there may exist a profitable deviation for a subset of voters from a deviation coordinated by a political party). Even so, one may argue that political parties are long-lived entities that can arrange transfers, or an intertemporal distribution of burdens, that will induce voters to conform to the SNE outcome. Thus, it may be possible to justify the use of the SNE equilibrium concept in this context. Nevertheless, in the next section, we analyze the postelection politics game using equilibrium concepts that consider only self-enforcing multilateral deviations.

5 Alternative Equilibrium Refinements

5.1 Coalition-Proof Equilibria

As noted above, the strong Nash equilibrium concept requires considering deviations by subsets of players that may not be self-enforcing. Thus, for certain deviations, it may be the case that some subset of the deviators would wish to deviate from the deviation. If so, then the original deviation is not consistent with the individual rationality of each player. In response, Bernheim, Peleg, and Whinston (1987) develop the notion of coalition-proof Nash equilibrium (hereafter, CPNE). Intuitively, a Nash equilibrium is *not* a CPNE if a subset of players can undertake a profitable joint deviation, *and* there are no further profitable deviations by subsets of the deviating coalition (holding the behavior of nondeviators fixed) that are self-enforcing.

Characterizing the set of coalition-proof Nash equilibria of Γ is difficult. We are only able to prove that some policies cannot be implemented in a coalition-proof equilibrium. (see Appendix 7.3). In particular, policies (x', r') such that $\left(x_m^*, r^{\min}\right)$ is unanimously preferred by all voters to (x', r') cannot be induced by a CPNE. It is important to clarify that all this result establishes is that certain policies are inconsistent with a CPNE. Nonetheless, it is possible to show that there exist some nonmedian policies (i.e. policies other than $\left(x_m^*, r^{\min}\right)$) that are not eliminated by imposing a requirement that there exist no self-enforcing deviations (i.e. by the notion of coalition-proofness). For this, we require further refinements, such as the

one introduced below.

5.2 Ideological Loyalty

Our basic model allows voters to choose any reservation utility $b_i \in \mathbb{R}$, even if this entails voting for the reelection of an incumbent whose ideology the voter finds repugnant. For example, if a "left" voter such as $i = 1$ sets $b_i = -\infty$, she will vote to reelect an incumbent who implements a policy on the extreme right, such as $x = x_N^*$. To address this issue, we introduce a fairly mild notion of "ideological loyalty" on the part of voters; this entails that voters on the left do not vote for incumbents who implement policies on the right, and *vice versa*:

Definition 5 *An "ideologically loyal" strategy is one for which the following holds: if $i \leq m$ (respectively, $i \geq m$), i never votes for the reelection of an incumbent who implements a policy $x > x_m^*$ (respectively, $x < x_m^*$). Thus, $b_i \geq u_i(x_m^*, r^{\min})$ for all $i \in \{1, ..., N\}$.*

An ideologically loyal Nash equilibrium (ILNE) is a Nash equilibrium in ideologically loyal strategies for all voters

An ideologically loyal CPNE (ILCPNE) is a CPNE in ideologically loyal strategies for all voters

In the game that we are analyzing here, a precommitment to following an ideologically loyal strategy does not involve any problems of credibility for voters. This is because, looking forward from the election, the incumbent and the challenger are identical from the voter's perspective. It cannot be denied, however, that in more general settings, ideological loyalty may entail a commitment by voters that is not always *ex post* optimal.

Refining the set of Nash equilibria (characterized in Section 3) using this notion of ideological loyalty leads to the following result:

Proposition 6 *If **b** is an ILNE profile of voter strategies, then the incumbent implements either $(x_m^*, r^{\min})$ or $(x^+, r^{\max})$ (where x^+ is determined by the incumbent's selection criterion over x, given $r^{\max}$).*

Moreover, in any ILNE in which the incumbent is reelected, the incumbent always implements $(x_m^*, r^{\min})$

This result establishes that imposing a requirement of ideological loyalty on voter strategies dramatically reduces the set of outcomes that can be sustained in a Nash

equilibrium. Essentially, either the median outcome will prevail, or the incumbent will eschew the possibility of reelection, and extract maximal rents. In the latter case, however, profitable multilateral deviations will generally be possible. For example, suppose that $N = 5$ and $b_i = +\infty, i = 1, ..., 5$; the incumbent will implement some policy $(x^+, r^{\max})$. If all voters deviate simultaneously to a strategic profile that induces the incumbents to implement $(x^+, r^{\min})$, they would all be better off. Of course, bearing in mind the reservations expressed earlier about the SNE concept, we wish to focus only on multilateral deviations that are self-enforcing, and hence on coalition-proof equilibria. The following result shows that there always exist self-enforcing deviations from any ILNE that leads to $r^{\max}$:

Proposition 7 *If* **b** *is an ILCPNE profile of voter strategies, then the incumbent always implements the policy* $(x_m^*, r^{\min})$

Thus, as long as voters' behavior conforms to the restriction embodied in the assumption of ideological loyalty, the median voter outcome is the unique coalition-proof equilibrium outcome. This implies that, as long as we are prepared to accept ideological loyalty as a reasonable constraint on voter strategies, nonmedian policy outcomes can be ruled out as being subject to self-enforcing profitable multilateral deviations. Thus, Proposition 7 provides an alternative formulation of a median voter theorem for postelection politics, without appealing to binding commitments or implicit conventions among voters (as long as they satisfy the condition of ideological loyalty).

