Observable Implications of Nash and Subgame-Perfect Behavior in Extensive Games*

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Abstract

We provide necessary and sufficient conditions for observed outcomes in extensive game forms, in which preferences are unobserved, to be rationalized first, partially, as a Nash equilibrium and then, fully, as the unique subgame-perfect equilibrium. Thus, one could use these conditions to find that play is (a) consistent with subgame-perfect equilibrium, or (b) not consistent with subgame-perfect behavior but is consistent with Nash equilibrium, or (c) consistent with neither.

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JEL Classification Numbers: C72, C92.

1 INTRODUCTION

How can one test whether play in a game is consistent with equilibrium when we cannot observe the players' preferences? As a number of recent papers (Zhou 1997, Sprumont 2000, Ray and Zhou 2001, Sprumont 2001, Bossert and Sprumont 2002, , Zhou 2002, Bossert and Sprumont 2003, Carvajal 2003) have discussed, one can observe the outcome in a variety of game forms and extend the lessons of revealed preference theory for individual choice to concepts of equilibrium play in games.

Sprumont (2000) has taken up the issue for normal form games. Sprumont considers finite sets of actions, A_i, one for each player, i; the product set, A, is called the set of joint actions. A joint choice function, f, assigns to every possible subset B of A a non-empty set. A data set is a realization of a joint choice function. A data set is Nash rationalizable if there exist preference orderings on A such that for every B, $f(B)$ coincides with the set of Nash equilibria for the game defined by the set of actions B with those preferences. Sprumont provides necessary and sufficient conditions (*Persistence under Expansion* and *Persistence under Contraction*) for a data set in a normal game form to be Nash rationalizable.

As a complement to the work of Sprumont, Ray and Zhou (2001) consider situations in which the players move sequentially with perfect information. They fix an extensive game form (tree) G with complete information. A reduced game form, G', is obtained from G by deleting branches of G. A unique outcome is observed for each reduced game form. For Ray and Zhou, the data are the outcomes of all possible reduced game forms. They provide necessary and sufficient conditions (*Acyclicity of the Base Relation*, *Internal Consistency* and *Subgame Consistency*) for a data set in an extensive game form to be rationalizable as the unique subgame-perfect equilibrium in every reduced game form.

We are interested in the differences between Nash and subgame-perfect behavior in extensive games. Notice that in extensive game forms, we assume that we observe outcomes and not strategies (complete plans of actions), whereas in the work on normal game forms, strategies (equivalently, actions) are assumed to be observed. Thus, a data set in an extensive game form has missing observations compared to the corresponding normal form data set. Therefore, one cannot use Sprumont's conditions for Nash rationalization in an extensive game form by testing the conditions in the corresponding normal game form. To see this, consider the data set from the game tree (and all reduced forms) as in Figure 1a. The tree has two choice nodes; player 1 moves in the first node and has two choices, namely L and R. Player 2 moves in the second (after player 1 moves L) and also has two choices, namely l and r. There are 3 possible (non-trivial) reduced game forms as shown in the figure.

[*Insert Figures 1a and 1b here*]

The corresponding normal game form obviously has a 2x2 structure as shown in Figure 1b. There are 4 possible (non-trivial) reduced normal forms. Clearly, if we observe the outcomes in the trees G, G_1, G_2, and G_3, we do not observe player 2's choice of action when player 1 chooses to play R in the corresponding normal game form G_4.

It is indeed possible to observe data in extensive game forms that are not rationalizable by subgame-perfect equilibrium, yet can still be rationalized as Nash behavior. Consider for example, the following two distinct data sets, as described in Figures 2a and 2b, on the same game trees as in Figure 1a.

[*Insert Figures 2a and 2b here*]

Neither of these data sets satisfies the subgame consistency condition of Ray and Zhou and therefore cannot be rationalized as a subgame-perfect equilibrium. The data in Figure 2a, however, can be rationalized by a Nash equilibrium. The choice of player 1 to play R in the game form G can be justified as a Nash behavior on his part that assumes that player 2 would play r (although actually, player 2 prefers to play l when given the choice).[1] The data in Figure

[1] This is precisely the case of "incredible threat" often used to show the difference between Nash and subgame-perfect equilibrium in extensive form games.

2b, however, cannot be rationalized even by Nash equilibrium as there is no choice of player 2 that could justify player 1's choice of playing R in the game form G.

