LM-TYPE TESTS FOR A UNIT ROOT ALLOWING FOR A BREAK IN TREND

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Abstract

We consider LM-type tests for a unit root allowing for a break in trend at an unknown date. In addition to the minimum LM test statistic, we propose new LM-type tests based on the least squares estimator of the break date under the null. We examine asymptotic behavior under the null hypothesis with and without a break. For all the endogenous break tests considered, the limiting distribution when there is a break in slope is not the same as when there is no break. Other authors have obtained similar results in the context of DF-type tests. Since this discrepancy is smaller for the LM-type based on the least squares estimator, smaller size distortions are to be expected when using this test statistic. Simulation experiments confirm the superiority in terms of size, power and break date estimation of the proposed method.

JEL Classification: C12, C22

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1 INTRODUCTION

Common procedures to test for the presence of a unit root are based on extensions of the statistical techniques proposed by Dickey and Fuller (1979). Following Perron (1989), an increased attention has been given to the possibility of the existence of a one-time change in the deterministic component of a time series. He shows that Dickey-Fuller (DF) type tests will have a tendency for not rejecting the null hypothesis of a unit root for series that are stationary around a breaking trend. To solve this problem, several authors have proposed tests for a unit root that allow for the presence of a break in the trend function at an unknown date. These tests are generally based on fitting DF-type regressions which include additional dummy variables capturing the change in the break function. Zivot and Andrews (1992) propose choosing the break date which minimizes the DF t-statistic across all possible regressions. Perron (1997) and Vogelsang and Perron (1998) also consider choosing the break date according to the significance of the trend-break dummy parameters. These authors further consider methods that allow for sudden breaks, or of the 'additive outlier' (AO) type, and breaks that evolve more slowly over time, or of the 'innovational outlier' (IO) type.

Vogelsang and Perron (1998) show that the distribution of DF-type unit root test statistics that allow for the presence of a break are asymptotically invariant to a break in the intercept under the null. However, Nunes, Newbold and Kuan (1997), Lee and Strazicich (2001) and Harvey, Leybourne and Newbold (2001) show that in finite samples this result may be illusory. When the break date is selected according to the least favorable DF t-statistic, a large break in the intercept under the null leads to strong spurious rejections of the unit root hypothesis. The same is true for the IO, but not AO, tests when the break date is based on the significance of the dummy variables.[1]

When there is a break in the slope under the null, Vogelsang and Perron (1998) show that the size of the minimum DF-type tests that allow for a change in slope will approach one asymptotically. In fact, they show that these size distortions can be explained by the wrong break date being selected.[2] For the tests based on the dummy variables significance,

[1] Lee and Strazicich (2001) and Harvey, Leybourne and Newbold (2001) show that this is due to incorrect choice of the break date. As a solution, Harvey, Leybourne and Newbold (2001) suggest moving the chosen break date one period ahead.

[2] Vogelsang and Perron (1998) show that in the IO case that allows for a break in slope only, even by choosing the true break date would not yield a valid test.

the same is true in the IO case, but not in the AO. In this last case, the estimated location of the break will approach the true one, so that the limiting distribution of these tests are equivalent to the case where the break date is known as in Perron (1989). However, this distribution differs from the case where no break is present under the null. If one chooses the suggestion in Vogelsang and Perron (1998) to use the critical values corresponding to the no break case then tests will be undersized if there is in fact a break.

Another approach to unit root testing based on the LM principle was proposed by Schmidt and Phillips (1992). As shown in Amsler and Lee (1995), the asymptotic distribution of the LM test for a unit root is invariant to a change in the intercept under the null.[3] These authors also propose a modification to the LM test that allows for a break in the intercept at some known date. They show that the limiting distribution of the test statistic under the null hypothesis of a unit root is the same as for the Schmidt and Phillips (1992) LM test where no break is considered. This equivalence holds irrespective of whether such break is present or not under the null.

In this paper we consider LM-type tests that allow for the presence of breaks in the intercept and slope at unknown dates. Lee and Strazicich (1999, 2002) propose estimating these dates by minimizing the LM test statistic over a range of possible break dates. Their tests are asymptotically invariant to a change in the intercept under the null. In fact, we further show that the null limiting distribution for the minimum LM test allowing for a break in the intercept is the same as for the Schmidt and Phillips (1992) LM test with no break. This is not true for a break in the slope. We show that in such a case, and unlike the minimum DF-type tests, the slope break date estimated by the minimum LM test converges to the true break date. Therefore, when a break in slope is present under the null, the asymptotic distribution of the endogenous break minimum LM test statistic is the same as the distribution of the corresponding exogenous break LM test. When no break in slope is present under the null, the distribution is different, which leads to the same dilemma regarding the choice of the appropriate critical value as in the DF-type tests.

We also propose additional LM-type tests for a unit root where the break date is chosen according to the regression that best fits the data under the null. This criteria coincides with choosing the break date that maximizes the dummy variables significance. We also consider the case where a one directional t-statistic is used when the direction of

[3]In fact, these authors also show that the same invariance result holds for the DF test.

the break is known *a priori*. We show that for these proposed alternatives, the estimated break date approaches the true one when the unit root null hypothesis holds with a break. It follows that the null limiting distributions of these tests also differ according to whether a break in the slope has occurred or not. However, such discrepancy is found to be smaller than in the minimum LM test case. Therefore, size distortions when using the proposed alternative tests will also be smaller if critical values corresponding to the no break case are used but there is in fact a break. Simulation results show that these additional tests perform better in terms of estimating the true break date than the minimum LM test, both under the null and the alternative, leading to better size and power properties.

The structure of the paper is as follows. The next section presents the models and statistics. In Section 3, limiting distributions of the statistics under the null are derived when there is a break as well as when there is no break. Finite sample critical values as well as several finite sample size and power simulations are presented in Section 4. Concluding remarks are given in Section 5. All proofs are relegated to the Appendix.

2 LM-TYPE TESTS FOR A UNIT ROOT

We consider the following data generating process (DGP):

$$y_t = \delta_1 + Z_t^0 \beta + x_t, \tag{1}$$

$$(1 - \alpha L) x_t = \epsilon_t. \tag{2}$$

As in Schmidt and Phillips (1992), we assume the same regularity conditions in Phillips and Perron (1988) that allow for some degree of heterogeneity and autocorrelation in the ϵ sequence. We also define the following nuisance parameters

$$\sigma_\epsilon^2 = \lim_{T \to \infty} T^{-1} E \left(\sum_{t=1}^{T} \epsilon_t^2 \right), \tag{3}$$

$$\sigma^2 = \lim_{T \to \infty} T^{-1} E \left(\sum_{t=1}^{T} \epsilon_t \right)^2 \tag{4}$$

and assume σ_ϵ^2, $\sigma^2 > 0$. We also define the ratio $\omega^2 = \sigma_\epsilon^2 / \sigma^2$.