6 Discussion and Conclusion

This paper has presented a median voter result for a model of postelection politics. This can serve as a counterpart to analogous results in the study of preelection politics, enabling us to answer the question of what policies we can expect to observe as the outcome of electoral politics. There are a number of significant lessons to be drawn from this analysis. Most fundamentally, the game of postelection politics that we analyze yields outcomes that converge on the center of the distribution of voters' preferences. Moreover, the rents that incumbent politicians can extract while in office are restricted to the minimal level consistent with the contractual incompleteness of political constitutions.

While these results are encouraging, it should be emphasized that they rely on stringent requirements in terms of the enforceability of agreements and the existence

of perfect communication or implicit conventions among voters. A Nash equilibrium outcome alone will not, in general, yield ideological policies that reflect the median voter's preferences, nor will it constrain the rent-seeking of incumbent politicians. Our results thus provide some grounds for pessimism about the ability of electoral politics to constrain politicians' opportunistic behavior in circumstances where policy platforms are not binding.

We hope that this analysis can serve as a basis for further research on models of postelection politics. To conclude, we discuss a few possible extensions. Firstly, we have analyzed a model with a large, but finite, number of voters. Assuming a continuum of voters would, by making each voter negligible, presumably reinforce the result that any policy can be sustained in a Nash equilibrium. Secondly, we have assumed purely rent-maximizing politicians. An interesting extension would be to consider policy-motivated candidates (as in Wittman (1977) and Calvert (1985)). Presumably, it would then be possible to sustain equilibria with $r < r^{\min}$ if the incumbent cares sufficiently about policy. Other possible extensions include considering multiple ideological dimensions (in addition to the rent dimension), and heterogeneous types of politicians. Finally, we have focused on the case where the "valence" characteristic of incumbents (rents) is a choice variable for the politician. In adverse selection models of postelection politics (for instance, in the substantial literature on political business cycles - see e.g. Rogoff and Sibert (1988) and Rogoff (1990)), this is instead an intrinsic characteristic, such as competence. It would be of interest to consider the implications of this analysis for such models.

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7 Appendix

7.1 Proof of Proposition 2

We prove Proposition 2 by means of two lemmas, as follows:

Lemma 8 *For any $x^+ \in [x_2^*, x_{N-1}^*]$ and any $r^+ \in \left[r^{\min}, r^{\max}\right]$, there exists a Nash equilibrium profile of reservation utilities $\mathbf{b}^+$ that supports the policy (x^+, r^+), where x_2^* and x_{N-1}^* are the ideal points of voters 2 and $N-1$, respectively*

The proof essentially generalize the example given in Figure 1. Consider an exhaustive and mutually exclusive partition $\left(\Sigma, \bar{\Sigma}\right)$ of the set of voters $\{1, ..., N\}$ (so that $\Sigma \cup \bar{\Sigma} = \{1, ..., N\}$ and $\Sigma \cap \bar{\Sigma} = \emptyset$), such that $\{1, 2, N-1, N\} \subset \Sigma$ and $|\Sigma| = \frac{N+1}{2} + 1 = \frac{N+3}{2} \implies |\bar{\Sigma}| = \frac{N-3}{2}$ (where $|\Sigma|$ denotes the cardinality of the set Σ). Thus, Σ consists of a minimal supermajority of voters (i.e. one more than required for a majority) and includes the two extreme leftist and extreme rightist voters. Now suppose that voters choose a vector of reservation utilities $\mathbf{b}$ where:

$$b_i = \begin{cases} u_i\left(x', r'\right) & \text{if } i \in \Sigma \\ \infty & \text{if } i \in \bar{\Sigma} \end{cases} \tag{13}$$

for some arbitrary policy (x', r'), where $x' \in [x_2^*, x_{N-1}^*]$ and $r' \in \left[r^{\min}, r^{\max}\right]$. Clearly, an incumbent who faces this $\mathbf{b}$ and implements (x', r') will be reelected, with the votes of all voters in Σ. We wish to prove that implementing (x', r') is indeed the best response to this $\mathbf{b}$ by the incumbent, and that the b_i specified above is a best response of each voter to the incumbent's strategy $(x', r'|\mathbf{b})$ and to b_{-i}.

