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Comparative Statics in a Herding Model of Investment

by

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Abstract

This paper is an adaptation of the Chamley-Gale endogenous-timing information-revelation model of investment (*Econometrica,* 1994). The paper models a game with pure informational externalities where agents can learn by observing others' actions. Observational learning about the value of the investment project can result in massive social imitation, possibly leading the society to the incorrect choice, to an inefficient cascade. While Chamley and Gale characterize the equilibrium of such a game, this paper yields an analytic approximation to the probability of inefficient cascades and allows for the derivation of comparative statics results. This is useful for two reasons: i) these results indicate that some of the findings from the exogenous-timing herding literature may not necessarily be generalizable to the endogenous-timing framework. ii) the study may be useful in the analysis of a wide variety of applied issues including IPO pricing, speculative attacks and adoption of new technology.

1. Introduction

People often observe other people's actions while making their own decisions. This might be due to positive social or network externalities and/or it might be due to social learning. This paper focuses solely on observational learning without any externalities. When signals are private, rational agents may be able to infer the nature of the state from the actions of their predecessors. Herd behavior or information cascade occurs when everyone is imitating the crowd, even when their private information suggests the opposite. In this social learning process, if early movers' signals happen to be incorrect then agents may settle on a common inefficient action, resulting in an inefficient cascade. This paper adapts the endogenous-timing information-revelation investment model of Chamley and Gale (1994) to study the factors that make inefficient cascades more likely.

In a survey study, Devenow and Welch (1996) give an extensive list of empirical phenomena that informational cascades may explain[1]. Examples come both from real markets such as R&D investment decisions and from financial markets; among others, analysts' recommendation of a particular stock, bank runs and managers decisions to pay dividends may have elements of herding behavior. It is often argued that conformist behavior in financial and real markets may lead to sudden booms and crashes. This paper studies the factors that influence the likelihood of erroneous mass behavior, either when there is an investment boom even though the true value of the project is low (inefficient positive cascade), or when there is an investment collapse even though the true value is high (inefficient negative cascade).

In seminal papers by Banerjee (1992) and Bikhchandani and Hirshleifer and Welch (1992) each person observes the behavior of the people who went before him where there is an exogenously determined sequence in the moves. These models show that society may settle in an inefficient outcome because valuable information gets trapped at some stage of social learning. Chamley and Gale (1994) prove the existence of herd behavior even when the timing of moves and information revelation is endogenous. In an endogenous-timing framework, the individual agent has an incentive to wait in order to observe the actions of other players. However if everyone were to wait, the agent would rather move early in order to avoid cost of delay. Hence the timing decision is strategic.

[1] Also see the "living "document by Bikhchandani, Hirshleifer and Welch (1996) for an overview of the theoretical and empirical literature on herding.

While Chamley and Gale characterize the equilibrium of such a game, this paper yields an analytic approximation to the probability of inefficient cascades and allows for the derivation of comparative statics results.

To the best of my knowledge, in a framework where agents do not have preferential access to information[2], this is the first endogenous-timing herding model that allows for the derivation of comparative statics results for the probability of negative and positive information cascades. The analysis will allow us to examine whether inefficient cascades are more or less likely as signal quality improves, as the observation period length increases and as there is more to lose or gain. This is useful for two reasons: First, it allows a deeper understanding of the relationship between exogenous and endogenous timing herding models. This paper shows that some of the results on exogenous-timing herding models do not necessarily generalize to models with endogenous timing. Secondly, the derivation of comparative static results in the Chamley-Gale model provides a framework that may be useful in the analysis of a wide variety of applied issues. Some of these will be discussed in the conclusion.

2. Framework

Each of the identical risk neutral agents with an investment option can exercise the option at any date T=0,1,2,...∞ of his choice. All options are identical and indivisible. The investment decision is irreversible. $\delta \in (0,1)$ is the common discount factor. Each player with an option chooses either to invest now or delay. If the player never invests the payoff is 0. Whether or not the player has an option is private information. Only if the option is exercised information is revealed. The true value of the investment is identical for all players and it is denoted by $V \in \{V^H, V^L\}$ where $V^H > 0$ and $V^L < 0$. $V = V^H$ with prior probability $q^* \in (0,1)$.

This paper adapts the r-Fold Replica Game of Chamley and Gale [3].This implies that the population is unboundedly large. While the population consists of rN agents, only rn of them

[2]Zhang (1997) provides a endogenous timing framework where the first mover is the agent with the highest precision of information. A cascade starts immediately after the first mover, all agents follow the expert leader.

[3]This corresponds to Section 6 in Chamley and Gale.

have an opportunity to undertake an investment project. The results will hold as $r \rightarrow \infty$. When the project value is high, more people are aware of the investment opportunity and hence more people have an option to invest.

$$n = \begin{cases} n^H \ \ when\, V = V^H \\ n^L \ \ when\, V = V^L \end{cases} \tag{1)a}$$

and $\psi \equiv \frac{n^L}{n^H} < 1$.

Chamley and Gale assume that the number of people with an option is stochastic but it is more likely to be high when the true value of the project is high. However, here the value of the project is either high or low and there is a one-to-one mapping between V and n. The restriction to only two possible project values will allow us to summarize agents' beliefs about the true state of the nature at time T in a single variable: the probability that the project value is high. This mapping will prove to be very convenient in eventually formulating the learning process in a linear fashion. So far this is a special case of Chamley and Gale.

Let us now introduce the changes to the Chamley and Gale framework. In Chamley and Gale both orders and processing of orders happen in discrete time. Whereas here, agents will place discrete-time state-contingent orders which get processed in continuous time. Players place their orders at the beginning of each period. Orders are processed randomly during the period – the exact time that an individual order is processed is distributed uniformly in the period. Since information on others' actions will be arriving during the period, players are permitted to make their orders (both invest and wait orders) contingent on the flow of information. Payoffs on all orders processed in a period are received at the end of the period. The benefit of moving to continuous-time order processing is that it will allow us to approximate a transformation of the agent's problem as a Wiener process with absorbing boundaries and hence derive the probability of inefficient cascades.

Each invest order comes with a state-contingent wait order. The investment cannot be reversed in case the invest order is already processed. During the interval $[T,T+1)$, if the state-contingent wait order is triggered, then at most M of the newly triggered wait orders are processed, where M is a large but finite number. The number of newly triggered wait orders W may exceed M. In that case, a randomly selected W-M of these newly triggered wait orders are ignored. These are simply continued to be processed as invest orders. One can interpret M as the maximum capacity of the processing agency to accommodate state-contingent orders. Each wait order comes with a state-contingent invest order. During the interval $[T, T+1)$, if the state of the state-contingent invest order is triggered, then at most M of the newly triggered invest orders are processes. If the number of the newly triggered invest orders Z is greater than M, then the remaining Z-M are not processed during the period. As will be shown, this form of contingent order will ensure that in equilibrium the expected payoffs from putting an invest or wait order will be the same as the expected payoffs from putting in an invest or wait order in the Chamley and Gale framework.