This specification allows for the presence of different deterministic mean components by different choices of the exogenous variables in Z_t^0. The case of a deterministic trend with no structural change considered in Schmidt and Phillips (1992) corresponds to

$$Z_t^0 = t.$$

As in Perron (1989) we consider models where a break has occurred in the trend function at some unknown date denoted by T_B^0, with $1 < T_B^0 < T$, where T is the sample size. The superscript 0 is used to denote the true break date. We consider three different models. Model 1 allows for a change in the intercept and corresponds to

$$Z_t^0 = (DU_t^0, t) \tag{5}$$

where $DU_t^0 = 1(t > T_B^0)$ and $1(\cdot)$ is the indicator function. In Model 2 there is a change in both intercept and slope specified as

$$Z_t^0 = (DU_t^0, t, DT_t^0) \tag{6}$$

where $DT_t^0 = 1(t > T_B^0)(t - T_B^0)$. Finally, Model 3 allows for a change in slope such that the two segments of the trend function are joined:

$$Z_t^0 = (t, DT_t^0). \tag{7}$$

Allowing for more than one break could be easily accommodated in this model specification by appropriate choices of Z_t^0. In this paper only the additive outlier (AO) versions of Perron's models are considered.

We consider testing the unit root null hypothesis

$$H_0 : \alpha = 1 \tag{8}$$

using LM-type test statistics based on the tests proposed by Schmidt and Phillips (1992). Assuming normality of the errors, the restricted maximum likelihood estimator of β, denoted by $\tilde{\beta}$, is obtained by estimating the following regression by OLS:

$$\Delta y_t = \Delta Z_t \beta + u_t, \tag{9}$$

where ΔZ_t denotes the first difference of the regressors Z_t based on an assumed break date denoted by T_B. If the assumed break date, T_B, differs from the true one, T_B^0, then Z_t may also differ from Z_t^0. Define the 'residuals'

$$\tilde{S}_t = y_t - \tilde{\delta}_1^* - Z_t \tilde{\beta} \tag{10}$$

where

$$\tilde{\delta}_1^* = y_1 - Z_1\tilde{\beta}. \tag{11}$$

LM-type tests for a unit root are then obtained by OLS estimation of the following test regression:

$$\Delta\tilde{S}_t = \Delta Z_t\beta + \phi\tilde{S}_{t-1} + e_t. \tag{12}$$

To allow for autocorrelated errors, an augmented regression could be estimated as in Amsler and Lee (1995) or Lee and Strazicich (1999, 2002):

$$\Delta\tilde{S}_t = \Delta Z_t\beta + \phi\tilde{S}_{t-1} + \sum_{j=1}^{k} c_j\Delta\tilde{S}_{t-j} + e_t, \tag{13}$$

where the choice of k could be based on a number of alternative procedures as in the case of the augmented DF-type tests (see for example Vogelsang and Perron, 1998). Alternatively, as in Schmidt and Phillips (1992), a simple correction of the test statistics could be used.

The LM-type test statistic for a unit root in these models is given by the t-statistic for testing $\phi = 0$ and is denoted by $t_\phi(j, T_B)$, where j denotes the model ($j = 1, 2, 3$) and T_B indicates the break date used. The Schmidt and Phillips (1992) t-statistic corresponding to the no break case, $Z_t = t$, will be denoted as t_ϕ.

To implement the tests allowing for a break, some choice of T_B must be made. Following Zivot and Andrews (1992), Perron (1997) and Vogelsang and Perron (1998), Lee and Strazicich (1999, 2002) propose the minimal t-statistic obtained over some range of break dates, i.e. $t_\phi(j, T_B(t_\phi)) = \inf_{\lambda\in\Lambda} t_\phi(j, [\lambda T])$, where $t_\phi(j, [\lambda T])$ denotes the t-statistic with a break at $T_B = [\lambda T]$, $[\lambda T]$ is the integer part of λT, and Λ is some compact subset of $[0, 1]$.

In this paper, we propose selecting the break date corresponding to the least squares estimator of T_B, that is, the date that minimizes the sum of squared residuals in the first step regression (9). We denote by $\hat{T}_B$ the value of T_B chosen in this way. This choice of the break date coincides with the one obtained by maximizing the F-statistic on the significance of the two dummy variables ΔDU_t and ΔDT_t in regression (9) for Model 2, and maximizing the absolute value of the t-statistic for ΔDU_t in Model 1 and for ΔDT_t in Model 3.

We also consider choosing the break date that maximizes (minimizes) the t-statistic for ΔDU_t in Model 1 and for ΔDT_t in Models 2 and 3, both in the first step regression

(9), when the direction of the change is known to be positive (negative) *a priori*. We denote these choices by $T_B(t_{\Delta Z})$. These procedures are similar to the ones discussed by Perron (1997) and Vogelsang and Perron (1998) in the context of DF-type tests.

3 ASYMPTOTIC DISTRIBUTIONS UNDER THE NULL

The results for the asymptotic distributions under the null hypothesis of a unit root are presented for two cases. First, we consider the case of no break in the DGP. Results for Model 1 assuming a fixed break date are given in Amsler and Lee (1995). Lee and Strazicich (1999) also consider Model 1 when the break date is given by $T_B(t_\phi)$. We further show that in this case, the limiting distribution is the same as in the no break case of Schmidt and Phillips (1992). Lee and Strazicich (2002) also give results for Models 1 and 2 allowing for two breaks with the break dates fixed or estimated as $T_B(t_\phi)$. We consider the case of just one break. We further consider using $T_B(t_{\Delta Z})$ and $\hat{T}_B$ for all three models.

In the second case there is a break under the null. Results for Model 1 assuming a fixed break date are given in Amsler and Lee (1995). We provide results for Models 1, 2 and 3 when the break date is chosen according to any of the criteria considered above.

Critical values for the limiting distributions were obtained using $T = 1{,}000$ and 10,000 replications. The RNDN normal pseudo-random number generator in GAUSS 3.5 was used in the simulations. We followed the same procedure described in Zivot and Andrews (1992) and set Λ equal to the largest possible window.

3.1 The case of no break

In this subsection we consider the case where no break has occurred under the null hypothesis of a unit root. The following DGP is assumed:

$$\begin{aligned} y_t &= \delta_1 + \gamma_1 t + x_t, \\ x_t &= x_{t-1} + \epsilon_t. \end{aligned}$$

For Model 1, Amsler and Lee (1995) show that when a fixed break date, $T_B = [\lambda T]$, is assumed, the asymptotic distribution of $t_\phi(1, T_B)$ is the same as the one obtained for t_ϕ, which doesn't allow for a break, as given by equation (22) in Schmidt and Phillips (1992). As mentioned in Amsler and Lee (1995), this is explained by the fact that the inclusion of ΔDU_t which equals 1 for only one observation, has no effect asymptotically. It follows that $t_\phi(1, T_B(t_\phi))$, $t_\phi(1, \hat{T}_B)$ and $t_\phi(1, T_B(t_{\Delta Z}))$ also have this same limiting distribution. The last rows in Tables 2, 5 and 8 give the asymptotic critical values using the case $T = 2{,}000$ in Schmidt and Phillips (1992). Of course, in finite samples this invariance result no longer holds. Finite sample critical values are presented in Section 4.