To prove this, note first that, given this $\mathbf{b}$, the incumbent will never choose any policy involving $r < r'$, as this is dominated by (x', r') (which satisfies the reelection constraint, and yields higher rent). Given this, we can then show that the incumbent will never choose a policy with $x \neq x'$ and $r \geq r'$. Consider without loss of generality $x < x'$: by the strict quasi-concavity of $u_i(.)$, $u_i(x, r) \leq u_i(x, r') < u_i(x', r') = b_i$ for $i = N-1, N$ (i.e. deviating to (x, r) leads to the reelection rules of voters N and $N-1$ being violated). Thus, at least 2 voters in Σ (voters N and $N-1$) will

vote against reelection, along, of course, with all voters in $\bar{\Sigma}$ (a total of at least $\frac{N+1}{2}$ voters). This is sufficient to cause the incumbent's defeat; thus, the incumbent will not deviate to any $x < x'$. A symmetric argument establishes that the incumbent will not deviate to any (x, r) with $x > x'$ and $r \geq r'$. Given that $x = x'$, note that the incumbent cannot implement any policy with $r > r'$ and be reelected. For all voters in Σ, $u_i(x', r) < u_i(x', r') = b_i$ when $r > r'$; thus, all voters would vote against the incumbent's reelection. Note also that $r' \geq r^{\min}$ implies that (x', r') is preferable (from the incumbent's point of view) to implementing $r^{\max}$ and foregoing reelection. Hence, implementing (x', r') is the incumbent's best response to the **b** specified above.

Now consider deviations by voter i from the specified b_i.

- Suppose $i \in \Sigma$. Setting a higher b_i will not change the incumbent's optimal action - there is still a majority of voters who will reelect if (x', r') is implemented, so the arguments above continue to hold. Voter i's deviation will lead to $\mathbb{I}_i = 0$ (i.e. to i voting against the incumbent), but as the outcome (x', r') does not change, i is not strictly better off by deviating. Unilaterally setting a lower b_i cannot lead to a strict increase in voter i's payoff.

- For $i \in \bar{\Sigma}$, setting a higher b_i keeps her outside the winning coalition. Setting a lower b_i can only change the policy implemented in a way that decreases voter i's payoff.

Lemma 9 *If $\mathbf{b}^+$ is a Nash equilibrium profile of reservation utilities that leads to reelection and supports policy (x^+, r^+), then $x^+ \in [x_2^*, x_{N-1}^*]$ and $r^+ \in [r^{\min}, r^{\max}]$*

Note first that $r^+ \notin [r^{\min}, r^{\max}]$ is inconsistent with the incumbent seeking reelection. Now, assume that $x_{N-1}^* < x^+$ (noting that a symmetric proof exists for $x^+ < x_2^*$). First, we establish that there exists some (x', r') such that $x' < x^+$ and (x', r') is strictly preferred to (x^+, r^+) by both voter $N-1$ and by the incumbent:

- suppose that $r^+ < r^{\max}$; as $x^+ > x_{N-1}^*$, (x^+, r^+) cannot be an ideal point for voter $N-1$. Hence, by the strict quasi-concavity of $u_{N-1}(.)$, there exists some (x', r') such that $r' > r^+$ and $u_{N-1}(x', r') > u_{N-1}(x^+, r^+)$. Furthermore, $x' < x^+$; otherwise, if $x' \geq x^+$, $u_{N-1}(x', r') < u_{N-1}(x', r^+) \leq u_{N-1}(x^+, r^+))$. Note that the incumbent strictly prefers (x', r') to (x^+, r^+), as $r' > r^+$.

- suppose that $r^+ = r^{\max}$, and consider $(x', r') = (x_{N-1}^*, r^{\max})$: as x_{N-1} is the ideal point of voter $N-1$, $u_{N-1}(x_{N-1}^*, r^{\max}) > u_{N-1}(x^+, r^+)$. Note that the incumbent also prefers $(x_{N-1}^*, r^{\max})$ to (x^+, r^+) by the assumption of nonextremist preferences.

By the SCP condition (6), $u_i(x', r') > u_i(x^+, r^+)$ for $i \in \{1, ..., N-1\}$. Since, (x^+, r^+) is implemented in a Nash equilibrium, at least $\frac{N+1}{2}$ voters must vote for reelection (i.e. set $b_i \leq u_i(x^+, r^+)$). Then, there are two possibilities:

1. at least $\frac{N+1}{2}$ voters in $\{1, ..., N-1\}$ vote for reelection. For these voters, $b_i \leq u_i(x^+, r^+) < u_i(x', r')$. But then, the incumbent would be reelected if she implements (x', r') instead of (x^+, r^+), and prefers the former. Thus, (x^+, r^+) cannot be implemented in equilibrium.

2. exactly $\frac{N-1}{2}$ voters in $\{1, ..., N-1\}$ vote for reelection, along with voter N. Then, consider a voter in $\{1, ..., N-1\}$ who votes against reelection (i.e. sets $b_i > u_i(x', r')$). A deviation by this voter to $b'_i = u_i(x', r')$ allows the incumbent to implement (x', r') instead of (x^+, r^+), which raises the deviator's payoff (and is preferred by the nonextreme incumbent). Thus, $\mathbf{b}^+$ cannot be a Nash equilibrium profile of reservation utilities.