3. Equilibrium

We will start out by conjecturing that the equilibrium of this new game mirrors the equilibrium in the game of Chamley and Gale. Then it will be shown that in this conjectured equilibrium the players' expected payoffs from their equilibrium strategies and from possible deviations are the same as those resulting to players in Chamley and Gale's game. And hence Chamley and Gale's proof of equilibrium will apply here as well[4].

Each player who receives an investment option faces a tradeoff between investing and delaying. If the player invests now he collects the undiscounted payoff but faces the risk of making a loss in case the true value is V^L. If the player delays he collects only discounted payoffs but he can make use of information revealed by other players' actions. If the agent knew how many people had the investment option he would know V. Hence observing the number of people who invest can help predict the true value of the project. The focus is only on the symmetric Perfect Bayesian Equilibria. Before describing the equilibrium strategies, let us first introduce some critical values.

3.1. Critical Values

The prior probability that $V=V^H$ is q^*. Denote q_t as the subjective probability at time t that the true value is high. Since orders are processed in continuous time, q_t evolves in continuous time. The index of time for discrete decision time nodes will be denoted by T. While $t\in\mathbb{R}^+$, the index $T\in\mathbb{N}$. So, at discrete time nodes when $t=T$, $q_t=q_T$. At the beginning of the game, the probability

[4]This approach cannot rule out the possibility that other equilibria may also exist.

that the project has a high value, q_T at time T=0, conditional on having received an investment opportunity, is given by:

$$q_0 = \frac{\frac{rn^H}{rN} q^*}{\frac{rn^H}{rN} q^* + \frac{rn^L}{rN}(1-q^*)} \tag{2)b$$

Since $\psi \equiv \frac{n^L}{n^H}$, (2) can be rewritten as,

$$q_0 = \frac{q^*}{q^* + \psi(1-q^*)} \tag{3)c$$

The game is of interest if initially the expected value of the project is positive. Otherwise each agent would strictly prefer to wait and the game would end immediately with an investment collapse[5].

It will be useful to introduce two critical values for the subjective probability. Define $\underline{q}$ as the probability where the expected value of the project is zero:

$$\underline{q}\, V^H + (1-\underline{q})\, V^L = 0 \tag{4)d$$

When $q_T < \underline{q}$, the expected value of investment is negative. So the agent will strictly prefer to wait. Since everyone who has not yet invested is identical they all prefer to wait and the game effectively ends. Investment stops for good.

Define $\overline{q}$ as the probability where the agent is just indifferent between investing now and waiting even though information about the true value of the project is to be fully revealed with certainty next period:

$$\overline{q}\, V^H + (1-\overline{q})\, V^L = \delta\, \overline{q}\, V^H \tag{5)e$$

The left hand side is the expected value from investing. The right hand side gives the expected value from waiting given that the true value of the project is to be revealed. When $q_T > \overline{q}$ the agent will strictly prefer to invest now. And so will all identical players, and the game ends where all players with an option invest. The game will be said to be *active* when $\underline{q} < q_T < \overline{q}$.

[5]This implies that $\psi \frac{(1-q^*)}{q^*} < -\frac{V^H}{V^L}$.

3.2. Learning

The player's actions depend on the publicly observed history of the game which is described by the sequence of the number of people who invested during each period. Following the notation in Chamley and Gale, for any history h, let $\lambda(h)$ denote the probability that a player who has not yet invested does so after observing the history h. In the *active* phase of the game, it must be that $0<\lambda(h)<1$. Assume for a moment that an agent expects all people with an investment opportunity to invest this period. Then he would strictly prefer to wait to be able to learn the value of the project for sure. But so would everyone else. Hence $\lambda(h)\neq 1$. If he expects nobody else to invest this period, there would be no learning this period, so as long as expected value from investment is positive he would strictly prefer to invest now.[6] But so would everyone else. Hence $\lambda(h)\neq 0$, by contradiction. In equilibrium, $0<\lambda(h)<1$, such that players are just indifferent between waiting and investing now. Notice that λ is the endogenous information revelation parameter. If λ were zero, no information would be revealed. If λ were equal to one, the number of people who invest would fully reveal information about the value of the project.

As $r\to\infty$, the number of people putting in invest orders at a decision node is given by the Poisson approximation to the binomial distribution. The parameter of the Poisson distribution is the mean number of invest orders, rn times $\lambda=\lambda(h)$. The probability that k players invest at a decision node given λ is:

$$f(k;\lambda)=\begin{cases}\dfrac{e^{-\lambda rn}(\lambda rn)^k}{k!} & \text{for } k\in\mathbb{N} \text{ and } k\in[0,rn]\\ 0 & \text{elsewhere}\end{cases} \tag{6)f}$$

Define $f^H(k;\lambda)\equiv f(k;\lambda)$ when $n=n^H$, and $f^L(k;\lambda)\equiv f(k;\lambda)$ when $n=n^L$. If n^L were equal to n^H, then $\psi=1$ and the two probability density functions would collapse together. In such an extreme case the quality of the signal k would be nil and the signal would not reveal any information. However as ψ decreases, the signal quality improves and an observation of the rate of investment provides valuable information in distinguishing between $f^H(k;\lambda)$ and $f^L(k;\lambda)$.

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[6]While intuitive, this argument assumes that is not optimal to wait for information that may arrive several periods later. This is in fact the case. The optimal program will have a "one-step property" where at any period the agent is willing to make a once and for all invest-not invest decision. See Chamley and Gale proposition 3 for the proof.

Define k_T as the number of invest orders put in at the decision node *T-1*. Assuming no contingencies are triggered, k_T is total number of people who invest in the time interval [*T-1,T*) and it is public knowledge at decision node *T*. The history up to time *T* is h_T. The λ that makes the agents indifferent between investing and waiting at the decision node *T* is λ_T. Bayesian learning suggests that at time *T*, when the agent observes k_T people investing, the subjective probability will evolve following:

$$q_T = \frac{q_{T-1} f^H(k_T;\lambda_{T-1})}{q_{T-1} f^H(k_T;\lambda_{T-1}) + (1-q_{T-1}) f^L(k_T;\lambda_{T-1})} \qquad (7)g$$

Chamley and Gale prove that in equilibrium λ is independent of both r and the total number of people who have already invested[7]. The basic intuition is that the individuals' learning is equivalent to learning from sequence of samples. Since $r \to \infty$, the rate of investment is very small compared to the size of the economy. Therefore one can think of the sampling simply as sampling with replacement. The equilibrium λ at the decision node *T*, will solely depend on history captured by q_{T-1} and k_T. In the *active* phase of the game, for each $q_T \in (\underline{q}, \overline{q})$, there will be a critical $\kappa = \kappa(q_T)$, such that q_{T+1} is just at or below $\underline{q}$. So the following equation implicitly defines λ_T where the agent is just indifferent between investing now and waiting.