Consider now Models 2 and 3. The limiting distribution of $t_\phi(j, T_B)$ $(j = 2, 3)$ in the case of a fixed break date $T_B = [\lambda T]$ is given by:

$$t_\phi(j, T_B) \Rightarrow R(\lambda) \qquad (j = 2, 3) \tag{14}$$

where

$$R(\lambda) = -\frac{1}{2}\omega \left(\int_0^1 \underline{V}(r, \lambda)^2 dr \right)^{-1/2}, \tag{15}$$

$\underline{V}(r, \lambda)$ denotes the residuals from the projection of the process $V(r, \lambda)$ onto the subspace generated by the functions $\{1, du(r, \lambda)\}$ with $du(r, \lambda) = 1(r > \lambda)$,

$$\begin{aligned} V(r, \lambda) &= \left[W(r) - \frac{r}{\lambda} W(\lambda) \right] 1(r \leq \lambda) \\ &\quad + \left[W(r) - W(\lambda) - \frac{r - \lambda}{1 - \lambda}(W(1) - W(\lambda)) \right] 1(r > \lambda) \end{aligned} \tag{16}$$

corresponds to a double standard Brownian bridge such that $V(0, \lambda) = V(\lambda, \lambda) = V(1, \lambda) = 0$, and $W(r)$ is a standard Wiener process. The symbol '$\Rightarrow$' in (14) denotes weak convergence of the associated probability measures. This result was obtained in Lee and Strazicich (2002) for Model 2 in the context of two structural breaks. Notice that the limiting distribution is the same for Models 2 and 3 because, as in Model 1, the regressor ΔDU_t is asymptotically negligible. Critical values for (14) when $\omega = 1$ are presented in Table 1 for several values of λ. They are also asymptotically valid for dependent and heterogeneous errors if $t_\phi(j, T_B)$ is multiplied by a consistent estimator of $1/\omega$ as in Schmidt and Phillips (1992).

When the break date is chosen to minimize the t-statistic, Lee and Strazicich (2002) using arguments similar to those in Zivot and Andrews (1992) prove that:

$$t_\phi(j, T_B(t_\phi)) \Rightarrow \inf_{\lambda \in \Lambda} R(\lambda) \qquad (j = 2, 3).$$

Critical values for this limiting distribution when $\omega = 1$ appear in the last row of Tables 3 and 4.[4]

We also obtain the limiting distributions of $t_\phi(j, T_B(t_{\Delta Z}))$ and $t_\phi(j, \hat{T}_B)$ $(j = 2, 3)$ where the break date T_B is chosen based on the maximal dummy variable t-statistic or the least squares estimator of the break date. It is shown in the Appendix that:

$$t_\phi(j, T_B(t_{\Delta Z})) \Rightarrow R(\tilde{\lambda}) \qquad (j = 2, 3) \tag{17}$$

and

$$t_\phi(j, \hat{T}_B) \Rightarrow R(\hat{\lambda}) \qquad (j = 2, 3) \tag{18}$$

where $\tilde{\lambda} = \arg\max_{\lambda \in \Lambda} Q(\lambda)$ and $\hat{\lambda} = \arg\max_{\lambda \in \Lambda} Q(\lambda)^2$, with

$$Q(\lambda) = \sqrt{\lambda(1-\lambda)} \left(\frac{W(1) - W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda} \right).$$

Critical values for (17) appear in the last row of Tables 6 and 7, while critical values for (18) appear in the last row of Tables 9 and 10.

3.2 The case of a break

In this subsection we derive the limiting distributions of the LM-type test statistics when a break is present under the null hypothesis of a unit root. We consider first the case of a break in the intercept occurring at date $T_B^0 = [\lambda_0 T]$. The DGP is given by

$$y_t = \delta_1 + \delta DU_t^0 + \gamma_1 t + x_t.$$

Amsler and Lee (1995) show that the LM-type t-statistic for Model 1, $t_\phi(1, T_B)$, assuming a break at $T_B = [\lambda T]$, has the same limiting distribution as the no break Schmidt and Phillips (1992) LM t-statistic, t_ϕ, independently of the break date being correctly placed $(\lambda = \lambda_0)$ or not $(\lambda \neq \lambda_0)$.[5] In fact, Amsler and Lee (1995) also show that other unit root tests that do not allow for a break, such as the Schmidt and Phillips (1992) LM test and the DF test, are also asymptotically invariant to a break in the intercept under the null. This asymptotic invariance property also holds for the LM-type tests for Models 2

[4]Lee and Strazicich (2002) only provide critical values for the case of two breaks with $\Lambda = [0.1, 0.9]$ and $T = 100$.

[5]If the break date is correctly placed then invariance to the value of δ also holds in finite samples.

and 3.[6] In summary, if there is a break in the intercept then the limiting distributions of $t_\phi(j, T_B(t_\phi))$, $t_\phi(j, \hat{T}_B)$ and $t_\phi(j, T_B(t_{\Delta Z}))$ $(j = 1, 2, 3)$ coincide with the corresponding ones described in the previous subsection for the no break case.

We consider now the consequences of the presence of a break in the slope under the null. For a break occurring at date $T_B^0 = [\lambda_0 T]$, the DGP is given by:

$$y_t = \delta_1 + \gamma_1 t + \gamma DT_t^0 + x_t.$$

In the Appendix we show that:

$$T^{1/2} t_\phi = O_p(1) \tag{19}$$

so that the Schmidt and Phillips (1992) LM t-statistic t_ϕ converges to zero as $T \to \infty$. It follows that the probability of rejecting the null hypothesis of a unit root approaches zero when there is a break in slope under the null. The same result is obtained for any of the LM-type tests allowing for a change in the intercept (namely $t_\phi(1, T_B)$, $t_\phi(1, T_B(t_\phi))$, $t_\phi(1, \hat{T}_B)$ and $t_\phi(1, T_B(t_{\Delta Z}))$) since the inclusion of ΔDU_t does not matter asymptotically.

