Knowledge Creation as a Square Dance on the Hilbert Cube*

Marcus Berliant** and Masahisa Fujita±

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Abstract

This paper presents a micro-model of knowledge creation and transfer in a small group of people. It is intended to contribute eventually to the development of microfoundations for aggregate models of knowledge externalities used in various literatures, such as those pertaining to endogenous growth theory, urban agglomeration and growth, organizational R&D and knowledge creation, and human capital accumulation. Our model incorporates two key aspects of the cooperative process of knowledge creation: (i) heterogeneity of people in their state of knowledge is essential for successful cooperation in the joint creation of new ideas, while (ii) the very process of cooperative knowledge creation affects the heterogeneity of people through the accumulation of knowledge in common. In the two person case, we show that the equilibrium process tends to result in the accumulation of too much knowledge in common compared to the most productive state. Unlike the two-person case, in the four person case we show that under certain conditions, the equilibrium process of knowledge creation by four persons may converge to the most productive state. Extensions of the basic model are discussed. JEL Classification Numbers: D83, O31, R11

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**Department of Economics, Washington University, Campus Box 1208, 1 Brookings Drive, St. Louis, MO 63130-4899 Phone: (314) 935-8486, Fax: (314) 935-4156, e-mail: berliant@economics.wustl.edu

±Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto, 606-01 Japan Phone: (81-75) 753-7122, Fax (81-75) 753-7198, e-mail: fujita@kier.kyoto-u.ac.jp

1 Introduction

We attempt to provide microfoundations for aggregate models of knowledge creation and transfer, for example the models used in the endogenous growth literature. How does knowledge creation occur? How does it perpetuate itself?

As people create and transfer knowledge, they change. Thus, the history of meetings and their content is important. If people meet for a long time, then their base of knowledge in common increases, and their partnership eventually becomes less productive. Similarly, if two persons have very different knowledge bases, they have little common ground for communication, so their partnership will not be very productive. In fact, whether a person is working alone or working with others, they could obtain a knowledge base that is not very compatible with that of another person who has not worked with them previously.

The basic framework that employs knowledge creation as a black box driving economic growth is usually called the endogenous growth model. Here we make a modest attempt to open that black box. The literature using this black box includes Shell (1966), Romer (1986, 1990), Lucas (1988), Jones and Manuelli (1990), and many papers building on these contributions. There are two key features of our model in relation to the endogenous growth literature. First, our agents are heterogeneous, and that heterogeneity is endogenous to the model. Second, the effectiveness of the externality between agents working together can change over time, and this change is endogenous.

Fujita and Weber (2003) consider a model where heterogeneity between agents is exogenous and discrete. They examine the effects of immigration policy on the productivity and welfare of workers. They note that progress in technology in a country where workers are highly trained is in small steps involving intensive interactions between workers and a relatively homogeneous work force, whereas countries that specialize in production of new knowledge have a relatively heterogeneous work force. This motivates our examination of how endogenous worker heterogeneity affects industrial structure, the speed of innovation, and the pattern of worker interaction.

The literature that motivated us to try to construct foundations for knowledge creation is the work in urban economics on cities as the factories of new ideas. In her classic work, Jane Jacobs (1969, p. 50) builds on Marshall (1890) when discussing innovation: "This process is of the essence in understanding cities because cities are places where adding new work to older work proceeds vigorously. Indeed, any settlement where this happens becomes a city." Lucas

(1988, p. 38) extends this:

> But, as Jacobs has rightly emphasized and illustrated with hundreds of concrete examples, much of economic life is 'creative' in much the same way as is 'art' and 'science'. New York City's garment district, financial district, diamond district, advertising district and many more are as much intellectual centers as is Columbia or New York University. The specific ideas exchanged in these centers differ, of course, from those exchanged in academic circles, but the process is much the same. To an outsider, it even *looks* the same: a collection of people doing pretty much the same thing, emphasizing his own originality and uniqueness.

Recent work in this line of research includes Fujita and Thisse (2002, chapter 11), Berliant *et al* (2003), Duranton and Puga (2001), and Helsley and Strange (2003). A contemporary empirical complement can be found in Greunz (2003).

A very interesting contribution that is related to our work is Keely (2003). It studies the formation of geographical clusters of innovative and knowledge sharing activity when ideas and productivity are related to the number of skilled workers in a cluster. There are two major differences between this work an ours. First, Keely (2003) employs exogenously given technology for the production of ideas and final good production as a function of (skilled) labor in a cluster. In contrast, we attempt to open the black box of ideas and productivity by modeling interactions, specifically knowledge sharing and creation, between pairs of agents. Second, in Keely (2003), the only source of heterogeneity in agents is their level of technology, represented by a coefficient on the final good production function. Here we use much richer form of heterogeneity, and thus a very different form of idea and goods production.

Differentiation of agents in terms of *quality* (or vertical characteristics) of knowledge is studied in Jovanovic and Rob (1989) in the context of a search model. In contrast, our model examines (endogenous) horizontal heterogeneity of agents and its effect on knowledge creation, knowledge transfer, and consumption. We employ myopic agents in a setting with no uncertainty (in particular about what other agents know), so search is unnecessary. They focus on the implementation of ideas as distinct from their conception, whereas we employ simultaneous knowledge creation, transfer, and good production.[1]

[1] Since agents are only differentiated vertically, although it is clear in this model how

Finally, due to the differences in the models, the results are different. They obtain a unique steady state independent of initial conditions, and efficiency of the steady state when there are no externalities in pairwise meetings. We obtain a steady state equilibrium that is highly dependent on initial conditions, and that can be efficient when there are externalities for a non-negligible set of initial conditions.

For simplicity, we employ a deterministic framework. It seems possible to add stochastic elements to the model, but at the cost of complexity. It should also be possible to employ the law of large numbers to a more basic stochastic framework to obtain equivalent results.

The analogy between partner dancing and working jointly to create and exchange knowledge is useful, so we will use terms from these activities interchangeably. We assume that it is not possible for more than two persons to meet or dance at one time, though more than one couple can dance simultaneously.

Our results are summarized as follows. First, in a two person model where myopic agents can decide whether or not to work with each other, there exist many sink points in the interaction game, depending on initial heterogeneity. The most interesting of these features too much homogeneity relative to the most productive state. In the four person model, where agents can choose to work alone or to collaborate with another (under certainty about everyone's state of knowledge), there is a unique sink point for each set of initial conditions. When the initial state features relative homogeneity of knowledge between agents, the sink will be the most productive state.

Section 2 gives the model and notation, Section 3 analyzes equilibrium in the case of two participants or dancers, Section 4 examines welfare in the two person model, Section 5 extends the model to four persons and analyzes equilibrium and welfare, whereas Section 6 provides our conclusions and suggestions for future dancing. Two appendices provide the proofs of key results.

2 The Model - Ideas and Knowledge

In this section, we introduce the basic concepts of our model of ideas and knowledge.

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knowledge transfer between an agent with more knowledge to an agent with less knowledge can occur when the two are matched, it is less obvious how the agent with a higher level of knowledge increases their knowledge level through a match with an agent with a lower level of knolwedge.

An *idea* is represented by a box. It has a label on it that everyone can read. This label describes the contents, and everyone can read it (the label is common knowledge in the game we shall describe). Each box contains an idea that is described by its label. Learning the actual contents of the box, as opposed to its label, takes time, so although anyone can read the label on the box, they cannot understand its contents without investing time. This time is used to open the box and to understand fully its contents. An example is a recipe for making "udon noodles as in Takamatsu." It is labelled as such, but would take time to learn. Another example is reading a paper in a journal. Its label or title can be understood quickly, but learning the contents of the paper requires an investment of time. Production of a new paper, which is like opening a new box, either jointly or individually, also takes time.

Suppose we have an infinite number of boxes, each containing a different piece of knowledge, which is what we call an idea. We put them in a row in an arbitrary order.

There are N persons in the economy, where N is a finite integer. People are indexed by i and j. We assume that each person has a replica of the infinite row of boxes introduced above, and that each copy of the row has the same order. Our model features continuous time. Fix time $t \in \mathbb{R}_+$ and consider any person i. A box is indexed by $k = 1, 2, ...$ Take any box k. If person i knows the idea inside that box, we put a sticker on it that says 1; otherwise, we put a sticker on it that says 0. That is, let $x_i^k(t) \in \{0, 1\}$ be the sticker on box k for person i at time t. The *state of knowledge*, or just *knowledge*, of person i at time t is thus defined to be $K_i(t) = (x_i^1(t), x_i^2(t), ...) \in \{0, 1\}^\infty$. The reason we use an infinite vector of possible ideas is that we are using an infinite time horizon, and there are always new ideas that might be discovered, even in the preparation of udon noodles. More formally, let $\mathcal{H}$ be the *Hilbert cube*; it consists of all real sequences with values in $[0, 1]$. That is, if $\mathbb{N}$ is the set of natural numbers, then $\mathcal{H} = [0, 1]^{\mathbb{N}}$. So the knowledge of person i at time t, $K_i(t)$, is a vertex of the Hilbert cube $\mathcal{H}$. Notice that given any vertex of $\mathcal{H}$, there exists an infinite number of adjacent vertices. That is, given $K_i(t)$ with only finitely many non-zero components, there is an infinite number of ideas that could be created in the next step.

Given $K_i(t) = (x_i^1(t), x_i^2(t), ...)$,

$$n_i(t) = \sum_{k=1}^{\infty} x_i^k(t) \tag{1}$$

represents the number of ideas known by person i at time t. Next, we will define the number of ideas that two persons, i and j, both know. Assume that $j \neq i$. Define $K_j(t) = (x_j^1(t), x_j^2(t), ...)$ and

$$n_{ij}^c(t) = \sum_{k=1}^{\infty} x_i^k(t) \cdot x_j^k(t) \tag{2}$$

So $n_{ij}^c(t)$ represents the number of ideas known by both persons i and j at time t. Notice that i and j are symmetric in this definition, so $n_{ij}^c(t) = n_{ji}^c(t)$. Define

$$n_{ij}^d(t) = n_i(t) - n_{ij}^c(t) \tag{3}$$

to be the number of ideas known by person i but not known by person j at time t.

Knowledge is a set of ideas that are possessed by a person at a particular time. However, knowledge is not a static concept. New knowledge can be produced either individually or jointly, and ideas can be shared with others. But all of this activity takes time.

Now we describe the components of the rest of the model. Consider first a model with just two agents, i and j. At each time, each faces a decision about whether or not to meet with the other. If both want to meet at a particular time, a meeting will occur. If either does not want to meet, then they do not meet. If the agents do not meet at a given time, then they produce separately and also create new knowledge separately. If the two persons do decide to meet at a given time, then they share older knowledge together and create new knowledge together.

So consider a given time t. In order to explain how knowledge creation, knowledge exchange, and commodity production work, it is useful for intuition (but not technically necessary) to view this time period of fixed length as consisting of subperiods of fixed length. Each individual is endowed with a fixed amount of labor that is supplied inelastically during the period. In the first subperiod, individual production takes place. We shall assume constant returns to scale in physical production, so it is not beneficial for individuals to collaborate in production. Each individual uses their labor during the first subperiod to produce consumption good on their own, whether or not they are meeting. We shall assume below that although there are no increasing returns to scale in production, the productivity of a person's labor depends on their stock of knowledge. Activity in the second subperiod depends on whether or not there is a meeting. If there is no meeting, then each person spends the second subperiod creating new knowledge on their own. Evidently, the new knowledge created during this subperiod can differ between the two persons, because they are not communicating. They open different boxes.[2] If there is a meeting, then the second subperiod is divided into two parts. In the first part, the two persons who are meeting spend their time (and labor) sharing old knowledge, boxes they have opened in previous time periods that the other person has not opened. In the second part, they create new knowledge *together*, so they open boxes together.[3]

What do the agents know when they face the decision about whether or not to meet at time t? Each person knows both $K_i(t)$ and $K_j(t)$. In other words, each person is aware of their own knowledge and is also aware of the other's knowledge. Thus, they also know $n_i(t)$, $n_j(t)$, $n_{ij}^c(t) = n_{ji}^c(t)$, $n_{ij}^d(t)$, and $n_{ji}^d(t)$ when they decide whether or not to meet at time t. The notation for whether or not a meeting actually occurs at time t is: $\delta_{ij}(t) \equiv \delta_{ji}(t) = 1$ if a meeting occurs and $\delta_{ij}(t) \equiv \delta_{ji}(t) = 0$ if no meeting occurs at time t. Meetings only occur if both persons agree that a meeting should take place.

Next, we must specify the dynamics of the knowledge system and the objectives of the people in the model in order to determine whether or not they decide to meet at a particular time. In order to accomplish this, it is easiest to abstract away from the notation for specific boxes, $K_i(t)$, and to focus on the dynamics of the quantity statistics related to knowledge, $n_i(t)$, $n_j(t)$, $n_{ij}^c(t) = n_{ji}^c(t)$, $n_{ij}^d(t)$, and $n_{ji}^d(t)$. Since we are treating ideas symmetrically, in a sense these quantities are sufficient statistics for our analysis.

The simplest piece of the model to specify is what happens if there is no meeting and the two people thus work in isolation. Let $a_i(t)$ be the rate of creation of new ideas created by person i and let $a_j(t)$ be the rate of creation of new ideas created by j, both at time t. Let $b_{ij}(t)$ and $b_{ji}(t)$ be the rate of transfer of ideas from i to j and from j to i, respectively, at time t.[4] Then we assume that the creation of new knowledge during isolation ($\delta_{ij}(t) = 0$) is governed by the following equations:

$$\begin{aligned} a_i(t) &= \alpha \cdot n_i(t) \text{ and } a_j(t) = \alpha \cdot n_j(t) \text{ when } \delta_{ij}(t) = 0. \\ b_{ij}(t) &= 0 \text{ and } b_{ji}(t) = 0 \text{ when } \delta_{ij}(t) = 0. \end{aligned} \tag{4}$$

So we assume that if there is no meeting at time t, individual knowledge grows

[2] Since there is an infinity of different boxes, the probability that the two agents will open the same box (even at different points in time) is assumed to be zero.

[3] Clearly, the creation of this paper is an example of the process described.

[4] In principle, all of these time-dependent quantities are positive integers. However, for simplicity we take them to be continuous (in $\mathbb{R}_+$) throughout the paper.

at a rate proportional to the knowledge already acquired by an individual. Meanwhile, knowledge held commonly by the two persons does not grow. In particular, ideas are not shared.

If a meeting does occur at time t ($\delta_{ij}(t) = 1$), then both knowledge exchange between the two persons and joint knowledge creation occur. When a meeting takes place, joint knowledge creation is governed by the following dynamics :

$$a_{ij}(t) = \beta \cdot [n_{ij}^c(t) \cdot n_{ij}^d(t) \cdot n_{ji}^d(t)]^{\frac{1}{3}} \tag{5}$$

So when two people meet, joint knowledge creation occurs at a rate proportional to the normalized product of their knowledge in common, the individual knowledge of i, and the individual knowledge of j. The rate of creation of new knowledge is highest when the proportions of ideas in common, ideas exclusive to person i, and ideas exclusive to person j are split evenly. Ideas in common are necessary for communication, while ideas exclusive to one person or the other imply more heterogeneity or originality in the collaboration. If one person in the collaboration does not have exclusive ideas, there is no reason for the other person to meet and collaborate.

Under these circumstances, no knowledge creation in isolation occurs. During meetings at time t, knowledge transfer can occur in addition to the creation of new knowledge. Knowledge transfer is governed by the following dynamics:

$$\begin{aligned} b_{ij}(t) &= \gamma \cdot [n_{ij}^d(t) \cdot n_{ij}^c(t)]^{\frac{1}{2}} \\ b_{ji}(t) &= \gamma \cdot [n_{ji}^d(t) \cdot n_{ij}^c(t)]^{\frac{1}{2}} \end{aligned} \tag{6}$$

So when a meeting occurs, knowledge transfer from i to j happens at a rate proportional to the normalized product of the number of ideas that person i has but that person j does not have, and the ideas common to both persons. The explanation is that communication is necessary for knowledge transfer, so the two persons must have some ideas in common ($n_{ij}^c(t)$). But in addition, person i must have some ideas that are not already possessed by person j ($n_{ij}^d(t)$). The same intuition applies to knowledge transfer in the opposite direction from j to i, represented by the second equation in (6). The change in the number of ideas that both persons have in common ($\dot{n}_{ij}^c(t)$) is the sum of knowledge transfers in both directions and the new ideas jointly created. From person i's perspective, the number of ideas that i has but j doesn't have ($b_{ij}^d(t)$) decreases with knowledge transfers from i to j. Finally, the change in the number of ideas possessed by person i is the sum of the ideas that are jointly created and the number of ideas transferred from j to i. The analogous statements hold for the variables associated with j.

Let us focus on agent i (the equations for agent j are analogous). With a meeting, we have the following dynamics incorporating both knowledge creation and transfer:

$$\begin{aligned}\dot{n}_i(t) &= a_{ij}(t) + b_{ji}(t) \\ \dot{n}_{ij}^c(t) &= a_{ij}(t) + b_{ij}(t) + b_{ji}(t) \\ \dot{n}_{ij}^d(t) &= -b_{ij}(t)\end{aligned}$$

Given this structure, we can define the rates of idea innovation and knowledge transfer at time t, depending on whether or not a meeting occurs.

$$\begin{aligned}\dot{n}_i(t) &= [1-\delta_{ij}(t)]\cdot\alpha\cdot n_i(t) + \\ &\quad \delta_{ij}(t)\cdot(\beta\cdot[n_{ij}^c(t)\cdot n_{ij}^d(t)\cdot n_{ji}^d(t)]^{\frac{1}{3}} + \gamma\cdot[n_{ji}^d(t)\cdot n_{ij}^c(t)]^{\frac{1}{2}}) \\ \dot{n}_{ij}^c(t) &= \delta_{ij}(t)\cdot(\beta\cdot[n_{ij}^c(t)\cdot n_{ij}^d(t)\cdot n_{ji}^d(t)]^{\frac{1}{3}} + \gamma\cdot[n_{ji}^d(t)\cdot n_{ij}^c(t)]^{\frac{1}{2}} \\ &\quad +\gamma\cdot[n_{ij}^d(t)\cdot n_{ji}^c(t)]^{\frac{1}{2}}) \\ \dot{n}_{ij}^d(t) &= [1-\delta_{ij}(t)]\cdot\alpha\cdot n_i(t) - \delta_{ij}(t)\cdot\gamma\cdot[n_{ij}^d(t)\cdot n_{ji}^c(t)]^{\frac{1}{2}}\end{aligned}$$

Whether a meeting occurs or not, there is production in each period for both persons. *Felicity in that time period is defined to be the quantity of output.*[5] Define $y_i(t)$ to be production output (or felicity) for person i at time t, and define $y_j(t)$ to be production output (or felicity) of person j at time t. Normalizing the coefficient of production to be 1, we take

$$y_i(t) = n_i(t) \tag{7}$$

so

$$\dot{y}_i(t) = \dot{n}_i(t)$$

By definition,

$$\frac{\dot{y}_i(t)}{y_i(t)} = \frac{\dot{n}_i(t)}{n_i(t)} \tag{8}$$

which represents the rate of growth of income.

Finally, we must define the rule used by each person to decide whether they want a meeting at time t or not. To keep the model tractable in this first analysis, we assume a myopic rule. So a person would like a meeting if and only if the increase in their rate of output with a meeting is higher than the increase

[5]Given that the focus of this paper is on *knowledge creation* rather than production, we use the simplest possible form for the production function.

in their rate of output without a meeting.[6] Note that we use the *increase in the rate of output* rather than the rate of output since in a continuous time model, the rate of output at time t is unaffected by the decision about whether to meet made at time t. Formally,

$$\begin{gathered}\delta_{ij}(t) = 1 \iff \\ \beta \cdot [n_{ij}^c(t) \cdot n_{ij}^d(t) \cdot n_{ji}^d(t)]^{\frac{1}{3}} + \gamma \cdot [n_{ji}^d(t) \cdot n_{ij}^c(t)]^{\frac{1}{2}} > \alpha \cdot n_i(t) \text{ and} \\ \beta \cdot [n_{ji}^c(t) \cdot n_{ji}^d(t) \cdot n_{ij}^d(t)]^{\frac{1}{3}} + \gamma \cdot [n_{ij}^d(t) \cdot n_{ji}^c(t)]^{\frac{1}{2}} > \alpha \cdot n_j(t)\end{gathered} \tag{9}$$

This completes the statement of the model. Dropping the time dependence of variables to analyze dynamics, we obtain the following equations of motion.

$$\begin{aligned}\dot{y}_i &= \dot{n}_i = [1-\delta_{ij}] \cdot \alpha \cdot n_i + \\ &\quad \delta_{ij} \cdot (\beta \cdot [n_{ij}^c \cdot n_{ij}^d \cdot n_{ji}^d]^{\frac{1}{3}} + \gamma \cdot [n_{ji}^d \cdot n_{ij}^c]^{\frac{1}{2}}) \\ \dot{n}_{ij}^c &= \delta_{ij} \cdot (\beta \cdot [n_{ij}^c \cdot n_{ij}^d \cdot n_{ji}^d]^{\frac{1}{3}} + \gamma \cdot [n_{ji}^d \cdot n_{ij}^c]^{\frac{1}{2}} + \gamma \cdot [n_{ij}^d \cdot n_{ji}^c]^{\frac{1}{2}}) \\ \dot{n}_{ij}^d &= [1-\delta_{ij}] \cdot \alpha \cdot n_i - \delta_{ij} \cdot \gamma \cdot [n_{ij}^d \cdot n_{ji}^c]^{\frac{1}{2}}\end{aligned} \tag{10}$$

This system, with analogous equations for agent j, represents a partner dance on the vertices of the Hilbert cube.

As we are attempting to model close interactions within small groups, we assume that at each time, the myopic persons interacting choose a core configuration. That is, we restrict attention to configurations such that at any point in time, no coalition of persons can get together and make themselves better off *in that time period*. In essence, our solution concept at a point in time is the myopic core.