$$q_T V^H + (1-q_T) V^L = \delta \sum_{k=\kappa(q_T)+1}^{rN} \left[\left(q_T f^H(k,\lambda_T) + (1-q_T) f^L(k,\lambda_T)\right) \left(\frac{q_T f^H(k,\lambda_T) V^H + (1-q_T) f^H(k,\lambda_T) V^L}{q_T f^H(k,\lambda_T) + (1-q_T) f^L(k,\lambda_T)} \right) \right] \qquad (8)h$$

The left hand side gives the expected payoff from investing now. The right hand side gives the discounted expected payoff from waiting. The first term of the brackets is the probability of observing a particular k at time *T+1*. The second term is the expected value of the project given that the particular k is observed.

3.3. Equilibrium Strategies

[7] See Proposition 8 and the proof in Chamley and Gale.

Let us first assume that the institutional setup restricts the agents to only use is $\underline{q}$ and $\overline{q}$ as their triggers for the contingency orders. In Appendix C, this assumption is relaxed. The equilibrium of the game with any finite set Γ of possible contingency trigger points with cardinality greater than one and which contains both $\overline{q}$ and $\underline{q}$ is shown to yield the same boundary crossing probabilities as the baseline model.

PROPOSITION: Let λ_T be described by Equation (8), the following equilibrium strategy supports a symmetric Perfect Bayesian Equilibrium:

a) If subjective probability is sufficiently low $q_T \leq \underline{q}$, put in a wait order with a state-contingent invest order. If in the time interval $[T,T+1)$, $q_t \geq \overline{q}$, the state-contingent invest order is triggered.

b) If subjective probability is sufficiently high, $q_T \geq \overline{q}$, put in an invest order with a state-contingent wait order. If in the time interval $[T,T+1)$, $q_t \leq \underline{q}$, the state-contingent wait order is triggered.

c)If subjective probability is $\underline{q}<q_T<\overline{q}$, with probability λ_T, put in an invest order with a stat-contingent wait order. If in the time interval $[T,T+1)$, $q_t \leq \underline{q}$, the state-contingent wait order is triggered. With probability $(1-\lambda_T)$ put in a wait order with a state-contingent invest order. If in the time interval $[T,T+1)$, $q_t \geq \overline{q}$ the state-contingent invest order is triggered.

PROOF: a) By equation (4), when the subjective probability is $\underline{q}$, the expected value of the project is just equal to zero. Hence the agent strictly prefers to wait when $q_T < \underline{q}$. If in the time interval $[T,T+1)$, $q_t \geq \overline{q}$ the expected value of the project would be so high that the agent would prefer to invest. Note that in this case, the contingency order will never be triggered in equilibrium. Once $q_T \leq \underline{q}$ all identical agents with an investment opportunity will prefer to wait. This becomes an absorbing state and the investment ends for good. No new information can be received in the time interval $[T,T+1)$ to increase q_t above $\overline{q}$.

b) By equation (5), when the subjective probability is $\overline{q}$, the expected value of investing now is just equal to waiting one more period assuming that information about the true value of the project were to be reveal for sure next period. Hence, when $q_t \geq \overline{q}$, the agent prefers to invest right away. If in the time interval $[T,T+1)$ new information were to arrive such that $q_t \leq \underline{q}$ the agent would prefer to wait. Notice that this is an absorbing state. When $q_t \geq \overline{q}$ all agents with an investment option would prefer to invest. Since $r \rightarrow \infty$, the rate of information flow would be a continuous variable and the true value of rn and hence V, would be revealed at once. If $V=V^H$, agents subjective probability would remain above $\overline{q}$. If $V=V^L$, the subjective probability would immediately drop down below $\underline{q}$. All agents state contingent orders would be triggered at once but only M of them would be able to stop the investment. The game would end with all investing except for those lucky M people.

c) If subjective probability is $\underline{q}<q_T<\overline{q}$, the expected value of investing is positive but the agent will also consider waiting in order to learn about the true value of the project. In equilibrium the agents is just indifferent between investing now and waiting. See the beginning of section 3.2 for the discussion of the non-existence of pure-strategy equilibrium.

i) The agent with an investment option who has not yet exercised his option will put an invest order at time T with probability λ_T. If however in the time interval $[T,T+1)$, q_t falls below $\underline{q}$, the agent would prefer to wait. Once the contingency is triggered all unprocessed invest orders would convert into wait orders. Since M is a very large number, investment would stop for good.

ii) The agent with an investment option who has not yet exercised her option will put a wait order at time T with probability $(1-\lambda_T)$. If however in the time interval $[T,T+1)$, q_t rises above $\overline{q}$ the agent would prefer to invest. In fact all agents would now prefer to invest all at once. M is very large but finite, whereas $r\rightarrow\infty$. Hence M newly arrived invest orders would be processed this period. All the rest would be processed next period. At time T, the agent realizes that the is an infinitely small probability that his invest order would be processed if the state is triggered. Hence equation (H) continues to define λ_T.

The equilibrium strategies and the possible deviations of this game yield the same payoffs as in Chamley and Gale.

4. Information Cascades

The subjective probability evolves as a result of observational learning from the rate of investment each period, which is a stochastic variable. Chamley and Gale prove that eventually the game will end with an information cascade[8]. If the subjective probability hits $\underline{q}$ before $\overline{q}$, the game ends with an investment collapse. If the subjective probability hits $\overline{q}$ before $\underline{q}$, the game ends with an investment boom. We are particularly interested in the probability of inefficient

[8]Proposition 8 in Chamley and Gale.

cascades. The measures of interest are then the probability that the process hits $\underline{q}$ before $\overline{q}$ when $V = V^H$, and the probability that the process hits $\overline{q}$ before $\underline{q}$ when $V = V^L$. The first would be an inefficient negative cascade and the latter would be an inefficient positive cascade.