For Models 2 and 3, if the break in slope is correctly placed ($T_B = T_B^0$) then $t_\phi(j, T_B)$ $(j = 2, 3)$ will be exactly invariant to the value of γ under the null hypothesis. It follows that, under the null hypothesis of a unit root, the limiting distribution of the exogenous break LM-type t-statistic for these two models when a break occurs will be the same as that obtained in (14) when no break has occurred:

$$t_\phi(j, [\lambda_0 T]) \Rightarrow R(\lambda_0) \qquad (j = 2, 3). \tag{20}$$

Invariance no longer holds when the break date is misplaced. In the Appendix we show that if $\lambda \neq \lambda_0$ then

$$T^{1/2} t_\phi(j, [\lambda T]) = O_p(1) \qquad (j = 2, 3), \tag{21}$$

so that $t_\phi(j, [\lambda T])$ converges to zero asymptotically. It follows that when the break date is incorrectly chosen, the probability of rejecting the null hypothesis, when it holds with a break in the slope, approaches zero as $T \to \infty$.

[6] There is exact invariance in the case of Model 2 when the break date is correctly placed. This point is also discussed in Lee and Strazicich (2002). Similar invariance results are obtained by Vogelsang and Perron (1998) in the context of DF-type tests that allow for a break in the trend.

Consider now the asymptotic behavior of the minimal LM-type tests $t_\phi(j, T_B(t_\phi))$ $(j = 2, 3)$. Since $t_\phi(j, [\lambda T])$ has a limiting distribution given by (20) that has support over the negative real line when $\lambda = \lambda_0$, but converges to zero when $\lambda \neq \lambda_0$, it follows that if $\lambda_0 \in \Lambda$ then $\arg\inf_{\lambda\in\Lambda} t_\phi(j, [\lambda T])$ $(j = 2, 3)$ converges to λ_0. Therefore, we obtain the following asymptotic result:

$$t_\phi(j, T_B(t_\phi)) \Rightarrow R(\lambda_0) \qquad (j = 2, 3). \tag{22}$$

It is interesting to note that the corresponding DF-type tests have quite different properties. As shown in Vogelsang and Perron (1998), minimal DF-type tests allowing for a break in slope diverge asymptotically because the estimated break date does not converge to the true one.

Finally, we consider the limiting distribution of $t_\phi(j, T_B(t_{\Delta Z}))$ and $t_\phi(j, \hat{T}_B)$ $(j = 2, 3)$. As shown in the Appendix, both $T_B(t_{\Delta Z})$ and $\hat{T}_B$ converge to the true break date asymptotically so that

$$t_\phi(j, T_B(t_{\Delta Z})) \Rightarrow R(\lambda_0) \qquad (j = 2, 3) \tag{23}$$

and

$$t_\phi(j, \hat{T}_B) \Rightarrow R(\lambda_0) \qquad (j = 2, 3). \tag{24}$$

As in the minimal LM-type test, the limiting distribution equals that obtained in the case where the break date is known. It follows that some size distortions will arise if critical values for the no break case are used but the unit root null hypothesis holds with a break in slope. A similar result was obtained by Vogelsang and Perron (1998) in the case of the DF-type tests allowing for a break in the trend. However, since the critical values for $t_\phi(j, T_B(t_{\Delta Z}))$ and $t_\phi(j, \hat{T}_B)$ are considerably closer to the fixed break critical values, size distortions will be larger when using $t_\phi(j, T_B(t_\phi))$. For example, for a 5% significance level, the fixed break critical values in Table 1 vary between -3.29 for $\lambda = 0.9$ and -3.66 for $\lambda = 0.5$, while from Table 3 the critical value for $t_\phi(2, T_B(t_\phi))$ equals -4.27. Closer to the fixed break case is the critical value for $t_\phi(2, T_B(t_{\Delta Z}))$ from Table 7 which equals -3.47, or for $t_\phi(2, \hat{T}_B)$ from Table 9 which equals -3.50. For instance, if there is in fact a break at $\lambda = 0.5$ and the asymptotic 5% critical values for the endogenous break tests are used, asymptotically the true size of $t_\phi(2, T_B(t_\phi))$ will be below 1% while for $t_\phi(2, T_B(t_{\Delta Z}))$ and $t_\phi(2, \hat{T}_B)$ the true size will be between 5% and 10%. This results in a loss in the power of $t_\phi(2, T_B(t_\phi))$.

4 FINITE SAMPLE SIMULATIONS

In this section, finite sample critical values as well as size and power simulations are presented for the statistics $t_\phi(j, T_B(t_\phi))$, $t_\phi(j, T_B(t_{\Delta Z}))$ and $t_\phi(j, \hat{T}_B)$ $(j = 1, 2, 3)$. We obtain these by simulating from the following DGP:

$$y_t = \delta DU_t^0 + \gamma DT_t^0 + x_t, \tag{25}$$

$$x_t = \alpha x_{t-1} + \rho \Delta x_{t-1} + e_t + \psi e_{t-1}, \tag{26}$$

where e_t are i.i.d. $N(0, 1)$ random deviates. Each simulation was based on 10,000 replications. We set Λ equal to the largest window possible for each sample size considered.

4.1 Critical values with no break

In this subsection we present finite sample critical values for the statistics $t_\phi(j, T_B(t_\phi))$, $t_\phi(j, T_B(t_{\Delta Z}))$ and $t_\phi(j, \hat{T}_B)$ $(j = 1, 2, 3)$ assuming no break, $\delta = \gamma = 0$, under the null $\alpha = 1$. We only present results for the case of no autocorrelation in the first difference of the errors: $\rho = \psi = 0$. Size distortions caused by the presence of a break, $\delta, \gamma \neq 0$, or by autocorrelation in the errors, $\rho, \psi \neq 0$, are considered in the next subsection. We have set $x_0 = 0$, $\delta_1 = 0$ and $\gamma_1 = 0$ without loss of generality since in this case the statistics are exactly invariant to these parameters.

For selecting the truncation lag parameter k in regression (13) we have considered two procedures. In the first case we set $k = 0$ which corresponds to regression (12). As an alternative, we also consider a data-dependent method as in Perron (1989, 1997) and Vogelsang and Perron (1998) denoted as $k(t - sig)$. For any given value of T_B, k is chosen so that the coefficient on the last included lagged first difference is significant at the 10% level, but insignificant in higher-order autoregressions up to some fixed maximum lag length denoted by $kmax$. We set $kmax = 5$ so that our results are comparable with those presented in Vogelsang and Perron (1998) for the augmented DF-type tests allowing for a break.

We present the results for $T = 50$, 100 and 150 in Tables 2–10.[7] In general, the asymptotic critical values provide reasonably good approximations to the finite sample critical values when $k = 0$. Only in the case of $t_\phi(1, T_B(t_\phi))$ does convergence to the

[7]Critical values for $t_\phi(1, T_B(t_\phi))$ when $T = 100$ and $k = 0$ differ slightly from the ones presented in Lee and Strazicich (2002) because of a different choice of Λ.

limiting critical values seem to be somewhat slower. When $k(t-sig)$ is used with $kmax = 5$, the critical values are much smaller than the asymptotic and $k = 0$ critical values. Similar results were found by Vogelsang and Perron (1998) for the augmented DF-type tests. These discrepancies seem to be larger for the tests based on $T_B(t_\phi)$. The simulations also suggest that in all cases these differences vanish asymptotically.