3 Equilibrium Dynamics: Two Persons

In order to analyze our system, we first divide all of our equations by the total number of ideas possessed by i and j:

$$n^{ij} = n_{ij}^d + n_{ji}^d + n_{ij}^c \tag{11}$$

and define new variables

$$\begin{aligned}m_{ij}^c &\equiv m_{ji}^c = \frac{n_{ij}^c}{n} = \frac{n_{ji}^c}{n} \\ m_{ij}^d &= \frac{n_{ij}^d}{n}, m_{ji}^d = \frac{n_{ji}^d}{n}\end{aligned}$$

[6] We will see that the rule used in the case of ties is not important.

From (11), we obtain

$$1 = m_{ij}^d + m_{ji}^d + m_{ij}^c \tag{12}$$

After some detailed calculations (see Appendix a of the technical appendix for all of the steps), we obtain $\dot{m}_{ij}^d$ and $\dot{m}_{ji}^d$ as functions of m_{ij}^d and m_{ji}^d *only*, as follows.

$$\begin{aligned} \dot{m}_{ij}^d &= [1-\delta_{ij}]\cdot\alpha\cdot\{(1-m_{ij}^d)(1-m_{ij}^d-m_{ji}^d)\} \\ &\quad -\delta_{ij}\cdot\{\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}+m_{ij}^d\cdot\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}\} \\ \dot{m}_{ji}^d &= [1-\delta_{ij}]\cdot\alpha\cdot\{(1-m_{ji}^d)(1-m_{ij}^d-m_{ji}^d)\} \\ &\quad -\delta_{ij}\cdot\{\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}+m_{ji}^d\cdot\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}\} \end{aligned} \tag{13}$$

To study this more, we must study (9) further. Deleting time indices and dividing by n,

$$\begin{aligned} &\delta_{ij} = 1 \iff \\ &\qquad \beta\cdot[m_{ij}^c\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot m^c]^{\frac{1}{2}} > \alpha\cdot m_i \\ &\qquad\qquad \text{and } \beta\cdot[m_{ij}^c\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot m^c]^{\frac{1}{2}} > \alpha\cdot m_j \end{aligned}$$

Substituting further,

$$\begin{aligned} &\delta_{ij} = 1 \iff \\ &\beta\cdot[(1-m_{ji}^d-m_{ij}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot(1-m_{ji}^d-m_{ij}^d)]^{\frac{1}{2}} > \alpha\cdot(1-m_{ji}^d) \\ &\text{and } \beta\cdot[(1-m_{ji}^d-m_{ij}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot(1-m_{ji}^d-m_{ij}^d)]^{\frac{1}{2}} > \alpha\cdot(1-m_{ij}^d) \end{aligned}$$

In other words, meetings occur when the rate of growth of income or utility of each person is higher with a meeting than without a meeting.

Define

$$\begin{aligned} F_i(m_{ij}^d, m_{ji}^d) &= \beta\cdot[(1-m_{ji}^d-m_{ij}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}} + \\ &\quad \gamma\cdot[m_{ji}^d\cdot(1-m_{ji}^d-m_{ij}^d)]^{\frac{1}{2}}-\alpha\cdot(1-m_{ji}^d) \\ F_j(m_{ij}^d, m_{ji}^d) &= \beta\cdot[(1-m_{ji}^d-m_{ij}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}} + \\ &\quad \gamma\cdot[m_{ij}^d\cdot(1-m_{ji}^d-m_{ij}^d)]^{\frac{1}{2}}-\alpha\cdot(1-m_{ij}^d) \end{aligned} \tag{14}$$

and

$$M_i = \{(m_{ij}^d, m_{ji}^d)\in\mathbb{R}_+^2 \mid m_{ij}^d+m_{ji}^d\le 1,\ F_i(m_{ij}^d, m_{ji}^d) > 0\} \tag{15}$$

$$M_j = \{(m^d_{ij}, m^d_{ji}) \in \mathbb{R}^2_+ \mid m^d_{ij} + m^d_{ji} \leq 1,\ F_j(m^d_{ij}, m^d_{ji}) > 0\} \tag{16}$$

whereas

$$M = M_i \cap M_j$$

The function $F_i(m^d_{ij}, m^d_{ji})$ represents the net benefit for i of meeting instead of isolation. Likewise for $F_j(m^d_{ij}, m^d_{ji})$. The set M_i represents those pairs (m^d_{ij}, m^d_{ji}) such that i wants to meet with j, since for these pairs, the rate of growth of i's utility or income with a meeting is higher than the rate of growth of i's utility or income without a meeting. The set M_j represents those pairs (m^d_{ij}, m^d_{ji}) such that j wants to meet with i. Of course, the set M represents those pairs (m^d_{ij}, m^d_{ji}) such that both persons want to meet with each other. Thus, meetings will occur at time t for pairs in M.

We represent our model in our Figures as a function of m^d_{ij} and m^d_{ji}; since $m^d_{ij} + m^d_{ji} + m^c_{ij} = 1$, we know that $1 - m^c_{ij} = m^d_{ij} + m^d_{ji} \leq 1$, where this inequality is represented by half of the unit square (a triangle) in $\mathbb{R}^2$. We put m^d_{ij} on the horizontal axis and m^d_{ji} on the vertical axis, omitting m^c.

Figure 1, panels (a) and (b) illustrate the sets M_i and M_j, respectively, for $\beta = \gamma = 1$ and for various values of α. Of course, panels (a) and (b) are mirror images of each other across the 45° line. Figure 2 illustrates M, the set of pairs where both persons want to meet, and its complement, where no meetings occur, for the same parameter values. When (m^d_{ij}, m^d_{ji}) is close to the boundary of the triangle, meetings do not occur. The reason is that the two persons have too little in common to interact effectively (near the diagonal) or someone has too little exclusive knowledge (near the axes) to interact effectively. Meetings only take place in the interior where the three components of knowledge are relatively balanced.

FIGURES 1 AND 2 GO HERE

In fact we can describe the properties of the set M in general. The set M has the shape depicted in Figure 2; see Appendix b of the technical appendix for proof. In particular, M is roughly the shape of an apple core aligned on the 45° line. As α increases, the productivity of creating ideas alone increases, so people are less likely to want to meet to create, implying that each M_i and M_j shrinks as α increases, as does M. If α is a little more than 1, M disappears. To be precise, let $M(\alpha)$ be the set M under the parameter value α. Then, whenever $\alpha_1 < \alpha_2$, the set $M(\alpha_2)$ is entirely contained in $M(\alpha_1)$. Thus, as shown in Figure 2, there is a unique point B contained in every $M(\alpha)$, provided $M(\alpha)$ is nonempty. We call B the *bliss point*, for the point B in Figure 2 is the point where the rate of increase in income or utility is maximized for each person, as we will explain in the next section (see also Lemma A6 in Appendix c of the technical appendix).

Next we discuss the dynamics of the system. Consider first the case where there is no meeting, so $\delta_{ij} = 0$ is fixed exogenously. Then from equations (13), the dynamics are given by the following equations:

$$\begin{aligned}
\dot{m}_{ij}^d &= \alpha \cdot (1 - m_{ij}^d)(1 - m_{ij}^d - m_{ji}^d) \\
\dot{m}_{ji}^d &= \alpha \cdot (1 - m_{ji}^d)(1 - m_{ij}^d - m_{ji}^d)
\end{aligned}$$

FIGURE 3 GOES HERE

Figure 3, panel (a) illustrates the gradient field assuming that $\delta_{ij} = 0$. Several facts follow quickly from these derivations. First, if there is no meeting ($\delta_{ij} = 0$), then both $\dot{m}_{ij}^d$ and $\dot{m}_{ji}^d$ are non-negative, and positive on the interior of the triangle. So if there is no meeting, the vector field points to the northeast. Furthermore, in the lower half of the triangle where $m_{ij}^d \geq m_{ji}^d$ (the other part is symmetric), we have

$$\frac{\dot{m}_{ji}^d}{\dot{m}_{ij}^d} = \frac{1 - m_{ji}^d}{1 - m_{ij}^d} \geq 1$$

where the inequality is strict off of the diagonal. Thus, when $\delta_{ij} = 0$, the vector field points northeast but toward the diagonal. Under the assumption of no meeting, the system tends to sink points along the diagonal line where $m_{ij}^d + m_{ji}^d = 1$, illustrated in Figure 3, panel (a) by a bold line.

Figure 3, panel (b) illustrates the gradient field assuming that $\delta_{ij} = 1$. Then (13) implies:

$$\begin{aligned}
\dot{m}_{ij}^d &= -\gamma \cdot [m_{ij}^d \cdot (1 - m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1 - m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}} \\
\dot{m}_{ji}^d &= -\gamma \cdot [m_{ji}^d \cdot (1 - m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ji}^d \cdot \beta \cdot [(1 - m_{ij}^d - m_{ji}^d) \cdot m_{ji}^d \cdot m_{ij}^d]^{\frac{1}{3}}
\end{aligned} \tag{17}$$

Both of these expressions are negative on the interior of the triangle and the vector field points southwest. Consider, for convenience, the lower half of the triangle where $m_{ij}^d \geq m_{ji}^d$; the other part is symmetric. Then

$$\begin{aligned}
\frac{\dot{m}_{ji}^d}{\dot{m}_{ij}^d} &= \frac{\gamma \cdot [m_{ji}^d \cdot (1 - m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ji}^d \cdot \beta \cdot [(1 - m_{ij}^d - m_{ji}^d) \cdot m_{ji}^d \cdot m_{ij}^d]^{\frac{1}{3}}}{\gamma \cdot [m_{ij}^d \cdot (1 - m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1 - m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}} \\
&\leq 1
\end{aligned}$$

where the inequality is strict off of the diagonal. Thus, the vector field points southwest but toward the diagonal, as illustrated in Figure 3, panel (b). The only sink is at $(0, 0)$, so the system eventually moves there under the assumption of a meeting.

Next, we combine the case where there is no meeting ($\delta_{ij} = 0$) with the case where there is a meeting ($\delta_{ij} = 1$), and let the agents choose whether or not to meet. This is illustrated in Figure 4.

FIGURE 4 GOES HERE

The model follows the dynamics for meetings ($\delta_{ij} = 1$) on M and the dynamics for no meetings ($\delta_{ij} = 0$) on the complement of M.

In general, there is a continuum of stable points of the system, corresponding to the points where $m_{ij}^d + m_{ji}^d = 1$. For these points, eventually the myopic return to no meeting dominates the returns to meetings, since eventually the two persons have almost nothing in common. These stable points, however, are not very interesting.

We have not completely specified the dynamics. This is especially important on the boundary of M, where at least one person is indifferent between meeting and not meeting. We take an arbitrarily small unit of time, Δt, and assume that if at least one person becomes indifferent between meeting and not meeting, but the two persons are currently meeting, then the meeting must continue for at least Δt units of time. Similarly, if the two persons are not meeting when one person becomes indifferent between meeting and not meeting while the other wants to meet or is indifferent, then they cannot meet for at least Δt units of time. So if a person becomes indifferent between meeting or not meeting at time t, the function $\delta_{ij}(t)$ cannot change its value until time $t + \Delta t$. Finally, when at least one person initially happens to be on the boundary of M (that is, at least one person is indifferent between meeting and not meeting), then they cannot meet for at least Δt units of time. Under this set of rules, we can be more specific about the dynamic process near the boundary of M.

In terms of dynamics, if the system does not evolve toward the uninteresting stable points where there are no meetings (and the two people have nothing in common), eventually the system reaches the southwest boundary of the set M. From there, the assumption that δ_{ij} is constant over time intervals of at least length Δt at the boundary of M will drive the system in a zig-zag process toward the place furthest to the southwest and on the diagonal that is a member of M. In other words, this is the point $J = (m^d, m^d) \in M$ with lowest norm. It is the remaining stable point of our model. Small movements around J will continue due to our assumption about the dynamics at the boundary of M, namely that meetings or isolation are sticky. As $\Delta t \to 0$, the process converges to the point J. The point J features symmetry between the two agents with a large degree of homogeneity relative to the remainder of the points in M and the other points in the triangle generally.

So given various initial compositions of knowledge (m_{ij}^d, m_{ji}^d), where will the system end up? If the initial composition of knowledge is relatively unbalanced, in other words near the boundary of the triangle, the sink will be a point on the diagonal where $m_{ij}^d + m_{ji}^d = 1$. If the initial composition of knowledge is relatively balanced, then the sink will be the point J.

Using the facts about the shape of M, the point J exists and is unique as long as $M \neq \emptyset$.

At the point $J = (m^J, m^J)$, $m^J \leq \frac{2}{5}$, for reasons explained in the next section.

Without loss of generality, we can allow δ_{ij} to take values in $[0, 1]$ rather than $\{0, 1\}$. The interpretation of a fractional δ_{ij} is that at each instant of time, a person divides their time between a meeting δ_{ij} proportion of that instant and isolation $(1 - \delta_{ij})$ proportion of that instant. All of our results concerning the model when δ_{ij} is restricted to $\{0, 1\}$ carry over to the case where $\delta_{ij} \in [0, 1]$. The reason is that except on the boundary of M, persons strictly prefer $\delta_{ij} \in \{0, 1\}$ to fractional values of δ_{ij}, as each person's objective function is linear in δ_{ij}. On the boundary of M, our rule concerning dynamics prevents δ_{ij} from taking on fractional values, as it must retain its value from the previous iteration of the process for at least time $\Delta t > 0$. So if the process pierces the boundary from inside M, it must retain $\delta_{ij} = 1$ for an additional time of at least Δt. If it pierces the boundary from outside M, it must retain $\delta_{ij} = 0$ for an additional time of at least Δt. It may seem trivial to allow fractional δ_{ij} when discussing equilibrium behavior, but allowing fractional δ_{ij} is crucial to the next section, where we consider efficiency.

4 Efficiency: Two persons

To construct an analog of Pareto efficiency in this model, we use a social planner who can choose whether or not people should meet in each time period. As noted above, we shall allow the social planner to choose values of δ_{ij} in $[0, 1]$, so that persons can be required to meet for a percentage of the total time in a period, and not meet for the remainder of the period. To avoid dependence of our notion of efficiency on a discount rate, we employ the following alternative concepts. The first is stronger than the second. A *path of* δ_{ij} is a piecewise continuous function of time (on $[0, \infty)$) taking values in $[0, 1]$. For each path of δ_{ij}, there corresponds a unique time path of m^d_{ij} determined by equation (17), respecting the initial condition, and thus a unique time path of income $y_i(t; \delta_{ij})$. We say that a path δ'_{ij} *(strictly) dominates* a path δ_{ij} if

$$y_i(t; \delta'_{ij}) \geq y_i(t; \delta_{ij}) \text{ and } y_j(t; \delta'_{ij}) \geq y_j(t; \delta_{ij}) \text{ for all } t \geq 0$$

with strict inequality for at least one over a positive interval of time. As this concept is quite strong, and thus difficult to use as an efficiency criterion, it will sometimes be necessary to employ a weaker concept, which we discuss next. We say that a path δ_{ij} *is overtaken* by a path δ'_{ij} if there exists a t' such that

$$y_i(t; \delta'_{ij}) > y_i(t; \delta_{ij}) \text{ and } y_j(t; \delta'_{ij}) > y_j(t; \delta_{ij}) \text{ for all } t > t'.$$

Two types of sink points were analyzed in the last section. First consider equilibrium paths that have m^J as the sink point; they reach m^J in finite time and stay there. Using Figure 5, we will construct an alternative path δ'_{ij} that dominates the equilibrium path δ_{ij}.

FIGURE 5 GOES HERE

In constructing this path, we will make use of income changes along the upward sloping diagonal in Figure 4. Setting

$$\begin{aligned} m^d_{ij} &= m^d_{ji} = m \\ y_i &= y_j = y \end{aligned} \tag{18}$$

we use (10) and (11) to obtain

$$\begin{aligned} \frac{\dot{y}(t)}{y(t)} &= \frac{\dot{y}(t)}{n_i(t)} = \frac{\dot{y}(t)}{n(t)[1-m]} \\ &= [1-\delta_{ij}] \cdot \alpha + \delta_{ij} \cdot \{\beta \cdot [(1 - \frac{m}{1-m}) \cdot (\frac{m}{1-m})^2]^{\frac{1}{3}} \\ &\quad + \gamma \cdot [(1 - \frac{m}{1-m}) \cdot (\frac{m}{1-m})]^{\frac{1}{2}})\} \end{aligned} \tag{19}$$

To simplify notation, we define the growth rate when the two persons meet, $\delta_{ij} = 1$, as

$$g(m) = \beta \cdot [(1 - \frac{m}{1-m}) \cdot (\frac{m}{1-m})^2]^{\frac{1}{3}} + \gamma \cdot [(1 - \frac{m}{1-m}) \cdot (\frac{m}{1-m})]^{\frac{1}{2}}) \quad (20)$$

Thus

$$\frac{\dot{y}(t)}{y(t)} = [1 - \delta_{ij}] \cdot \alpha + \delta_{ij} \cdot g(m) \quad (21)$$

Figure 5 illustrates the graph of the function $g(m)$ as a bold line for $\beta = \gamma = 1$. We can show[7] that $g(m)$ is strictly quasi-concave on $[0, \frac{1}{2}]$, achieving its maximal value at $m^B \in [\frac{1}{3}, \frac{2}{5}]$. We can also show (see Lemma A6 of the technical appendix) that $m = m^B$ corresponds to the bliss point B in Figure 2. In other words, whenever $M \neq \emptyset$, $B = (m^B, m^B) \in M$, so the point $J = (m^J, m^J)$ defined in Figure 4 and in the previous section has the property that $m^J \leq 2/5$, as it is defined to be the point in M on the diagonal and closest to the origin. We define the point $I = (m^I, m^I)$ in Figure 4 to be the point in M on the diagonal and farthest from the origin.

Let t' be the time at which the equilibrium path reaches (m^J, m^J). Let the planner set $\delta'_{ij}(t) = \delta_{ij}(t)$ for $t \leq t'$, taking the same path as the equilibrium path until t'. From this time on, the planner uses only symmetric points, namely those on the upward sloping diagonal in Figure 4; these points comprise the horizontal axis in Figure 5. At time t', the planner takes $\delta'_{ij}(t) = 0$ until (m^I, m^I) is attained, prohibiting meetings so that the dancers can profit from ideas created in isolation. Then the planner sets $\delta'_{ij}(t) = 1$ until (m^J, m^J) is attained, permitting meetings and the development of more knowledge in common. The last two phases are repeated as necessary.

From Figure 5, the income paths $y_i(t; \delta'_{ij})$ and $y_j(t; \delta'_{ij})$ generated by the path δ'_{ij} clearly dominates the income paths $y_i(t; \delta_{ij})$ and $y_j(t; \delta_{ij})$generated by the equilibrium path δ_{ij}. Thus, the equilibrium is far from the most productive path in the two person model.

Next consider equilibrium paths $\delta_{ij}(t)$ that end in sink points on the downward sloping diagonal in Figure 4. Our dominance criterion cannot be used in this situation, since in potentially dominating plans, the planner will need to force the couple to meet outside of region M in Figure 4 in early time periods. During this time interval, the dancers could do better by not meeting, and thus a comparison of the income derived from the paths would rely on the discount

[7] See Lemma A6 of the technical appendix.

rate, something we are trying to avoid. So we will use our weaker criterion here, that of overtaking.

Given an equilibrium path $\delta_{ij}(t)$ with sink point on the diagonal, the planner can construct an overtaking path $\delta'_{ij}(t)$ as follows. The first phase is to construct a path $\delta'_{ij}(t)$ that reaches a point in region M in finite time. Such a path can readily be constructed using Figures 3 and 4.[8] After reaching region M, the second and third phases are the same as described above for the construction of a path that dominates one ending with m^J. Since the paths with sinks on the downward sloping diagonal have income growth α at every time, whereas the new path $\delta'_{ij}(t)$ features income growth that exceeds α whenever the couple is meeting. Thus, $\delta'_{ij}(t)$ overtakes $\delta_{ij}(t)$.

The most productive state m^B is characterized by less homogeneity than the stable point m^J. Of course, attaining m^B requires the social planner to force the two persons not to meet some of the time. Otherwise, the system evolves toward more homogeneity.

5 Equilibrium Dynamics: Four Persons

5.1 The General Framework

The model with only two people is very limited. Either two people are meeting or they are each working in isolation. With four people, the dancers can be partitioned into two sets of dance partners. Within each pair, the two dancers are working together, but both pairs of partners are working simultaneously. This creates more possibilities in our model, as the knowledge created within a dance pair is not known to the other pair. Thus, knowledge differentiation can evolve between the two pairs of dance partners. Furthermore, the option of switching partners is available with four dancers, but not with two.

To begin the analysis, consider the case $N = 4$. This is a square dance on the vertices of the Hilbert cube. We consider four people and impose symmetry conditions to avoid messy technical issues that would be present with three persons or without symmetry. Furthermore, to keep the analysis tractable, we assume that there is no knowledge transfer during meetings or

[8] Such a path can be constructed as follows. In Figure 2 or Figure 4, take the union of all closed, one dimensional intervals parallel to the 45° line with one endpoint on an axis and the other endpoint a member of M. Call this set M'. From time 0, take $\delta = 1$. Using Figure 3(b), the path hits M' in finite time. From this time on, take $\delta = 0$. Using Figure 3(a), the path hits M in finite time.

dancing, so $\gamma = 0$.[9] That is, here we deal with the case in which the knowledge at issue is so *sticky* that new ideas are kept forever by the creator (or by the pair of joint creators) as a tacit knowledge.

At this point, it is useful to remind the reader that we are using a myopic core concept to determine equilibrium at each point in time. In fact, it is necessary to sharpen that concept in the model with 4 persons. When there is more than one vector of strategies that is in the myopic core at a particular time, namely more than one vector of joint strategies implies the same, highest first derivative of income for all persons, the one with the highest second derivative of income is selected. The justification for this assumption is that at each point in time, people are attempting to maximize the flow of income.