Also, notice that, under the (revealed) preferences that rationalize the outcomes in Figure 2a, the game G has multiple Nash equilibria. There is a Nash equilibrium (indeed, subgame-perfect) outcome (L, l) in the game, which however is not observed, as we assume that only one outcome is observed in each reduced game form.

In this paper, we first provide a necessary and sufficient condition for partial Nash rationalization; i.e., we rationalize the data in each reduced game as one of the possibly multiple Nash equilibria. For each game form G', we consider strategies that are consistent with the observed outcome in the reduced game. If there exist strict preferences such that any one of these strategies can be shown to be a best response for each player i, given that the other players' strategies are fixed, then clearly the observed outcome is consistent with Nash behavior. This motivates our necessary and sufficient condition, called *Extensive Form Consistency*, which compares the outcomes of a set of reduced extensive form games, varying the set of feasible strategies for one player while the other players' strategies are fixed. For example, in the data in Figure 2b, there are two strategies consistent with the given outcome in the game form G, namely, (R, l) and (R, r); if we fix player 2's strategy at either l or r, we see from the outcomes of the reduced games G_2 and G_3, that player 1 prefers to play L. Our extensive form consistency is not satisfied here and R cannot be rationalized as Nash behavior in game G. In the data set in Figure 2a, the condition is satisfied and the outcome in game G can be rationalized using the strategy profile (R, r), as from G_3, player 1 prefers to play R.

We then provide a condition, *Subgame-Perfect Consistency*, which uses observations of reduced game outcomes that are proper subgames below a node with at least one active player other than the one at that node, to ensure that the strategies played are consistent with not only Nash but also with subgame-perfect behavior. The data set in Figure 2a does not satisfy this condition because player 2 is active in G_1, which is a proper subgame of G, and is observed to move l; under this circumstance, we know from G_2, player 1 prefers L to R. Thus the outcome R in G violates subgame-perfect consistency.

2 ANALYSIS

2.1 Set-up

We study n-person extensive form games with perfect information. The structure is identical to that in Ray and Zhou (2001). We therefore maintain their terminologies and the notations as much as possible.

An extensive game form G is a finite rooted tree with set of nodes, X, with a distinct initial node x_0, and a *precedence function* $p : X/x_0 \to X$. If $p(y) = x$, then x is called an *immediate predecessor* of y. Also y is called an *immediate successor* of x, or $y \in s(x)$. Let $S(x)$ denote the set of all successors of x. A node z is called a terminal node, or an outcome, if there exists no $x \in X$ such that $p(x) = z$. The set of all terminal nodes is Z. A path ρ is a finite sequence of nodes: $(x_k : k = 0, \ldots, m)$ where $x_k = p(x_{k+1})$ for each k and x_m is a terminal node. A path leading to a terminal node x_m, $\rho(x_m)$, can be uniquely identified.

The set of non-terminal nodes, X/Z, are partitioned into n subsets, $\{X_1, X_2, \ldots, X_n\}$, where X_i, called the player i's partition, is the set of nodes at which player i moves; player i's moves determine one $y \in s(x)$ for each $x \in X_i$. A pure strategy t_i for player i specifies a unique choice at each node in X_i. The set of pure strategies available to player i is T_i.

Definition 1 *A reduced extensive game form G' of an extensive game form G is an extensive game form consisting of (i) terminal nodes $Z' \subseteq Z$ and (ii) all the non-terminal nodes that belong to $\rho(z')$ for any $z' \in Z'$.*

Thus, any set of terminal nodes Z'uniquely refers to the reduced game form G'. As with G, the set of non-terminal nodes in G' can also be partitioned into n many player-partitions, $\{X'_1, \ldots, X'_n\}$.

Let Γ be the set of all possible reduced extensive game forms of an extensive game form G.

Player i is *active* in any (reduced) game form G' if X'_i is non-empty with at least one node $x \in X'_i$ such that $|s(x)| \geq 2$.

Definition 2 *For each reduced extensive game form G' and a non-terminal node $x \in X'/Z'$, the subgame form beginning at x, G'_x, is the reduced extensive game form consisting of (i) terminal nodes $Z'(x) = Z' \cap S(x)$ and (ii) all the non-terminal nodes that belong to $\rho(z')$ for any $z' \in Z' \cap S(x)$.*[2]

A pure strategy t'_i for player i in G' specifies a unique choice of an immediate successor $y \in s(x)$ at each node x in X'_i. The set of pure strategies available to player i is T'_i. Clearly, although $Z' \subseteq Z$, T'_i may not be a subset of T_i.