Thus, (x^+, r^+), where $x^+ > x^*_{N-1}$ and $r^+ \in [r^{\min}, r^{\max}]$, cannot be sustained as a Nash equilibrium. A symmetric argument establishes that (x^+, r^+), where $x^+ < x^*_2$ and $r^+ \in [r^{\min}, r^{\max}]$, cannot be sustained as a Nash equilibrium.

7.2   Proof of proposition 4

We prove Proposition 4 by means of two lemmas, as follows:

Lemma 10 *Let $b^*_i = u_i(x^*_m, r^{\min})$ for voters $i = 1, ..., N$. Then, $\mathbf{b}^* = (b^*_1, ..., b^*_N)$ is a strong (and Majoritarian) Nash equilibrium profile of voter strategies*

First, we establish that, when faced with $\mathbf{b}^* = (b^*_1, ..., b^*_N)$, the incumbent will implement $(x^*_m, r^{\min})$. Noting that the incumbent will never implement $r < r^{\min}$, consider a policy $(x', r') \neq (x^*_m, r^{\min})$ where $r' \geq r^{\min}$. Then, $u_m(x', r') < u_m(x^*_m, r^{\min}) = b^*_m$ by the uniqueness of m's ideal point; thus, m will vote against reelection. For the other voters, there are two possible cases: $x' \geq x^*_m$ and $x' \leq x^*_m$. Consider $x' \geq x^*_m$: for $i \in \{1, ..., m-1\}, u_i(x', r') < u_i(x^*_m, r^{\min}) = b^*_i$ by the SCP condition (6). Thus, all these voters (in addition to m, a total of $\frac{N+1}{2}$) will vote against reelection. Consider $x' \leq x^*_m$: for $i \in \{m+1, ..., N\}, u_i(x', r') < u_i(x^*_m, r^{\min}) = b^*_i$ by the SCP condition (6). Thus, all these voters (in addition to m, a total of $\frac{N+1}{2}$) will vote against reelection. Hence, any policy (x', r') will lead to defeat. In contrast, $(x^*_m, r^{\min})$ satisfies the reelection constraints of all voters, and hence leads to reelection. Thus, the incumbent will implement $(x^*_m, r^{\min})$.

Next, we show that, given that it leads to $(x^*_m, r^{\min})$ being implemented, $\mathbf{b}^* = (b^*_1, ..., b^*_N)$ is a strong Nash equilibrium profile of voter strategies. Specifically, we consider multilateral deviations by a subset of voters from $\mathbf{b}^*$, and show that, if the deviation leads to some other policy $(x^d, r^d) \neq (x^*_m, r^{\min})$ being implemented, it cannot make every deviating voter strictly better off. Let $\Sigma_d \subseteq \{1, ..., N\}$ be a coalition of voters who deviate from $\mathbf{b}^*$. Suppose that this deviation induces the incumbent to implement $(x^d, r^d) \neq (x^*_m, r^{\min})$; noting that the incumbent will never implement $r < r^{\min}$, it must be that case that $r^d \geq r^{\min}$. Let Σ_L be the subset of voters who are worse off as a result of the deviation i.e. $i \in \Sigma_L$ if $u_i(x^d, r^d) < u_i(x^*_m, r^{\min})$. Using the SCP condition (6), if $x^d \geq x^*_m$, then $\{m, ..., N\} \subseteq \Sigma_L$, while if $x^d \leq x^*_m$, then $\{1, ..., m\} \subseteq \Sigma_L$; in either case, $|\Sigma_L| \geq \frac{N+1}{2}$ (i.e. a majority of voters are worse off as a result of the deviation).

Now suppose that the deviating coalition Σ_d contains no members of Σ_L (i.e. $\Sigma_L \cap \Sigma_d = \emptyset$). Then, it follows that $|\Sigma_d| < \frac{N+1}{2}$, and that the reelection constraints for a majority of voters are not satisfied if (x^d, r^d) is implemented (as $|\Sigma_L| \geq \frac{N+1}{2}$, and $b^*_i = u_i(x^*_m, r^{\min}) > u_i(x^d, r^d)$ for $i \in \Sigma_L$). But, this implies that the incumbent

will not be reelected by implementing (x^d, r^d), and hence contradicts the premise that the deviation induces the incumbent to implement $(x^d, r^d) \neq (x_m^*, r^{\min})$. Now suppose instead that the deviating coalition Σ_d contains at least one member of Σ_L (i.e. $\Sigma_L \cap \Sigma_d \neq \emptyset$). But then at least one member of the deviating coalition is not strictly better off as a result of the deviation (i.e. the deviation is not strictly beneficial for all deviators).

Thus, there is no feasible deviation from $\mathbf{b}^*$ by a subset of voters that makes each deviator strictly better off and induces the incumbent to implement a policy other than $(x_m^*, r^{\min})$. This implies that $\mathbf{b}^* = (b_1^*, ..., b_N^*)$ is a strong Nash equilibrium profile of voter strategies. Furthermore, b^* is also a Majoritarian Nash equilibrium profile of voters strategies.