The initial state of knowledge is symmetric among the four dancers, and given by

$$n_{ij}^c(0) = n^c(0) \text{ for all } i \neq j \tag{22}$$

$$n_{ij}^d(0) = n^d(0) \text{ for all } i \neq j \tag{23}$$

At the initial state, each pair of dancers has the same number of ideas, $n^c(0)$, in common. Moreover, for any pair of dancers, the number of ideas that one dancer has but the other does not have is the same and equal to $n^d(0)$.

Next, we examine possible equilibrium configurations, noting that the equilibrium configuration can vary with time. Figure 6 gives the possibilities at any given time. Given that the initial state of knowledge is symmetric among the four dancers, it turns out that the equilibrium configuration at any time also maintains the basic symmetry among the dancers.

FIGURE 6 GOES HERE

Panel (a) in Figure 6 represents the case in which each of the four dancers is working alone, creating new ideas in isolation. Panels (b-1) to (b-3) represent the three possible configurations of partner dancing, in which two couples each dance separately but simultaneously. In panel (b-1), for example, 1 and 2 dance together. At the same time, 3 and 4 dance together.

Although panels (a) to (b-3) represent the basic forms of dance with four persons, it turns out that the equilibrium path often requires a mixture of these

[9] Qualitatively, all of the results of the previous sections (assuming that $N = 2$) hold if $\gamma = 0$ and $\beta > 0$, for the following reason. As γ tends to zero, in Figure 5, m^B tends to $2/5$ from the left, but the general shape of $g(m)$ in remains the same. This observation will be useful for making comparisons of the results for four dancers with the results for two dancers.

basic forms. That is, on the equilibrium path, people wish to change partners as frequently as possible. The purpose is to balance the number of different and common ideas with partners as best as can be achieved. This suggests a square dance with rapidly changing partners on the equilibrium path.

Please refer to panels (c-1) to (c-3) in Figure 6. Each of these panels represents square dancing where a dancer rotates through two fixed partners as fast as possible in order to maximize the instantaneous increase in their income. In panel (c-1), for example, dancer 1 chooses dancers 2 and 3 as partners, and rotates between the two partners under equilibrium values of δ_{12} and δ_{13} such that $\delta_{12} + \delta_{13} = 1$. Dancers 2, 3 and 4 behave analogously. In order for this type of square dance to take place, of course, all four persons must agree to follow this pattern.[10] Finally, panel (d) depicts square dancing in which each dancer rotates though all three possible partners as fast as possible.

To identify which form of square dancing will take place on the equilibrium path, we derive several preliminary expressions. Analogous to (7) for the two-person case, we define the size of knowledge for person i at time t to be $n_i(t)$ and the income for person i at time t to be $y_i(t) = n_i(t)$. Given any pair of persons, i and j, let $n_{ij}^c(t)$ be the number of ideas that are possessed by both i and j at time t, let $n_{ij}^d(t)$ be the number of ideas that person i has but person j does not have at time t, and let $n_{ji}^d(t)$ be the number of ideas which person j has but person i does not have at time t. Then, it holds by definition that

$$n_i(t) = n_{ij}^c(t) + n_{ij}^d(t) \tag{24}$$

Define $n^{ij}(t)$ be the total number of ideas possessed by persons i and j together at time t. Then, tautologically

$$n^{ij}(t) = n_{ij}^c(t) + n_{ij}^d(t) + n_{ji}^d(t) \tag{25}$$

Similar to notation used in the previous sections, define

$$m_{ij}^c(t) = \frac{n_{ij}^c(t)}{n^{ij}(t)},\ m_{ij}^d(t) = \frac{n_{ij}^d(t)}{n^{ij}(t)} \tag{26}$$

Next, let $\delta_{ij}(t)$ represent the meeting index for dancers i and j at time t, where $\delta_{ij}(t) = 1$ means that person i dances exclusively with j at time t, $\delta_{ij}(t) = 0$ means that i and j are not meeting at time t, whereas $0 < \delta_{ij}(t) < 1$ means that person i dances with j at time t for the fraction of time $\delta_{ij}(t)$.[11]

[10] In square dancing terminology, this is the "call."

[11] Hereafter, we assume that for each pair i and j, $\delta_{ij}(t)$ is a piecewise continuous function of time, taking values in $[0, 1]$.

Let us focus on the payoffs and decisions of any dancer, say i. Setting $\delta_{ij} = 0$ in (10), we define the increase in income when person i is dancing in isolation as follows:

$$\dot{y}_i(t) = \alpha n_i(t) \quad \text{when } i \text{ works alone.} \tag{27}$$

Likewise, setting $\gamma = 0$ and $\delta_{ij}(t) = 1$ in (10), we define the increase in income when dancer i is paired exclusively with dancer $j \neq i$ at time t as

$$\dot{y}_i^{ij}(t) = \dot{n}_i(t) = \beta \cdot \left[n_{ij}^c(t) \cdot n_{ij}^d(t) \cdot n_{ji}^d(t)\right]^{1/3} \quad \text{when } \delta_{ij}(t) = 1 \tag{28}$$

In general, when $\delta_{ij}(t)$ takes positive values for more than one j, namely when person i is dancing with more than one partner, the instantaneous increase in income of dancer i at time t is defined as

$$\begin{aligned} \dot{y}_i(t) = \dot{n}_i(t) &= \sum_{j \neq i} \delta_{ij}(t) \cdot \dot{y}_i^{ij}(t) \\ &= \sum_{j \neq i} \delta_{ij}(t) \cdot \beta \cdot \left[n_{ij}^c(t) \cdot n_{ij}^d(t) \cdot n_{ji}^d(t)\right]^{1/3} \end{aligned} \tag{29}$$

At each time t, person i wants to maximize the instantaneous rate of increase in income. Hence, given two potential partners, j and k, by identifying the sign of the difference in the change in income

$$\dot{y}_i^{ij}(t) - \dot{y}_i^{ik}(t), \tag{30}$$

we can identify whether person i prefers j or k as a partner at time t. Since income $y_i(t) = n_i(t)$ at time t is a fixed number at time t, the sign of (30) is the same as the sign of

$$\frac{\dot{y}_i^{ij}(t) - \dot{y}_i^{ik}(t)}{y_i(t)} \equiv \frac{\dot{y}_i^{ij}(t)}{y_i(t)} - \frac{\dot{y}_i^{ik}}{y_i(t)} \tag{31}$$

In general, the identification of this sign is not easy. However, in a special case which is relevant to our analysis, this task becomes easier. Suppose that the knowledge composition of i and j is symmetric at time t:

$$n_{ij}^d(t) = n_{ji}^d(t) \tag{32}$$

That is, at time t, persons i and j have the same number of exclusive ideas. Equations (25) and (32) together imply

$$1 = m_{ij}^c(t) + 2m_{ij}^d(t) \tag{33}$$

Since $y_i(t) = n_i(t)$, using (24), (25) and (32) yields

$$\begin{aligned} y_i(t) &= n^c_{ij}(t) + n^d_{ij}(t) \\ &= n^{ij}(t) - n^d_{ij}(t) \\ &= n^{ij}(t) \cdot \left[1 - m^d_{ij}(t)\right] \end{aligned} \tag{34}$$

Furthermore, substituting (32) into (28) gives

$$\dot{y}^{ij}_i(t) = \beta \cdot \left[n^c_{ij}(t) \cdot n^d_{ij}(t)^2\right]^{1/3} \quad \text{when } \delta_{ij}(t) = 1 \tag{35}$$

So, when $\delta_{ij}(t) = 1$, equations (33) to (35) yield

$$\begin{aligned} \frac{\dot{y}^{ij}_i(t)}{y_i(t)} &= \frac{\beta \cdot \left[n^c_{ij}(t) \cdot n^d_{ij}(t)^2\right]^{1/3}}{n^{ij}(t) \cdot \left[1 - m^d_{ij}(t)\right]} \\ &= \frac{\beta \cdot \left[m^c_{ij}(t) \cdot m^d_{ij}(t)^2\right]^{1/3}}{1 - m^d_{ij}(t)} \\ &= \frac{\beta \cdot \left[\left(1 - 2m^d_{ij}(t)\right) \cdot m^d_{ij}(t)^2\right]^{1/3}}{1 - m^d_{ij}(t)} \\ &= \beta \left[\left(1 - \frac{m^d_{ij}(t)}{1 - m^d_{ij}(t)}\right) \cdot \left(\frac{m^d_{ij}(t)}{1 - m^d_{ij}(t)}\right)^2\right]^{1/3} \end{aligned}$$

Since we assume $\gamma = 0$ in this section, setting $\gamma = 0$ in the definition (20) of g, we know

$$g(m) \equiv \beta \left[\left(1 - \frac{m}{1-m}\right)\left(\frac{m}{1-m}\right)^2\right]^{1/3} \tag{36}$$

Thus

$$\frac{\dot{y}^{ij}_i(t)}{y_i(t)} = g(m^d_{ij}(t)) \tag{37}$$

The $g(m)$ curve defined by (36) is depicted in Figure 7 with $\beta = 1$. The bliss point is at $m^B = 2/5$.

FIGURE 7 GOES HERE

Similarly, suppose that the knowledge bases of persons i and k are pairwise-symmetric at time t:

$$n^d_{ik}(t) = n^d_{ki}(t) \tag{38}$$

Then defining

$$m^d_{ik}(t) \equiv \frac{n^d_{ik}(t)}{n^{ik}(t)}, \; m^c_{ik}(t) \equiv \frac{n^c_{ik}(t)}{n^{ik}(t)}$$

we can show that when $\delta_{ik}(t) = 1$,

$$\frac{\dot{y}_i^{ik}(t)}{y_i(t)} = g\left(m_{ik}^d(t)\right) \tag{39}$$

So when conditions (32) and (41) both hold at time t, equation (31) reduces to

$$\frac{\dot{y}_i^{ij}(t) - \dot{y}_i^{ik}(t)}{y_i(t)} = g(m_{ij}^d(t)) - g(m_{ik}^d(t)) \tag{40}$$

If this expression is positive, then person i prefers j to k as his or her partner at time t.

Suppose that all four dancers are pairwise symmetric at time t, so condition (32) holds for all pairs (i, j) where $i, j = 1, \cdots, 4$ $(i \neq j)$. Then, by using the $g(m)$ curve depicted in Figure 7, we can identify the best partner (or partners) for each person at time t. If person i dances alone at time t, the rate of income increase is given by (27) as

$$\frac{\dot{y}_i(t)}{y_i(t)} = \frac{\dot{y}_i(t)}{n_i(t)} = \alpha \tag{41}$$

Thus, if

$$\alpha > \max_{k \neq i} g(m_{ik}^d(t)) \tag{42}$$

then person i will choose to dance alone at time t. Hence, a necessary and sufficient condition for person i to choose j as a best partner at time t (or one of the best partners for a mixed dance) is:

$$g(m_{ij}^d(t)) = \max\left\{\max_{k \neq i} g\left(m_{ik}^d(t)\right), \alpha\right\} \tag{43}$$

Furthermore, for i and j to dance as partners at time t, of course, condition (43) must hold when the roles of i and j are switched in the expression. We will show below that such a reciprocal relation holds always on the equilibrium path starting with the symmetric initial conditions (22) and (23).

Now we are ready to investigate the actual equilibrium path, depending on the given initial composition of knowledge,

$$m_{ij}^d(0) = m^d(0) = \frac{n^d(0)}{n^c(0) + 2n^d(0)}$$

which is common for all pairs i and j $(i \neq j)$. In Figure 7, let m^J and m^I be defined on the horizontal axis at the left intersection and the right intersection between the $g(m)$ curve and the horizontal line at height α, respectively.

5.2 The Main Result

In the remainder of this paper, we assume that

$$\alpha < g(m^B) \tag{44}$$

so as to avoid the trivial case of all agents always working in isolation.

Figure 8 provides a diagram explaining our main result.

FIGURE 8 GOES HERE

The top horizontal line represents the initial common state $m^d(0)$, while the bottom horizontal line represents the final common state or sink point, $m^d(\infty)$. There are four regions of the initial state that result in four different sink points. To be precise:

The Main Result: *The equilibrium path and sink point depend discontinuously on the initial condition, namely the initial value of the proportion of ideas held by each person but nobody else, $m^d(0)$, that is assumed to be the same for all agents.*[12] *The pattern of interaction between persons and the sink point as a function of the initial condition are given in Figure 8 and as follows.*

(i) For $0 \leq m^d(0) \leq 2/5 = m^B$, the equilibrium path consists of an initial time interval (possibly the empty set) in which all four persons work independently (form (a)), followed by an interval in which all persons work with another but trade partners as rapidly as possible (form (d)). The sink point is the bliss point, 2/5.

(ii) There exists a certain $\widehat{m}$ with $m^B < \widehat{m} < m^I$, such that when $m^B < m^d(0) \leq \widehat{m}$, the equilibrium path consists of three phases. First, the four persons are paired arbitrarily and work with their partners (form (b-1)). Second, they switch to new partners and work with their new partners for a time (form (b-2)). Finally, each person works alternately with partners with whom they worked in the first two phases, but not with the person with whom they have not worked previously (form (c-1)). The sink point is 1/3.[13]

(iii) For $\widehat{m} < m^d(0) \leq m^I$, the equilibrium path pairs the 4 persons into two couples arbitrarily, and each person dances exclusively with the same partner forever (form (b-1)). The sink point is m^J.

[12] To be precise, we also assume that the initial values of the number of ideas held commonly by each pair of agents are the same, and the number of ideas that an agent holds but another agent does not hold are the same for all pairs of agents. The precise statement of these assumptions can be found in (22) and (23).

[13] Here we are assuming that $g(1/3) > \alpha$. If $g(1/3) \leq \alpha$, then the sink point is m^J.

(iv) For $m^I < m^d(0) \leq 1/2$, each person dances alone forever (form (a)). The sink point is $1/2$.

5.2.1 Case (i): $0 < m^d(0) \leq 2/5 = m^B$

First suppose that the initial state is such that

$$m^J \leq m^d(0) \leq m^B$$

Then, since $g(m^d_{ij}(0)) = g\left(m^d(0)\right) > \alpha$ for any possible dance pairs of i and j, no person wishes to dance alone at the start. However, since the value of $g(m^d_{ij}(0))$ is the same for all possible pairs, all forms of (b-1) to (d) in Figure 6 are possible equilibrium dance configurations at the start. To determine which one of them will actually take place on the equilibrium path, we must consider the dynamics of dancing immediately after the start. We can demonstrate the following (see Appendix 1 for proof).

Lemma 1: *Under the symmetry assumptions on initial conditions for four persons, suppose that $m^J < m^d(0) < 2/5$. If all persons have partners at time* 0, *then person* 1 *prefers (and thus all four persons prefer) to change partners immediately.*

The intuition behind this result is as follows. The condition $m^d(0) < 2/5 (= m^B)$ means that the four dancers have relatively too many ideas in common initially, and thus they wish to have partners who have relatively more ideas that are different from theirs. When dancing starts in the form of panel (b-1) in Figure 6, dancers 1 and 2 are producing more common ideas; in contrast, *from the view point of dancers 1 and 2*, dancers 3 and 4 are accumulating new ideas that are different from theirs. In fact, let $\Delta n^c_{12}(t)$ be the number of ideas created by the partnership of 1 and 2 from time 0 to time t given by (72). Differentiating (74) and (77) we obtain

$$\dot{m}^d_{12}(t) = -\frac{n^d(0)}{\left[n^c(0) + \Delta n^c_{12}(t) + 2n^d(0)\right]^2} \cdot \Delta\dot{n}^c_{12}(t) < 0 \tag{45}$$

$$\dot{m}^d_{13}(t) = \frac{n^c(0)}{\left(n^c(0) + 2\left[n^d(0) + \Delta n^c_{12}(t)\right]\right)^2} \cdot \Delta\dot{n}^c_{12}(t) > 0 \tag{46}$$

where, from equation (79),

$$\Delta\dot{n}^c_{12}(t) = \beta\left[n^c(0)^{2/3} + \frac{2}{3}\beta n^d(0)^{2/3} t\right]^{1/2} n^d(0)^{2/3} > 0$$

From (45) we can see that the proportion of ideas that are not common for partners 1 and 2 is decreasing with time, while the proportion of ideas that are not common for partners 1 and 3 is increasing with time. Since g is monotonically increasing on the domain $(m^J, 2/5)$ (see Figure 7), the value $g(m_{12}^d(t))$ of the dance partnership $\{1,2\}$ is decreasing with time, while the value $g\left(m_{13}^d(t)\right)$ of the dance partnership $\{1,3\}$ is increasing with time. Hence, given the symmetric situation of the four dancers, everyone wants to change partners immediately.

Lemma 1 implies that when $m^J < m^d(0) < 2/5$, people wish to change partners as frequently as possible. This suggests, on the equilibrium path, a square dance with rapidly changing partners represented by one of panels (c-1) to (d) in Figure 6 at the start. Actually, we can show that the square dance configurations (c-1) to (c-3) cannot occur on the equilibrium path. For example, suppose that the dancing in the form of panel (c-1) occurs at the start, where $\delta_{12} = \delta_{13} = 1/2$, $\delta_{14} = 0$ and so forth. Then, analogous to (78), we can show that for any sufficiently small $t > 0$,

$$m_{14}^d(t) > m_{12}^d(t) = m_{13}^d(t)$$

and hence $g\left(m_{14}^d(t)\right) > g\left(m_{12}^d(t)\right) = g\left(m_{13}^d(t)\right)$ for any sufficiently small $t > 0$. Thus, dancer 1 wants to change partners from 2 and 3 to 4 immediately. The intuition behind this result is the same as that behind Lemma 1.

Therefore, when $m^J < m^d(0) < 2/5 (= m^B)$, on the equilibrium path, only configuration (d) in Figure 6 can take place at the start, where $\delta_{ij} = 1/3$ for all $i \neq j$. The dynamics for this square dance are as follows. The creation of new ideas always takes place in pairs. Pairs are cycling rapidly. Dancer 1 spends 1/3 of each period with dancer 2, and 2/3 of the time dancing with other partners. Given the symmetric initial conditions and the symmetry of the equilibrium path, the number of common ideas are the same for every pair of dancers.

Omitting the time index, we define

$$n^c = n_{ij}^c \text{ for all } i \neq j.$$

Similarly, the number of ideas not held in common are the same for every pair of dancers:

$$n^d = n_{ij}^d \text{ for all } i \neq j.$$

Therefore the total number of ideas for any pair of partners is given by

$$n = n^c + 2n^d \tag{47}$$

So the dynamics of the system are described as follows.

$$\begin{aligned}\dot{n}^c &= \frac{1}{3}\beta \cdot [n^c \cdot (n^d)^2]^{\frac{1}{3}} \\ \dot{n}^d &= \frac{2}{3}\beta \cdot [n^c \cdot (n^d)^2]^{\frac{1}{3}}\end{aligned}$$

Defining

$$m^c = \frac{n^c}{n},\ m^d = \frac{n^d}{n}$$

we can use (47) to obtain

$$\begin{aligned}\dot{m}^d &= \frac{d(\frac{n^d}{n})}{dt} = \frac{\dot{n}^d}{n} - \frac{n^d}{n} \cdot \frac{\dot{n}}{n} \\ &= \frac{2}{3}\beta \cdot [(1-2m^d) \cdot (m^d)^2]^{\frac{1}{3}} - m^d \cdot \frac{\dot{n}}{n}\end{aligned} \tag{48}$$

$$\begin{aligned}\frac{\dot{n}}{n} &= \frac{\dot{n}^c}{n} + \frac{2\dot{n}^d}{n} \\ &= \frac{5}{3} \cdot \beta \cdot [(1-2m^d) \cdot (m^d)^2]^{\frac{1}{3}}\end{aligned}$$

so

$$\dot{m}^d = \frac{2-5m^d}{3} \cdot \beta \cdot [(1-2m^d) \cdot (m^d)^2]^{\frac{1}{3}} \tag{49}$$

This expression is positive when $m^d < m^B = 2/5$, and zero if $m^d = 2/5$. Hence, beginning at any point $m^d(0) < 2/5$, the system moves to the right, eventually settling at the bliss point B.

When $0 \leq m^d(0) < m^J$, it is obvious that the four persons work alone until they reach m^J.[14] Then they follow the path explained above, eventually reaching m^B.

5.2.2 Case (ii): $m^B < m^d(0) \leq \widehat{m}$ [15]

Next, let us consider the dynamics of the system when it begins to the right of $m^B = 2/5$(but to the left of $\widehat{m}$, which will be defined soon), for example at m_0^din Figure 9, where the $g(m)$curve from Figure 7 is duplicated in the top part. In other words, the initial state reflects a higher degree of heterogeneity than the bliss point, so the dancers want to increase the knowledge they have

[14] Movement to the right beyond m^J requires application of the second order conditions for equilibrium selection.

[15] Please note that we have not yet defined $\widehat{m}$. It will appear soon.

in common through couple dances.[16] Suppose that the four persons initiate pairwise dancing, for example in form (b-1) of Figure 6. Then, as in Case (i), $m_{12}^d(t)$and $m_{13}^d(t)$are given by (74) and (77), respectively. Thus, as shown in (45) and (46), we have $\dot{m}_{12}^d < 0$and $\dot{m}_{13}^d > 0$at such points in the domain, and $g(m)$is downward sloping in this part of the domain. Thus, the value of retaining the dance partnerships $\{1,2\}$and $\{3,4\}$is higher than the value of switching partners to, for example, $\{1,3\}$and $\{2,4\}$, so the original pairs will continue to dance for at least a short while. This contrasts with the behavior of the system when $m^J < m^d < 2/5$.