For any (reduced) extensive game form G' a strategy profile $t' = (t'_1, \ldots, t'_n)$ determines an outcome $\Omega(t') = z'$, where $\Omega : \Pi_i T'_i \to Z'$.

Definition 3 *For any $G' \in \Gamma$ and the corresponding pure strategy sets $\langle T'_1, ..., T'_n \rangle$, let $T''_i \subseteq T'_i$ for all i be non-empty sets of pure strategies. A strategy-reduced extensive game form G'' is an extensive game form consisting of (i) terminal nodes $Z'' \subseteq Z'$ with $z'' \in Z''$ such that $z'' = \Omega(t'')$ for some $t'' \in \Pi_i T''_i$ and (ii) all the non-terminal nodes that belong to $\rho(z'')$ for any $z'' \in Z''$.*

Clearly, a strategy-reduced extensive game form G'' is a reduced game form (of the original game G).[3] Starting from $G' \in \Gamma$ and a fixed strategy profile t', we then look at a set of strategy-reduced extensive game forms in which the

[2] The subgame form G'_x is thus the reduced game form consisting of the path from x_0 to x and the subgame below the node x.

[3] Another way to look at the strategy-reduced extensive game forms is to consider the corresponding normal form representations. Formally, from a reduced extensive game form G', one can uniquely define a normal game form H' as the set of players $(1, \ldots, n)$, the set of strategies for each player T'_i, and the function $\Omega : (t'_1, \ldots, t'_n) \to Z'$. A reduced normal game form H'' of a normal game form H' consists of a list $\langle T''_1, \ldots, T''_n \rangle$ of nonempty subsets $T''_i \subseteq T'_i$ for all i and the corresponding outcomes $\Omega(t'')$. From every H'' one can then uniquely define a corresponding extensive game form G'' defined by $Z'' \subseteq Z$ with $z'' \in Z''$ iff $z'' = \Omega(t'')$ for some $t'' \in \langle T''_1, \ldots, T''_n \rangle$.

other players' strategies are fixed, while varying the set of feasible strategies for player k maintaining the strategy t'_k feasible.

Definition 4 *For any $G' \in \Gamma$ and the corresponding pure strategy sets $\langle T'_1, ..., T'_n \rangle$, given a $t' = (t'_1, \ldots, t'_n)$ where $t'_i \in T'_i$, and a particular player k, an individually-strategy-reduced extensive game form $G''(t', k)$ is a strategy-reduced extensive game form with $t'_k \in T''_k \subseteq T'_k$, and $T''_j = t'_j$ for all $j \neq k$.*

Definition 5 *A binary individually-strategy-reduced extensive game form $G''(t', k; 2)$ is an individually-strategy-reduced extensive game form consisting of $|Z''| = 2$.*

Suppose each player i has preferences over Z described as a strict ordering Q^*_i over Z. Let the players play reduced games $G'(Q^*)$ for every $G' \in \Gamma$. Let $O : \Gamma \to Z$ be the outcome function. We observe $O(G') \in Z'$ and thus the unique path $\rho(O(G'))$ for every $G' \in \Gamma$. We do not observe strategies; thus players' intended moves off the path cannot be observed.

Definition 6 *An outcome function O is partially rationalized by Nash equilibrium in strict preferences if for all i, there exists Q_i over Z such that $O(G')$ coincides with a Nash equilibrium of the game $G'(Q)$ for every $G' \in \Gamma$.*

Similarly, an outcome function O is fully rationalized by subgame-perfect Nash equilibrium in strict preferences if for all i, there exists Q_i over Z such that $O(G')$ coincides with the unique subgame-perfect Nash equilibrium of the game $G'(Q)$ for every $G' \in \Gamma$.

2.2 Conditions

Condition 1 *Extensive Form Consistency (XC): For any $G' \in \Gamma$ and the corresponding pure strategy sets $\langle T'_1, \ldots, T'_n \rangle$ with the outcome $O(G') = z'$, there exists a $t^* = (t^*_1, \ldots, t^*_n)$ with $t^*_i \in T'_i$ for all i and $\Omega(t^*) = z'$ such that for all i, for all binary individually-strategy-reduced extensive game forms $G''(t^*, i; 2)$, $O(G''(t^*, i; 2)) = z'$.*

Condition 2 *Subgame-perfect Consistency (SPC): For each game G', consider each non-terminal node $x \in X'/Z'$ such that $x \in \rho(O(G'))$ with player i such that $x \in X_i$. For each non-terminal node $y \in s(x)$ such that (i) $y \notin \rho(O(G'))$, and (ii) there is at least one active player other than i in G'_y, $O(O(G'), O(G'_y)) = O(G')$.*