4.2 Finite sample size and power

We now present the results of several finite sample size and power simulations using $T = 100$. We considered several values of δ and γ, both under the null, $\alpha = 1$, and under the alternative $\alpha = 0.8$. For the cases where a break occurs, δ, $\gamma \neq 0$, the true break date was set to $T_B^0 = 50$ ($\lambda_0 = 0.5$).

In the first set of simulations we considered i.i.d. errors, $\rho = \psi = 0$, and set $k = 0$ in order to isolate the effects of the breaks from the effects of autocorrelation. We used the 5% critical values described in the previous subsection for the case of no break and presented in Tables 2–10 for $T = 100$ and $kmax = 0$. Results appear in Table 11.

We first discuss the results obtained for the Schmidt and Phillips (1992) LM test t_ϕ. A break in the intercept has a minor impact on size. However, it may lead to a severe decrease in power. These results are in line with the findings in Amsler and Lee (1995). For a break in slope, size is practically zero, confirming the asymptotic result in subsection 3.2. A break in slope also drives power to zero.

Consider now the results for the test statistics using Model 1 which allows for a break in intercept. When the unit root null hypothesis, $\alpha = 1$, holds with a break in the intercept only, $\delta \neq 0$ and $\gamma = 0$, $t_\phi(1, T_B(t_\phi))$ becomes slightly undersized. Size equals 4.7% for $\delta = 5$ and 3.6% for $\delta = 10$. For $t_\phi(1, T_B(t_{\Delta Z}))$ and $t_\phi(1, \hat{T}_B)$, the exact size nearly matches nominal size. This better performance is explained by the fact that the correct break date is almost always correctly identified using these two procedures. On the other hand, whenever a change in slope occurs, $\gamma \neq 0$, all test statistics become severely undersized. This result confirms the asymptotic findings in subsection 3.2. When the alternative holds without a break, $\alpha = 0.8$, we see that $t_\phi(1, T_B(t_\phi))$ performs better than $t_\phi(1, T_B(t_{\Delta Z}))$ and $t_\phi(1, \hat{T}_B)$. However, in the presence of a break in the intercept, the reverse occurs. When the alternative holds with a break in the slope, all tests have power close to zero. In all the cases, we see that the correct break date is more frequently selected when using $t_\phi(1, T_B(t_{\Delta Z}))$ and $t_\phi(1, \hat{T}_B)$.

Next, we discuss the results obtained for the tests based on Model 2. Consider first the results under the null. As the break in the intercept gets larger, $t_\phi(2, T_B(t_\phi))$ and $t_\phi(2, T_B(t_{\Delta Z}))$ become more undersized. On the other hand, the size for $t_\phi(2, \hat{T}_B)$ is always close to 6%. For a change in slope, $t_\phi(2, T_B(t_\phi))$ is undersized while $t_\phi(2, T_B(t_{\Delta Z}))$ is oversized. The size for $t_\phi(2, \hat{T}_B)$ is again only slightly above nominal size. When both a break in intercept and slope occur, $t_\phi(2, \hat{T}_B)$ performs better than the other test statistics. Under the alternative, when there is no break, the three tests considered have similar powers. When a break occurs we see that $t_\phi(2, T_B(t_\phi))$ always performs much worse than $t_\phi(2, \hat{T}_B)$. As expected, $t_\phi(2, T_B(t_{\Delta Z}))$ performs better when there is a break in slope only. As in Model 1, the correct break date is always more frequently selected when using $t_\phi(2, \hat{T}_B)$ relative to $t_\phi(2, T_B(t_\phi))$.

Finally, we address the results for the tests based on Model 3. This model is designed to cope with a change in slope only. When a break in the intercept occurs under the null, all test statistics become undersized. For a break in the slope, $t_\phi(3, T_B(t_\phi))$ is also undersized, while the size of $t_\phi(3, \hat{T}_B)$ is closer to the nominal size. Under the alternative hypothesis, when there is no break, $t_\phi(3, T_B(t_\phi))$ and $t_\phi(3, T_B(t_{\Delta Z}))$ perform better than $t_\phi(3, \hat{T}_B)$. It is interesting to note that in this case the power of $t_\phi(2, \hat{T}_B)$ is larger than the power of $t_\phi(3, \hat{T}_B)$. When there is a break in the intercept under the alternative, all test statistics have low power. For a break in the slope, as expected $t_\phi(3, T_B(t_{\Delta Z}))$ performs better, followed closely by $t_\phi(3, \hat{T}_B)$. Power for $t_\phi(3, T_B(t_\phi))$ is lower mainly because the correct break date is selected less often. Finally, when both a break in the intercept and in the slope occur under the alternative, $t_\phi(3, T_B(t_\phi))$ performs better, but still distant from the power achieved using the Model 2 test statistics $t_\phi(2, T_B(t_{\Delta Z}))$ and $t_\phi(2, \hat{T}_B)$.

Overall, the results suggest that the minimum LM tests suffer from size distortions in the presence of a break under the null. In contrast, for the $t_\phi(2, \hat{T}_B)$ test statistic, true size is always close to nominal size. This test also revealed good power properties for the different types of breaks considered making it particularly attractive when it is not possible to restrict the break to occur only in the intercept or only in the slope. When it is known that only a break in intercept may have occurred, the corresponding test for Model 1, $t_\phi(1, \hat{T}_B)$, has more power. However, all tests based on Model 1 are severely undersized when a break in slope occurs. If it is known that there is a break in slope only, and its direction is known *a priori*, some gain in power may also be obtained by using $t_\phi(3, T_B(t_{\Delta Z}))$. Then again, if there is a break in intercept, all tests based on Model 3 are severely affected in terms of power. The only case where a minimum LM test statistic is superior to other tests in terms of size and power is when using Model 1 and when no break has occurred. However, when one is sure that there is no break, then the Schmidt and Phillips (1992) LM test, t_ϕ, would be preferred. Finally, we note that when a break occurs under the null or under the alternative, tests based on $T_B(t_{\Delta Z})$ or $\hat{T}_B$ seem to select the correct break date more often than tests based on $T_B(t_\phi)$.

In a second set of simulations we allow for autocorrelated errors and set $kmax = 5$. In this case, we use the 5% finite sample critical values for $T = 100$ and $kmax = 5$. Also, as in Vogelsang and Perron (1998), we consider the following five error specifications: (1) $\rho = 0$, $\psi = 0$, (2) $\rho = 0.6$, $\psi = 0$, (3) $\rho = -0.6$, $\psi = 0$, (4) $\rho = 0$, $\psi = 0.5$ and (5) $\rho = 0$, $\psi = -0.5$. Results for the LM-type test statistics based on Model 2 appear in Table 12.