FIGURE 9 GOES HERE

To see if the pairs continue to be stable or if they eventually switch partners, we calculate the relative speeds of $\dot{m}_{12}^d$and $\dot{m}_{13}^d$. For person 1, $m_{12}^d$is the knowledge differential when they are paired with their current partner, person 2, while m_{13}^d is the knowledge differential when they are paired with a potential partner, person 3. We refer to $g(m_{13}^d)$as the "shadow value" of the potential partnership between persons 1and 3.

Taking the ratio of (46) to (45),

$$\frac{\dot{m}_{13}^d(t)}{\dot{m}_{12}^d(t)} = -\frac{n^c(0)}{n^d(0)}[1 - \frac{\Delta n_{12}^c(t)}{n^c(0) + 2[n^d(0) + \Delta n_{12}^c(t)]}]^2 \tag{50}$$

Using (74), and setting $\Delta n_{12}^c(0) = 0$,

$$m^d(0) = \frac{n^d(0)}{n^c(0) + 2n^d(0)}$$

Hence

$$m^d(0) > \frac{2}{5} \text{ if and only if } \frac{n^c(0)}{n^d(0)} < \frac{1}{2}.$$

So from (50)

$$-\frac{\dot{m}_{13}^d(t)}{\dot{m}_{12}^d(t)} < \frac{1}{2} \text{ when } m_{12}^d(0) > \frac{2}{5}$$

The important implication is that $m_{12}^d(t)$ is decreasing at a rate at least twice the speed of increase of $m_{13}^d(t)$. Provided that $m^d(0)$ is sufficiently close to 2/5, eventually there will be a time t' such that $g(m_{12}^d(t')) = g(m_{13}^d(t'))$ and partners change from $\{1,2\}$ and $\{3,4\}$ to, for example, $\{1,3\}$ and $\{2,4\}$. The intuition

[16] In fact, dancing forms (b-1), (c-1) and (d) are all possible first dances from these initial conditions. They all have the same $\dot{y}$. After some calculations, we will show in the next footnote that indeed pattern (b-1) features the highest d^2y/dt^2.

is that initially $m^d(0) > 2/5$, so there are few common ideas within initial ideas partnerships. In the partnership $\{1,2\}$, for instance, a common idea base is built for the initial time interval beginning at time 0, and productivity increases. This can be seen in Figure 9 as a move by the partnership $\{1,2\}$ left from m_0^d. When common ideas become too numerous (or m decreases beyond m^B), productivity decreases. These dynamics occur quickly. On the other hand, the shadow value of the partnership $\{1,3\}$ must also be considered. Since dancers 1 and 3 are not partners, $m_{13}^d(t)$ is increasing, and thus $g(m_{13}^d(t))$ is decreasing. Its value is decreasing relatively slowly, as the percentage of ideas in common between persons 1 and 3 declines. Eventually, the values of the two partnerships coincide, and the dancers switch partners.

Indeed, we can show the following (see Appendix 2 for the proof):

Lemma 2: *Assuming symmetry of initial conditions for four persons, suppose that $2/5 < m^d(0) < 1/2$. If they form partnerships $\{1,2\}$ and $\{3,4\}$ initially, and keep the same partnerships, then there exists a time t' such that for $t > 0$,*

$$g(m_{12}^d(t)) \underset{<}{\overset{>}{=}} g(m_{13}^d(t)) \quad \text{as} \quad t \underset{>}{\overset{<}{=}} t' \tag{51}$$

and the following relationship holds at time t':

$$m_{13}^d(t') = \frac{2}{5} + \frac{\left(m^d(0) - \frac{2}{5}\right)\left(1 - m^d(0)\right)}{m^d(0)^2\left[2 - \left(\frac{1}{m^d(0)} - 2\right)\left(4 - \frac{1}{m^d(0)}\right)\right]} \tag{52}$$

By symmetry, similar relationships hold for other combinations of actual and shadow partners. We can readily see from (52) that

$$m_{13}^d(t') > m^d(0) \quad \text{for} \quad \frac{2}{5} < m^d(0) < \frac{1}{2} \tag{53}$$

and $m_{13}^d(t')$ increases continuously from 2/5 to 1/2 as $m^d(0)$ moves from 2/5 to 1/2. Furthermore, we can see by (77) and (79) that the value of $m_{13}^d(t)$ increases continuously from $m^d(0)$ to 1/2 as t increases from 0 to ∞. Hence, equation (52) defines uniquely the switching time t' as a function of $m^d(0)$, which is denoted by $t^s\left[m^d(0)\right]$. By construction, t^s is an increasing function of $m^d(0)$ such that

$$t^s[2/5] = 0 \quad \text{and} \quad \lim_{m^d(0) \to 1/2} t^s\left[m^d(0)\right] = \infty$$

Setting $t' = t^s\left[m^d(0)\right]$ in (51) and (52), and denoting

$$m_{12}^d\left(t^s\left[m^d(0)\right]\right) \equiv m_{12}^d\left[m^d(0)\right], \, m_{13}^d\left(t^s\left[m^d(0)\right]\right) \equiv m_{13}^d\left[m^d(0)\right]$$

we have that

$$g\left(m_{12}^d\left[m^d(0)\right]\right) = g\left(m_{13}^d\left[m^d(0)\right]\right) \tag{54}$$

and

$$m_{13}^d\left[m^d(0)\right] = \frac{2}{5} + \frac{\left(m^d(0) - \frac{2}{5}\right)\left(1 - m^d(0)\right)}{m^d(0)^2\left[2 - \left(\frac{1}{m^d(0)} - 2\right)\left(4 - \frac{1}{m^d(0)}\right)\right]} \tag{55}$$

which defines the positions of the initial partnerships at which switching occurs. In Figure 9, we draw the $m_{13}^d\left[m^d(0)\right]$ curve in the bottom part (with the bold line). For an illustration, we take m_0^d as the initial value of $m^d(0)$, and using the real lines with arrows, we show how to determine the switching positions $m_{13}^d\left[m_0^d\right]$ and $m_{12}^d\left[m_0^d\right]$.

Let $\hat{m}$ be the critical value of $m^d(0)$ such that

$$m_{13}^d\left[\hat{m}\right] = m^I \tag{56}$$

Using Figure 9, we can readily show that $2/5 < \hat{m} < m^I$. Suppose that $2/5 < m^d(0) \leq \widehat{m}$. Then, under the partnership $\{1,2\}$ and $\{3,4\}$, it holds that

$$g\left(m_{12}^d(t)\right) > g\left(m_{13}^d(t)\right) > \alpha \quad \text{for} \quad 0 < t < t'$$

and hence partnerships $\{1,2\}$ and $\{3,4\}$ continue until time t'. However, if they maintained the same partnerships longer, then

$$g\left(m_{12}^d(t)\right) < g\left(m_{13}^d(t)\right) \quad \text{for} \quad t > t'$$

This implies that the original partnership cannot be continued beyond time t', and that the dancers switch to the new partnerships, say $\{1,3\}$ and $\{2,4\}$, at time t', where

$$g\left(m_{12}^d(t')\right) = g\left(m_{13}^d(t')\right) \tag{57}$$

These new partnerships last only for a limited time. Indeed, we can show the following (see Appendix 3 for the proof):

Lemma 3: *In the context of Lemma 2, suppose that the initial partnerships $\{1,2\}$ and $\{3,4\}$ switch to the new partnerships $\{1,3\}$ and $\{2,4\}$ at time t' where*

$$g\left(m_{12}^d(t')\right) = g(m_{13}^d(t'))$$

and

$$m_{12}^d(t') = m_{34}^d(t') < m^B < m_{13}^d(t') = m_{14}^d(t') \tag{58}$$

Assuming that the new partnerships are kept after time t', let t'' be the time at which $m_{12}^d(t)$ and $m_{13}^d(t)$ become the same:

$$m_{12}^d(t'') = m_{13}^d(t'') \tag{59}$$

Then, it holds for $t > t'$,

$$g\left(m_{12}^d(t)\right) \lesseqgtr g(m_{13}^d(t)) \quad \text{as} \quad t \lesseqgtr t'' \tag{60}$$

and

$$g\left(m_{13}^d(t)\right) > g(m_{14}^d(t)) \quad \text{for} \quad t' < t \leq t'' \tag{61}$$

Hence, indeed, the new partnerships $\{1,3\}$ and $\{2,4\}$ formed at time t' can be sustained until time t''. This second switching-time, t'', is uniquely determined by solving the following relationship,

$$\Delta n_{13}^c(t', t'') = n_{13}^d(t') - n_{12}^d(t') \equiv \Delta n_{12}^c(t') \tag{62}$$

where $\Delta n_{13}^c(t', t)$ is the number of ideas created under the partnership $\{1,3\}$ from time t' to time $t \geq t'$, which is given by (86). The position where $m_{12}^d(t)$ meets $m_{13}^d(t)$ is given by

$$m_{12}^d(t'') = m_{13}^d(t'') = \frac{2}{5} - \frac{m^d(0) - \frac{2}{5}}{5m^d(0) - 1} \tag{63}$$

By symmetry, similar relationships hold for other combinations of actual and shadow partners. In particular, it holds that

$$m_{12}^d(t'') = m_{13}^d(t'') = m_{34}^d(t'') = m_{24}^d(t'') \equiv m^d(t'') \tag{64}$$

Since equations (86) and (90) together imply that the value of $m_{13}^d(t)$ decreases continuously from $m_{13}^d(t') > m^d(0)$ to 0 as t increases from t' to ∞, and since by equation (52) $m_{13}^d(t')$ is a function only of $m^d(0)$, equation (63) defines uniquely the time t'' as a function of $m^d(0)$, which is denoted by $\tilde{t}^s\left[m^d(0)\right]$. Setting $t'' = \tilde{t}^s\left[m^d(0)\right]$ in (64), we denote

$$\tilde{m}^d\left[m^d(0)\right] \equiv m_{12}^d\left(\tilde{t}^s\left[m^d(0)\right]\right) = m_{13}^d\left(\tilde{t}^s\left[m^d(0)\right]\right) = m_{34}^d\left(\tilde{t}^s\left[m^d(0)\right]\right) = m_{24}^d\left(\tilde{t}^s\left[m^d(0)\right]\right) \tag{65}$$

where, using (63), $\tilde{m}^d\left[m^d(0)\right]$ is defined as follows:

$$\tilde{m}^d\left[m^d(0)\right] = \frac{2}{5} - \frac{m^d(0) - \frac{2}{5}}{5m^d(0) - 1} \tag{66}$$

which represents the position of the second partnerships at which switching occurs.

In Figure 9, the $\tilde{m}^d\left[m^d(0)\right]$ curve is drawn in the bottom part by a bold, broken line. And, taking m_0^d as the initial value of $m^d(0)$, and using the broken lines with arrows, we demonstrate how to determine the second switching position $\tilde{m}^d\left[m^d(0)\right]$. It follows from (63) that the value of $\tilde{m}^d\left[m^d(0)\right]$ decreases continuously, where

$$\tilde{m}^d\left[\frac{2}{5}\right] = \frac{2}{5} > \tilde{m}^d\left[m^d(0)\right] > \frac{1}{3} = \tilde{m}^d\left[\frac{1}{2}\right] \quad \text{for } \frac{2}{5} < m^d(0) < \frac{1}{2}$$

If partnerships $\{1,3\}$ and $\{2,4\}$ were maintained beyond time t'', then it would follow from (60) that

$$g\left(m_{12}^d(t)\right) > g\left(m_{13}^d(t)\right) \quad \text{for } t > t'' \tag{67}$$

This implies that the same partnerships cannot be continued beyond t''. To see what form of dancing will take place after t'', first note that dancers cannot go back to the previous form of partnerships $\{1,2\}$ and $\{3,4\}$. If they did so, then the proportion of the knowledge in common for the actual partners $\{1,2\}$ would increase, while the proportion of the differential knowledge for the shadow partnership $\{3,4\}$ would increase. This means that the following relationship,

$$m_{12}^d(t) < m^d(t'') < m_{13}^d(t) < m^B$$

holds immediately after t'', and thus

$$g\left(m_{12}^d(t)\right) < g\left(m_{13}^d(t)\right) \tag{68}$$

which contradicts with the assumption that $\{1,2\}$ is the actual partnership. Furthermore, relation (61) implies that under any possible partnership, the following inequality

$$g\left(m_{13}^d(t)\right) > g\left(m_{14}^d(t)\right) \tag{69}$$

holds immediately after t''. Thus, immediately after time t'', equilibrium dancing cannot include partnerships $\{1,4\}$ and $\{2,3\}$. Hence, provided that $g(1/3) > \alpha$, we can see from Figure 6 that the only possible equilibrium configuration immediately after t'' is the square dance in the form of (c-1), involving a rapid rotation of non-diagonal partnerships, $\{1,2\}$, $\{1,3\}$, $\{2,4\}$ and $\{3,4\}$. Indeed, as shown below, this form of dancing will continue on the equilibrium path forever after t''.

The dynamics for this square dance are as follows. Using (62) together with relations (88) through (92), and by symmetry, we obtain the following initial conditions at time t'':

$$\begin{aligned} n_{ij}^c(t'') = n_{ji}^c(t'') = \; & n^c(0) + \Delta n_{12}^c(t') \equiv n^c(t'') \\ & \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3) \end{aligned}$$

$$n^d_{ij}(t'') = n^d_{ji}(t'') = \begin{array}{l} n^d(0) + \Delta n^c_{12}(t') \equiv n^d(t'') \\ \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3) \end{array}$$

which are symmetric for all pairs of non-diagonal dancers. Starting with this symmetric state, dancer 1, for example, spends 1/2 of each period with dancers 2 and 3, respectively. Given the symmetric initial conditions at t'' and the symmetry of the equilibrium path, the number of common ideas and the number of uncommon ideas are respectively the same for every pair of non-diagonal dancers.

Omitting the time index, we define

$$n^c = n^c_{ij} = n^c_{ji} \quad \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3)$$

$$n^d = n^d_{ij} = n^d_{ji} \quad \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3)$$

and hence the total number of ideas for any pair of non-diagonal partners is given by

$$n = n^c + 2n^d$$

So, setting

$$\delta_{ij} = \delta_{ji} = \frac{1}{2} \quad \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3)$$

the dynamics of the system are given by

$$\dot{n}^c = \dot{n}^d = \frac{1}{2}\beta \cdot \left[n^c \cdot \left(n^d\right)^2\right]^{1/3}$$

Define

$$m^c = \frac{n^c}{n}, m^d = \frac{n^d}{n}$$

Then, analogous to the derivation of (49), we can obtain

$$\dot{m}^d = \frac{1 - 3m^d}{2} \cdot \beta \cdot \left[\left(1 - 2m^d\right) \cdot \left(m^d\right)^2\right]^{1/3} \tag{70}$$

which is negative when $m^d > \frac{1}{3}$, and zero if $m^d = \frac{1}{3}$. Thus, beginning at any point $m^d(t'') > \frac{1}{3}$, the system moves to the left, eventually settling at $m^d = \frac{1}{3}$.[17]

We can readily show that, along the path above, relation (69) holds for all $t \geq t''$ where $m^d_{13}(t) \equiv m^d(t)$. Hence, starting at time t'', no dancer wishes to

[17] At this point, we have enough machinery to demonstrate that pattern (b-1) is chosen as the first dance, as promised at the beginning of this Case and in the previous footnote. The dynamics for the three configurations (b-1), (c-1), and (d) are given by the following equations, respectively. For configuration (b-1), analogous to the derivation of (49), we can obtain:

$$\dot{m}^d_{12} = -m^d_{12}(1 - m^d_{12})g(m^d_{12}).$$

deviate from the square dance in the form of (c-1) in Figure 6. Thus, we can conclude as follows:

Lemma 4: *In the context of Lemmas 2 and 3, let t'' be the time defined by the relation (62). At this time, the partnerships $\{1,3\}$ and $\{2,4\}$, which started at t' by switching from the initial partnerships $\{1,2\}$ and $\{3,4\}$, reach the symmetric state such that*

$$m_{ij}^d(t'') = m_{ji}^d(t'') = m_{12}^d(t'') \quad \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3),$$

where $1/3 < m_{12}^d(t'') < m^B$. Then, the four persons together start the square dance in the form (c-1) of Figure 6 where $\delta_{ij} = \delta_{ji} = 1/2$ for all $i \neq j$, $(i,j) \neq (1,4) \neq (2,3)$. This square dance continues forever after time t'', while maintaining the symmetric state such that

$$m_{ij}^d(t) = m_{ji}^d(t) \equiv m^d(t) \quad \text{for all } i \neq j, (i,j) \neq (1,4) \neq (2,3)$$

and eventually $m^d(t)$ reaches $1/3$.

It is interesting to observe that, in the entire equilibrium process starting with the symmetric state of knowledge such that $m_i^d(0) = m^d(0) > m^B$ for all i, partnerships $\{1,4\}$ and $\{2,3\}$ never coalesce. That is, given that the proportion of differential knowledge for all pairs of dancers at the start exceeds the most productive point m^B, they try to increase the proportion of knowledge in common as quickly as possible through partner dancing. This initial stages of building up knowledge in common through partner dancing, however, divides all possible pairs of partners, who were symmetric at the start, into two heterogenous groups: those pairs that developed a sufficient proportion of knowledge in common through actual meetings, and those pairs that increased further the proportion of uncommon knowledge because they did not have a chance to work together. Since the latter group of potential partners is excluded from the square dance in the last stage, the equilibrium process of the four-person system ends up with a state of knowledge that is less than the most productive.

For configuration (c-1), see equation (70) . For configuration (d), see equation (49). Note that at time 0, $m_{12}^d(0) = m^d(0)$. Substituting and comparing these equations, we find that configuration (b-1) generates the highest value of $-\dot{m}^d(0)$ and thus the highest value of $\frac{dg(m^d(0))}{dt} = g'(m^d(0)) \cdot \dot{m}^d(0)$ (for $g'(m^d(0)) < 0$). So configuation (b-1) generates the highest value of $\frac{d(\dot{y}/y)}{dt} = \frac{dg(m^d(0))}{dt}$. Note that $\frac{d(\dot{y}/y)}{dt} = \frac{d^2y/dt^2}{y} - g(m^d(0))^2$, so configuration (b-1) also generates the highest value of d^2y/dt^2.

5.2.3 Case (iii): $\widehat{m} < m^d(0) \leq m^I$

Next suppose $m^d(0)$ is such that $\hat{m} < m^d(0) \leq m^I$. As in Case (ii), dancers are more heterogeneous than at the bliss point, so they would like to increase the knowledge they hold in common through couple dancing, for example using configuration (b-1) in Figure 6. The initial phase of Case (iii) is the same as the initial phase of Case (ii). However, using (55), we know that $m^d_{13}[m^d(0)] > m^I$. Thus, $g\left(m^d_{12}(t)\right) > g\left(m^d_{13}(t)\right)$ for all t before $m^d_{12}(t)$ reaches m^J, whereas $g\left(m^d_{12}(t)\right) > \alpha > g\left(m^d_{13}(t)\right)$ when $m^d_{12}(t)$ reaches m^J. So each dancer keeps their original partner as the system climbs up to B and on to J. When the system reaches $m^d(t) = m^J$, each dancer uses fractional δ_{ij} to attain m^J by switching between working in isolation and dancing with their original partner.

5.2.4 Case (iv): $m^I < m^d(0) \leq 1/2$

Finally, suppose $m^d(0) > m^I$. Then, $g\left(m^d(0)\right) < \alpha$, and hence there is no chance for four persons to make any partnership. Thus, each dances in solo forever, and eventually reaches $m^d = 1/2$.

Compiling all four cases, the *Main Result* follows.

There are several important remarks to be made about our *Main Result*. First, the sink point changes discontinuously with changes in the initial conditions. Second, unlike the model with two persons, the sink point is efficient for a large set of initial conditions. Third, from one set of initial conditions (Case (iii)), the four persons divide into two separate groups between which no interaction occurs. Thus, from an initial state that is symmetric, we obtain an equilibrium path with an asymmetry.

5.3 Efficiency: Four Persons

Finally, we consider the welfare properties of the equilibrium path. We examine each of the cases enumerated above, beginning with Case (iii). This Case is quite analogous to the two person model with sink point m^J, and essentially the same argument implies that the equilibrium path can be dominated. What distinguishes this case is the fact that at the sink point, meeting and not meeting have the same one period payoff for all persons, namely the percentage change in income. Thus, the social planner can change δ_{ij} for a length of time without changing payoffs, but after this length of time, payoffs can be made higher, as illustrated in Section 4.

Next consider Case (iv). The equilibrium cannot be dominated. It has each person always working in isolation. Thus, $m^d(0)$ lies in $(m^I, \frac{1}{2}]$ and m^d moves right with time. If there were a dominating path, then the social planner must force some pair to work together over a non-trivial interval of time. The first such interval of time will have values of m^d in $(m^I, \frac{1}{2}]$, so the persons working together will have lower income during this interval, contradicting the assumption of domination.

Consider Case (i). Let $\delta_{ij}(t)$ be the equilibrium path. When $m^d(0) > m^J$, $\delta_{ij}(t) = 1/3$ for all t and for all pairs i and j, and the payoffs from meeting always exceed not meeting for any person. We show in Appendix d of the technical appendix that this is the unique path of meetings that maximizes the income over each non-negligible interval of time. So the equilibrium path is not dominated by any other feasible path. Furthermore, the equilibrium path approaches the most productive state, m^B. When $m^d(0) \leq m^J$, similar to Case (iv), strict domination cannot occur when $m^d \leq m^J$. The equilibrium path begins at $m^d(0)$ and reaches m^J in finite time. Combining this with what we have determined about the equilibrium path starting at $m^d(0) > m^J$, we obtain that the equilibrium path is not dominated, and approaches the most productive state.

Finally, consider Case (ii), when $m^B < m^d(0) \leq \widehat{m}$. We also show in Appendix d of the technical appendix that the equilibrium path is not dominated by any other feasible path, but unlike Case (i), it approaches $m^d = 1/3$, that is not the most productive state.

