The γ-Core and Coalition Formation[1]

Parkash Chander[2]

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Abstract

This paper reinterprets the γ-core (Chander and Tulkens (1995, 1997)) and justifies it as well as its prediction that the efficient coalition structure is stable in terms of the coalition formation theory. It is assumed that coalitions can freely merge or break apart, are farsighted (that is, it is the final and not the immediate payoffs that matter to the coalitions) and a coalition may deviate *if and only if* it stands to gain from it. It is then shown that subsequent to a deviation by a coalition, the nonmembers will have incentives to break apart into singletons, as is assumed in the definition of the γ-characteristic function, and that the grand coalition is the only stable coalition structure.

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[2] Indian Statistical Institute, and National University of Singapore, 1 Arts Link, AS 2, Level 6, Singapore 117570. Email: ecsparka@nus.edu.sg.

The γ-Core and Coalition Formation

1. Introduction

The concept of a characteristic function which specifies the worth of each coalition is central to the theory of cooperative games. The worth of a coalition is what it can achieve on its own without the cooperation of the nonmembers. If there are no externalities, i.e., if the payoffs to the members of a coalition do not depend on the actions of the nonmembers, then the worth can be defined without specifying the actions of the nonmembers. But if externalities are present, then in order to calculate the worth of a coalition one must also predict the actions of the nonmembers. This has been however a disputed issue and alternative assumptions in this respect lead to different concepts of characteristic functions such as the α-, β-, and γ - characteristic functions.[3]

The γ- characteristic function was introduced most recently by Chander and Tulkens (1995, 1997) based on an assumption concerning the behavior of the nonmembers which is more plausible than those considered previously. They consider a game in strategic form with transferable utility in which it is efficient for the grand coalition to form and choose the strategy profile that maximizes the joint surplus. They assume that when a coalition forms it neither takes as given the strategies of its complement, as in the case of strong and coalition proof Nash equilibria, nor does it presume that the complement would follow minimax or maximin strategies, as in the case of the α - and β - characteristic functions, instead it looks forward to the best reply payoff corresponding to the equilibrium that its actions would induce. More specifically, when a coalition *S* forms it assumes that the nonmembers would not take any particular coalitional action against it, but would adopt only their individually best reply strategies. This results into a Nash equilibrium between the coalition *S* and the nonmembers acting individually, with the

[3] These have been studied and contrasted to each other in various externalities contexts by Scarf (1971), Starret (1972), and Maler (1989) and in the public goods context by Foley (1970), Roberts (1974), Moulin (1987), and Chander (1993) among others. It is well-known that because of the underlying minimax or maximin assumptions, the α - and β - characteristic functions may lead to large cores. In fact, as noted by Maler (1989) and Ray and Vohra (1997), in some cases the α - and β - cores may include the whole set of Pareto optima.

members of S playing their joint best response strategies to the individual best response strategies of the nonmembers. The assumption that the nonmembers act individually is justified by Chander and Tulkens (1997, fn.6, p.387) by showing that if the nonmembers form one or more non-singleton coalitions then the payoff of S defined by the resulting Nash equilibrium across the coalitions would only be higher. Therefore, the assumption that the nonmembers act individually is equivalent to granting S a certain degree of pessimism.[4] In other words, S presumes the worst possible coalition structure which is that the nonmembers will not form any non-singleton coalitions.[5] Thus, the assumption is not concerning which coalitions will form, but rather which coalitions the coalition S thinks will or will not form. The uncertainty regarding the emerging coalition structure after the deviation is thus resolved by assuming that the coalition S presumes the coalition structure which is worst from its point of view. Chander and Tulkens (1997) show that the so-defined γ-characteristic function implies stability of the grand coalition in at least the games they consider.

This stability result has been contrasted with the inefficiency results obtained in the theory of coalition formation (see e.g. Ray and Vohra (1997), and Yi (1997)). In particular, this stability result has been attributed to the γ-theory assumption that the nonmembers do not form any non-singleton coalitions. It has been claimed that this is an arbitrary assumption and that which coalitions will be formed by the nonmembers after the deviation by a coalition should be determined endogenously, as in the coalition formation theory, and not assumed exogenously.

The purpose of this paper is to reinterpret the γ-core and justify it as well as its prediction that the efficient coalition structure is stable in terms of the coalition formation theory.

[4] This is pessimism of a different sort: it is not concerning the strategies that will be adopted by the non-members (as in the case of α - and β - characteristic functions), but about the coalition structures that will emerge.