The first thing to notice is that although true size depends on the correlation structure considered, in general size distortions caused by the autocorrelation in the errors are not too large. The only exception where tests are largely oversized is in experiment (5) where the errors have a negative MA(1) component. Similar findings were obtained in the context of the AO DF-type tests in Vogelsang in Perron (1998). On the other hand, regardless of the correlation structure of the errors, the consequences of a break on true size are similar to those obtained above for the case of no autocorrelation and $kmax = 0$. A break in the intercept or the slope usually leads to a decrease in the size of $t_\phi(2, T_B(t_\phi))$ and to an increase in the size of $t_\phi(2, \hat{T}_B)$. The size of $t_\phi(2, T_B(t_{\Delta Z}))$ tends to decrease with a break in the intercept and to increase with a break in the slope.

We consider now the results in terms of power. In the case of a negative AR(1) component in the errors, all tests have low power. Vogelsang in Perron (1998) found the same behavior for AO DF-type tests. Regardless of the autocorrelation pattern considered, $t_\phi(2, T_B(t_\phi))$ tends to be superior to $t_\phi(2, \hat{T}_B)$ when there is no break. However, the power of $t_\phi(2, T_B(t_\phi))$ is reduced in the presence of a break in the intercept or in the slope. Again, Vogelsang and Perron (1998) report a similar result for the AO DF-type tests. In contrast, the power of $t_\phi(2, \hat{T}_B)$ tends to be larger in the presence of a break. This is explained by the fact that $\hat{T}_B$ selects the true break date more often than $T_B(t_\phi)$, and when the wrong break date is selected the tests have a tendency to underreject the null. Finally, as expected, $t_\phi(2, T_B(t_{\Delta Z}))$ is preferred to $t_\phi(2, \hat{T}_B)$ when the break occurs only in the slope.

5 CONCLUSION

This paper considers LM-type tests for a unit root allowing for the presence of a break in the trend function at an unknown date. Three possible cases are considered: a change in intercept, a change in slope, and both. In addition to the minimum LM test statistic, we propose tests where the break date is estimated using the significance of the trend break parameter or the least squares estimator of the break date under the null.

We examine the asymptotic behavior of the LM-type tests when the null hypothesis of a unit root holds with a break as well as when there is no break. The test statistics are asymptotically invariant to the magnitude of the intercept change. However, they are not invariant to the magnitude of the slope change. For all the endogenous break tests considered, the null limiting distribution when there is a break in slope is not the same as when there is no break. Since the discrepancy is larger for the minimum LM-type unit root test, smaller size distortions are to be expected when using the other proposed test statistics.

A Monte Carlo study compares the finite sample performance of the alternative endogenous break LM-type tests. Results suggest the superiority in terms of size, power and break date estimation of our proposed methods relative to the minimum LM-type unit root tests when there is a break.

APPENDIX

In this appendix, we prove the asymptotic results presented in the text by employing the functional central limit theorem (FCLT) used in Phillips and Perron (1988). Limiting results for the minimal and maximal test statistics are obtained by first establishing weak convergence for a fixed λ and then applying the continuous mapping theorem as in Zivot and Andrews (1992). Throughout the appendix $\Rightarrow$ denotes weak convergence in distribution and $\xrightarrow{p}$ convergence in probability.

When the null hypothesis of a unit root holds with a possible break in the slope at date $T_B^0 = [\lambda_0 T]$ we have that

$$y_t = \delta_1 + \gamma_1 t + \gamma DT_t^0 + x_t \tag{A.1}$$

and

$$\Delta y_t = \gamma_1 + \gamma DU_t^0 + \epsilon_t. \tag{A.2}$$

We can write (A.2) in matrix notation as $\Delta Y = \Delta Z^0 \beta + \epsilon$ where $\Delta Y = (\Delta y_2, \ldots, \Delta y_T)'$, $\Delta Z^0 = (\Delta Z_2^{0\prime}, \ldots, \Delta Z_T^{0\prime})'$, $\Delta Z_t^0 = (1, DU_t^0)$, $\epsilon = (\epsilon_2, \ldots, \epsilon_T)'$, and $\beta = (\gamma_1, \gamma)'$.

The first step regression (9) can also be written in matrix notation as

$$\Delta Y = \Delta Z \beta + U \tag{A.3}$$

where $\Delta Z = (\Delta Z_2', \ldots, \Delta Z_T')'$ and $U = (u_2, \ldots, u_T)'$. The least squares estimator of β is given by $\tilde{\beta} = (\Delta Z' \Delta Z)^{-1} \Delta Z' \Delta Y$.

Define the following $(T-1 \times 1)$ vectors $\tilde{S}_{-1} = (\tilde{S}_1, \ldots, \tilde{S}_{T-1})'$ and $\Delta \tilde{S} = (\Delta \tilde{S}_2, \ldots, \Delta \tilde{S}_T)'$. Using (10) and (11) we get that $\Delta \tilde{S} = \Delta Y - \Delta Z \tilde{\beta}$, so that $\Delta \tilde{S}$ is the vector of residuals from regression (A.3). Define the orthogonal projection matrix $M = I - \Delta Z (\Delta Z' \Delta Z)^{-1} \Delta Z'$. We have that $\Delta \tilde{S} = M \Delta Y$ and

$$M \Delta \tilde{S} = \Delta \tilde{S}. \tag{A.4}$$

It also follows that

$$\begin{aligned} \tilde{S}_{-1}' M \Delta \tilde{S} &= \tilde{S}_{-1}' \Delta \tilde{S} \\ &= \sum_{t=2}^{T} \tilde{S}_{t-1}' \Delta \tilde{S}_t' \\ &= -\frac{1}{2} \Delta \tilde{S}' \Delta \tilde{S} \end{aligned} \tag{A.5}$$

where the last equality follows as in Lemma 1 in Schmidt and Phillips (1992).