2.3 Revealed Preferences

Given an outcome function O, following Ray and Zhou (2001), one can construct incomplete preference orderings for players over the terminal nodes. Consider the paths that lead to two different terminal nodes u and v. Take the player i who has to play at the node where these two paths diverge. Player i's preference over u and v can be determined by his choice in the reduced game form G' which has only two terminal nodes, u and v. This incomplete order, P_i, for player i, is known as the *revealed base relation.* Formally, for any u, $v \in Z$, let x be the node at which the paths to u and v diverge. If $x \in X_i$, then uP_iv if and only if $u = O(G')$, where G' is the reduced game form which has only two terminal nodes, u and v.

Lemma 1 *If XC is satisfied, then the revealed base relation is acyclic.*[4]

Proof. Suppose we have a cycle in the revealed base relation for some player i involving the terminal nodes z_1, z_2, ..., z_k such that $z_1P_iz_2P_i \dots z_kP_iz_1$. Consider the reduced extensive game form G' characterized by the set of terminal nodes $Z' = (z_1, z_2, \dots, z_k)$. This is clearly a game form where only player i is active and chooses among the nodes in Z'. Wlog, suppose, $O(G') = z_1$. Now XC implies that the outcome in the individually-strategy-reduced extensive game form consisting only of z_1 and z_k is z_1, which contradicts $z_kP_iz_1$. Hence we cannot have a cycle in the revealed base relation. ■

Lemma 2 *An acyclic base relation can be extended to a strict ordering on Z which is complete and acyclic (equivalently, transitive, for a complete ordering)*

[4] Ray and Zhou (2001) take acyclicity as one of their conditions.

for each player i.

Proof. We are omitting the proof here. It follows from a routine argument using Zorn's lemma (cf. Richter 1966, Theorem 1).[5] ■

2.4 Results

Theorem 1 *XC is necessary and sufficient for partial Nash rationalization in strict preferences.*[6]

Proof. Necessity is straightforward and hence we only show sufficiency here. From the previous lemmas we know that if XC is satisfied, we can define a complete transitive strict ordering Q_i on Z for all i that is consistent with the base preference relation P_i. We will show, for each game G', there exists a strategy profile such that the outcome corresponding to the profile is the observed outcome $O(G')$ and that the strategy profile is a Nash equilibrium of the game $G'(Q)$. We know, for each game G', there exists a $t^* = (t_1^*, \ldots, t_n^*)$ with $t_i^* \in T_i'$ for all i and $\Omega(t^*) = O(G')$ satisfying XC. If every player follows this strategy, then the outcome is $O(G')$. Let us show that these strategies indeed constitute a Nash equilibrium for every G'. Suppose any player i deviates and plays any other strategy $\widetilde{t_i'}$ to induce a different outcome $\widetilde{z'}$. By XC, the outcome of the binary individually-strategy-reduced extensive game form $G''(t^*, i; 2)$ with $Z'' = \{\widetilde{z'}, z'\}$ is z'. Hence, by the revealed base relation, $z' P_i \widetilde{z'}$ implying $z' Q_i \widetilde{z'}$. Therefore player i cannot deviate and be better off. ■

Theorem 2 *XC and SPC together are necessary and sufficient for full rationalization by subgame-perfect Nash equilibrium in strict preferences.*

Proof. Once again, necessity is straightforward and hence we only show sufficiency here. From the previous theorem we know that for each game G', there exists a $t^* = (t_1^*, \ldots, t_n^*)$ with $t_i^* \in T_i'$ for all i and $\Omega(t^*) = O(G')$ that

[5] See the first part of the proof of the main theorem in Ray and Zhou (2001).