6 Conjectures and Conclusions

We have considered a model of knowledge creation and exchange that is based on individual behavior, allowing myopic agents to decide whether joint or individual production is best for them at any given time. In the case of four agents, we have allowed them to choose their best partner or to work in isolation. This is a pure externality model of knowledge creation, with no markets.

In the case of two people, there are a continuum of sink points (equilibria) for the knowledge accumulation process. Every state where the two agents have a negligible proportion of ideas in common is attainable as an equilibrium for some initial condition. There is one additional and more interesting sink, involving a large degree of homogeneity as well as symmetry of the two agents, and this is attainable from a non-negligible set of initial conditions. Relative to the efficient state, the first set of sink points has agents that are too heterogeneous, while the second sink point has agents that are too homogeneous.[18]

With four persons, we analyze the special case where there is only joint creation of new knowledge but no knowledge transfer. We find that, surprisingly, for a range of initial conditions that imply a large degree of homogeneity among agents, the sink is the efficient state. If agents begin with a large degree of heterogeneity, then the sink is inefficient, and it can be one of several points, including the analog of the relatively homogeneous sink in the two person case. Despite a symmetric set of initial conditions, asymmetries can arise endogenously in our structure. In particular, each agent might communicate pairwise with some, but not all other, agents in equilibrium. The asymmetries that arise can partition the agents endogenously into different groups, giving rise to an asymmetric interaction structure from a situation that is initially symmetric.

Many extensions of our work come to mind, though we note that the most important tool we have used in the analysis is symmetry. Thus, if one wants to extend the model to include more people, it is likely important that the number of dancers N be even. It is important and interesting to add knowledge transfer to the model with more than 2 people. Then we can study comparative statics with respect to speeds of knowledge transfer and knowledge creation on the equilibrium outcome and on its efficiency. It would also be interesting to add knowledge transfer without meetings, similar to a public good. For instance, agents might learn from publicly available sources of information, like newspapers or the web. Markets for ideas would also be a nice feature.

One set of extensions would allow agents to decide, in addition to the people they choose with whom to work, the intensity of knowledge creation and exchange.

We note that what we have done, in essence, is to open the "black box" of knowledge externalities in more aggregate models to find smaller "black boxes" inside that we use in our model. These "black boxes" are given by the exogenous functions representing knowledge transfer and creation within a meeting of two agents. It will be important to open these "black boxes" as well. That is, the microstructure of knowledge creation and transfer within meetings must be explored.

Another set of extensions would be to add stochastic elements to the model, so the knowledge creation and transfer process is not deterministic. As re-

[18] The proximate cause is agent myopia.

marked in the introduction, probably our framework can be developed from a more primitive stochastic model, where the law of large numbers is applied to obtain our framework as a reduced form.[19]

Eventually, we must return to our original motivation for this model, as stated in the introduction. Location seems to be an important feature of knowledge creation and transfer, so regions and migration are important, along with urban economic concepts more generally. It would be very useful to extend the model to more general functional forms. Finally, it would be interesting to proceed in the opposite direction by putting more structure on our concept of knowledge, allowing asymmetry or introducing notions of distance, such as a metric, on the set of ideas[20] or on the space of knowledge.[21] Finally, it would be useful to add vertical differentiation of knowledge, as in Jovanovic and Rob (1989), to our model of horizontally differentiated knowledge.

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7 Appendix 1: Proof of Lemma 1

To begin the analysis, let us assume that one of the configurations (b-1) to (b-3) occurs at time 0, and examine conditions under which the dancers will continue to dance with the same partner or change to another configuration. Since all four dancers are symmetric at time 0, without loss of generality, let us assume that configuration (b-1) takes place at the start; persons 1 and 2 dance together, while persons 3 and 4 do the same separately but simultaneously. We focus on the payoffs and decisions of dancer 1; the calculations for the other dancers are similar. Suppose that configuration (b-1) began at time 0 and has continued up to time t. Since dancers 3 and 4 are identical from the viewpoint of dancer 1, we examine the motivation of dancer 1 to change partners from 2 to 3 at time t.

The calculations of Section 3 apply to each pair of dancers independently in this case, since it is as if each pair of partners were isolated from the other pair. What we must examine is the possibility that dancer 1 may want to change partners from 2 to 3. This task can be achieved by applying relation (40) where we set $i = 1$, $j = 2$, and $k = 3$. When we calculate the values of $m^d_{12}(t)$ and $m^d_{13}(t)$ by using equations (25) and (26), please note that γ is set 0, so there is no transfer of knowledge, only creation of knowledge.

First, to obtain the value of $m^d_{12}(t)$ at time $t > 0$, we calculate the value of each component in the right side of equation (25) for $i = 1$ and $j = 2$. Since there is no transfer of knowledge when 1 and 2 are dancing together, it holds at time t that

$$n^d_{12}(t) = n^d_{21}(t) = n^d(0) \tag{71}$$

Thus, using (5), the number of ideas created by the partnership of 1 and 2 from time 0 to time t is given by

$$\Delta n^c_{12}(t) = \int_0^t \beta[n^c_{12}(s) \cdot n^d(0)^2]^{\frac{1}{3}} ds \tag{72}$$

and hence

$$n^c_{12}(t) = n^c_{21}(t) = n^c(0) + \Delta n^c_{12}(t) \tag{73}$$

Using (25),

$$\begin{aligned} n^{12}(t) &= n^c_{12}(0) + 2n^d_{12}(t) \\ &= n^c(0) + \Delta n^c_{12}(t) + 2n^d(0) \end{aligned}$$

So, applying (26) for $i = 1$ and $j = 2$ yields

$$m^d_{12}(t) = \frac{n^d(0)}{n^c(0) + \Delta n^c_{12}(t) + 2n^d(0)} \tag{74}$$

Of course, dancer 1 could switch partners at time $t > 0$, say from dancer 2 to dancer 3.[22] Then we have

$$n^c_{13}(t) = n^c(0) \tag{75}$$

[22] Symmetry implies that if dancer 1 wants to switch partners, then so do the other dancers. This is consistent with the rules of square dancing.

Since dancers 1 and 3 have not met prior to time t, the number of ideas they have in common is the number they had in common initially. Moreover,

$$n_{13}^d(t) = n^d(0) + \Delta n_{12}^c(t)$$

The number of ideas that dancer 1 knows but dancer 3 does not know at time t is the number of ideas that dancer 1 knows but dancer 3 does not know initially, plus the number of ideas that dancers 1 and 2 created during their partnership from time 0 to time t. Similarly,

$$n_{31}^d(t) = n^d(0) + \Delta n_{34}^c(t) = n^d(0) + \Delta n_{12}^c(t) = n_{13}^d(t) \tag{76}$$

By symmetry, $\Delta n_{12}^c(t) = \Delta n_{34}^c(t)$, so $n_{31}^d(t) = n_{13}^d(t)$. Define the total number of ideas possessed by partners 1 and 3 at time t under the assumption that dancer 1 switches partners from dancer 2 to dancer 3 exactly at time t to be

$$\begin{aligned} n^{13}(t) &= n_{13}^c(t) + n_{13}^d(t) + n_{31}^d(t) \\ &= n_{13}^c(t) + 2n_{13}^d(t) \\ &= n^c(0) + 2\left[n^d(0) + \Delta n_{12}^c(t)\right] \end{aligned}$$

Thus, using (26),

$$m_{13}^d(t) = \frac{n^d(0) + \Delta n_{12}^c(t)}{n^c(0) + 2[n^d(0) + \Delta n_{12}^c(t)]} \tag{77}$$

Subtracting (74) from (77), we have

$$m_{13}^d(t) - m_{12}^d(t) = \frac{\Delta n_{12}^c(t)\left[n^c(0) + \Delta n_{12}^c(t) + n^d(0)\right]}{(n^c(0) + \Delta n_{12}^c(t) + 2n^d(0)) \cdot (n^c(0) + 2\left[n^d(0) + \Delta n_{12}^c(t)\right])} > 0 \tag{78}$$

which is positive because $\Delta n_{12}^c(t)$ is positive for any $t > 0$.

Here, we note that substituting (73) into (72) yields the integral equation

$$\Delta n_{12}^c(t) = \int_0^t \beta\left[\left(n^c(0) + \Delta n_{12}^c(s)\right) n^d(0)^2\right]^{1/3} ds$$

and its solution is given by

$$\Delta n_{12}^c(t) = \left[n^c(0)^{2/3} + \frac{2}{3}\beta n^d(0)^{2/3} t\right]^{3/2} - n^c(0) \tag{79}$$

Thus, by substituting (79) into each equation from (73) to (78), all variables can be expressed as explicit functions of time t.

Now suppose that the initial value of m, $m^d(0)$, is such that $m^J < m^d(0) < m^B = 2/5$, as illustrated in Figure 7. Then, equation (78) implies that for any sufficiently small $t > 0$,

$$m^J < m_{12}^d(t) < m_{13}^d(t) < 2/5.$$

Thus, $g\left(m_{12}^d(t)\right) < g\left(m_{13}^d(t)\right)$. So, using (40),

$$\frac{\dot{y}_1^{12}(t) - \dot{y}_1^{13}(t)}{y_1(t)} = g\left(m_{12}^d(t)\right) - g\left(m_{13}^d(t)\right) < 0 \tag{80}$$

This means that as soon as the dancing in the form of (b-1) in Figure 6 starts at time 0, the value $g\left(m_{13}^d(t)\right)$ of the *shadow partnership* between dancers 1 and 3 exceeds the value $g\left(m_{12}^d(t)\right)$ of the *actual partnership* between 1 and 2. Thus, dancer 1 wants to change partners from 1 to 3 immediately. Since all four persons are in the symmetric situation when the dancing form (b-1) in Figure 6 is initiated, everyone wants to change partners immediately.

8 Appendix 2: Proof of Lemma 2

Under the partnership $\{1,2\}$ and $\{3,4\}$, first we show that there exists uniquely the time $t' > 0$ such that

$$g(m_{12}^d(t')) = g(m_{13}^d(t')) \tag{81}$$

Since condition (32) holds for all $i, j = 1, 2, 3, 4 (i \neq j)$ at time t', using (35) and (40), we can see that the relation (81) holds if and only if

$$n_{12}^c(t')n_{12}^d(t')^2 = n_{13}^c(t')n_{13}^d(t')^2 \tag{82}$$

or, using (71), (73), (75) and (76), if and only if

$$\left[n^c(0) + \Delta n_{12}^c(t')\right] \cdot n^d(0)^2 = n^c(0) \cdot \left[n^d(0) + \Delta n_{12}^c(t')\right]^2$$

which can be rewritten as follows:

$$\Delta n_{12}^c(t') \cdot n^d(0)^2 \left\{1 - \frac{2n^c(0)}{n^d(0)} - \frac{n^c(0)}{n^d(0)}\frac{\Delta n_{12}^c(t')}{n^d(0)}\right\} = 0$$

Since $\Delta n_{12}^c(t') \cdot n^d(0)^2 > 0$ for any $t' > 0$, this means that the terms inside the braces be zero, or

$$\frac{\Delta n_{12}^c(t')}{n^d(0)} = \frac{n^d(0)}{n^c(0)} - 2 \tag{83}$$

On the other hand, setting $t = t'$ in equation (77) and arranging the terms yields

$$n^c(0) + 2\left[n^d(0) + \Delta n^c_{12}(t')\right] = \frac{n^d(0)}{m^d_{13}(t')} + \frac{\Delta n^c_{12}(t')}{m^d_{13}(t')}$$

or

$$\frac{n^c(0)}{n^d(0)} + 2 - \frac{1}{m^d_{13}(t')} = \frac{\Delta n^c_{12}(t')}{n^d(0)}\left(\frac{1}{m^d_{13}(t')} - 2\right)$$

Substituting (83) into the right hand side of this equation and arranging the terms yields

$$\begin{aligned} m^d_{13}(t') &= \frac{\frac{n^d(0)}{n^c(0)} - 1}{\frac{n^c(0)}{n^d(0)} + \frac{2n^d(0)}{n^c(0)} - 2} \\ &= \frac{1 - \frac{n^c(0)}{n^d(0)}}{\left(\frac{n^c(0)}{n^d(0)}\right)^2 + 2 - \frac{2n^c(0)}{n^d(0)}} \end{aligned} \tag{84}$$

Setting $t = 0$ in (77) and using $m^d_{13}(0) = m^d(0)$, we have

$$m^d(0) = \frac{n^d(0)}{n^c(0) + 2n^d(0)}$$

or

$$\frac{n^c(0)}{n^d(0)} = \frac{1}{m^d(0)} - 2 \tag{85}$$

Substituting (85) into (84) yields

$$\begin{aligned} m^d_{13}(t') &= \frac{3 - \frac{1}{m^d(0)}}{\left(\frac{1}{m^d(0)} - 2\right)^2 + 2 - 2\left(\frac{1}{m^d(0)} - 2\right)} \\ &= \frac{3 - \frac{1}{m^d(0)}}{2 - \left(\frac{1}{m^d(0)} - 2\right)\left(4 - \frac{1}{m^d(0)}\right)} \end{aligned}$$

Deducting 2/5 from the both sides of this equation, we can obtain

$$m^d_{13}(t') - \frac{2}{5} = \frac{\left(m^d(0) - \frac{2}{5}\right)\left(1 - m^d(0)\right)}{m^d(0)^2\left[2 - \left(\frac{1}{m^d(0)} - 2\right)\left(4 - \frac{1}{m^d(0)}\right)\right]}$$

which leads to the equation (52) in Lemma 2. Hence, relation (81) holds if and only if equation (52) holds. We can readily see that the right hand side of equation (52) increases continuously from 2/5 to 1/2 as $m^d(0)$ moves from 2/5 to 1/2. On the other hand, using (77) and (79), we can see that the value of $m^d_{13}(t)$ increases continuously from $m^d(0)$ to 1/2 as t increases from 0 to ∞. Therefore, for any $m^d(0) \in (2/5, 1/2)$, relation (52) defines uniquely the time $t' > 0$ at which the equality (81) holds. Finally, since $m^d_{12}(t)$ decreases and $m^d_{13}(t)$ increases with time t and since the function $g(m)$ is single-peaked at $m = 2/5$, we have the relation (51).■

9 Appendix 3: Proof of Lemma 3

To examine how long the new partnerships will be maintained, let us focus on the partnership $\{1,3\}$. Let $\Delta n_{13}^c(t',t)$ be the number of ideas created under the partnership $\{1,3\}$ from time t' to time $t \geq t'$, which is given by

$$\Delta n_{13}^c(t',t) = \int_{t'}^{t} \beta \left[n_{13}^c(s) \cdot n_{13}^d(s)^2 \right]^{1/3} ds \tag{86}$$

where, using (75) and (76), for $t \geq t'$

$$\begin{aligned} n_{13}^c(t) \quad = n_{31}^c(t) \quad &= n_{13}^c(t') + \Delta n_{13}^c(t',t) \\ &= n^c(0) + \Delta n_{13}^c(t',t) \end{aligned} \tag{87}$$

$$n_{13}^d(t) = n_{31}^d(t) = n_{13}^d(t') = n^d(0) + \Delta n_{12}^c(t') \tag{88}$$

Substituting (87) and (88) into (86) and solving the integral equation yields

$$\Delta n_{13}^c(t',t) = \left[n^c(0)^{2/3} + \frac{2}{3}\beta n_{13}^d(t')^{2/3}(t-t') \right]^{3/2} - n^c(0) \tag{89}$$

Using (25), (87) and (88),

$$\begin{aligned} n^{13}(t) &= n_{13}^c(t) + 2n_{13}^d(t) \\ &= n^c(0) + 2n_{13}^d(t') + \Delta n_{13}^c(t',t) \end{aligned}$$

So,

$$m_{13}^d(t) = \frac{n_{13}^d(t')}{n^c(0) + 2n_{13}^d(t') + \Delta n_{13}^c(t',t)} \tag{90}$$

At any time $t > t'$, dancer 1 could switch from the present partner 3 to the previous partner 2 who has been dancing with person 4 after time t'. Then,

$$n_{12}^c(t) = n_{12}^c(t') \tag{91}$$

$$n_{12}^d(t) = n_{12}^d(t') + \Delta n_{13}^c(t',t) \tag{92}$$

$$n_{21}^d(t) = n_{12}^d(t) \quad \text{by symmetry}$$

so

$$n^{12}(t) = n_{12}^c(t') + 2\left[n_{12}^d(t') + \Delta n_{13}^c(t',t) \right]$$

which leads to

$$m_{12}^d(t) = \frac{n_{12}^d(t') + \Delta n_{13}^c(t',t)}{n_{12}^c(t') + 2\left[n_{12}^d(t') + \Delta n_{13}^c(t',t) \right]} \tag{93}$$

Likewise, at any time $t > t'$, dancer 1 could switch from the present partner 3 to person 4 (instead of person 2). Then, since persons 1 and 4 never danced together previously,

$$n_{14}^{c}(t) = n^{c}(0) \tag{94}$$

$$\begin{aligned} n_{14}^{d}(t) &= n^{d}(0) + \Delta n_{12}^{c}(t') + \Delta n_{13}(t', t) \\ &= n_{13}^{d}(t') + \Delta n_{13}(t', t) \end{aligned} \tag{95}$$

$$n_{41}^{d}(t) = n_{14}^{d}(t) \quad \text{by symmetry}$$

so

$$n^{14}(t) = n^{c}(0) + 2\left[n_{13}^{d}(t') + \Delta n_{13}(t', t)\right]$$

and hence

$$m_{14}^{d}(t) = \frac{n_{13}^{d}(t') + \Delta n_{13}(t', t)}{n^{c}(0) + 2\left[n_{13}^{d}(t') + \Delta n_{13}(t', t)\right]} \tag{96}$$

By differentiating (90), (93) and (96), we have

$$\dot{m}_{12}^{d}(t) = \frac{n_{12}^{c}(t')}{\left(n_{12}^{c}(t') + 2\left[n_{12}^{d}(t') + \Delta n_{13}^{c}(t', t)\right]\right)^{2}} \cdot \Delta \dot{n}_{13}^{c}(t', t) > 0 \tag{97}$$

$$\dot{m}_{13}^{d}(t) = -\frac{n_{13}^{d}(t')}{\left[n^{c}(0) + 2n_{13}^{d}(t') + \Delta n_{13}^{c}(t', t)\right]^{2}} \cdot \Delta \dot{n}_{13}^{c}(t', t) < 0 \tag{98}$$

$$\dot{m}_{14}^{d}(t) = \frac{n^{c}(0)}{\left(n^{c}(0) + 2\left[n_{13}^{d}(t') + \Delta n_{13}(t', t)\right]\right)^{2}} \cdot \Delta \dot{n}_{13}^{c}(t', t) > 0 \tag{99}$$

where, from (89),

$$\Delta \dot{n}_{13}^{c}(t', t) = \beta \left[n^{c}(0)^{2/3} + \frac{2}{3}\beta n_{13}^{d}(t')^{2/3}(t - t')\right]^{1/2} n_{13}^{d}(t')^{2/3} > 0$$

Hence, under the partnerships $\{1, 3\}$ and $\{1, 4\}$, both $m_{12}^{d}(t)$ and $m_{14}^{d}(t)$ increase while $m_{13}^{d}(t)$ decreases with time t. Let t'' be the time at which $m_{12}^{d}(t)$ becomes equal to $m_{13}^{d}(t)$:

$$m_{12}^{d}(t'') = m_{13}^{d}(t'') \tag{100}$$

Then, since $m_{12}^{d}(t') < m^{B} < m_{13}^{d}(t') = m_{14}^{d}(t')$ and since $g(m)$ is single-peaked at m^{B}, it holds that

$$\min\left\{g(m_{12}^{d}(t)), g(m_{13}^{d}(t))\right\} > g(m_{12}^{d}(t')) > g(m_{14}^{d}(t)) \quad \text{for } t' < t \leq t'' \tag{101}$$

Hence, in the time interval $(t', t'']$, dancer 1 never desires to switch partners from person 3 to person 4. It is, however, not *a priori* obvious which of

$g(m_{12}^d(t))$ and $g(m_{13}^d(t))$ is greater in the interval (t', t''). However, given that function $g(m)$ is steeper on the right of bliss point m^B in Figure 9, we can guess that the value of $g(m_{13}^d(t))$ is increasing faster (initially, at least) than the value of $g(m_{12}^d(t))$, and hence the partnership $\{1, 3\}$ will continue until $m_{13}^d(t)$ crosses the bliss point and then becomes the same as $m_{12}^d(t)$. Indeed, we prove this next.