The estimator of ϕ in the second step regression (12) can be written as

$$\begin{aligned}\hat{\phi} &= (\tilde{S}'_{-1}M\tilde{S}_{-1})^{-1}\tilde{S}'_{-1}M\Delta\tilde{S} \\ &= -\frac{1}{2}(\tilde{S}'_{-1}M\tilde{S}_{-1})^{-1}\Delta\tilde{S}'\Delta\tilde{S}\end{aligned} \tag{A.6}$$

with the second equality following from (A.5). The t-statistic for testing $\phi = 0$ can also be written as

$$\begin{aligned}t(\phi = 0) &= (s^2\tilde{S}'_{-1}M\tilde{S}_{-1})^{-1/2}\tilde{S}'_{-1}M\Delta\tilde{S} \\ &= -\frac{1}{2}(s^2\tilde{S}'_{-1}M\tilde{S}_{-1})^{-1/2}\Delta\tilde{S}'\Delta\tilde{S},\end{aligned} \tag{A.7}$$

where s^2 is the estimated variance of the errors given by

$$s^2 = \frac{1}{T-1}(\Delta\tilde{S} - \tilde{S}_{-1}\hat{\phi})'M(\Delta\tilde{S} - \tilde{S}_{-1}\hat{\phi}). \tag{A.8}$$

Proof of (19) in the text. We show that when there is a break in the slope under the null hypothesis of a unit root, so that (A.1)–(A.2) hold with $\gamma \neq 0$, then we have the result in (19). To obtain t_ϕ, the first step regression (9) must be estimated assuming no break, which corresponds to $Z_t = t$, $\Delta Z_t = 1$, and can be written as

$$\Delta y_t = \gamma_1 + u_t.$$

The least squares estimator of γ_1 is given by

$$\tilde{\gamma}_1 = \frac{1}{T-1}\sum_{t=2}^{T}\Delta y_t. \tag{A.9}$$

For a break in slope at date $T_B^0 = [\lambda_0 T]$, it follows from (A.2) and (A.9) that $\tilde{\gamma}_1 = \gamma_1 + \gamma\frac{T-T_B^0}{T-1} + \frac{1}{T-1}\sum_{t=2}^{T}\epsilon_t$. By the FCLT we obtain

$$T^{1/2}(\tilde{\gamma}_1 - \gamma_1^*) \Rightarrow \sigma W(1) \tag{A.10}$$

where $\gamma_1^* = \gamma_1 + \gamma(1-\lambda_0)$.

From (10) and (11) we have that $\tilde{S}_t = y_t - y_1 - \tilde{\gamma}_1(t-1)$ and $\Delta\tilde{S}_t = \Delta y_t - \tilde{\gamma}_1$. Using (A.1) and (A.2) we get

$$\begin{aligned}\tilde{S}_t &= (x_t - x_1) - (\tilde{\gamma}_1 - \gamma_1)(t-1) + \gamma DT_t^0 \\ &= (x_t - x_1) - (\tilde{\gamma}_1 - \gamma_1^*)(t-1) - (\gamma_1^* - \gamma_1)(t-1) + \gamma DT_t^0,\end{aligned} \tag{A.11}$$

and

$$\Delta\tilde{S}_t = \epsilon_t - (\tilde{\gamma}_1 - \gamma_1) + \gamma DU_t^0. \tag{A.12}$$

The first term in (A.11) is $O_p(T^{1/2})$ by the FCLT. By (A.10) the second term is also $O_p(T^{1/2})$. The last two terms are $O(T)$. It follows that

$$T^{-1}\tilde{S}_{[rT]} \xrightarrow{p} \gamma f(r) \tag{A.13}$$

where

$$f(r) = -(1-\lambda_0)r + 1(r > \lambda_0)(r - \lambda_0).$$

From (A.13) we get

$$\begin{aligned} T^{-3}\tilde{S}'_{-1}M\tilde{S}_{-1} &= \sum_{t=2}^{T}\frac{1}{T}\left(\frac{\tilde{S}_{t-1}}{T} - \frac{1}{T-1}\sum_{t=2}^{T}\frac{\tilde{S}_{t-1}}{T}\right)^2 \\ &\xrightarrow{p} \int_0^1\left(\gamma f(r) - \int_0^1 \gamma f(s)ds\right)^2 dr. \end{aligned} \tag{A.14}$$

Computing the integrals in (A.14) we arrive at

$$T^{-3}\tilde{S}'_{-1}M\tilde{S}_{-1} \xrightarrow{p} \gamma^2\frac{\lambda_0^2(1-\lambda_0)^2}{12}. \tag{A.15}$$

From (A.12) and after rearranging terms we have that

$$\begin{aligned} \Delta\tilde{S}'\Delta\tilde{S} &= \sum_{t=2}^{T}\left(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1) + \gamma DU_t^0\right)^2 \\ &= \sum_{t=2}^{T}\left(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1)(1 - DU_t^0) - (\tilde{\gamma}_1 - \gamma_1 - \gamma)DU_t^0\right)^2. \end{aligned}$$

Since by (A.10) we have that $\tilde{\gamma}_1 \xrightarrow{p} \gamma_1^* = \gamma_1 + \gamma(1-\lambda_0)$ it follows that

$$T^{-1}\Delta\tilde{S}'\Delta\tilde{S} \xrightarrow{p} \sigma_\epsilon^2 + \gamma^2(1-\lambda_0)^2\lambda_0 + \gamma^2\lambda_0^2(1-\lambda_0) = \sigma_\epsilon^2 + \gamma^2\lambda_0(1-\lambda_0). \tag{A.16}$$

Using (A.15) and (A.16) in (A.6) we obtain:

$$T^2\hat{\phi} \xrightarrow{p} -\frac{1}{2}\left(\gamma^2\frac{\lambda_0^2(1-\lambda_0)^2}{12}\right)^{-1}\left(\sigma_\epsilon^2 + \gamma^2\lambda_0(1-\lambda_0)\right). \tag{A.17}$$

Using (A.15), (A.16) and (A.17) in (A.8) we also get that

$$s^2 \xrightarrow{p} \sigma_\epsilon^2 + \gamma^2 \lambda_0 (1 - \lambda_0). \tag{A.18}$$

Finally, using (A.15), (A.16) and (A.18) in (A.7), we get

$$\begin{aligned} T^{1/2} t_\phi &= -\frac{1}{2} (s^2 T^{-3} \tilde{S}'_{-1} M \tilde{S}_{-1})^{-1/2} T^{-1} \Delta \tilde{S}' \Delta \tilde{S} \\ &\xrightarrow{p} -\frac{1}{2} \left(\gamma^2 \frac{\lambda_0^2 (1 - \lambda_0)^2}{12} \right)^{-1/2} \left(\sigma_\epsilon^2 + \gamma^2 \lambda_0 (1 - \lambda_0) \right)^{1/2} \end{aligned}$$

proving the result. □

In what follows, we present proofs of the results for Model 3 only. All the corresponding proofs for Model 2 would follow along similar lines since it differs from Model 3 only by the inclusion of an asymptotically negligible one-time dummy variable.