This establishes the existence of a policy that can be supported by a strong Nash equilibrium profile of voter strategies. Next, we turn to the issue of uniqueness:

Lemma 11 *If* $\mathbf{b}' = (b_1', ..., b_N')$ *is a strong (or Majoritarian) Nash equilibrium profile of voter strategies, then the policy* $(x_m^*, r^{\min})$ *is always implemented*

Suppose that $b' = (b_1', ..., b_N')$ is a Majoritarian Nash equilibrium profile of voter strategies, (remember that if b' is a SNE, it is *a fortiori* a MNE) and let (x', r') be the policy implemented by the incumbent when faced with $\mathbf{b}'$. Then, suppose that $(x', r') \neq (x_m^*, r^{\min})$ (bearing in mind that $r' \geq r^{\min}$ by the incumbent's incentive compatibility constraint). We proceed by showing that for any $(x', r') \neq (x_m^*, r^{\min})$ there exists a profitable deviation by a subset of voters (thereby contradicting the premise that $\mathbf{b}'$ is a Majoritarian Nash equilibrium profile of voter strategies).

Consider the case where $(x', r') \neq (x_m^*, r^{\min})$, with $x' \geq x_m^*$ and $r' \geq r^{\min}$ (note that a symmetric argument is available for $x' \leq x_m^*$). We partition the set of voters into 2 mutually exhaustive and exhaustive subsets, based on whether the voter would vote for reelection if the policy $(x_m^*, r^{\min})$ were implemented. Let Σ_A be the subset of voters who choose $b_i' > u_i(x_m^*, r^{\min})$ (and would hence vote against reelection if $(x_m^*, r^{\min})$ were implemented), and $\bar{\Sigma}_A$ be the subset of voters who choose $b_i' \leq u_i(x_m^*, r^{\min})$ (and would hence vote for reelection if $(x_m^*, r^{\min})$ were implemented). Note that $\Sigma_A \cap \bar{\Sigma}_A = \emptyset$ and $\Sigma_A \cup \bar{\Sigma}_A = \{1, ..., N\}$.

Consider the subset of voters $i > m$ (i.e. $\{m+1, ..., N\}$): we partition these voters into 2 mutually exclusive and exhaustive subsets. Let Σ_3 denote the set of voters $i > m$ who are in $\bar{\Sigma}_A$, and let Σ_4 denote the set of voters $i > m$ who are in Σ_A:

$$\Sigma_3 = \{m+1, ..., N\} \cap \bar{\Sigma}_A = \{i > m \,|\, b_i' \leq u_i(x_m^*, r^{\min})\} \tag{14}$$

$$\Sigma_4 = \{m+1, ..., N\} \cap \Sigma_A = \{i > m \,|\, b_i' > u_i(x_m^*, r^{\min})\} \tag{15}$$

Let $k = |\Sigma_3|$, noting that $0 \leq k \leq \frac{N-1}{2}$ and that $|\Sigma_4| = \frac{N-1}{2} - k$.

Now consider the subset of voters $\{1, ..., m\}$. We partition this set into 2 mutually exclusive and exhaustive subsets. Let Σ_1 be a set consisting of the median voter m and of $(\frac{N+1}{2} - k - 1)$ other (arbitrarily chosen) members of $\{1, ..., m\}$, and let Σ_2 be

the complement of Σ_1 in $\{1,..,m\}$ (so that it includes the remainder of $\{1,...,m\}$). Note that $|\Sigma_1| = \frac{N+1}{2} - k$ and $|\Sigma_2| = k$.

Given that $\mathbf{b}'$ is a strong Nash equilibrium profile of voter strategies that induces policy (x',r'), consider the following joint deviation by all voters belonging to the set $\Sigma_1 \cup \Sigma_2$ from b'_i to b^d_i, where

$$b^d_i = \begin{cases} u_i\left(x^*_m, r^{\min}\right) & \text{if } i \in \Sigma_1 \\ \infty & \text{if } i \in \Sigma_2 \end{cases} \tag{16}$$

(with the strategies of all voters $i \in \Sigma_3 \cup \Sigma_4$ held fixed at b'_i). Faced with this deviation, the incumbent will implement $\left(x^*_m, r^{\min}\right)$. To show this, note first that implementing $\left(x^*_m, r^{\min}\right)$ induces all voters in $\Sigma_1 \cup \Sigma_3$ (a total of $\frac{N+1}{2} - k + k = \frac{N+1}{2}$) to vote for reelection, and thus leads to the incumbent being reelected. Moreover, implementing any $(x^+, r^+) \neq \left(x^*_m, r^{\min}\right)$ with $r^+ \geq r^{\min}$ will lead to defeat. To establish this point, note first that $u_m\left(x^*_m, r^{\min}\right) > u_m(x^+, r^+)$ (by the uniqueness of m's ideal point). For other voters, there are 3 cases to consider:

(i) Suppose that $x^+ > x^*_m$: $u_m(x^+, r^+) < u_m\left(x^*_m, r^{\min}\right) \implies u_i(x^+, r^+) < u_i\left(x^*_m, r^{\min}\right)$ for all $i < m$ (by the SCP condition (6)). In particular, $u_i(x^+, r^+) < u_i\left(x^*_m, r^{\min}\right) = b^d_i$ for all $i \in \Sigma_1$, so all voters in $\Sigma_1 \cup \Sigma_2$ (a total of $\frac{N+1}{2}$) will vote against reelection, and the incumbent cannot implement this type of policy and be reelected.