In the context of Lemma 2, suppose that the initial partnerships $\{1, 2\}$ and $\{3, 4\}$ switch to the new partnerships $\{1, 3\}$ and $\{2, 4\}$ at time t', when condition (57) holds. And assume that the new partnerships are kept after time t'. Then, since each of $\{1, 2\}$ and $\{1, 3\}$ is pairwise symmetric, applying (35) and (40) in the present context, for $t \geq t'$ we have

$$g\left(m_{13}^d(t)\right) \gtreqless g\left(m_{12}^d(t)\right) \quad \text{as} \quad n_{13}^c(t) n_{13}^d(t)^2 \gtreqless n_{12}^c(t) n_{12}^d(t)^2 \tag{102}$$

Using (87), (88), (91) and (92), it follows that

$$\begin{aligned} & n_{13}^c(t) n_{13}^d(t)^2 - n_{12}^c(t) n_{12}^d(t)^2 \\ = \;& \left[n_{13}^c(t') + \Delta n_{13}^c(t', t)\right] n_{13}^d(t')^2 - n_{12}^c(t') \left[n_{12}^d(t') + \Delta n_{13}^c(t', t)\right]^2 \\ = \;& \Delta n_{13}^c(t', t) n_{13}^d(t')^2 \left\{1 - \frac{2 n_{12}^c(t') n_{12}^d(t')}{n_{13}^d(t')^2} - \frac{n_{12}^c(t')}{n_{13}^d(t')^2} \cdot \Delta n_{13}^c(t', t)\right\} \end{aligned}$$

Hence, for $t \geq t'$, it holds that

$$g\left(m_{13}^d(t)\right) \gtreqless g\left(m_{12}^d(t)\right) \quad \text{as} \quad \Delta n_{13}^c(t', t) \lesseqgtr \frac{n_{13}^d(t')^2}{n_{12}^c(t')} - 2 n_{12}^d(t') \tag{103}$$

To simplify the expression above, we derive a useful expression. By definition, the following identity holds at any time t:

$$n_1(t) = n_{12}^c(t) + n_{12}^d(t) = n_{13}^c(t) + n_{13}^d(t) \tag{104}$$

Setting $t = t'$ in (104) and using (82) yields

$$n_{12}^c(t') = \frac{n_{13}^d(t')^2}{n_{12}^d(t') + n_{13}^d(t')} \tag{105}$$

$$n_{13}^c(t') = \frac{n_{12}^d(t')^2}{n_{12}^d(t') + n_{13}^d(t')} \tag{106}$$

Substituting (105) into the last term in (103) gives

$$\begin{aligned} \frac{n_{13}^d(t')^2}{n_{12}^c(t')} - 2 n_{12}^d(t') &= n_{13}^d(t') - n_{12}^d(t') \\ &= \left(n^d(0) + \Delta n_{12}^c(t')\right) - n^d(0) \\ &= \Delta n_{12}^c(t') \end{aligned}$$

using (71) and (88) at $t = t'$. Thus, we can conclude that

$$g\left(m^d_{13}(t)\right) \gtreqless g\left(m^d_{12}(t)\right) \quad \text{as} \quad \Delta n^c_{13}(t',t) \lesseqgtr \Delta n^c_{12}(t') \tag{107}$$

Let t'' be the time such that

$$\Delta n^c_{13}(t',t'') = \Delta n^c_{12}(t') \tag{108}$$

Since equation (89) implies that $\Delta n^c_{13}(t',t') = 0$ and since $\Delta n^c_{13}(t',t)$ increases continuously to ∞ as t tends to ∞, equation (108) uniquely defines $t'' > t'$. Hence, we can conclude from (107) that for $t \geq t'$,

$$g\left(m^d_{13}(t)\right) \gtreqless g\left(m^d_{12}(t)\right) \quad \text{as} \quad t \lesseqgtr t'' \tag{109}$$

Substituting (105) into (93) and setting $t = t''$ and using $\Delta n^c_{13}(t',t'') = \Delta n^c_{12}(t') = n^d_{13}(t') - n^d_{12}(t')$ yields

$$m^d_{12}(t'') = \frac{n^d_{13}(t')}{\frac{n^d_{13}(t')^2}{n^d_{12}(t')+n^d_{13}(t')} + 2n^d_{13}(t')}$$

Likewise, using (87) to set $n^c_{13}(t') = n^c(0)$ in (90) and using (106) also yields

$$m^d_{13}(t'') = \frac{n^d_{13}(t')}{\frac{n^d_{13}(t')^2}{n^d_{12}(t')+n^d_{13}(t')} + 2n^d_{13}(t')}$$

Hence, rewriting the expression above, and using the relations $n^d_{13}(t') = n^d(0) + \Delta n^c_{12}(t')$ and $n^d_{12}(t') = n^d(0)$, we have

$$\begin{aligned}
m^d_{12}(t'') = m^d_{13}(t'') &= \frac{1}{\frac{n^d_{13}(t')}{n^d_{12}(t')+n^d_{13}(t')}+2} \\
&= \frac{1}{\frac{n^d(0)+\Delta n^c_{12}(t')}{2n^d(0)+\Delta n^c_{12}(t')}+2} \\
&= \frac{1}{\frac{1+\frac{\Delta n^c_{12}(t')}{n^d(0)}}{2+\frac{\Delta n^c_{12}(t')}{n^d(0)}}+2} \\
&= \frac{1}{3-\frac{n^c(0)}{n^d(0)}} \quad \text{(using (83))} \\
&= \frac{1}{5-\frac{1}{m^d(0)}} \quad \text{(using (85))}
\end{aligned}$$

which can be rewritten as (63). Thus,

$$m^d_{12}(t'') = m^d_{13}(t'') < m^B = 2/5 \tag{110}$$

This gives the alternative definition of time t'', which has been introduced in (59).Thus, (107) and (108) imply (60) and (62) in Lemma 3. Finally, relation (61) follows immediately from (101).■

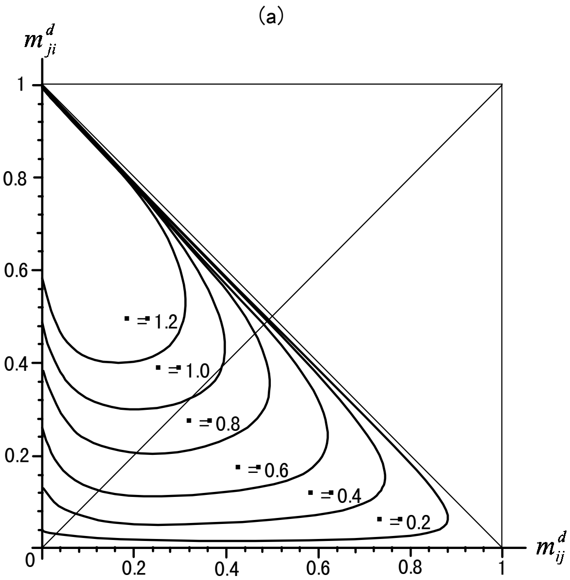

Figure 1: The sets M_i and M_j under various values of α.

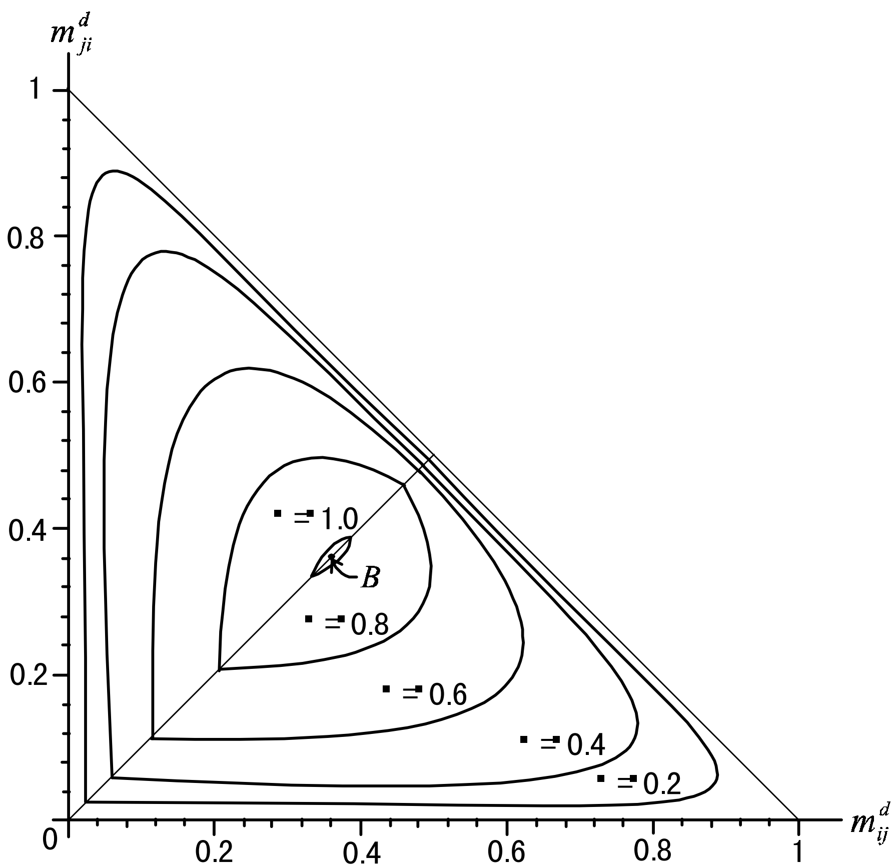

Figure 2: The set M under various values of α, and the bliss point B.

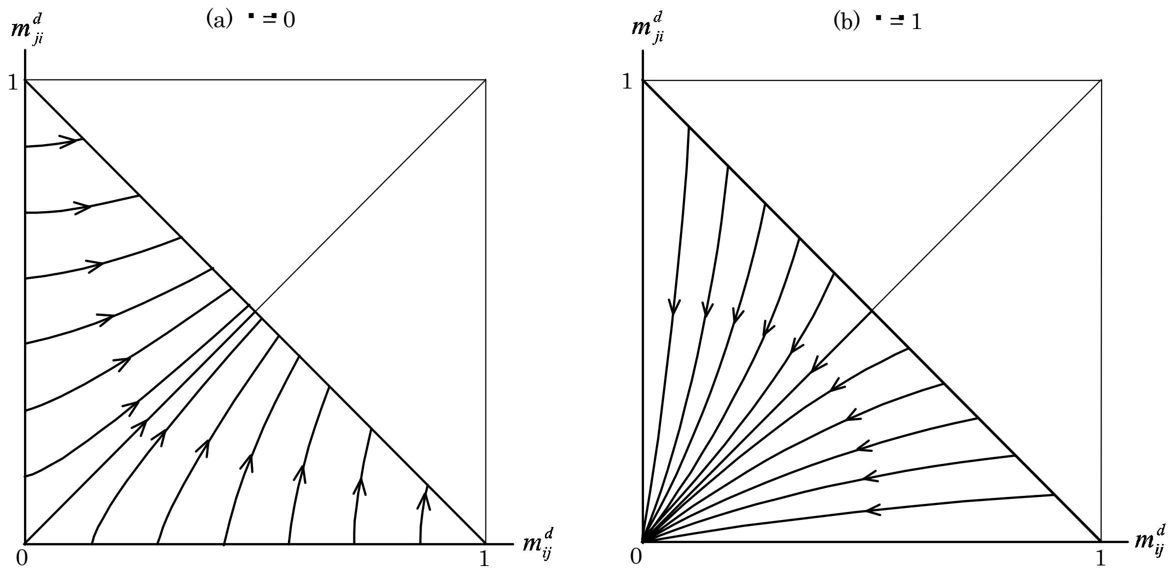

Figure 3: Dynamics under fixed value of δ.

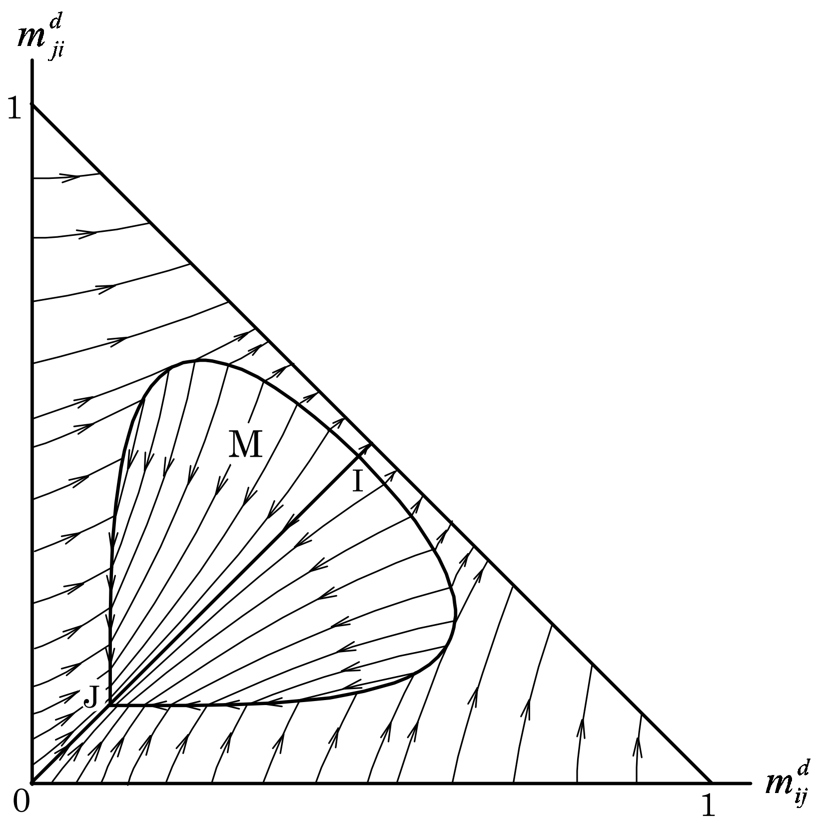

Figure 4: Dynamics with δ endogenous.

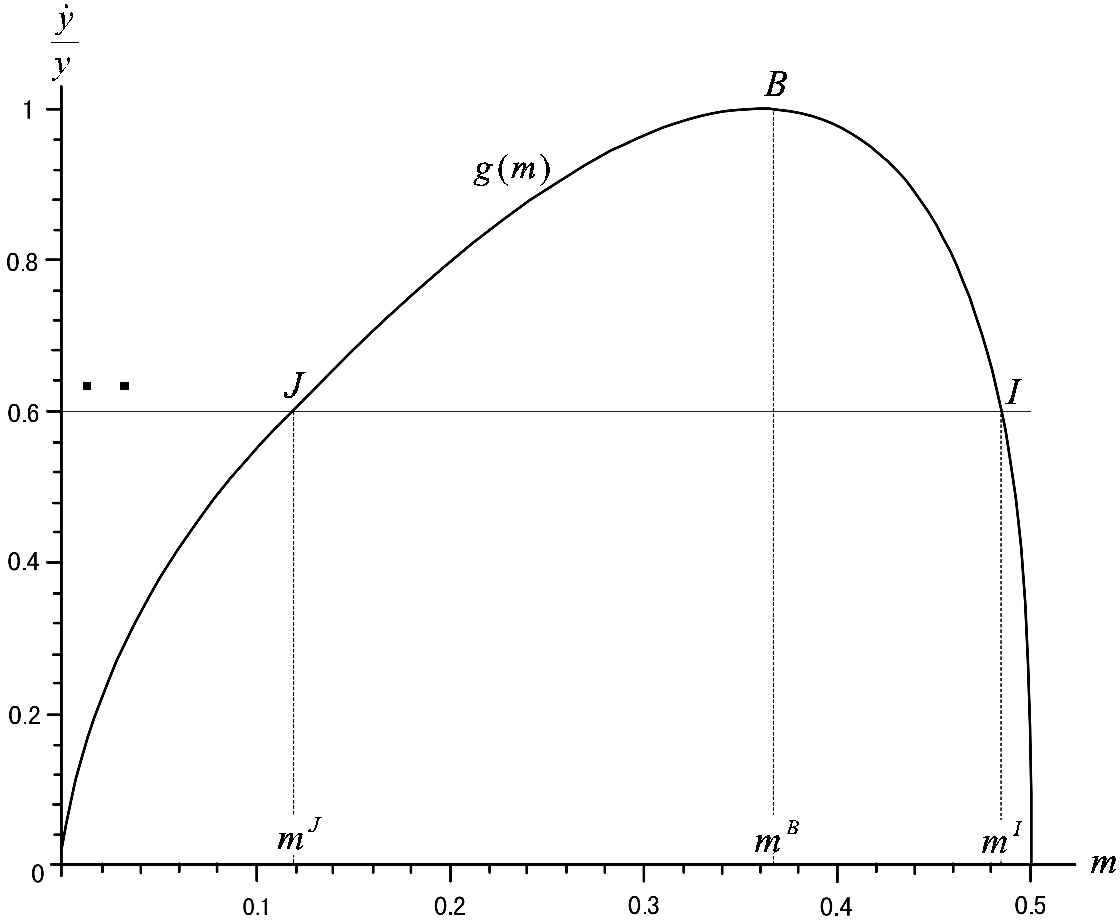

Figure 5: Efficiency and the bliss point

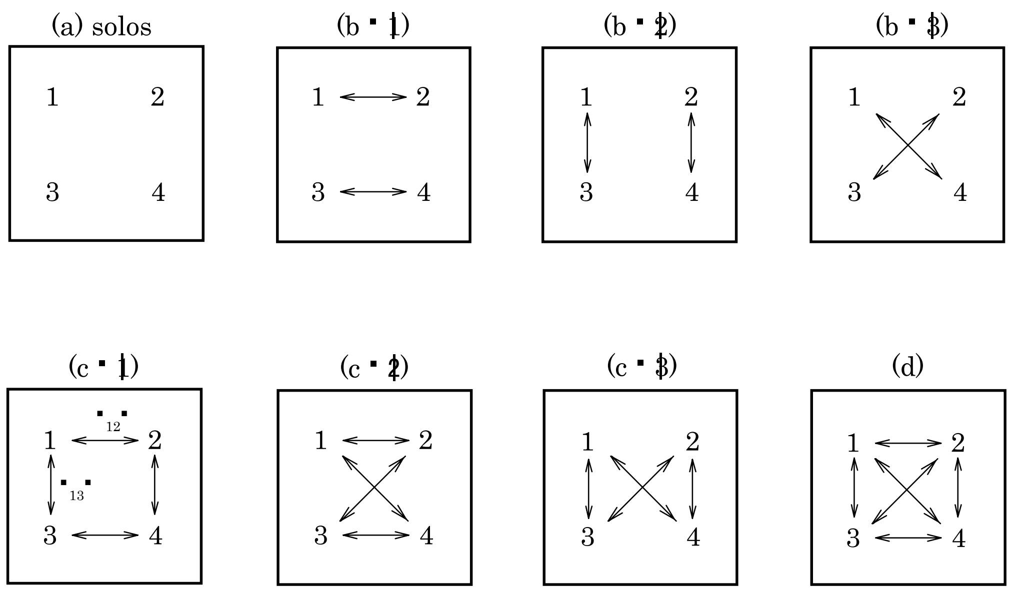

Figure 6: Possible equilibrium configurations

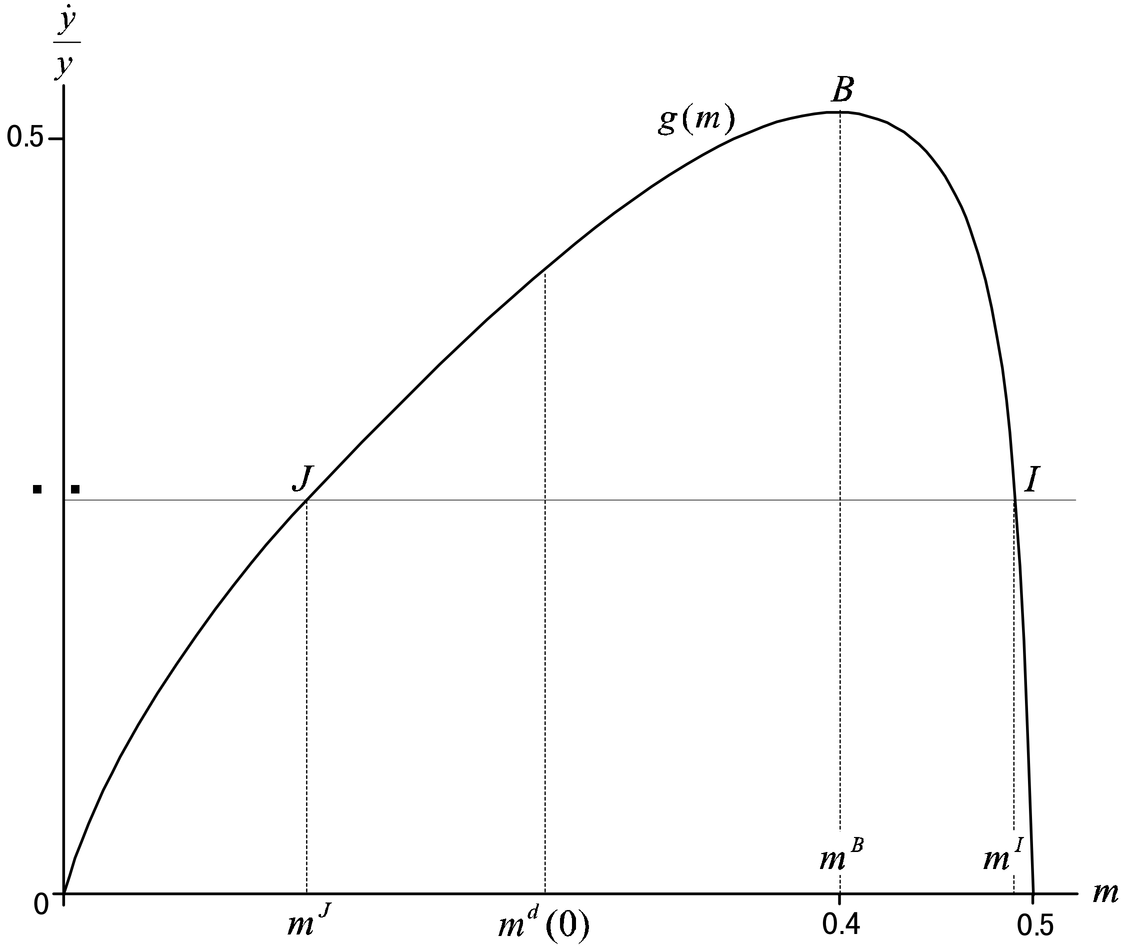

Figure 7: The $g(m)$ curve and the bliss point when $\gamma = 0$ and $\beta = 1$.