[5] Chander and Tulkens (1995, 1997) consider games that imply positive externalities from coalition formation. For games that imply negative externalities from coalition formation, the corresponding assumption underlying the γ-characteristic function is that S assumes that the coalition $N \setminus S$ would form and its members would adopt the best response joint strategies. See Yi (1997) for the classification of games that imply positive or negative externalities from coalition formation.

Though the γ-core has been applied to games that are not necessarily symmetric, we consider here, as in the coalition formation theory, mainly symmetric games. Furthermore, in order to be more concrete, we consider the original economic model of Chander and Tulkens (1995, 1997) with identical agents. We show that if coalitions can freely merge or break apart, are farsighted (that is, it is the final and not the immediate payoffs that matter to the coalitions) and a coalition may deviate *if and only if* it stands to gain from it, then the nonmembers will actually not form any non-singleton coalitions, as is assumed in the definition of the γ- characteristic function, and the grand coalition is the only stable coalition structure. We thus justify the γ-core by showing that both its assumptions and predictions are consistent with the coalition formation theory.

The contents of this paper are as follows. We begin in section 2 with a simple example that illustrates the γ-theory and its relationship with the coalition formation theory. In section 3, we introduce the general model and the concept of indirect dominance. Section 4 presents the main results. Section 5 draws the conclusion.

2. The Illustrative Example

Consider an economy consisting of three identical countries or agents. Let $N = \{1,2,3\}$ denote the set of these agents. There are two kinds of commodities: a standard private good, whose quantities are denoted by y, and an environmental good (in fact, a bad), whose quantities are denoted by z. The private and the environmental good can be produced by the agents according to the following rules:

$$y_i = e_i^{\frac{1}{2}},\ i \in N; \text{ and } z = \sum_{i \in N} e_i\ ,$$

where e_i is to be interpreted as the emissions of country i. The preferences of country i are represented by the utility function:

$$u_i(y_i, z) = y_i - z,\ i \in N.$$

Let $T_i = \{e_i : e_i \geq 0\}, T = T_1 \times T_2 \times T_3$, and $u = (u_1, u_2, u_3)$.

We consider the strategic form game $[N,T,u]$. We **assume** that there are no transfers among the members of a coalition and they all get equal payoffs.[6] We also **assume** that a coalition deviates if and only if it stands to strictly gain from it.

Standard arguments show that the game $[N,T,u]$ has a unique Nash equilibrium which induces the following state of the economy

$$\bar{e}_i = \frac{1}{4};\ \bar{y}_i = \frac{1}{2},\ i \in N;\ \bar{z} = \frac{3}{4};\ \text{and}\ \bar{u}_i = -\frac{1}{4},\ i \in N, \tag{1}$$

where $(\bar{e}_1, \bar{e}_2, \bar{e}_3) \in T$ are the Nash equilibrium strategies. It is easily seen that a Pareto efficient state of the economy is given by

$$e_i^* = \frac{1}{36},\ y_i^* = \frac{1}{6},\ i \in N;\ z^* = \frac{1}{12};\ \text{and}\ u_i^* = \frac{1}{12};\ i \in N, \tag{2}$$

and that the emission levels are the same for all Pareto efficient states. We claim that the strategies (e_1^*, e_2^*, e_3^*) belong to the γ-core. Since the players are identical, we need to consider only two types of deviations, namely: a deviation by a coalition of any two players, say $\{1,2\}$ and a deviation by a coalition of any single player, say $\{3\}$.

Define $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ such that $\tilde{e}_1, \tilde{e}_2$ = argmax $(e_1^{\frac{1}{2}} + e_2^{\frac{1}{2}} - 2e_1 - 2e_2 - 2\tilde{e}_3)$ and $\tilde{e}_3$ = argmax $(e_3^{\frac{1}{2}} - \tilde{e}_1 - \tilde{e}_2 - e_3)$. Then,

$$\tilde{e}_1 = \tilde{e}_2 = \frac{1}{16},\ \tilde{e}_3 = \frac{1}{4},\ \tilde{u}_1 = \tilde{u}_2 = -\frac{1}{8}\ \text{and}\ \tilde{u}_3 = \frac{1}{8}. \tag{3}$$

The strategies $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ represent the Nash equilibrium between the coalitions $\{1,2\}$ and $\{3\}$. Comparing the payoffs of the coalition $\{1,2\}$ under the strategies $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$