(ii) Suppose that $x^+ < x^*_m$: $u_m(x^+, r^+) < u_m\left(x^*_m, r^{\min}\right) \implies u_i(x^+, r^+) < u_i\left(x^*_m, r^{\min}\right)$ for all $i > m$ (by the SCP condition (6)) i.e. for $i \in \Sigma_3 \cup \Sigma_4$. Thus, implementing (x^+, r^+) will lead m, all voters in Σ_2, and all voters in Σ_4 (who set $b'_i > u_i\left(x^*_m, r^{\min}\right)$) will vote against reelection. This is a total of $1 + k + \frac{N-1}{2} - k = \frac{N+1}{2}$ voters, so the incumbent cannot implement this type of policy and be reelected.

(iii) Suppose that $x^+ = x^*_m$ and $r^+ > r^{\min}$: then, $u_i(x^+, r^+) < u_i\left(x^*_m, r^{\min}\right)$ for all i. All voters in $\Sigma_1 \cup \Sigma_2 \cup \Sigma_4$ will vote against reelection (a total of $\frac{N+1}{2} + \frac{N-1}{2} - k$ voters, and hence a majority), so the incumbent cannot implement this type of policy and be reelected.

Thus, the incumbent cannot implement any $(x^+, r^+) \neq \left(x^*_m, r^{\min}\right)$ with $r^+ \geq r^{\min}$ and be reelected. Note also that by the SCP condition (6), $u_i\left(x^*_m, r^{\min}\right) > u_i(x', r')$ for all $i \in \Sigma_1 \cup \Sigma_2$, so every member of the deviating coalition is strictly better off under the policy implemented when they deviate (i.e. $\left(x^*_m, r^{\min}\right)$) than under the policy induced by $\mathbf{b}'$ (i.e. (x', r')). But, this contradicts the premise that $\mathbf{b}'$ is a strong Nash equilibrium profile of voter strategies.

A symmetric argument can be made for the case where $(x', r') \neq \left(x^*_m, r^{\min}\right)$, with $x' \leq x^*_m$ and $r' \geq r^{\min}$, with:

$$\Sigma_1 = \left\{i \geq m \mid b^d_i = u_i\left(x^*_m, r^{\min}\right)\right\} \qquad |\Sigma_1| = \frac{N+1}{2} - k$$

$$\Sigma_2 = \left\{i > m \mid b^d_i = \infty\right\} \qquad |\Sigma_2| = k$$

$$\Sigma_3 = \left\{i < m \mid b'_i \leq u_i\left(x^*_m, r^{\min}\right)\right\} \qquad |\Sigma_3| = k$$

$$\Sigma_4 = \left\{i < m \mid b'_i > u_i\left(x^*_m, r^{\min}\right)\right\} \qquad |\Sigma_4| = \frac{N-1}{2} - k$$

Thus, if $\mathbf{b}'$ is a Majoritarian Nash equilibrium profile of voter strategies, the policy implemented is always $\left(x_m^*, r^{\min}\right)$.

7.3 Coalition-Proof Equilibria

It is possible, to characterize certain subsets of policies that cannot be sustained in any CPNE. Consider the following subset of the policy space:

Definition 12 *Let* $\Sigma_1 = \left\{ i \mid b_i < u_i\left(x_m^*, r^{\min}\right) \right\} \subseteq \Sigma = \{1, ..., N\}$ *be the subset of voters who set* $b_i < u_i\left(x_m^*, r^{\min}\right)$. *Then, let* $\mathnormal{}_1$ *be the subset of the policy space defined by*

$$\mathnormal{}_1 = \left\{ (x, r) \mid \forall i \in \Sigma_1,\ u_i\left(x, r\right) < u_i\left(x_m^*, r^{\min}\right) \right\}$$

If Σ_1 *is empty, we adopt the convention that* $\mathnormal{}_1 = \emptyset$.

Thus, $\mathnormal{}_1$ is the set of all policies that are inferior to the median policy $\left(x_m^*, r^{\min}\right)$ from the point of view of all voters in Σ_1 (note that Σ_1 includes only a subset of those voters who would vote for an incumbent who implements $\left(x_m^*, r^{\min}\right)$, as it excludes those who set $b_i = u_i\left(x_m^*, r^{\min}\right)$). Note that $\mathnormal{}_1$ is nonempty - for instance, any policy $\left(x_m^*, r'\right)$, where $r' > r^{\min}$ is an element of $\mathnormal{}_1$, as $u_i\left(x_m^*, r'\right) < u_i\left(x_m^*, r^{\min}\right)$ for all voters (and hence for all voters in Σ_1). Using this definition, we can show that no CPNE involves a policy in the set $\mathnormal{}_1$ (and, moreover, this result always rules out some feasible policies, as $\mathnormal{}_1$ is nonempty):