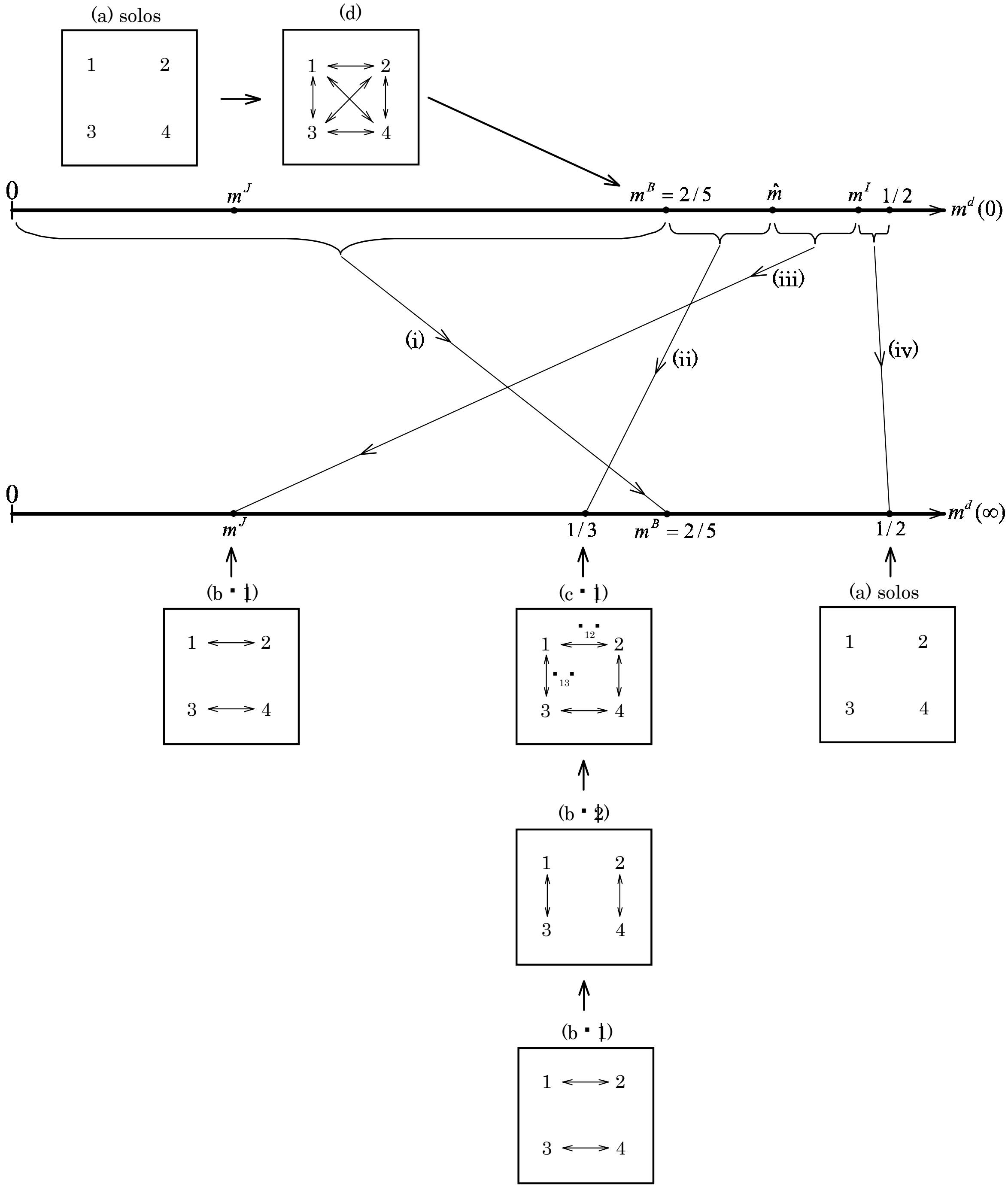

Figure 8: Correspondence between the initial point $m^d(0)$ and the long-run equilibrium point $m^d(\infty)$.

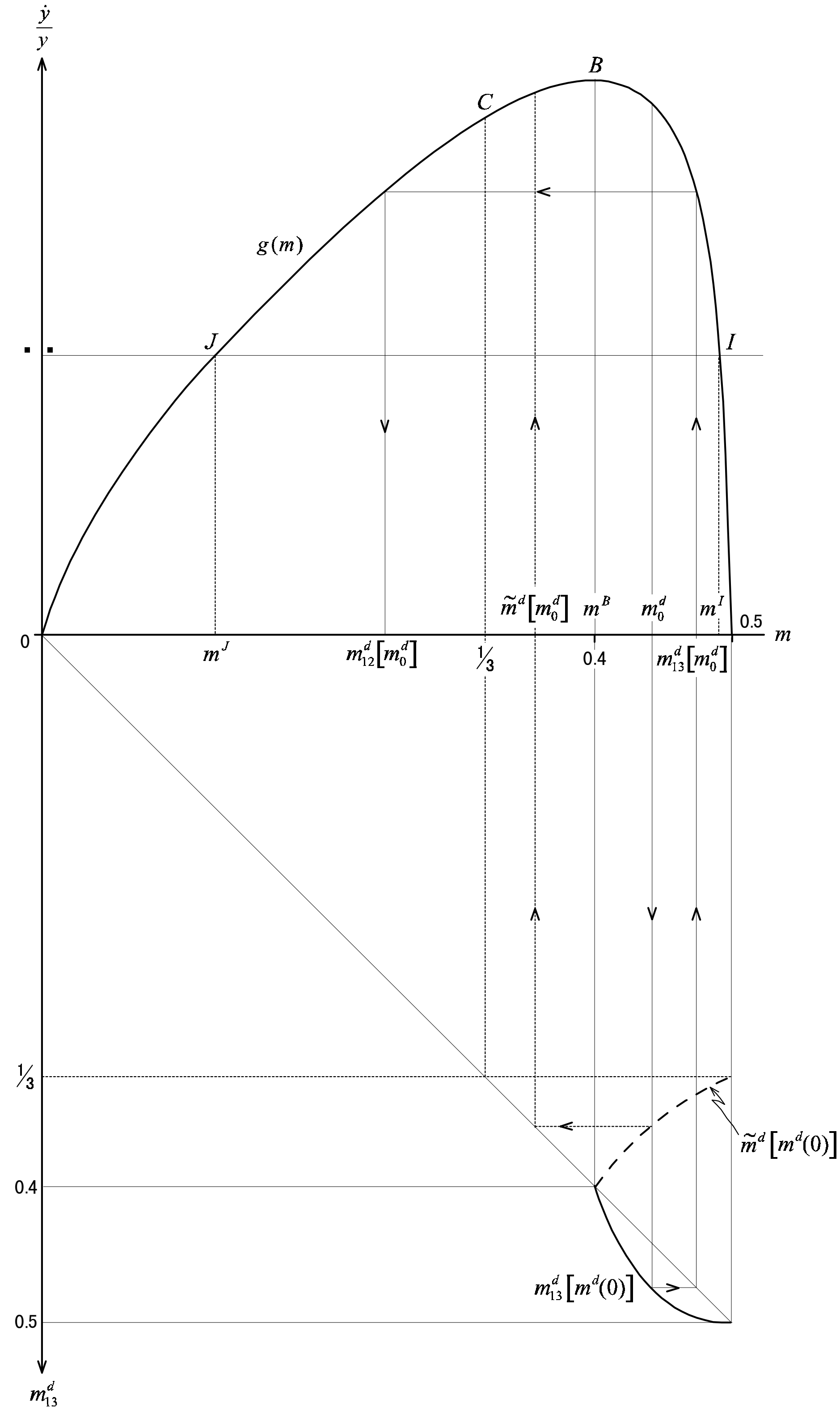

Figure 9: (a) Real lines with arrows: the $m_{13}^d\left[m^d(0)\right]$ curve and the determination of the switching positions $m_{13}^d\left[m_0^d\right]$ and $m_{12}^d\left[m_0^d\right]$. (b) Broken lines with arrows: the $\tilde{m}^d\left[m^d(0)\right]$ curve and the switching position $\tilde{m}^d\left[m_0^d\right]$.

10 Technical Appendix

10.1 Appendix a

Theorem A1: *Knowledge dynamics evolve according to the system:*

$$\begin{aligned}
\dot{m}_{ij}^d &= [1-\delta_{ij}]\cdot\alpha\cdot\{(1-m_{ij}^d)(1-m_{ij}^d-m_{ji}^d)\} \\
&\quad -\delta_{ij}\cdot\{\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}+m_{ij}^d\cdot\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}\} \\
\dot{m}_{ji}^d &= [1-\delta_{ji}]\cdot\alpha\cdot\{(1-m_{ji}^d)(1-m_{ij}^d-m_{ji}^d)\} \\
&\quad -\delta_{ji}\cdot\{\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}+m_{ji}^d\cdot\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}\}
\end{aligned}$$

Proof of Theorem A1: Dividing the system by n yields

$$\begin{aligned}
\frac{\dot{y}_i}{n} &= \frac{\dot{n}_i}{n}=[1-\delta_{ij}]\cdot\alpha\cdot m_i+ \\
&\quad \delta_{ij}\cdot(\beta\cdot[m^c\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot m^c]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ij}^c}{n} &= \delta_{ij}\cdot(\beta\cdot[m^c\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot m^c]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ij}^d\cdot m^c]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ij}^d}{n} &= [1-\delta_{ij}]\cdot\alpha\cdot m_i-\delta_{ij}\cdot\gamma\cdot[m_{ij}^d\cdot m^c]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
\frac{\dot{y}_j}{n} &= \frac{\dot{n}_j}{n}=[1-\delta_{ji}]\cdot\alpha\cdot m_j+ \\
&\quad \delta_{ji}\cdot(\beta\cdot[m^c\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot m^c]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^c}{n} &= \delta_{ji}\cdot(\beta\cdot[m^c\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot m^c]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ji}^d\cdot m^c]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^d}{n} &= [1-\delta_{ji}]\cdot\alpha\cdot m_j-\delta_{ji}\cdot\gamma\cdot[m_{ji}^d\cdot m^c]^{\frac{1}{2}}
\end{aligned}$$

Substituting (12) for m^c,

$$\begin{aligned}
\frac{\dot{y}_i}{n} &= \frac{\dot{n}_i}{n}=[1-\delta_{ij}]\cdot\alpha\cdot m_i+ \\
&\quad \delta_{ij}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ij}^c}{n} &= \delta_{ij}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ij}^d}{n} &= [1-\delta_{ij}]\cdot\alpha\cdot m_i-\delta_{ij}\cdot\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\dot{y}_j}{n} &= \frac{\dot{n}_j}{n} = [1-\delta_{ji}]\cdot\alpha\cdot m_j + \\
&\quad \delta_{ji}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^c}{n} &= \delta_{ji}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^d}{n} &= [1-\delta_{ji}]\cdot\alpha\cdot m_j-\delta_{ji}\cdot\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}
\end{aligned}
$$

Recalling (3),

$$
\begin{aligned}
m_i &= m_{ij}^d+m^c = 1-m_{ji}^d \\
m_j &= m_{ji}^d+m^c = 1-m_{ij}^d
\end{aligned}
$$

so

$$
\begin{aligned}
\frac{\dot{y}_i}{n} &= \frac{\dot{n}_i}{n} = [1-\delta_{ij}]\cdot\alpha\cdot(1-m_{ji}^d) + \\
&\quad \delta_{ij}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ij}^c}{n} &= \delta_{ij}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ij}^d\cdot m_{ji}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}})) \\
\frac{\dot{n}_{ij}^d}{n} &= [1-\delta_{ij}]\cdot\alpha\cdot(1-m_{ji}^d)-\delta_{ij}\cdot\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\dot{y}_j}{n} &= \frac{\dot{n}_j}{n} = [1-\delta_{ji}]\cdot\alpha\cdot(1-m_{ij}^d) + \\
&\quad \delta_{ji}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^c}{n} &= \delta_{ji}\cdot(\beta\cdot[(1-m_{ij}^d-m_{ji}^d)\cdot m_{ji}^d\cdot m_{ij}^d]^{\frac{1}{3}}+\gamma\cdot[m_{ij}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}} \\
&\quad +\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}) \\
\frac{\dot{n}_{ji}^d}{n} &= [1-\delta_{ji}]\cdot\alpha\cdot(1-m_{ij}^d)-\delta_{ji}\cdot\gamma\cdot[m_{ji}^d\cdot(1-m_{ij}^d-m_{ji}^d)]^{\frac{1}{2}}
\end{aligned}
$$

Now

$$
\begin{aligned}
\dot{m}_{ij}^d &= \frac{d(n_{ij}^d/n)}{dt} \\
&= \frac{\dot{n}_{ij}^d}{n} - \frac{n_{ij}^d \cdot \dot{n}}{n^2} \\
&= \frac{\dot{n}_{ij}^d}{n} - \frac{n_{ij}^d}{n} \cdot \frac{\dot{n}}{n} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ji}^d) - \delta_{ij} \cdot \gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} - m_{ij}^d \cdot (\frac{\dot{n}_{ij}^d}{n} + \frac{\dot{n}_{ji}^d}{n} + \frac{\dot{n}_{ij}^c}{n}) \\
&= [1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ji}^d) - \delta_{ij} \cdot \gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} \\
&\quad -m_{ij}^d \cdot \{[1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ji}^d) - \delta_{ij} \cdot \gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + [1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ij}^d) \\
&\quad -\delta_{ij} \cdot \gamma \cdot [m_{ji}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + \delta_{ij} \cdot (\beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}} \\
&\quad +\gamma \cdot [m_{ji}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + \gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}})\} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ji}^d) - \delta_{ij} \cdot \gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} \\
&\quad -m_{ij}^d \cdot \{[1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ji}^d) + [1-\delta_{ij}] \cdot \alpha \cdot (1-m_{ij}^d) \\
&\quad +\delta_{ij} \cdot \beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}\} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot \{1 - m_{ji}^d - 2m_{ij}^d + (m_{ij}^d)^2 + m_{ij}^d \cdot m_{ji}^d\} \\
&\quad -\delta_{ij} \cdot \{\gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}\} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot \{(1-m_{ij}^d)(1-m_{ji}^d) - m_{ij}^d + (m_{ij}^d)^2\} \\
&\quad -\delta_{ij} \cdot \{\gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}\} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot \{(1-m_{ij}^d)(1-m_{ji}^d) - m_{ij}^d(1-m_{ij}^d)\} \\
&\quad -\delta_{ij} \cdot \{\gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}\} \\
&= [1-\delta_{ij}] \cdot \alpha \cdot \{(1-m_{ij}^d)(1-m_{ji}^d - m_{ij}^d)\} \\
&\quad -\delta_{ij} \cdot \{\gamma \cdot [m_{ij}^d \cdot (1-m_{ij}^d - m_{ji}^d)]^{\frac{1}{2}} + m_{ij}^d \cdot \beta \cdot [(1-m_{ij}^d - m_{ji}^d) \cdot m_{ij}^d \cdot m_{ji}^d]^{\frac{1}{3}}\}
\end{aligned}
$$

The fourth line follows from (11), that implies

$$\frac{\dot{n}}{n} = \frac{\dot{n}_{ij}^d}{n} + \frac{\dot{n}_{ji}^d}{n} + \frac{\dot{n}_{ij}^c}{n} \tag{111}$$

Symmetric calculations hold for $\dot{m}_{ji}^d$. ■

10.2 Appendix b

Theorem A2: *Suppose that* $(m_{ij}^d, m_{ji}^d) \in M$. *Then* $(m_{ji}^d, m_{ij}^d) \in M$ *and the line segment* $[(m_{ij}^d, m_{ji}^d), (m_{ji}^d, m_{ij}^d)] \subseteq M$. *In particular, if* $M \neq \emptyset$, *then it contains a point on the diagonal segment* $[(0,0),(1,1)]$. *Moreover, the diagonal intersected with* M *is a convex set. In fact, every line parallel to the diagonal* *intersected with M is a convex set. Finally, every point in $M \cap ((0,0),(1,1))$ has a neighborhood contained in M.*

To prove Theorem A2, we proceed with a sequence of lemmata. First we need some definitions to make notation easier.

Definitions:

$$\begin{aligned} f(m,m') &= \beta \cdot [(1-m-m') \cdot m \cdot m']^{\frac{1}{3}} \\ h(m,m') &= \gamma \cdot [(1-m-m') \cdot m']^{\frac{1}{2}} \end{aligned}$$

With these definitions, the equations defining M_i (15) and M_j (16) become:

$$f(m_{ij}^d, m_{ji}^d) + h(m_{ij}^d, m_{ji}^d) - \alpha \cdot (1 - m_{ji}^d) > 0 \tag{112}$$

$$f(m_{ji}^d, m_{ij}^d) + h(m_{ji}^d, m_{ij}^d) - \alpha \cdot (1 - m_{ij}^d) > 0 \tag{113}$$

Lemma A1: *$(m_{ij}^d, m_{ji}^d) \in M_i$ and $m_{ij}^d \geq m_{ji}^d$ imply $(m_{ji}^d, m_{ij}^d) \in M_i$. $(m_{ij}^d, m_{ji}^d) \in M_j$ and $m_{ij}^d \leq m_{ji}^d$ imply $(m_{ji}^d, m_{ij}^d) \in M_j$.*

Proof of Lemma A1: $f(m_{ij}^d, m_{ji}^d) = f(m_{ji}^d, m_{ij}^d)$. $\frac{h(m_{ji}^d, m_{ij}^d)}{h(m_{ij}^d, m_{ji}^d)} = [\frac{m_{ij}^d}{m_{ji}^d}]^{\frac{1}{2}} \geq 1$, since $m_{ij}^d \geq m_{ji}^d$. $(m_{ij}^d, m_{ji}^d) \in M_i$ implies $f(m_{ij}^d, m_{ji}^d) + h(m_{ij}^d, m_{ji}^d) - \alpha(1 - m_{ji}^d) > 0$. Since $h(m_{ji}^d, m_{ij}^d) \geq h(m_{ij}^d, m_{ji}^d)$ and $m_{ij}^d \geq m_{ji}^d$, $f(m_{ji}^d, m_{ij}^d) + h(m_{ji}^d, m_{ij}^d) - \alpha(1 - m_{ij}^d) > 0$. Hence, $(m_{ji}^d, m_{ij}^d) \in M_i$. A symmetric argument works for the second part of the lemma. ■

Lemma A2: *Suppose that $m_{ij}^d \geq m_{ji}^d$. Then $(m_{ij}^d, m_{ji}^d) \in M$ if and only if $(m_{ij}^d, m_{ji}^d) \in M_i$.*

Proof of Lemma A2: It is obvious that $(m_{ij}^d, m_{ji}^d) \in M$ implies $(m_{ij}^d, m_{ji}^d) \in M_i$. So suppose that $(m_{ij}^d, m_{ji}^d) \in M_i$. Then by symmetry of the definitions of M_i and M_j, $(m_{ji}^d, m_{ij}^d) \in M_j$. By Lemma A1, $(m_{ji}^d, m_{ij}^d) \in M_i$. Applying symmetry of the definitions again yields $(m_{ij}^d, m_{ji}^d) \in M_j$. Hence $(m_{ij}^d, m_{ji}^d) \in M_j \cap M_i = M$. ■

Lemma A3: *Suppose that $(m_{ij}^d, m_{ji}^d) \in M$. Then $(m_{ji}^d, m_{ij}^d) \in M$ and the line segment $[(m_{ij}^d, m_{ji}^d), (m_{ji}^d, m_{ij}^d)] \subseteq M$. In particular, if $M \neq \emptyset$, then it contains a point on the diagonal segment $[(0,0),(1,1)]$.*

Proof of Lemma A3: First, if $(m_{ij}^d, m_{ji}^d) \in M$, then $(m_{ji}^d, m_{ij}^d) \in M$ by symmetry of the definitions of M_i and M_j. Now consider the line segment $[(m_{ij}^d, m_{ji}^d), (m_{ji}^d, m_{ij}^d)]$. In particular, consider the case $m_{ij}^d \geq m_{ji}^d$ and the line segment between (m_{ij}^d, m_{ji}^d) and the point (m, m) on the diagonal, $[(m_{ij}^d, m_{ji}^d), (m, m)] \subseteq [(m_{ij}^d, m_{ji}^d), (m_{ji}^d, m_{ij}^d)]$ (the line segment $[(m, m), (m_{ji}^d, m_{ij}^d)]$ can be covered with a symmetric argument). Since for all $(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) \in [(m_{ij}^d, m_{ji}^d), (m, m)]$,

$\widehat{m}_{ij}^d \geq \widehat{m}_{ji}^d$, by Lemma A2 it suffices to show that $(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) \in M_i$. We must verify the equation stating that $(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) \in M_i$, namely

$$f(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) + h(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) - \alpha \cdot (1 - \widehat{m}_{ji}^d) > 0 \tag{114}$$

Now for all $(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) \in [(m_{ij}^d, m_{ji}^d), (m, m)]$, there exists an $x \geq 0$ with $\widehat{m}_{ij}^d = m_{ij}^d - x \geq m_{ji}^d + x = \widehat{m}_{ji}^d$, since the line segment lies below the diagonal. Now

$$\begin{aligned} & f(m_{ij}^d - x, m_{ji}^d + x) - f(m_{ij}^d, m_{ji}^d) \\ = \ & \beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d - x) \cdot (m_{ji}^d + x)]^{\frac{1}{3}} - \beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d) \cdot (m_{ji}^d)]^{\frac{1}{3}} \\ = \ & \beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d) \cdot (m_{ji}^d) + (1 - m_{ji}^d - m_{ij}^d) \cdot x \cdot (m_{ij}^d - m_{ji}^d - x)]^{\frac{1}{3}} \\ & -\beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d) \cdot (m_{ji}^d)]^{\frac{1}{3}} \\ \geq \ & \beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d) \cdot (m_{ji}^d) + (1 - m_{ji}^d - m_{ij}^d) \cdot x^2]^{\frac{1}{3}} \\ & -\beta \cdot [(1 - m_{ji}^d - m_{ij}^d) \cdot (m_{ij}^d) \cdot (m_{ji}^d)]^{\frac{1}{3}} \\ \geq \ & 0 \end{aligned}$$

$$\begin{aligned} & h(m_{ij}^d - x, m_{ji}^d + x) - h(m_{ij}^d, m_{ji}^d) \\ = \ & \gamma \cdot [(m_{ji}^d + x) \cdot (1 - m_{ji}^d - m_{ij}^d)]^{\frac{1}{2}} - \gamma \cdot [m_{ji}^d \cdot (1 - m_{ji}^d - m_{ij}^d)]^{\frac{1}{2}} \geq 0 \end{aligned}$$

Finally,

$$\alpha \cdot (1 - m_{ji}^d - x) \leq \alpha \cdot (1 - m_{ji}^d)$$

Hence,

$$\begin{aligned} & f(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) + h(\widehat{m}_{ij}^d, \widehat{m}_{ji}^d) - \alpha \cdot (1 - \widehat{m}_{ji}^d) \\ = \ & f(m_{ij}^d - x, m_{ji}^d + x) + h(m_{ij}^d - x, m_{ji}^d + x) - \alpha \cdot (1 - m_{ji}^d - x) \\ \geq \ & f(m_{ij}^d, m_{ji}^d) + h(m_{ij}^d, m_{ji}^d) - \alpha \cdot (1 - m_{ji}^d) > 0 \end{aligned}$$

The last line follows because $(m_{ij}^d, m_{ji}^d) \in M$. ■

Lemma A4: For any constant $a \in (-1, 1)$ the intersection of the set M and the line $\{(m_{ij}^d, m_{ji}^d) \in \mathbb{R}_+^2 \mid m_{ij}^d + m_{ji}^d \leq 1,\ m_{ji}^d = m_{ij}^d - a\}$ is a convex set.