[6] Ray and Vohra (1999) provide a justification for the assumption of equal sharing of coalition gain among the coalition members. In an infinite-horizon model of coalition formation among symmetric players with endogenous bargaining, they show that in any equilibrium without delay there is equal sharing.

and (e_1^*, e_2^*, e_3^*), it is seen that the coalition {1, 2} will not gain from the deviation.[7]

We now consider deviation by {3} which really brings into focus the said γ-theory assumption. When $S = \{3\}$ deviates it presumes that $N \setminus S = \{1,2\}$ will break up into singletons and the resulting equilibrium will be the Nash equilibrium between {3}, {1} and {2}, which leads to the same payoffs as in (1). From a comparison of the payoffs in (1) and (2), it follows that coalition {3} will also not gain from its deviation. This shows that the strategies (e_1^*, e_2^*, e_3^*) are in the γ-core of the strategic form game $[N, T, u]$.

But why should the coalition $\{1,2\}$ break up into singletons when {3} deviates? The stability of the grand coalition depends crucially on the answer to this question, as {3} will gain from its deviation if {1, 2} did not break up (see the payoffs of {3} in (3) and (2)) and hence would engage in deviation.

Let us consider first the argument against breaking up of {1, 2}: If {3} deviates and {1, 2} does not breakup into singletons, then the resulting equilibrium and the corresponding payoff of {1, 2} is as given in (3), which as seen from (1) is higher than what its payoff would be if it were to break up and induce the coalition structure ({1}, {2}, {3}). The coalition {1, 2} therefore should not break up. This argument however assumes implicitly either that the coalition structure ({1}, {2}, {3}) which emerges after the coalition {1, 2} breaks apart is final or that the coalition {1, 2} is myopic and is concerned only with its payoff at the next step.

Ray and Vohra (1997) assume the coalitions to be farsighted, but preclude the possibility of coalition merging, that is, coalitions can only become finer and not coarser. This means that the coalition structure ({1}, {2}, {3}) is final as further deviations are, by assumption, ruled out. Their analysis, therefore, implies stability of the coalition structure

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[7] It is worth noting that alternatively the α - and β - strategies imply $\tilde{e}_3 \to \infty$ and $\tilde{u}_1, \tilde{u}_2 \to -\infty$.

({1, 2}, {3}) and not of the grand coalition {1, 2, 3}. The coalition structure ({1, 2}, {3}) is also stable in terms of the coalition formation games considered by Carraro and Siniscalco (1993) who assume the coalitions to be myopic.[8] The Ray and Vohra assumption that coalitions cannot merge, therefore, has similar implications as the assumption of myopia.

Let us now introduce the possibility of coalition merging (Chander (1999) and Diamantoudi and Xue (2002)) in the above story. This creates the possibility of further continuations after the coalition {1, 2} breaks apart and the coalition structure ({1}, {2}, {3}) emerges. Indeed, the coalitions {1}, {2}, and {3} may merge and form the grand coalition {1, 2, 3} which gives to each merging coalition a higher payoff (compare the payoffs in (1) and (2)). Contrary to the argument discussed above, the farsighted coalition {1, 2} has an incentive to break up and induce the temporary structure ({1}, {2}, {3}) as that would then lead to the formation of the grand coalition and to payoffs for its members which are strictly higher than if it did not break up[9] (compare the payoff of {1, 2} in (3) and (2)). Since the initial deviating coalition {3} is farsighted, it would realize that its deviation will not benefit its members, as it would only lead back to the grand coalition via the breaking up of {1, 2} into singletons, and hence it will not engage in deviation. This implies stability of the grand coalition.

The coalition formation theory considers alternative approaches to deal with the issue of multiple continuations after a single deviation (see e.g. Greenberg (1990), Chwe (1994), Ray and Vohra (1997) and Xue (1998)): the initial deviating coalitions may evaluate the subsequent other deviations in optimistic or pessimistic ways. For example, in the definition of the von Neumann and Morgenstern abstract stable set, a coalition deviates as long as this deviation might lead to **some final** outcome that benefits its members. In contrast, in the definition of the largest consistent set (Chwe (1994), a coalition deviates

[8] See also d'Aspremont and Gabszewicz (1986) and Barrett (1994; 2003).
[9] Emergence of this type of coalition structures has been observed empirically. For instance, the Comprehensive Test Ban Treaty can come into force *only* if all the current and potential nuclear powers sign it. Another case in point is the Kyoto Protocol. After the US refusal to be a party to the protocol, will the rest of the countries implement or abandon it? See Tulkens (1998) for an interesting discussion on the possibility of such coalition structures.

only if its members benefit from **all final** outcomes that its deviation may lead to. It is clear from the discussion above that the assumption underlying the γ-theory is in the same vein, as it is equivalent to assuming that a coalition engages in deviation *if and only if* its members strictly benefit from the outcome that gives the least payoffs from among all the final outcomes its deviation may lead to.[10] Any other final outcomes, even if they strictly benefit the deviating coalition, are irrelevant and the possibility of just one final outcome which does not strictly benefit the members of the deviating coalition is enough to deter it from deviation.