[6] This theorem is the extensive game form analog of Sprumont's Theorem 3 for normal form games.

constitutes a Nash equilibrium for every G'. We will prove that these outcomes coincide with the outcomes of the subgame perfect Nash equilibrium that can be constructed using the complete transitive revealed strict ordering Q_i as in Lemma 2. Suppose this is not true. Then there must exist a reduced game G' in which there exists a node x such that these outcomes do not constitute a subgame perfect equilibrium for the subgame form beginning at x, G'_x, but they do for G'_w, for all $w \in s(x)$. For, if such an x does not exist, we would be able to find an infinite sequence of nodes $\{x_k\}$ with $x_k = p(x_{k+1})$, for each k, which contradicts the assumption that the game always ends. Suppose at G'_x, player i is active, that is, $x \in X'_i$. As play at G'_x is not subgame-perfect but is for all subgames succeeding x, then it must be true that, given Q_i, player i can deviate at x from the outcome path $\rho(O(G'))$ and obtain an outcome that he prefers to $O(G')$. If $x \notin \rho(O(G'))$ then player i cannot change the outcome by deviating at x. So let us assume $x \in \rho(O(G'))$. Suppose player i deviates and moves to a successor $y \in s(x)$ such that $y \notin \rho(O(G')$. If y is a terminal node then consider the binary individually-strategy-reduced extensive game form $G''(t^*, i; 2)$ with $Z'' = \{O(G'),\, y\}$. If y is a non-terminal node and the subgame G'_y has player i as the only active player then consider the binary individually-strategy-reduced extensive game form $G''(t^*, i; 2)$ with $Z'' = \{O(G'),\, O(G'_y)\}$. By XC, the outcome of either binary individually-strategy-reduced extensive game form is $O(G')$. Hence, by the revealed base relation, player i cannot deviate and be better off. Now suppose y is a non-terminal node and the subgame G'_y has at least one active player other than i. Then by SPC, $O(O(G'), O(G'_y)) = O(G')$. Therefore, again by the revealed base relation, player i cannot deviate and be better off.[7] ■

[7] As in Ray and Zhou's (2001) proof, this argument uses the *one deviation property* (as in Lemma 98.2 of Osborne and Rubinstein, 1994) which is a necessary and sufficient condition for subgame-perfect equilibrium.

3 CONCLUSION

In this paper we provide separate testable restrictions for Nash and subgame-perfect equilibrium. Our two conditions together are equivalent to the three conditions proposed by Ray and Zhou for subgame-perfect rationalization. The advantage however is that our conditions can be used to test for Nash behavior alone and also to distinguish between Nash and subgame-perfect behavior.

Our conditions are also constructed in such a way that violations of these conditions refer specifically to players and nodes. Checking these conditions can help identify the players and the nodes where subgame-perfect or Nash behavior are not observed. Thus, even though the data come from a collective choice situation of a multi-player game, we can recover information about individual rationality. This could be relevant to obtain results to rationalize observed outcomes using other notions of rationality such as multiple rationales (Kalai, Rubinstein and Spiegler, 2002).

One possible criticism of our test for Nash behavior could be that the restrictions are described over observable outcomes and unobservable strategies. Note that, however, for the class of games we consider, the set of unobservable strategies that are consistent with an observed outcome is finite. Thus, the tests can be carried out in finitely many steps for a given data set. Tests of this form have been used in the previous literature. For example, Diewert and Parkan (1985) developed nonparametric tests that require checking whether there exists a real solution to a (linear) programming problem defined over observed and unobserved variables.

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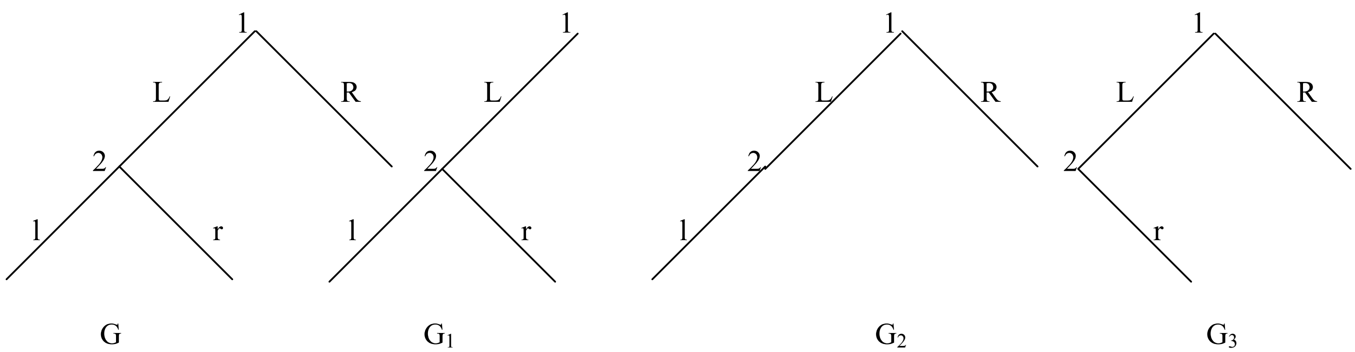