4.1. Transformation

In order to obtain the boundary crossing probabilities, we will need to transform the problem into an equivalent problem that is tractable. Subjective probabilities evolve following (7), substitute $f^H(k_T;\lambda_{T-1})$ and $f^L(k_T;\lambda_{T-1})$ into (7). Cancel out k_T factorial from the numerator and denominator. Take the inverse of both the left and right hand side of the equation and subtract one from each side. Now plugging in ψ for $\frac{n^L}{n^H}$ yields,

$$\frac{1-q_T}{q_T} = \frac{1-q_{T-1}}{q_{T-1}} e^{(1-\psi)\lambda_{T-1} rn^H} \psi^{k_T} \qquad \text{(9)k}$$

Taking the natural logarithm of both sides yields:

$$ln\left(\frac{1-q_T}{q_T}\right) = ln\left(\frac{1-q_{T-1}}{q_{T-1}}\right) + (1-\psi)\lambda_{T-1} rn^H + ln(\psi)k_T \qquad \text{(10)m}$$

where k_T is distributed Poisson with the parameter $\lambda_{T-1} rn^H$ when the true value of the project is high and it is distributed Poisson with the parameter $\lambda_{T-1} rn^L$ when the true value of the project is low. For large λrn, the Poisson distribution can be approximated by the normal distribution. Notice that $k_T \geq 0$. However the normal distribution assigns positive probability to events with $k_T < 0$. Hence this approximation is less than perfect for small λrn. Define w_T as:

$$w_T = ln\left(\frac{1-q_T}{q_T}\right) \qquad \text{(11)p}$$

Notice that w_T is an increasing monotonic transformation of q_T. Plugging (p) into (m), we get a transformed problem:

$$w_T = w_{T-1} + k_T^* \qquad \text{(12)zh}$$

where k_T^* is distributed normal with mean μ and variance σ^2:

$$k_T^* \sim \begin{cases} N\left(\mu^H = (\psi - 1 - \ln\psi)\lambda_T n^H,\ (\sigma^2)^H = (\ln\psi)^2 \lambda_T n^H\right) & \text{when } V = V^H \\ N\left(\mu^L = (\psi - 1 - \psi\ln\psi)\lambda_T n^H,\ (\sigma^2)^L = \psi(\ln\psi)^2 \lambda_T n^H\right) & \text{when } V = V^L \end{cases} \quad . \qquad \text{(14)zo}$$

$\mu^H>0$ by Appendix Claim A1. And $\mu^L<0$ by Appendix Claim A3.

Individual learning is a stochastic process with independent increments. This process is a well known description of individual learning in cognitive psychology. In much of that literature individuals are modeled as learning through random sampling with exogenously determined "response thresholds." This characterization of the learning process is used to explain laboratory evidence on individual response times and error rates. The present paper shows that even with fully rational agents group behavior will resemble individual behavior with boundedly rational agents of the type used in cognitive psychology.[9]

The transformation (p) of the lower bound given by (4), of the upper bound given by (5) and of the starting point given by (3) yield :

The lower bound:	$\underline{q} \Rightarrow \underline{w}$	$\underline{w} = ln(-V^L) - lnV^H$	(15)q
The upper bound:	$\overline{q} \Rightarrow \overline{w}$	$\overline{w} = ln(-V^L) - lnV^H - ln(1-\delta)$	(16)r
The starting point:	$q_0 \Rightarrow w_0$	$w_0 = lnq^* - ln(1-q^*) - ln\psi$	(17)s

Notice that for the game to be *active*, $\underline{w} < w_0$ since initially the expected value of the project is positive (see footnote 5). And $w_0 < \overline{w}$ examining (e) and (c) together.

4.2. Boundary Crossing Probabilities with constant λ

The individual learning process follows the equation (14) where the error term is distributed approximately normal with mean μ and variance σ^2. Both the mean and the variance of the process depend λ_T and hence they depend on the history of the game. They are not constant.

Now we are going to examine a different process. In this modified problem, we will examine the process described by equation (zh) and (zo) yet with a constant $\lambda \in (0,1)$, implying

[9]See Luce (1986) for an introduction to this literature. MORE?

a constant drift and variance. In section 4.3, we will prove that the process with the endogenously determined λ_T will yield identical boundary crossing probabilities as in the modified problem with fixed λ.

Note that orders are processed in continuous time and the processing time of each order is distributed uniformly over the period [T,T+1). So we can define w_t as a continuous variable which coincides with w_T when t=T. Denote k_t^{**} as the stochastic term which is distributed normal where μ and σ^2 (given by Equation (zo)) are respectively the drift velocity and the power of the noise of the process. Assuming $\lambda_{T-1}= \lambda_T$., w_t can be approximated by a Wiener process[10]:

$$w_t = w_{t-1} + k_t^{**} \tag{Az}$$

Equations (17), (15) and (16) give the starting point and the bounds. We can easily compute the boundary crossing probabilities.

i) Probability of hitting $\underline{w}$ before $\overline{w}$ when $V=V^H$ and $\lambda_{T-1}= \lambda_T$: In this case, the drift is positive, $\mu^H>0$. The probability of hitting $\underline{w}$ before $\overline{w}$ is given by[11]:

$$Prob(\underline{w}\ before\ \overline{w} \,|\, V = V^H and\ \lambda_T = \lambda_{T-1}) = 1 - \left[\left(e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H w_0}{(\sigma^2)^H}}\right) \Big/ \left(e^{\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}} - e^{\frac{-2\mu^H \overline{w}}{(\sigma^2)^H}}\right)\right] \tag{18)t$$

The system is defined by six equations: (8), (15), (16), (17), (18) and (12) that defines μ^H. and $(\sigma^2)^H$. Combining the six equations, one can find a closed form solution for the probability of hitting the lower bound before the upper bound. Divide the numerator and the denominator of (18) by $exp\left(\frac{-2\mu^H \underline{w}}{(\sigma^2)^H}\right)$ and plug in the values of $\underline{w}$, $\overline{w}$, w_0 and μ^H and $(\sigma^2)^H$:

$$Prob(\underline{w}\ before\ \overline{w} \,|\, V = V^H and\ \lambda_{T-1} = \lambda_T) = 1 - \left(\frac{1-e^{-2\varphi\left[\delta \ln(1-q^H) - \ln q^H + \ln\psi - \ln V^H + \ln(-V^L)\right]}}{1-e^{-2\varphi\ln(1-\delta)}}\right) \tag{19)u$$

where $\varphi = -\frac{(\psi - 1 + ln\psi)}{(ln\psi)^2}$. φ<0 by Claim A1 in the Appendix. **Notice that this probability is independent of λ.**

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[10]FOOTNOTE ?????