We have illustrated by means of a simple example why after the deviation by a coalition S, the complement $N \setminus S$ may break up into singletons and deter the deviation. We now extend this argument to the general model. Some new complications arise that have been intentionally avoided by constructing a suitable example so that we could focus on the basic argument. However, our conclusions remain the same.

3. The General Model

Let $N = \{1,2,...,n\}$ be the set of identical agents and suppose that the private good and the environmental good can be produced by the agents according to the following general rules:

$$y_i = g(e_i), i \in N, \text{and } z = \sum_{i \in N} e_i, \tag{4}$$

where $g'(e_i) > 0$ and $g''(e_i) < 0.$ As before, e_i denotes the emissions of agent i. The preferences of agent i are represented by the utility function:

$$u_i(y_i, z) = y_i - v(z), i \in N, \tag{5}$$

where $v'(z) > 0$ and $v''(z) \geq 0.$

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[10] There might seem to be an additional assumption underlying the γ - core which is that the deviating coalition is able to foresee the final outcome after the complement breaks apart. But this is not an additional assumption at all as the coalitions are assumed to be farsighted to begin with.

We **assume** that there exists a finite e^0 such that $g'(e^0) < v'(e^0)$ and $g'(0) > nv'(e^0)$. This assumption rules out corner solutions and ensures that the emissions of the utility maximizing agents are no higher than e^0.

Let $(e_1^*, e_2^*, ..., e_n^*)$ be the Pareto efficient emissions. Then, the first order conditions imply

$$g'(e_i^*) = nv'(z^*), \text{ and } e_i^* = e_j^*, \; i \in N. \tag{6}$$

Let $u_i^* = g(e_i^*) - v(z^*), \; i \in N,$ be the corresponding payoffs.

Let $T_i = \{e_i : 0 \le e_i \le e^0\}$, $T = T_1 \times T_2 \times \cdots \times T_n$, and $u = (u_1, u_2, \ldots, u_n)$. We consider the strategic form game $[N, T, u]$.

Let $(\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n)$ denote the Nash equilibrium of the game $[N, T, u]$. Then,

$$g'(\bar{e}_i) = v'(\sum_{j \in N} \bar{e}_j), i = 1, 2, \ldots, n. \tag{7}$$

Let $(\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n)$ denote the payoffs corresponding to the Nash equilibrium, i.e., $\bar{u}_i = g(\bar{e}_i) - v(\bar{z}), \bar{z} = \sum_{j \in N} \bar{e}_j$.

A *partition* of *N* is $P = (S_1, S_2, ..., S_m)$ such that $\bigcup_{j=1}^m S_j = N$ and for all $i \ne j$, $S_i \cap S_j = \Phi$. Let n_j denote the cardinality of S_j, i.e., $n_j = |S_j|$. Since the players are identical, we may denote coalition S_j interchangeably by n_j. Those partitions of *N* that consist of a possibly non-singleton coalition S followed by one or more coalitions of singletons are of particular interest. We denote such a partition simply by $(S, 1, 1, \ldots, 1)$. The finest partition of *N* consisting of all singletons is denoted by $(1, 1, \ldots, 1)$. We shall interchangeably refer to the partition $P = (S_1, S_2, \ldots, S_m)$ as the set of coalitions $S_1, S_2, \ldots, S_m$, i.e., as $\{S_1, S_2, \ldots, S_m\}$.

A partition P is called a *coalition structure* and let $\wp$ be the *set of all coalition structures.* The idea of non-cooperative play across coalitions in a coalition structure is captured in the following definition.