Proof of Lemma A4: Since M is symmetric with respect to the diagonal $m_{ij}^d = m_{ji}^d$, let us consider $a \geq 0$. Setting $m_{ji}^d = m_{ij}^d - a$ in (14), define

$$\begin{aligned} k(m_{ij}^d) \ &\equiv \ F_i(m_{ij}^d, m_{ij}^d - a) \\ &= \ \beta \left[(1 + a - 2m_{ij}^d) m_{ij}^d (m_{ij}^d - a)\right]^{1/3} \\ &\quad + \gamma \left[(1 + a - 2m_{ij}^d)(m_{ij}^d - a)\right]^{1/2} - \alpha(1 + a - m_{ij}^d) \end{aligned}$$

Since $m_{ji}^d = m_{ij}^d - a \geq 0$ and $1 \geq m_{ij}^d + m_{ji}^d = 2m_{ij}^d - a$, the domain of the function k is

$$a \leq m_{ij}^d \leq \frac{1 + a}{2} \quad \text{where} \quad 0 \leq a < 1$$

By Lemma A2, the intersection of the set M and the line $m^d_{ji} = m^d_{ij} - a$ is the set of points satisfying

$$k(m^d_{ij}) > 0.$$

We show that function $k(m^d_{ij})$ is strictly concave on $(a, \frac{1+a}{2})$, and thus the set of points satisfying the inequality is convex. Differentiation of the function k yields

$$k'(m^d_{ij}) = A(m^d_{ij}) + B(m^d_{ij}) + \alpha$$

where

$$A(m^d_{ij}) \equiv \frac{\beta}{3}\left[(1+a-2m^d_{ij})m^d_{ij}(m^d_{ij}-a)\right]^{-2/3}\left[-6(m^d_{ij})^2 + 2m^d_{ij}(1+3a) - a(1+a)\right]$$

$$B(m^d_{ij}) \equiv \frac{\gamma}{2}\left[(1+a-2m^d_{ij})(m^d_{ij}-a)\right]^{-1/2}(1+3a-4m^d_{ij})$$

The second derivative of k is

$$k''(m^d_{ij}) = A'(m^d_{ij}) + B'(m^d_{ij})$$

where

$$\begin{aligned} A'(m^d_{ij}) &= -\frac{2\beta\left\{(m^d_{ij})^2(1+3a^2) - a(1+a)(1+3a)m^d_{ij} + a^2(1+a)^2\right\}}{9\left[(1+a-2m^d_{ij})m^d_{ij}(m^d_{ij}-a)\right]^{5/3}} \\ &= -\frac{2\beta\left\{\left[m^d_{ij}(1+3a^2) - \frac{a(1+a)(1+3a)}{2}\right]^2 + \frac{3a^2(1+a)^2(1-a)^2}{4}\right\}}{9\left[(1+a-2m^d_{ij})m^d_{ij}(m^d_{ij}-a)\right]^{5/3}(1+3a^2)} \end{aligned}$$

$$B'(m^d_{ij}) = -\frac{\gamma(1-a)^2}{4\left[(1+a-2m^d_{ij})(m^d_{ij}-a)\right]^{3/2}}$$

implying that $k''(m^d_{ij}) = A'(m^d_{ij}) + B'(m^d_{ij}) < 0$ on $(a, \frac{1+a}{2})$, so k is strictly concave on $(a, \frac{1+a}{2})$. Thus, $\{m^d_{ij} \in (a, \frac{1+a}{2}) \mid k(m^d_{ij}) > 0\}$ is convex, and the proof of the lemma is complete.■

Lemma A5: *Every point in $M \cap ((0,0),(1,1))$ has a neighborhood contained in M.*

Proof of Lemma A5: This follows directly from the definition of M; it implies that M is an open set.■

Theorem A2 follows directly from the combination of all of the Lemmata in this section.

10.3 Appendix c

Lemma A6: *The function* $g(m)$ *defined by (20) has the following properties:*

(i) $g(m)$ *is strictly quasi-concave on* $\left[0, \frac{1}{2}\right]$.

(ii) $g(m)$ *achieves its maximal value at* $m^B \in [\frac{1}{3}, \frac{2}{5}]$.

(iii) The point (m^B, m^B) *corresponds to the bliss point* B *in Figure 2, which is the unique point contained in every* M *that is nonempty.*

Proof of Lemma A6: (i) and (ii): For $m \in \left[0, \frac{1}{2}\right]$, let

$$x(m) \equiv \frac{m}{1-m} \text{ or } m(x) = \frac{x}{1+x}$$

and define

$$G(x) \equiv \beta \left[(1-x)x^2\right]^{1/3} + \gamma \left[(1-x)x\right]^{1/2} \text{ for } x \in [0,1] \qquad (115)$$

Then, using definition (20)

$$g(m) = G(x(m))$$

Hence,

$$g'(m) = G'(x(m)) \cdot x'(m)$$

Notice that

$$x'(m) = 1 + \frac{m}{(1-m)^2} > 0$$

so

$$g'(m) \gtreqless 0 \text{ exactly as } G'(x(m)) \gtreqless 0.$$

Now

$$G'(x) = C(x) + D(x)$$

where

$$C(x) \equiv \tfrac{\beta}{3}\left[(1-x)x^2\right]^{-2/3}(2-3x)x$$
$$D(x) \equiv \tfrac{\gamma}{2}\left[(1-x)x\right]^{-1/2}(1-2x)$$

Taking the derivatives of C and D respectively yields

$$C'(x) = -\tfrac{2\beta}{9}(1-x)^{-5/3}x^{-4/3} < 0$$
$$D'(x) = -\tfrac{\gamma}{4}(1-x)^{-3/2}x^{-3/2} < 0$$

Therefore, considering that

$$C(x) \gtreqless 0 \text{ as } x \lesseqgtr 2/3$$
$$D(x) \gtreqless 0 \text{ as } x \lesseqgtr 1/2$$

we can conclude that there exists a unique $x^* \in [1/2, 2/3]$ such that

$$G'(x) \gtreqless 0 \text{ as } x \lesseqgtr x^*$$

meaning that G is strictly single peaked and strictly quasi-concave, achieving its maximum value exactly at x^*. Hence, the function $g(m)$ also is strictly single peaked and strictly quasi-concave, achieving its maximum value at

$$m^B \equiv m(x^*) = \frac{x^*}{1+x^*} \in [1/3, 2/5]$$

(iii) To show that the point (m^B, m^B) corresponds to the bliss point B in Figure 2, let us recall how the bliss point has been defined. Let $M(\alpha)$ be the set M under the parameter value $\alpha > 0$. Then, a point $({}^*m^d_{ij}, {}^*m^d_{ji}) \in \mathbb{R}^2$ is called a bliss point if it holds that for any $\alpha > 0$,

$$M(\alpha) \neq \emptyset \Longrightarrow ({}^*m^d_{ij}, {}^*m^d_{ji}) \in M(\alpha) \tag{116}$$

To show the existence and the uniqueness of such a point, since $M(\alpha)$ is symmetric to the diagonal, let us focus on the lower half of $M(\alpha)$, and define

$$M^L(\alpha) = \left\{(m^d_{ij}, m^d_{ji}) \in M(\alpha) \mid m^d_{ij} \geq m^d_{ji}\right\}$$

Then, by Lemma A2, $M^L(\alpha)$ coincides with the lower part of M_i associated with α:

$$\begin{aligned} M^L(\alpha) &= \left\{(m^d_{ij}, m^d_{ji}) \in M_i(\alpha) \mid m^d_{ij} \geq m^d_{ji}\right\} \\ &= \{(m^d_{ij}, m^d_{ji}) \in \mathbb{R}^2 \mid m^d_{ij} \geq m^d_{ji} \geq 0, m^d_{ij} + m^d_{ji} \leq 1, \\ &\quad f(m^d_{ij}, m^d_{ji}) + h(m^d_{ij}, m^d_{ji}) - \alpha(1 - m^d_{ji}) > 0\} \end{aligned}$$

When $m^d_{ij} + m^d_{ji} = 1$ or $m^d_{ji} = 0$, we have $f(m^d_{ij}, m^d_{ji}) = h(m^d_{ij}, m^d_{ji}) = 0$, implying that $M^L(\alpha)$ does not contain any point (m^d_{ij}, m^d_{ji}) such that $m^d_{ij} + m^d_{ji} = 1$ or $m^d_{ji} = 0$. Thus, we can rewrite $M^L(\alpha)$ as follows:

$$\begin{aligned} M^L(\alpha) &= \left\{(m^d_{ij}, m^d_{ji}) \in \mathbb{R}^2 \mid m^d_{ij} \geq m^d_{ji} > 0, m^d_{ij} + m^d_{ji} < 1, \tfrac{f(m^d_{ij}, m^d_{ji})}{1-m^d_{ji}} + \tfrac{h(m^d_{ij}, m^d_{ji})}{1-m^d_{ji}} > \alpha\right\} \\ &= \{(m^d_{ij}, m^d_{ji}) \in \mathbb{R}^2 \mid m^d_{ij} \geq m^d_{ji} > 0, m^d_{ij} + m^d_{ji} < 1, \\ &\quad \beta\left[\left(1 - \tfrac{m^d_{ij}}{1-m^d_{ji}}\right)\tfrac{m^d_{ij}}{1-m^d_{ji}}\tfrac{m^d_{ji}}{1-m^d_{ji}}\right]^{1/3} + \gamma\left[\left(1 - \tfrac{m^d_{ij}}{1-m^d_{ji}}\right)\tfrac{m^d_{ji}}{1-m^d_{ji}}\right]^{1/2} > \alpha\} \end{aligned} \tag{117}$$

Given any $(m^d_{ij}, m^d_{ji}) \in M^L(\alpha)$ such that $m^d_{ij} > m^d_{ji}$, define

$$m \equiv \frac{m^d_{ij} + m^d_{ji}}{2}$$

Then, $m^d_{ij} > m > m^d_{ji}$, and $(m,m) \in M^L(\alpha)$ by Lemma A3. Furthermore,

$$\begin{aligned}
& \left(1-\frac{m}{1-m}\right)\left(\frac{m}{1-m}\right)^2 - \left(1-\frac{m^d_{ij}}{1-m^d_{ji}}\right)\frac{m^d_{ij}}{1-m^d_{ji}}\frac{m^d_{ji}}{1-m^d_{ji}} \\
= & \frac{(1-m^d_{ij}-m^d_{ji})m^2}{(1-m)^3} - \frac{(1-m^d_{ij}-m^d_{ji})m^d_{ij}m^d_{ji}}{(1-m^d_{ji})^3} \\
> & \frac{(1-m^d_{ij}-m^d_{ji})}{(1-m^d_{ji})^3}(m^2 - m^d_{ij}m^d_{ji}) \\
= & \frac{(1-m^d_{ij}-m^d_{ji})}{(1-m^d_{ji})^3}\frac{(m^d_{ij}-m^d_{ji})^2}{4} > 0
\end{aligned}$$

Likewise,

$$\begin{aligned}
& \left(1-\frac{m}{1-m}\right)\frac{m}{1-m} - \left(1-\frac{m^d_{ij}}{1-m^d_{ji}}\right)\frac{m^d_{ji}}{1-m^d_{ji}} \\
= & \frac{(1-m^d_{ij}-m^d_{ji})m}{(1-m)^2} - \frac{(1-m^d_{ij}-m^d_{ji})m^d_{ji}}{(1-m^d_{ji})^2} \\
> & \frac{(1-m^d_{ij}-m^d_{ji})}{(1-m^d_{ji})^2}(m - m^d_{ji}) > 0
\end{aligned}$$

Therefore, using the function $g(m)$ defined by (20), we can conclude that when $m^d_{ij} > m^d_{ji}$ and $m \equiv (m^d_{ij}+m^d_{ji})/2$,

$$g(m) > \beta\left[\left(1-\frac{m^d_{ij}}{1-m^d_{ji}}\right)\frac{m^d_{ij}}{1-m^d_{ji}}\frac{m^d_{ji}}{1-m^d_{ji}}\right]^{1/3} + \gamma\left[\left(1-\frac{m^d_{ij}}{1-m^d_{ji}}\right)\frac{m^d_{ji}}{1-m^d_{ji}}\right]^{1/2} \quad (118)$$

Moreover, (i) and (ii) of Lemma A6 mean that

$$g(m^b) > g(m) \text{ for any } m \neq m^b \quad (119)$$

Combining (117), (118) and (119), we can conclude that given any (m^d_{ij}, m^d_{ji}) such that $m^d_{ij} \geq m^d_{ji}$

$$(m^d_{ij}, m^d_{ji}) \in M^L(\alpha) \Longrightarrow (m^B, m^B) \in M^L(\alpha).$$

That is,

$$M^L(\alpha) \neq \emptyset \Longrightarrow (m^B, m^B) \in M^L(\alpha) \quad (120)$$

Hence, the point (m^B, m^B) is a bliss point. Finally, to show that the bliss point is unique, take any $\overline{\alpha} > 0$ such that $M^L(\overline{\alpha}) \neq \emptyset$, and take any $(\overline{m}^d_{ij}, \overline{m}^d_{ji}) \in M^L(\overline{\alpha})$ such that $(\overline{m}^d_{ij}, \overline{m}^d_{ji}) \neq (m^B, m^B)$. If $\overline{m}^d_{ij} > \overline{m}^d_{ji}$, then the inequality

(118) holds when (m_{ij}^d, m_{ji}^d) is replaced with $(\overline{m}_{ij}^d, \overline{m}_{ji}^d)$. If $\overline{m}_{ij}^d = \overline{m}_{ji}^d$, then $g(m^B) > g(\overline{m}_{ij}^d)$ by (119). Hence, if we define

$$\varepsilon \equiv g(m^B) - \left\{ \beta \left[\left(1 - \frac{\overline{m}_{ij}^d}{1-\overline{m}_{ji}^d} \right) \frac{\overline{m}_{ij}^d}{1-\overline{m}_{ji}^d} \frac{\overline{m}_{ji}^d}{1-\overline{m}_{ji}^d} \right]^{1/3} + \gamma \left[\left(1 - \frac{\overline{m}_{ij}^d}{1-\overline{m}_{ji}^d} \right) \frac{\overline{m}_{ji}^d}{1-\overline{m}_{ji}^d} \right]^{1/2} \right\}$$

then ε is positive. Replacing α with $g(m^B) - \frac{\varepsilon}{2}$ and (m_{ij}^d, m_{ji}^d) with $(\overline{m}_{ij}^d, \overline{m}_{ji}^d)$ in (117), we can see that

$$(\overline{m}_{ij}^d, \overline{m}_{ji}^d) \notin M^L \left(g(m^B) - \frac{\varepsilon}{2} \right)$$

whereas $(m^B, m^B) \in M^L \left(g(m^B) - \frac{\varepsilon}{2} \right)$. Thus, the point $(\overline{m}_{ij}^d, \overline{m}_{ji}^d)$ is not contained in the nonempty set $M^L \left(g(m^B) - \frac{\varepsilon}{2} \right)$, implying that the point $(\overline{m}_{ij}^d, \overline{m}_{ji}^d) \neq (m^B, m^B)$ is not a bliss point.■

10.4 Appendix d

First, notice that any feasible path $\delta_{ij}(\cdot)$ satisfies the following conditions:

$$\delta_{ij}(t) = \delta_{ji}(t) \text{ for all } t \text{ and for all } i \neq j.$$

Focus on person i; the equations for the other persons are analogous. For the remainder of this section, restrict attention to the situation when $\max_{j \neq i} g(m_{ij}^d(t)) > \alpha$. Then, for each i, any myopic core path or any solution to the planner's optimization problem requires that

$$\sum_{j \neq i} \delta_{ij}(t) = 1.$$

Using these two conditions, we can easily obtain:

$$\begin{aligned} \delta_{14}(t) &= \delta_{23}(t) = 1 - \delta_{12}(t) - \delta_{13}(t), \\ \delta_{24}(t) &= \delta_{13}(t),\ \delta_{34}(t) = \delta_{12}(t). \end{aligned}$$

In other words, the positions of all four persons can be exchanged. In the following, we focus on person 1, without loss of generality. Using equation (29), and analogous to the derivation of (37), omitting t we obtain

$$\frac{\dot{y}_1}{y_1} = \delta_{12} \cdot g(m_{12}^d) + \delta_{13} \cdot g(m_{13}^d) + \delta_{14} \cdot g(m_{14}^d) \tag{121}$$

Of course, for a path to be undominated in any interval of time, $\dot{y}_1/y_1$, or equivalently $\dot{y}_1$, must be maximal at any t (over δ_{12} and δ_{13}). So we examine the first order condition for maximizing $\dot{y}_1/y_1$. But when more than one combination of δ_{12} and δ_{13} maximize (121), then the second order conditions must be examined. To state the second order condition we calculate

$$\begin{aligned}\frac{d(\frac{\dot{y}_1}{y_1})}{dt} &= \dot{\delta}_{12}\cdot g(m_{12}^d)+\dot{\delta}_{13}\cdot g(m_{13}^d)+\dot{\delta}_{14}\cdot g(m_{14}^d)\\ &\quad +\delta_{12}\cdot g'(m_{12}^d)\cdot \dot{m}_{12}^d+\delta_{13}\cdot g'(m_{13}^d)\cdot \dot{m}_{13}^d+\delta_{14}\cdot g'(m_{14}^d)\cdot \dot{m}_{14}^d\end{aligned}$$

Suppose that $m_{12}^d = m_{13}^d = m_{14}^d \equiv m^d$. We also know that except on a set of measure zero, $\dot{\delta}_{12}+\dot{\delta}_{13}+\dot{\delta}_{14}=0$. Thus,

$$\frac{d(\frac{\dot{y}_1}{y_1})}{dt} = g'(m^d)\{\delta_{12}\cdot \dot{m}_{12}^d+\delta_{13}\cdot \dot{m}_{13}^d+\delta_{14}\cdot \dot{m}_{14}^d\}$$

Similar to the derivation of (49), we calculate:

$$\begin{aligned}\dot{m}_{12} &= g(m^d)\cdot\{(1-2m^d)\cdot\delta_{13}+(1-2m^d)\cdot\delta_{14}-m^d\cdot\delta_{12}\}\\ \dot{m}_{13} &= g(m^d)\cdot\{(1-2m^d)\cdot\delta_{12}+(1-2m^d)\cdot\delta_{14}-m^d\cdot\delta_{13}\}\\ \dot{m}_{14} &= g(m^d)\cdot\{(1-2m^d)\cdot\delta_{12}+(1-2m^d)\cdot\delta_{13}-m^d\cdot\delta_{14}\}\end{aligned}$$

Defining

$$\begin{aligned}\Gamma(\delta_{12},\delta_{13}) &\equiv \frac{1}{g(m^d)}\cdot\{\delta_{12}\cdot \dot{m}_{12}^d+\delta_{13}\cdot \dot{m}_{13}^d+\delta_{14}\cdot \dot{m}_{14}^d\}\\ &= 2(1-2m^d)\cdot\{\delta_{12}+\delta_{13}-\delta_{12}\cdot\delta_{13}-(\delta_{12})^2-(\delta_{13})^2\}\\ &\quad -m^d\{(\delta_{12})^2+(\delta_{13})^2+(1-\delta_{12}-\delta_{13})^2\}\end{aligned}$$

we have

$$\frac{d(\frac{\dot{y}_1}{y_1})}{dt} = g'(m^d)\cdot g(m^d)\cdot\Gamma(\delta_{12},\delta_{13})$$

Recalling from footnote 17 that maximizing d^2y_1/dt^2 is the same as maximizing $d(\frac{\dot{y}_1}{y_1})/dt$. Thus, the second order conditions require that

(a) if $g'(m^d) > 0$, then $\Gamma(\delta_{12},\delta_{13})$ must be maximized.

(b) if $g'(m^d) < 0$, then $-\Gamma(\delta_{12},\delta_{13})$ must be maximized.

By differentiating Γ, it is easy to see that Γ is strictly concave under any fixed $m^d < 1/2$, achieving a unique maximum at

$$\delta_{12}=\delta_{13}=\delta_{14}=1/3$$

Turning next to Case (i), starting from a common initial condition $m^d(0) > m^J$, since $g'(m^d(0)) > 0$ we can readily see that the unique undominated path is $\delta_{ij}(t) = 1/3$ for all t and for all $j \neq i$, which is the same as the equilibrium path for Case (i).

For Case (ii), $g'(m^d(0)) < 0$, any undominated path must start with a corner solution. For example,

$$\delta_{12} = \delta_{21} = 1, \; \delta_{34} = \delta_{43} = 1.$$

This is identical to the first phase of the equilibrium path of Case (ii). The proofs that the last two phases of the equilibrium path for Case (ii) are undominated follow similar logic.