[11]See Karlin and Taylor (1975).

ii) Probability of hitting $\underline{w}$ before $\overline{w}$ when $V=V^L$ and $\lambda_{T-1}=\lambda_T$: In this case the drift is negative, $\mu^L<0$. The probability of by of hitting $\underline{w}$ before $\overline{w}$ when $V=V^L$:

$$Prob(\underline{w}\ before\ \overline{w}\,|\,V=V^L\ and\ \lambda_{T-1}=\lambda_T)=\left(e^{\frac{-2\mu^L\underline{w}}{(\sigma^2)^L}}-e^{\frac{-2\mu^L w_0}{(\sigma^2)^L}}\right)\Bigg/\left(e^{\frac{-2\mu^L\underline{w}}{(\sigma^2)^L}}-e^{\frac{-2\mu^L\overline{w}}{(\sigma^2)^L}}\right) \tag{20}$$

The system is thus defined by six equations: (8), (15), (16), (17) , (20) and (12) describing μ^L and $(\sigma^2)^L$. Combining these six equations, one can find a closed form solution for the probability of by of hitting $\underline{w}$ before $\overline{w}$ when $V=V^L$:

$$Prob(\underline{w}\ before\ \overline{w}\,|\,V=V^L and\lambda_{T-1}=\lambda_T)=\frac{1-e^{-2\gamma\left[(\ln(1-q^*)-\ln q^*+\ln\psi-\ln(V^H)+\ln(-V^L)\right]}}{1-e^{-2\gamma\ln(1-\delta)}} \tag{21}$$

where $\gamma=-\dfrac{(\psi-1-\psi\ln\psi)}{\psi(\ln\psi)^2}$. $\gamma>0$ by Claim A3 in the Appendix.

4.3. Inefficient Cascade Probabilities for the Original Problem

Proposition 1: *The boundary crossing probabilities of the original problem are equal to the boundary crossing probabilities found using a Wiener process, (19) and (21) of the modified problem.*

Proof: In the actual learning process the parameter λ is updated via equation (8) at each $T\in N$. As long as contingencies are not triggered λ stays constant during the interval *[T, T+1).* The boundary crossing probabilities for this process can be reconstructed iteratively using the Lemma in Appendix C1. Starting with the Wiener process with absorbing boundaries defined in (*), (**), (15), (16) and (17), create a process where the parameter λ changes to λ' (which is stochastic) at t=1 and stays constant thereafter. From the lemma this new process has the same transition probabilities as the original process. Since after t=1 the process is a Wiener process we can do this again after one more period and the new process will also have the same transition probabilities. Iterating this argument yields the result. □

5. Comparative Statics

Since we have approximated closed-form solutions for the probability of inefficient cascades we can examine the comparative statics. We will start with the comparative statics that behave same as in exogenous-timing models and then move on to the new comparative statics results we learn from this endogenous-timing framework.

5.1. The Prior:

As one would expect, increasing the ex-ante probability that $V=V^H$ decreases the probability of an inefficient negative cascade.

$$\frac{dProb(INC)}{dq^*} = \frac{2\varphi e^{-2\varphi\left[\ln(1-q^*)-\ln q^*+\ln\psi-\ln V^H+\ln(-V^L)\right]}}{(1-q^*)q^*(1-e^{-2\varphi\ln(1-\delta)})} < 0 \qquad (22)x$$

since $\varphi<0$ and $\left(1-e^{-2\varphi\ln(1-\delta)}\right)>0$.

The likelihood of getting into into a positive cascade even though $V=V^L$, increases as the prior goes up.

$$\frac{dProb(IPC)}{dq^*} = -\frac{2\varphi e^{-2\gamma\left[\ln(1-q^*)-\ln q^*+\ln\varphi-\ln V^H+\ln(-V^L)\right]}}{(1-q^*)q^*(1-e^{-2\gamma\ln(1-\delta)})} > 0 \qquad (23)y$$

since $\gamma>0$ and $\left(1-e^{-2\gamma\ln(1-\delta)}\right)<0$. The prior probability q^* can take the interpretation of reputation. The comparative statics results indicate that the better the initial reputation of the investment project, the higher chances it will have to be undertaken by masses even when the true value of the project is low.

5.2. Project Value:

In an exogenous-timing herding framework Welch (1992) shows that as the expected value from investment goes up, early movers are more likely to invest. Hence the is a higher change that the society ends up with a positive cascade. In our endogenous-timing framework, we get the similar comparative statics for different reasons. If there is more to gain from successful investment, the probability of inefficient negative cascade goes down.

$$\frac{dProb(INC)}{dV^H} = \frac{2\varphi e^{-2\varphi\left[\ln(1-q^*)-\ln q^*+\ln\psi-\ln V^H+\ln(-V^L)\right]}}{(1-e^{-2\varphi\ln(1-\delta)})V^H} < 0 \qquad (24)ze$$

and likewise if there is less to lose from investing the probability of an inefficient negative cascade goes down.

$$\frac{dProb(INC)}{dV^L} = -\frac{2\varphi e^{-2\varphi\left[\ln(1-q^*)-\ln q^*+\ln\psi-\ln V^H+\ln(-V^L)\right]}}{(1-e^{-2\varphi\ln(1-\delta)})V^L} < 0 \qquad \text{(25)zf}$$

There are three forces. When the expected value of the project goes up either due to an increase in V^H or V^L, the agent is more inclined to move now rather than delay. Hence the equilibrium rate of information flow goes up, making the agent just indifferent between waiting and not. With a higher information flow it becomes less likely to fall into a negative cascade since the true value is high (a stronger μ^H due to a higher λ). However as the information flow goes up, so does the power of the noise of the learning process. The noise makes it more likely to fall into a negative cascade when $V=V^H$. These two forces exactly cancel each other out since λ cancels out from the probability of inefficient negative cascade. Meanwhile as the expected value of the project goes up, the upper bound $\overline{q}$ and the lower bound $\underline{q}$ both decrease, see equations (5) and (4), while the starting point is unchanged. Therefore the probability of hitting the lower bound before hitting the upper bound decreases, making an inefficient negative cascade less likely.

When the expected value of the project goes up, either due to an increase in V^H or in V^L, the likelihood of an inefficient positive cascade goes up.

$$\frac{dProb(IPC)}{dV^H} = -\frac{2\gamma e^{-2\gamma\left[\ln(1-q^*)-\ln q^*+\ln\psi-\ln V^H+\ln(-V^L)\right]}}{(1-e^{-2\gamma\ln(1-\delta)})V^H} > 0 \qquad \text{(26)zg}$$

and

$$\frac{dProb(IPC)}{dV^L} = \frac{2\gamma e^{-2\gamma\left[\ln(1-q^*)-\ln q^*+\ln\psi-\ln V^H+\ln(-V^L)\right]}}{(1-e^{-2\gamma\ln(1-\delta)})V^L} > 0 \qquad \text{(27)zk}$$

As the expected value of the project goes up, both the lower bound and the upper bounds goes down. The probability of hitting the lower bound before hitting the upper bound decreases, making an inefficient positive cascade more likely.