Given a coalition structure $P \in \wp$, the corresponding *coalitional equilibrium* is $(\widetilde{e}_1, \widetilde{e}_2, ..., \widetilde{e}_n)$ defined as

$$(\widetilde{e}_i)_{i \in S_j} = \text{argmax} \left(\sum_{i \in S_j} \left[\ g(e_i) - v(\sum_{i \in S_j} e_i + \sum_{k \in N \setminus S_j} \widetilde{e}_k) \ \right], j = 1,2,..., m. \right.$$

First order conditions imply

$$g'(\widetilde{e}_i) = n_j v'(\sum_{k \in N} \widetilde{e}_k), i \in S_j, j = 1,2,..., m. \tag{8}$$

Since g is concave, $\widetilde{e}_i < \widetilde{e}_j$ if $i \in S_k$ and $j \in S_l$ with $n_k > n_l$. Let $\widetilde{z} = \sum_{j \in N} \widetilde{e}_j$ denote the total emissions corresponding to the coalitional equilibrium. Then, $\widetilde{u}_i \equiv g(\widetilde{e}_i) - v(\widetilde{z}) < \widetilde{u}_j$ $\equiv g(\widetilde{e}_j) - v(\widetilde{z})$ if $i \in S_k$ and $j \in S_l$ with $n_k > n_l$, since $\widetilde{e}_i < \widetilde{e}_j$. So the payoffs of the members of larger coalitions are lower. Furthermore, by comparing (8) with the optimality condition (6), it follows that $\sum_{i \in N} \widetilde{u}_i < \sum_{i \in N} u_i^*$ if $P \neq N$. This leads to the following lemma

Lemma 1: The payoffs of the members of the largest coalition in a coalitional equilibrium are lower than their payoffs in the grand coalition, i.e., $\widetilde{u}_i < u_i^*$ for all $i \in S_k$ such that $n_k \geq n_j$ for all *j*.

This lemma has the following two important implications: (a) Let $(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$ denote the coalitional equilibrium corresponding to the partition $(S,1,1,\ldots,1)$ of N with $S \neq N$ and let $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ be the corresponding payoffs. Then, $\hat{u}_i < u_i^*$ for $i \in S,$ and (b) $\overline{u}_i < u_i^*, i \in N,$ where, as above, $\overline{u}_i$'s are the payoffs corresponding to the Nash equilibrium.

The existence and uniqueness of Nash equilibrium as well as coalitional equilibria with respect to any given partition of *N* follow from standard arguments as the strategy sets are compact and convex and the payoff functions are concave (see e.g. Chander and Tulkens (1997))

Since the payoff of each player in a coalitional equilibrium depends on the entire coalition structure *P*, let $u_i : \wp \to R$ denote *i's* payoff.

Given a coalition structure, either a coalition may break up into smaller coalitions or several coalitions may merge into a larger coalition. This is expressed formally as follows:

Given a coalition structure $P = \{S_1, S_2, ..., S_m\} \in \wp$, a collection of coalitions $T = \{T_1, T_2, ..., T_k\}$ *induce* a coalition structure $P' \in \wp$ such that

(i) either $T \subset P'$ and

$\exists S \in P$ such that $(\bigcup_{i=1}^{k} T_i) \subset S$ and if $(S \setminus \bigcup_{i=1}^{k} T_i) \neq \Phi$, then $(S \setminus \bigcup_{i=1}^{k} T_i) \in P'$.

(ii) or $T \subset P$ and $\bigcup_{i=1}^{k} T_i \in P'$.

This means that either P is coarser than P' or P' is coarser than P. Note that any given coalition structure P' can be induced from any other coalition structure P through a sequence of steps such as (i) and (ii).

We will write $P \xrightarrow{T} P'$ to denote "T induces P' from P".

Indirect Dominance[11]: P' *indirectly dominates* P, if there exists a sequence of coalition structures $P^1, P^2, ..., P^s \in \wp$, where $P^1 = P$ and $P' = P^s$, and a sequence of collection of coalitions $T^1, T^2, ..., T^{s-1}$ such that for all $j = 1,2,...,s-1$

[11] See also Harsanyi (1974), Chwe (1994) and Xue (1998).

(i) $P^j \xrightarrow{T^j} P^{j+1}$ and

(ii) $u_i(P^j) < u_i(P')$ for each $i \in T_l^j$ and $T_l^j \in T^j$.

The indirect dominance relation captures the idea that farsighted coalitions consider the final coalition structure that their deviations may lead to, and that only those deviations that strictly benefit their members in the end are carried out. A coalition structure P' indirectly dominates P if P' can replace P through a sequence of deviations such that at each step all deviators would be better-off at the final coalition structure P' compared to the status-quo they face.