Lemma 13 *i) If* $\mathbf{b} = (b_1, ..., b_N)$ *is a CPNE profile of voter strategies that induces the incumbent to implement* (x', r'), *then* $(x', r') \notin \mathnormal{}_1$.
ii) If $b = (b_1, ..., b_N)$ *is a CPNE profile of voter strategies that induces the incumbent to implement* (x', r'), *then for at least one voter* j, $u_j\left(x', r'\right) > u_j\left(x_m^*, r^{\min}\right)$.

i) Suppose that $(x', r') \in \mathnormal{}_1$. Without loss of generality, we can consider the case $x' \leq x_m^*$, where the single crossing condition (6) implies that $\forall\, i \in \{m, ..., N\}$, $u_i\left(x', r'\right) < u_i\left(x_m^*, r^{\min}\right)$. Let $\bar{\Sigma}_1 = \Sigma_1 \cup \{m, ..., N\}$ (note that $\bar{\Sigma}_1$ always includes a majority of voters, even when Σ_1 is empty). For $i \in \Sigma_1$, $(x', r') \in \mathnormal{}_1 \Rightarrow u_i\left(x', r'\right) < u_i\left(x_m^*, r^{\min}\right)$ (by the definition above), so that:

$$\forall i \in \bar{\Sigma}_1 \qquad u_i\left(x', r'\right) < u_i\left(x_m^*, r^{\min}\right) \tag{17}$$

Let $\bar{\Sigma}_2$ denote the complement of $\bar{\Sigma}_1$ in Σ such that $\bar{\Sigma}_1 \cap \bar{\Sigma}_2 = \emptyset$ and $\bar{\Sigma}_1 \cup \bar{\Sigma}_2 = \{1, ..., N\}$. Hence,

$$\bar{\Sigma}_2 = \left\{ i \in \{1, ..., m-1\} \text{ such that } b_i \geq u_i\left(x_m^*, r^{\min}\right) \right\}$$

Consider the following two cases:

1. All voters in $\bar{\Sigma}_1$ set $b_i = u_i\left(x_m^*, r^{\min}\right)$ (this can only be true if Σ_1 is empty, and all $i \in \{m, ..., N\}$ set $b_i = u_i\left(x_m^*, r^{\min}\right)$). Then all $i \in \{m, ..., N\}$ set $b_i = u_i\left(x_m^*, r^{\min}\right)$ and all $i \in \{1, ..., m-1\}$ set $b_i \geq u_i\left(x_m^*, r^{\min}\right)$. The incumbent will implement $\left(x_m^*, r^{\min}\right)$ (any policy other than $\left(x_m^*, r^{\min}\right)$ will result in at least m voters voting against reelection). Clearly, $\left(x_m^*, r^{\min}\right) \notin \mathnormal{}_1 = \emptyset$, so the proposition is true in this case.

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2. At least one voter in $\bar{\Sigma}_1$ sets $b_i \neq u_i\left(x_m^*, r^{\min}\right)$ (this will always hold whenever Σ_1 is nonempty). Consider a deviation by all those members of $\bar{\Sigma}_1$ who do not set $b_i = u_i\left(x_m^*, r^{\min}\right)$ to $b_i' = u_i\left(x_m^*, r^{\min}\right)$. We show first that this deviation induces the incumbent to implement $\left(x_m^*, r^{\min}\right)$. If the incumbent implements:

   (a) $\left(x_m^*, r^{\min}\right)$ she receives the votes of all voters in $\bar{\Sigma}_1$, and so is reelected.

   (b) (x^+, r^+) with $x^+ \leq x_m^*$ and $r^+ > r^{\min}$, all voters in $\{m, ..., N\}$ (i.e. at least m voters) vote against reelection because $b_i' = u_i\left(x_m^*, r^{\min}\right) > u_i\left(x^+, r^+\right)$.

   (c) (x^+, r^+) with $x^+ \geq x_m^*$ and $r^+ > r^{\min}$, voters in $\{1, .., m\} \cap \bar{\Sigma}_1$ vote against reelection because $b_i' = u_i\left(x_m^*, r^{\min}\right) > u_i\left(x^+, r^+\right)$, and voters in $\bar{\Sigma}_2$ vote against reelection because $b_i \geq u_i\left(x_m^*, r^{\min}\right) > u_i\left(x^+, r^+\right)$. Thus, at least the m voters in $\{1, ..., m\}$ vote against reelection.

The deviation therefore unambiguously induces $\left(x_m^*, r^{\min}\right)$. Note also that it is strictly profitable for all deviators, as they all belong to $\bar{\Sigma}_1$, so that $u_i\left(x', r'\right) < u_i\left(x_m^*, r^{\min}\right)$.