5.3. Discounting:

Discounting doesn't play a role in exogenous timing models. Examination of this issue requires an endogenous timing model. To my knowledge, This is the first endogenous-timing paper with comparative statics results on discounting. The agent makes a choice between investing now or later. If the agent waits, he can learn by observing other people's actions, however the payoff gets discounted. All else constant, as people get more patient, δ goes up, they will be more willing to wait. Since waiting induces learning, one might be tempted to conclude that higher δ would be associated with a smaller probability of an inefficient negative cascade. However this is not the case.

$$\frac{dProb(INC)}{d\delta} = -(1 - Prob(INC))\frac{2\varphi e^{-2\varphi x(1-\delta)}}{(1-\delta)(1-e^{-2\varphi x(1-\delta)})} > 0 \qquad (28)z$$

With a higher δ, at the ongoing rate of information flow agents would strictly prefer to wait, λ would be equal to zero. However as argued earlier, $\lambda=0$ cannot be sustained in equilibrium. So the rate of information flow goes down. In other words, in equilibrium, people are just indifferent between waiting and moving, hence a higher δ induces a smaller rate of information flow. Since $V=V^H$, a weaker information flow simply increasing the likelihood of a negative cascade due to a weaker μ^H. However at the same time the weaker information flow would increase the noise in the learning process. And these two opposing effects cancel each other out since λ cancels out from the probability of inefficient cascade. Meanwhile, a higher δ yields a higher upper bound $\overline{q}$, leaving the starting point and the lower bound unchanged. This also makes the inefficient negative cascade more likely.

On the other hand, the probability of an inefficient positive cascade goes down as δ goes up.

$$\frac{dProb(IPC)}{d\delta} = Prob(IPC)\frac{2\gamma e^{-2\gamma x(1-\delta)}}{(1-\delta)(1-e^{-2\gamma x(1-\delta)})} < 0 \qquad (29)za$$

A higher δ induces a higher upper bound $\overline{q}$. The subjective probability that $V=V^H$ must be higher for a patient agent to prefer to invest now when she is to find out the true value of the project for sure next period. This makes the inefficient positive cascade less likely.

5.4. Quality of information:

In the exogenous-timing model of Bikhchandani, Hirshleifer and Welch (1996) as quality of information goes up the likelihood of incorrect cascades unambiguously goes down. However in our endogenous-timing framework the effect of signal quality on the probability of inefficient herding is not monotone. As the signal quality improves, the likelihood of inefficient cascades may go up or down depending on the parameter values. (NELSON?, Decamps?)

A decrease in the signal quality (an increase in ψ) leads to a decrease in q_0, leaving the upper bound and the lower bound unchanged. An decrease in q_0 increase the probability of a positive cascade and decreases the probability of a negative cascade. Meanwhile, an increase in ψ (decrease in signal quality) also affects the drift velocity and the power of the noise of the stochastic learning process:

$$\frac{\partial \mu^H}{\partial \psi} = \frac{\psi - 1}{\psi} < 0 \quad \text{and} \quad \frac{\partial (\sigma^2)^H}{\partial \psi} = \frac{2\ln\psi}{\psi} < 0$$

$$\frac{\partial \mu^L}{\partial \psi} = -\ln\psi > 0 \quad \text{and} \quad \frac{\partial (\sigma^2)^L}{\partial \psi} = -\ln\psi(\ln\psi + 2)$$

Denote:

$$\beta = \ln(1 - q^*) - \ln q^* - \ln V^H + \ln(-V^L) \qquad \text{(30)zb}$$

then,

$$\frac{dProb(INC)}{d\psi} = \frac{2e^{-2\varphi(\beta + \ln\psi)}}{(1 - e^{-2\varphi\ln(1-\delta)})}\left(\frac{\varphi}{\psi} + (\beta + \ln\psi)\frac{d\varphi}{d\psi}\right) - \frac{2e^{-2\varphi\ln(1-\delta)}(1 - e^{-2\varphi(\beta + \ln\psi)})}{(1 - e^{-2\varphi\ln(1-\delta)})^2}\ln(1-\delta)\frac{d\varphi}{d\psi} \ \substack{> \\ <} \ 0 \qquad \text{(31)zc}$$

The effect of an increase in ψ on the probability of an inefficient negative cascade may be positive or negative depending on the value $\left(\frac{\varphi}{\psi} + (\beta + \ln\psi)\frac{d\varphi}{d\psi}\right)$ takes (See Appendix A3). Appendix A4 shows that the increase in ψ may result in an increase or in a decrease in the probability of an inefficient positive cascade depending on the parameter values.

$$\frac{dProb(IPC)}{d\psi} = -2\ln(1-\delta)e^{-2\gamma\ln(1-\delta)}\frac{(1-e^{-2\gamma(\beta+\ln\psi)})}{(1-e^{-2\gamma\ln(1-\delta)})^2}\frac{d\gamma}{d\psi} + \frac{2e^{-2\gamma(\beta+\ln\psi)}}{(1-e^{-2\gamma\ln(1-\delta)})}\left(\frac{\gamma}{\psi}+(\beta+\ln\psi)\frac{d\gamma}{d\psi}\right) \gtrless 0 \tag{32)zd}$$

This suggest that the results on signal quality of the exogenous timing herding literature may not be applicable to the endogenous timing herding. When the sequence of moves is endogenous there are three forces to be studied; Higher quality information (smaller ψ) increases the chances of efficient learning, and hence decreases the probability of an inefficient cascade. But as quality of information goes up, the rate of information arrival goes down. This increases the probability of an inefficient cascade. The third factor is the starting point. When ψ goes up, the starting point $q_{T=1}$ goes down. This makes the probability of a positive cascade go down and it makes the probability of a negative cascade go up. Exactly which force overwhelms the other/s depends on the parameter values of the problem. **Examples?**

6. Discussion

The comparative statics results from this endogenous timing herding model may be able to shed some light on a variety of questions from different fields of economics. The parameters of the model, the discount factor, the prior beliefs, the signal quality and the expected value of the project can take different interpretations depending on the market under consideration.

6.1. Initial Public Offerings

The IPO market is a fixed-price common-value good market where later potential investors can observe the investment decisions of early investors. One of the puzzles in this market is the strong documented underpricing [12]. And casual observation of the IPO market shows that offerings occasionally fail because there is too low of a demand. Both these features are consistent with our herding model of investment. A lower offering price increases the expected value to potential investors. This model would predict that a lower offering price would be associated with a lower probability of a negative cascade where the offering fails. Welch (1992) examines the price

[12]See Beatty and Ritter (1986).

setting by an informed seller of an IPO where buyers cascade. When there is inside information, Welch (1992) can explain why an optimally priced IPO might fail. This paper, however, predicts positive probability of a negative cascade for any price the seller picks even when there is no inside information.

6.2. Financial versus Real Markets

While δ simply represents the discount factor, it may also be regarded as capturing the time required to process and react to information. Keeping the rate of time preference constant, as the time to process information increases so does the distance between the time periods in the model, leading to a lower δ. In financial markets agents tend to attain and process information very quickly. In real investment, however, there is often a non-negligible time gap between the moment of a decision to undertake an investment project and the visibility of that decision. Hence in financial markets the relevant δ would be larger than in real markets. The paper suggests that as δ goes up the rate of information flow goes down and hence, the likelihood of an inefficient collapse would be higher in markets with quick information dissemination and processing even though one might be tempted to think that financial markets would have more information efficiency.