Lemma 2: Let $P = (S_1, S_2, \ldots, S_m)$ be some coalition structure and let $P' = P$ be a coalition structure induced from P by a collection of coalitions $T = \{T_1, T_2, \ldots, T_k\}$, i.e., $P \xrightarrow{T} P'$ such that $T \subset P'$ and $\bigcup_{i=1}^{k} T_i \in P$, i.e., P is coarser than P'. Then, $u_i(P') < u_i(P)$ for all $i \in S_j \neq \bigcup_{i=1}^{k} T_i,\ j \in \{1, 2, \ldots m\}$.

Proof of Lemma 2: Let $(\tilde{e}_1, \tilde{e}_2, ..., \tilde{e}_n)$ and $(e'_1, e'_2, \ldots, e'_n)$ be the coalitional equilibrium strategies corresponding to P and P', respectively. Let $\tilde{z} = \sum_{i \in N} \tilde{e}_i$ and $z' = \sum_{i \in N} e'_i$. We claim that $z' > \tilde{z}$. Suppose not, i.e., let $z' \leq \tilde{z}$. Then, from (8) and convexity of v, $e'_i \geq \tilde{e}_i$ for each $i \in S_j \neq \bigcup_{i=1}^{k} T_i,\ j \in \{1, 2, \ldots m\}$, and $e'_i > \tilde{e}_i$ for each $i \in \bigcup_{j=1}^{k} T_j$, since $|T_j| < \left|\bigcup_{i}^{k} T_i\right|$ for each j. But this contradicts our supposition that $z' \leq \tilde{z}$. Hence $z' > \tilde{z}$ and $e'_i \leq \tilde{e}_i$ for each $i \in S_j \neq \bigcup_{i=1}^{k} T_i,\ j \in \{1, 2, \ldots m\}$. This proves the lemma as this means $u_i(P') < u_i(P)$ for $i \in S_j \neq \bigcup_{i=1}^{k} T_i,\ j \in \{1, 2, \ldots m\}$.

Lemma 2 implies that coalition formation results into positive externalities for other coalitions. Intuitively, when a coalition forms its members internalize the effect of their emissions on each other resulting into lower emissions which benefits the other coalitions as well.

Proposition 1: Let $P=(S, N \setminus S)$ for some $S \neq N$. Then, $P' = N$ indirectly dominates P.

Proof of Proposition 1: There are two possible cases: either $|S| \leq |N \setminus S|$ or $|S| > |N \setminus S|$. Consider first $|S| \leq |N \setminus S|$. Let $(\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_n)$ be the coalitional equilibrium payoffs corresponding to the partition $(S, N \setminus S)$. Then, in view of Lemma 1, $\tilde{u}_i < u_i^*$ for all $i \in N \setminus S$. Furthermore, let $(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_n)$ be the coalitional equilibrium payoffs corresponding to the partition $(S, 1, 1, \ldots, 1)$. Then, $\hat{u}_i < u_i^*$ for $i \in S$. Similarly, $\bar{u}_i < u_i^*$ for $i \in N$ where $\bar{u}_i$'s are the Nash equilibrium payoffs. Consider the sequence of partitions P^1, P^2, P^3, P^4 with $P^1 = P (= (S, N \setminus S))$, $P^2 = (S, 1, 1, \ldots, 1)$, $P^3 = (1, 1, \ldots, 1)$ and $P^4 = N$. Let $P' = P^4$, $T^1 = \{\{i\} : i \in N \setminus S\}$, $T^2 = \{\{i\} : i \in S\}$, and $T^3 = \{\{i\} : i \in N\}$. Then, (i) $P^j \xrightarrow{T^j} P^{j+1}$ and $u_i(P^j) < u_i(P')$ for each $\{i\} \in T^j$, $j = 1, 2, 3$. This proves that $P' = N$ indirectly dominates $P = (S, N \setminus S)$.

If $|S| > |N \setminus S|$, then by interchanging S and $N \setminus S$ and applying the same argument as in the preceding paragraph, S may break up into singletons first, followed by the breaking up of $N \setminus S$, which would then lead to the formation of N. This completes the proof.