Figure 1a

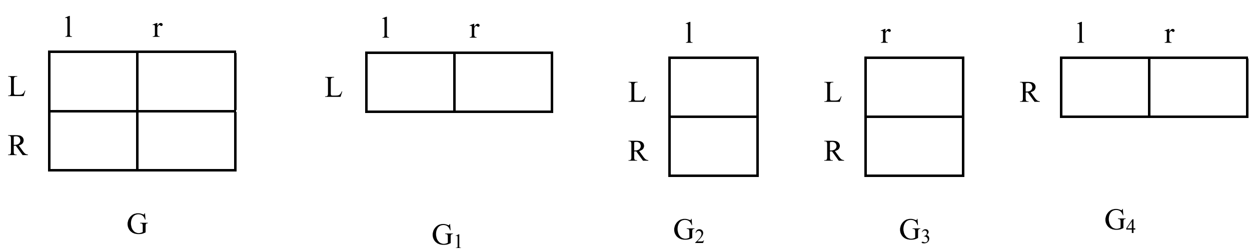

Figure 1b

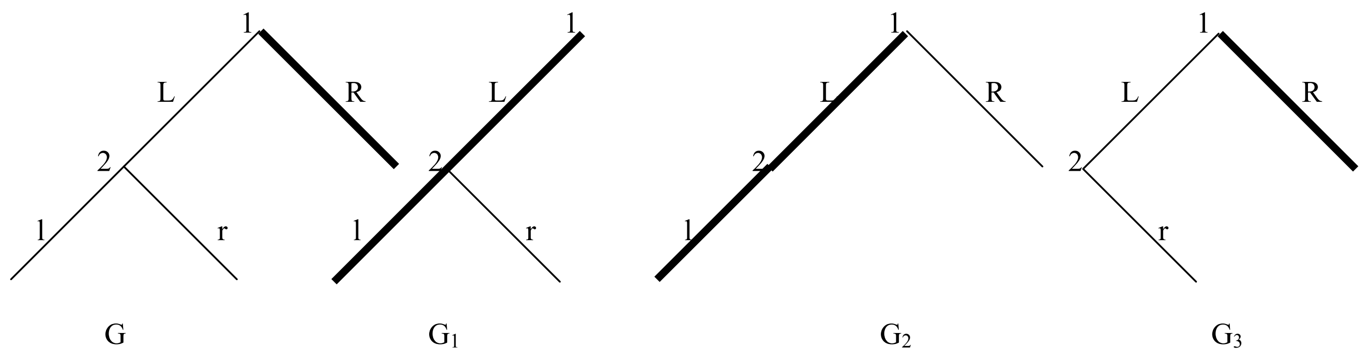

Figure 2a

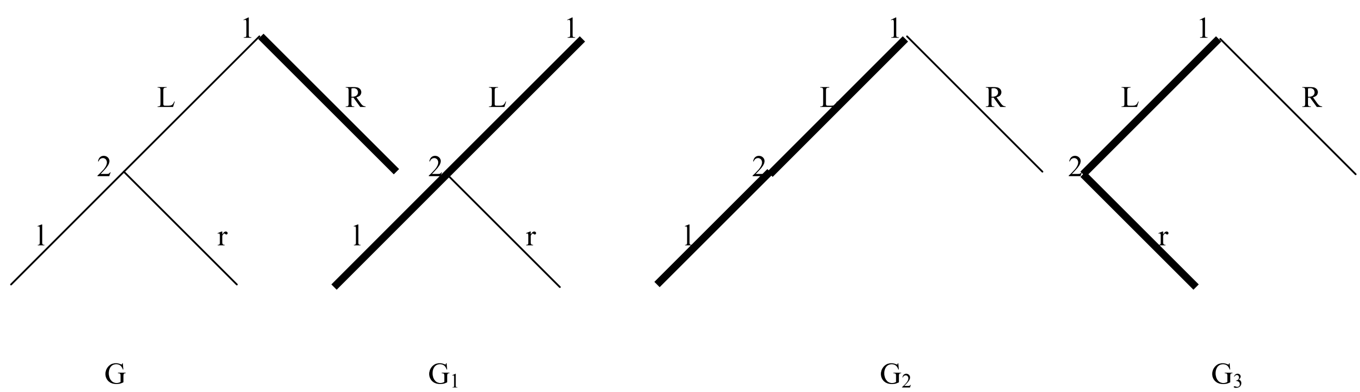

Figure 2b