In a more fully developed model for the purposes, one could analyze the effect of liquidity on the probability of inefficient collapses. The more liquid market might imply a higher δ since expected time to trade would be shorter. Hence a financial market that is open to the world markets and hence with higher liquidity might be more prone to inefficient collapse. This possibility is often suggested in the discussion of hot money and exchange rate/debt crises and the model presented here may be adaptable to give some meat to that discussion.

6.3. Speculative Attacks

The model may help to gain further understanding of the importance of the reputation of a government pursuing a fixed exchange rate regime. Suppose that an agent invests in foreign currency, the agent will have a low expected payoff if in reality the fundamentals of the economy are bad. The agent will have a high expected payoff if in reality the fundamentals of the economy are good. Each agent is aware of the potential speculative gains and has a one unit of domestic currency for possible investment in the foreign exchange market. Agents can observe the amount of speculative purchases from the monetary authority each period. The model would suggest that it is possible that a speculative attack is staged even when economic fundamentals are good. The possibility of such an inefficient cascade would decline however with the good reputation of the government.

6.4. Advertising, Warrantees and Buy-Back Options

This paper suggests that firms producing an identical high quality product will face different chances of falling into a negative cascade depending on their reputation q*. While the firm with a good reputation might have its product be purchased by masses, the firm with a lesser reputation has a higher chance of not being able to take off. This presents two questions to be further investigated: In a market with social learning would firms be tempted to overinvest in reputation possibly in advertising in order to avoid falling into a negative cascade?

Another key variable in the analysis is the expected value of the project. Warranty and buy back options are important elements of marketing new products as better warranty and buy back options signal higher product quality. Hence these options increase the expected value from investing in the product both directly and indirectly through signaling. This model suggests that in markets where there is social learning these marketing tools will have even a bigger significance. By offering warranty and buy back options, firms can increase the chances of positive cascades where purchases of the product booms. All else equal, firms that do not offer these options will have a relatively high probability of facing a collapse of purchases.

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Appendix

Appendix A:

Claim A1 : $(\psi - 1 - \ln\psi) > 0$

Proof: Define $f(\psi) = \psi - 1 - \ln\psi$. *Note that* $f(1) = 0$. *Since*

$f'(\psi) = 1 - \frac{1}{\psi} = \frac{\psi - 1}{\psi} < 0 \quad \Rightarrow \quad f(\psi) > 0$ *for* $0 < \psi < 1$. *So,* $(\psi - 1 - \ln\psi) > 0$ □

Claim A2: $\frac{d\varphi}{d\psi} < 0$

Proof: $\frac{d\varphi}{d\psi} = \frac{(1-\psi)(\ln\psi - 2) - 2\ln\psi}{\psi\left(\ln\psi\right)^3}$

Define f(ψ)=(1-ψ)(lnψ-2)-2lnψ. Note that f(1)=0. Since

$$f'(\psi) = 1 - ln\psi - \frac{1}{\psi} = -\frac{1}{\psi}(1 + \psi ln\psi - \psi) < 0 \qquad \Rightarrow f(\psi) > 0 \text{ for } 0<\psi<1 \qquad \square$$

Claim A3: $(\psi - 1 - \psi ln\psi) < 0$

Proof: Define f(ψ)=ψ-1-ψlnψ . Note that f(1)=0. Since f '(ψ)= 1-lnψ-1=-lnψ>0 $\Rightarrow$ *f(ψ) <0 for 0<ψ<1. So* $(\psi - 1 - \psi ln\psi) < 0$ □

Claim A4: $\frac{d\gamma}{d\psi} < 0$

Proof: $\gamma = -\frac{(\psi - 1 - \psi ln\psi)}{\psi(ln\psi)^2}$, so $\frac{d\gamma}{d\psi} = \frac{2\psi - 2 - \psi\ln\psi - \ln\psi}{\psi^2(\ln\psi)^3} < 0$

Define f(ψ)=2ψ-2-ψlnψ-lnψ. Note that f(1)=0. Since

$f'(\psi) = 1 - ln\psi - \frac{1}{\psi} = -\frac{1}{\psi}(1 - \psi + \psi ln\psi) < 0$ *by A1. Hence f(ψ)>0. So* $\frac{d\gamma}{d\psi} < 0$. □

Appendix B:

Claim B1: $\frac{dProb(INC)}{d\psi} > 0$ *or* $\frac{dProb(INC)}{d\psi} < 0$ *depending on parameter values.*

Proof: In equation (31), the first term on the right hand side is positive. (β+lnψ)<0, for $q_{T=1}>0$*.*

φ is given by (xx). $\frac{d\varphi}{d\psi} < 0$ *by Appendix A1. Examining (xx), for 0<α<1, 0>(β+lnψ)>ln(1-δ)>-∞. This condition is equivalent to* $\underline{q}<q_0<\overline{q}$ *of the original problem. Taking the limit when (β+lnψ)⇒0,*

$$\frac{d\varphi}{d\psi} \Rightarrow -\frac{2\varphi}{\psi(1 - e^{-2\varphi\ln(1-\delta)})} > 0 .$$

Taking the limit when ln(1-δ)⇒- ∞ first term of (???) goes to zero using the L'Hopital's Rule,

$$\frac{d\varphi}{d\psi} \Rightarrow -2e^{-2\varphi(\beta+\ln\psi)}\left(\frac{\varphi}{\psi} + (\beta + \ln\psi)\frac{d\varphi}{d\psi}\right) \quad For \quad \left(\frac{\varphi}{\psi} + (\beta + \ln\psi)\frac{d\varphi}{d\psi}\right) > 0$$

$$\beta + \ln\psi < \frac{-\frac{\varphi}{\psi}}{\frac{d\varphi}{d\psi}} = \frac{\ln\psi(1-\psi+\ln\psi)}{(1-\psi+\ln\psi)+(1-\psi+\psi\ln\psi)} < 0$$

This is also equivalent to,

$$\beta < \frac{\ln\psi(1-\psi+\psi\ln\psi)}{(1-\psi+\ln\psi)+(1-\psi+\psi\ln\psi)}. \qquad \square$$

Claim B2: $\frac{d\text{Prob(IPC)}}{d\psi} > 0$ *or* $\frac{d\text{Prob(IPC)}}{d\psi} < 0$ *depending on parameter values.*

Proof: Examining (32), for $0<\beta<1$, $0>(T+\ln\psi)>\ln(1-\delta)>-\infty$. *This condition is equivalent to* $\underline{q} < q_0 < \overline{q}$ *of the original problem. Taking the limit when* $(T+\ln\psi)\Rightarrow 0$,

$$\frac{d\beta}{d\psi} \Rightarrow \frac{2\phi}{(1-e^{-2\phi\ln(1-\delta)})\psi} < 0$$

since $1-e^{-2\phi\ln(1-\delta)} < 0$.