Notice that in the first case $N \setminus S$ has incentives to break up into singletons before S does. This is consistent with the γ-assumption that $N \setminus S$ breaks up into singletons after S deviates. In the second case however $N \setminus S$ breaks up into singletons after S does and this may seem to be inconsistent with the γ-assumption, but it is not. The reason is that when

$|S| > |N \setminus S|$ it does not matter whether or not $N \setminus S$ breaks up into singletons before S does. If $N \setminus S$ is assumed to breakup into singletons before S does then, in view of Lemma 2, the payoffs of members of S corresponding to the resulting coalition structure $(S,1,1,\ldots,1)$ would be even lower and S would still have incentives to break up into singletons. Therefore, assuming that $N \setminus S$ breaks up into singletons before S does is inconsequential when $|S| > |N \setminus S|$ as it does not affect the final outcome. In other words, there is no loss of generality in assuming, as the γ - theory does, that $N \setminus S$ breaks up into singletons.

As mentioned earlier the γ -theory is applicable to games that are not necessarily symmetric. Now in the asymmetric games, unlike the symmetric games, there is no relationship between the size of the coalition and the payoffs of its members, that is, 'the size does not matter'. Therefore, in the asymmetric games there are no two distinct cases as in the proof of Proposition 1. There is just one case to deal with and only one assumption to make that can be applied uniformly to all deviating coalitions irrespective of their size. It therefore makes sense to assume that $N \setminus S$ breaks up into singletons no matter whether S is larger or smaller than $N \setminus S$.

We postpone further discussion of the asymmetric games to later sections. For the time being we return to symmetric games. We show that there is in fact no coalition structure other than N which is stable.

Proposition 2: Let $P \in \wp$ be any coalition structure such that $P \neq N$. Then, $P' = N$ indirectly dominates P and N is the only stable coalition structure.

Proof of Proposition 2: Let $P \in \wp$ be some coalition structure. If $P = (1,1,\ldots,1)$, then $u_i(P) < u_i^*$ for all $i \in P$ and therefore $P \xrightarrow{\{\{i\}:i \in N\}} N$. If not, let $P^1 \equiv P$ and let S_1 be the largest coalition in P^1. Then, $|S_1| > 1$ and, in view of Lemma 1, $u_i(P^1) < u_i^*$ for all $i \in S_1$. Now $P^1 \xrightarrow{\{\{i\}:i \in S_1\}} P^2$, where P^2 is the coalition structure in which S_1 has been replaced by the coalitions $\{\{i\}: i \in S_1\}$. If $P^2 \neq (1,1,\ldots,1)$, then let S_2 be the largest coalition in P^2. Then, $|S_2| > 1$ and $u_i(P^2) < u_i^*$ for $i \in S_2$. Thus, $P^2 \xrightarrow{\{\{i\}: i \in S_2\}} P^3$, where P^3 is the coalition structure obtained by replacing S_2 by $\{\{i\}: i \in S_2\}$. We can continue in this fashion until we obtain $P^k = (1,1,\ldots,1)$ for some k. This defines a sequence of collections of coalitions $T^1, T^2, \ldots, T^k$ such that $P^1 \xrightarrow{T^1} P^2 \xrightarrow{T^2} P^3 \cdots \xrightarrow{T^{k-1}} P^k$ and $u_i(P^j) < u_i^*$ for each $\{i\} \in T^j$, $j = 1,2,\ldots,k$. Since $P^k \xrightarrow{\{\{i\}: i \in N\}} N$ and $u_i(P^k) < u_i^*$ for each $i \in N$, N indirectly dominates P^1.

We have shown that given any coalition structure $P \neq N$, there will always be a deviation and that N indirectly dominates every coalition structure $P \neq N$. This proves that N is the only stable coalition structure.

Some remarks are in order. Chander (1999) proves the stability of the grand coalition for games with three players. Diamantoudi and Xue (2002) prove the stability of the grand coalition for games with more than three players. However, they obtain multiple vN-M stable sets and stability of some coalition structures other than the grand coalition. In their framework a farsighted coalition may not engage in deviation even if it stands to strictly gain from the deviation.

4. The Asymmetric Case

We show that the γ-core can be justified similarly in the context of the asymmetric games. We first prove a lemma and then discuss this issue in the concluding section.

If the agents are not identical, equations (4) and (5) are modified as follows:

$$y_i = g_i(e_i), i \in N, \text{and } z = \sum_{i \in N} e_i, \tag{9}$$

where $g_i'(e_i) > 0$ and $g_i''(e_i) < 0$. As before, e_i denotes the emissions of agent i. The preferences of agent i are represented by the utility function:

$$u_i(y_i, z) = y_i - v_i(z), i \in N, \tag{10}$$

where $v_i'(z) > 0$ and $v_i''(z) \geq 0.$ The subscripts in the functions g_i and v_i capture the fact that the agents are not necessarily identical.