Now consider further deviations by subsets of the deviators. A subset of $\bar{\Sigma}_1 \cap \{m+1, N\}$ may wish to deviate to induce a higher x. But then, by the preceding argument (c) above, all voters in $\{1, ..., m\}$ vote against the incumbent's reelection. Similarly, a subset of $\bar{\Sigma}_1 \cap \{1, m-1\}$ may wish to deviate to induce a lower x. But then, by the preceding argument (b) above, all voters in $\{m, ..., N\}$ vote against the incumbent's reelection. There is therefore no further successful profitable deviation.

Thus, if $\mathbf{b}$ is a CPNE profile of voter strategies that induces the incumbent to implement (x', r'), and if $(x', r') \in \quad _1$ with $x' \leq x_m^*$, then there exists a self-enforcing profitable deviation by a subset of voters. Moreover, this deviation is immune to further successful deviations by subsets of the deviators. This contradicts the premise that $\mathbf{b}$ is a CPNE profile of voter strategies. An analogous argument is possible for $x' \geq x_m^*$. Thus, it follows that if $\mathbf{b}$ is a CPNE profile of voter strategies, then $(x', r') \notin \quad _1$.

ii) Define $\quad _2 = \left\{(x, r) \mid \forall i \in \{1, ..., N\}, u_i(x, r) < u_i\left(x_m^*, r^{\min}\right)\right\}$. Then clearly $_2 \subseteq \quad _1$. So by i), one has $(x', r') \notin \quad _2$. This implies that for at least one voter $j \in \{1, ..., N\}$ $u_j(x', r') \geq u_j\left(x_m^*, r^{\min}\right)$

7.4   Proof of Proposition 6

Let $\mathbf{b}$ be an ILNE profile of voter strategies (i.e. $b_i \geq u_i\left(x_m^*, r^{\min}\right)$ for all i). Suppose that $\mathbf{b}$ induces the incumbent to seek reelection. Then, assume that the incumbent implements $(x', r') \neq (x_m^*, r^{\min})$. Consider the case where $x' \leq x_m^*$ (an analogous argument exists for the case where $x' \geq x_m^*$). Clearly, $u_m(x', r') < u_m(x_m^*, r^{\min})$. By the SCP and the definition of ideological loyalty, it follows that for all $i \in \{m, ..., N\}$

$$u_i(x', r') < u_i(x_m^*, r^{\min}) \leq b_i \tag{18}$$

Thus, if the incumbent implements (x', r'), a majority of voters $\{m, ..., N\}$ will vote against reelection. But, this contradicts the premise that $\mathbf{b}$ induces the incumbent

to seek reelection and that the incumbent implements $(x', r') \neq (x_m^*, r^{\min})$ in equilibrium. Thus, any ILNE in which the incumbent is reelected must lead to the implementation of the policy $(x_m^*, r^{\min})$.

In any equilibrium in which the incumbent is not reelected, the incumbent will always extract maximal rents $r^{\max}$, and implement an x^+ determined by the selection criterion over x (given $r^{\max}$). An ILNE profile of voter strategies **b** does not eliminate the possibility that the incumbent will not seek reelection (e.g. consider $N = 5$ and $b_i = +\infty, i = 1, ..., 5$). Thus, $(x^+, r^{\max})$ is also an outcome that can be sustained in an ILNE.

7.5 Proof of proposition 7

Suppose that **b** is an ILCPNE profile of voter strategies, and that the incumbent implements $(x', r') \neq (x_m^*, r^{\min})$. Consider the case where $x' \geq x_m^*$, with $r' \geq r^{\max}$. Now, consider the deviation in equation (16) in the proof of Proposition 4 (note that here, voters in $\bar{\Sigma}_A$ set $b_i = u_i (x_m^*, r^{\min})$ to satisfy ideological loyalty). As we showed there, this deviation induces the incumbent to implement $(x_m^*, r^{\min})$, and is strictly profitable for all deviators. Note that the deviators adopt an ideological loyal strategy. For a CPNE, we also need to ask whether the deviation is self-enforcing.

We show that when voters are ideologically loyal, no further successful deviation is feasible. While m cannot benefit by deviating, some subset of $\{1, ..., m-1\}$ could potentially benefit by a deviation that induces the incumbent to move x to the left (i.e. $x < x_m^*$). This subset, however, includes at most $\frac{N-1}{2}$ voters, so, for the deviation to be successful, it would require at least one voter $i > m$ to vote for reelection when $x < x_m^*$ is implemented. But, by the assumption that all voters are playing ideologically loyal strategies, $\forall i > m, b_i \geq u_i(x_m^*, r^{\min})$. Thus, no $i \geq m$ will vote for reelection when $x < x_m^*$, so a deviation from the equation (16) deviation by a subset of $\{1, ..., m-1\}$ cannot be successful.

An analogous argument holds for $x' \leq x_m^*$. Therefore, any equilibrium involving a policy $(x', r') \neq (x_m^*, r^{\min})$ is subject to a deviation that is not subject to any further deviations. This contradicts the premise that **b** is an ILCPNE profile of voter strategies; hence, the incumbent always implements the policy $(x_m^*, r^{\min})$.