THE OTHER SIDE MISSING

Appendix C:

Lemma C1: *Let* $w_t(\lambda)$ *be a Wiener process with absorbing boundaries as defined in (*), (**), (15), (16) and with starting point* $w_T \in (\underline{w}, \overline{w})$. *Let* $\tilde{w}_t$ *be another process with the same form and parameters as* w_t *up to some possibly stochastic time* $\tau > T$ *at which time the parameter* λ *is replaced by* λ', *which may also be stochastic. Both* $w_t(\lambda)$ *and* $\tilde{w}_t$ *yield the same probabilities of hitting the boundaries which are given by * and **.*

Proof: Define $b(\tau, w_\tau, \lambda')$ *as the joint p.d.f. of* τ, w_τ *and* λ' *conditional on not hitting either boundary in* $t \le \tau$. *Define* $P_{w_T(\lambda)\to\overline{w}}$ *as the probability starting from* w_T *that process hits the boundry* $\overline{w}$ *before* $\underline{w}$. *Since* $w_t(\lambda)$ *is a standard Wiener process* $P_{w_0(\lambda)\to\overline{w}}$ *is given by*

$$P_{w_0(\lambda)\to\overline{w}} = 1 - \left[\left(e^{\frac{-2\mu\underline{w}}{\sigma^2}} - e^{\frac{-2\mu w_0}{\sigma^2}}\right) \Big/ \left(e^{\frac{-2\mu^1\underline{w}}{\sigma^2}} - e^{\frac{-2\mu^1\overline{w}}{\sigma^2}}\right)\right] \qquad \text{(33)zp}$$

and $P_{w_\tau(\lambda)\mapsto \underline{w}} = 1 - P_{w_\tau(\lambda)\mapsto \overline{w}}$. *These depend on λ only through the ratio* μ / σ^2. *From (12) this ratio is given by:*

$$\frac{\mu}{\sigma^2} = \begin{cases} \dfrac{1-\psi+\ln\psi}{\ln\psi} & \text{if } V = V^H \\ \dfrac{1-\psi+\psi\ln\psi}{\psi\ln\psi} & \text{if } V = V^L \end{cases} \qquad \text{(34)zq}$$

Hence the probabilities of $w_t(\lambda)$ *hitting the boundaries do not depend on λ. Although the date* τ *as no special relevance to this process we can still decomposed this probability into the probability that it transitions before or at* τ *and the probability it transitions after* τ :

$$\begin{aligned} P_{w_0(\lambda)\to\overline{w}} = {} & P_{w_0(\lambda)\mapsto\overline{w}|t\le\tau} \\ & + \left[1 - P_{w_0(\lambda)\mapsto\overline{w}|t\le\tau} - P_{w_0(\lambda)\to\underline{w}|t\le\tau}\right] \int_\tau \int_{w_\tau} \int_{\lambda'} P_{w_\tau(\lambda)\mapsto\overline{w}}\, \mathrm{b}(\tau, w_\tau, \lambda') \cdot d\tau \cdot dw_\tau \cdot d\lambda' \end{aligned} \qquad \text{(35)zm}$$

While we know the left-hand side of this, the formulas for the conditional probabilities and p.d.f.s on the right-hand side are unknown. However, since $\tilde{w}_t$ *starts off as the same process we can similarly decompose its probability as:*

$$\begin{aligned} P_{\tilde{w}_0\to\overline{w}} = {} & P_{w_0(\lambda)\mapsto\overline{w}|t\le\tau} \\ & + \left[1 - P_{w_0(\lambda)\mapsto\overline{w}|t\le\tau} - P_{w_0(\lambda)\mapsto\underline{w}|t\le\tau}\right] \int_\tau \int_{w_\tau} \int_{\lambda'} P_{w_\tau(\lambda')\mapsto\overline{w}}\, \mathrm{b}(\tau, w_\tau, \lambda') \cdot d\tau \cdot dw_\tau \cdot d\lambda' \end{aligned} \qquad \text{(36)zn}$$

Here both the left and right-hand side probabilities are unknown. Nevertheless, since it is the same process up to τ *these conditional probabilities and p.d.f.s are the same as in (35) with the exception of the continuation probabilities in the integrals. Note however these are simply the probabilities for the Wiener process starting from* w_τ *with parameter* λ' *and hence for each potential realization of* w_τ *and* λ' *the probability can be found from (33) by substituting* w_τ *for* w_0. *As before* λ' *cancels out from these probabilities. Therefore each* $P_{w_\tau(\lambda')\to\overline{w}}$ *in equation (36) is equal to the corresponding* $P_{w_\tau(\lambda)\to\overline{w}}$ *in (35) and hence* $P_{w_0(\lambda)\mapsto\overline{w}} = P_{\tilde{w}_0\to\overline{w}}$. *The same argument shows that* $P_{w_0(\lambda)\mapsto\underline{w}} = P_{\tilde{w}_0\to\underline{w}}$, *which completes the proof of the lemma.*

□

<u>Proposition C1:</u> The equilibrium of the game with any finite set Γ *of possible contingency trigger points with cardinality greater than one and which contains both* $\overline{q}$ *and* $\underline{q}$ *will yield the same transition probabilities as the baseline model.*

Proof: From the baseline model where $\Gamma = \{\underline{q}, \overline{q}\}$ *add one contingency trigger point* q'*. If* $q' \notin (\underline{q}, \overline{q})$ *then the state* q' *would never be reached in the baseline equilibrium and hence we can construct a parallel equilibrium where no agent chooses to have a contingency triggered at* q'*. Hence the edition of* q' *will not change the transition probabilities.*

If $q' \in (\underline{q}, \overline{q})$ *then some agents may choose to set contingencies there. Let* $\pi(q_T, t)$*, henceforth* π*, be the probability that an individual agent chooses to set a contingency trigger at* q'*. This may be either to buy or to cancel an impending order. Note that* π *may depend on t since for a given number of impending orders the time remaining in the period will determine the rate of information flow which in tern influences the expected value of waiting. By the same argument used for* λ *it is straightforward to show that* $\pi<1$*. If* q' *is a buy trigger and* $\pi=1$ *then each individual would prefer to wait since* $q' < \overline{q}$*. If it is a wait trigger and* $\pi=1$ *then each individual would prefer to buy since* $q' > \underline{q}$*.*

So either the addition of q' *has no effect on the outcome in the period (*$\pi=0$*) or in equilibrium each individual will be indifferent between using it as a trigger or not.*