Given a coalitional structure $P = (S_1, S_2, \ldots, S_m)$, let $(\widetilde{e}_1, \widetilde{e}_2, ..., \widetilde{e}_n)$ be the corresponding coalitional equilibrium. Then from the first order conditions

$$g_i'(\widetilde{e}_i) = \sum_{l \in S_j} v_l'(\sum_{k \in N} \widetilde{e}_k), \; i \in S_j, j = 1,2,...,m. \tag{11}$$

Lemma 3: Suppose the agents are not identical. Let $P = (S, 1, 1, \ldots, 1)$ be some coalition structure and let $P' = (S, T_1, T_2, \ \ldots, T_m)$ be a coalition structure such that $|T_k| > 1$ for at least some $k \in \{1, 2, \ldots, m\}$. Then, $u_i(P') > u_i(P)$ for all $i \in S.$

Proof of Lemma 3: Let $(\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_n)$ and $(e_1', e_2', \ldots, e_n')$ be the coalitional equilibrium strategies corresponding to P and P', respectively. Let $\hat{z} = \sum_{i \in N} \hat{e}_i$ and $z' = \sum_{i \in N} e_i'.$ We claim that $z' < \hat{z}.$ Suppose not, i.e., let $z' \geq \hat{z}.$ Then, as seen from (11), convexity of each v_l and $|T_j| \geq 1$ for each $j = 1, 2, \ldots, m$, imply that $e_i' \leq \hat{e}_i$ for each $i \in N$ and $e_i' < \hat{e}_i$ for each $i \in T_k$ such that $|T_k| > 1.$ But this contradicts our supposition that $z' \geq \hat{z}.$ Hence $z' < \hat{z}$ and $e_i' \geq \hat{e}_i$ for each $i \in S$. But this proves the lemma as this means $u_i(P') > u_i(P)$ for $i \in S.$

Furthermore, as shown in Chander and Tulkens (1995, 1997),[12] there exists an imputation $(x_1, x_2, \ldots, x_n)$ possibly involving transfers among the members of the grand coalition such that for all $P = (S, 1, 1, \ldots, 1), S \subset N,$

$$\sum_{i \in N} x_i = \sum_{i \in N} u_i(N) \text{ and } \sum_{i \in S} x_i \geq \sum_{i \in S} u_i(P).$$

[12] See also Helms (2001).

This means that if, as in the symmetric case, the deviating coalition S presumes that $N \setminus S$ will break up into singletons, then it will have no incentive to engage in the deviation, which implies stability of the grand coalition.

5. Conclusions

The purpose of this paper is not to review the coalition formation theory, but to justify in terms of this theory the γ - core and its prediction that the efficient coalition structure is stable. Some comparisons are however unavoidable. This has reference to the fact that the γ -theory has been developed and applied to asymmetric games whereas the coalition formation theory has been shown to be applicable to only symmetric games. For example, if the players are not identical, what would be the equilibrium coalition structure in our game with three players after $\{3\}$ deviates? Since the payoffs of the players across the coalitions can no longer be compared, the coalition formation theory is unable to make any prediction in such cases.[13] It is therefore reasonable to assume that in asymmetric games the deviating coalitions are also not able to predict the coalition structure subsequent to their deviations. In the face of such uncertainty, a coalition may deviate only if its members would benefit in all subsequent coalition structures. But this is precisely the assumption underlying the γ -core, as Lemma 3 shows for the asymmetric case that the payoffs of the members of the deviating coalition S are minimized when $N \setminus S$ breaks up into singletons.[14]

We have restricted our analysis to a class of games that imply positive externalities from coalition formation. For games with negative externalities, the corresponding assumption underlying the γ -characteristic function is that S assumes that the coalition $N \setminus S$ would form and its members would adopt the best response *joint* strategies. We however do not pursue this here as it is the subject matter of another paper.

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[13] An additional complication is that of transfers among the members of a coalition, which has also not been satisfactorily resolved in the coalition formation theory for asymmetric games.

[14] It also clarifies that it is transfers among the members of the grand coalition and not among the members of the deviating coalitions that play an important role.

Overall, our analysis shows that in order to obtain stability of coalition structures other than the grand coalition or the efficient coalition structure one must either invoke the assumption of myopia or place some exogenous restrictions on the process of coalition formation.

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