The first step regression (9) for Model 3 assuming a break occurring at $T_B = [\lambda T]$ corresponds to $Z_t = (t, DT_t)$, $\Delta Z_t = (1, DU_t)$, and can be written as

$$\Delta y_t = \gamma_1 + \gamma DU_t + u_t. \tag{A.19}$$

The least squares estimators of γ_1 and γ can be written as

$$\tilde{\gamma}_1 = \frac{1}{T_B - 1} \sum_{t=2}^{T_B} \Delta y_t, \tag{A.20}$$

$$\tilde{\gamma} = \frac{1}{T - T_B} \sum_{t=T_B+1}^{T} \Delta y_t - \frac{1}{T_B - 1} \sum_{t=2}^{T_B} \Delta y_t. \tag{A.21}$$

We also have from (10) and (11) that

$$\tilde{S}_t = y_t - y_1 - \tilde{\gamma}_1 (t - 1) - \tilde{\gamma} DT_t$$

and

$$\Delta \tilde{S}_t = \Delta y_t - \tilde{\gamma}_1 - \tilde{\gamma} DU_t.$$

Using (A.1) and (A.2) we obtain

$$\tilde{S}_t = (x_t - x_1) - (\tilde{\gamma}_1 - \gamma_1)(t - 1) - \tilde{\gamma} DT_t + \gamma DT_t^0 \tag{A.22}$$

and

$$\Delta \tilde{S}_t = \epsilon_t - (\tilde{\gamma}_1 - \gamma_1) - \tilde{\gamma} DU_t + \gamma DU_t^0. \tag{A.23}$$

Lemma 1. If $\gamma = 0$, or if $\gamma \neq 0$ and $\lambda = \lambda_0$, then the following holds:

$$T^{1/2}(\tilde{\gamma}_1 - \gamma_1) \Rightarrow \sigma \frac{W(\lambda)}{\lambda}, \tag{A.24}$$

$$T^{1/2}(\tilde{\gamma} - \gamma) \Rightarrow \sigma \left[\frac{W(1) - W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda} \right], \tag{A.25}$$

$$T^{-1/2} \tilde{S}_{[rT]} \Rightarrow \sigma V(r, \lambda), \tag{A.26}$$

where $V(r, \lambda)$ is defined in (16).

Proof of Lemma 1. If $\gamma = 0$, or if $\gamma \neq 0$ and $\lambda = \lambda_0$, then we have from (A.2) and (A.20) that $\tilde{\gamma}_1 = \gamma_1 + \frac{1}{T_B - 1} \sum_{t=2}^{T_B} \epsilon_t$. From (A.21) we also get $\tilde{\gamma} = \gamma + \frac{1}{T - T_B} \sum_{t=T_B+1}^{T} \epsilon_t - \frac{1}{T_B - 1} \sum_{t=2}^{T_B} \epsilon_t$. By the FCLT we arrive at (A.24) and (A.25). From (A.22) and using (A.24) and (A.25) we obtain:

$$T^{-1/2} \tilde{S}_{[rT]} \Rightarrow \sigma \left[W(r) - r \frac{W(\lambda)}{\lambda} - 1(r > \lambda)(r - \lambda) \left(\frac{W(1) - W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda} \right) \right].$$

After rearranging terms we finally obtain (A.26). □

Lemma 2. Suppose that $\gamma \neq 0$ and $\lambda \neq \lambda_0$. If $\lambda < \lambda_0$ then

$$T^{1/2}(\tilde{\gamma}_1 - \gamma_1) \Rightarrow \sigma \frac{W(\lambda)}{\lambda}, T^{1/2}(\tilde{\gamma} - \gamma') \Rightarrow \sigma \left[\frac{W(1) - W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda} \right] \tag{A.27}$$

where $\gamma' = \gamma \frac{1-\lambda_0}{1-\lambda}$. If $\lambda > \lambda_0$ then

$$T^{1/2}(\tilde{\gamma}_1 - \gamma_1') \Rightarrow \sigma \frac{W(\lambda)}{\lambda}, T^{1/2}(\tilde{\gamma} - \gamma'') \Rightarrow \sigma \left[\frac{W(1) - W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda} \right] \tag{A.28}$$

where $\gamma_1' = \gamma_1 + \gamma \frac{\lambda - \lambda_0}{\lambda}$ and $\gamma'' = \gamma \frac{\lambda_0}{\lambda}$. Finally we have that

$$T^{-1} \tilde{S}_{[rT]} \xrightarrow{p} \gamma f(r, \lambda) \tag{A.29}$$

where

$$f(r,\lambda)=\begin{cases} -\frac{1-\lambda_0}{1-\lambda}1(r>\lambda)(r-\lambda)+1(r>\lambda_0)(r-\lambda_0) & \text{if} \quad \lambda<\lambda_0, \\ -\frac{\lambda-\lambda_0}{\lambda}r-\frac{\lambda_0}{\lambda}1(r>\lambda)(r-\lambda)+1(r>\lambda_0)(r-\lambda_0) & \text{if} \quad \lambda>\lambda_0. \end{cases} \tag{A.30}$$

Proof of Lemma 2. Consider first the case $\lambda<\lambda_0$. From (A.2) and (A.20) we have that $\tilde{\gamma}_1=\gamma_1+\frac{1}{T_B-1}\sum_{t=2}^{T_B}\epsilon_t$. Similarly, using (A.2) and (A.21) we get $\tilde{\gamma}=\gamma\frac{T-T_B^0}{T-T_B}+\frac{1}{T-T_B}\sum_{t=T_B+1}^{T}\epsilon_t-\frac{1}{T_B-1}\sum_{t=2}^{T_B}\epsilon_t$. By the FCLT we obtain (A.27). Rearranging the terms in (A.22) we obtain $\tilde{S}_t=(x_t-x_1)-(\tilde{\gamma}_1-\gamma_1)(t-1)-(\tilde{\gamma}-\gamma')DT_t-\gamma' DT_t+\gamma DT_t^0$. The first term is $O_p(T^{1/2})$ by the FCLT. By (A.27) the second and third terms are also $O_p(T^{1/2})$. The last two terms are $O(T)$. Therefore we arrive at (A.29)–(A.30) for the case $\lambda<\lambda_0$.

For $\lambda>\lambda_0$ we have that $\tilde{\gamma}_1=\gamma_1+\gamma\frac{T_B-T_B^0}{T_B-1}+\frac{1}{T_B-1}\sum_{t=2}^{T_B}\epsilon_t$ and $\tilde{\gamma}=\gamma\frac{T_B^0-1}{T_B-1}+\frac{1}{T-T_B}\sum_{t=T_B+1}^{T}\epsilon_t-\frac{1}{T_B-1}\sum_{t=2}^{T_B}\epsilon_t$. By the FCLT we obtain (A.28). Rearranging (A.22) we have $\tilde{S}_t=(x_t-x_1)-(\tilde{\gamma}_1-\gamma_1')(t-1)-(\tilde{\gamma}-\gamma'')DT_t-(\gamma_1'-\gamma_1)(t-1)-\gamma'' DT_t+\gamma DT_t^0$. The first term is $O_p(T^{1/2})$ by the FCLT. The second and third terms are also $O_p(T^{1/2})$ by (A.28). The last three terms are $O(T)$. Therefore we arrive at (A.29)–(A.30) for the case $\lambda>\lambda_0$. □

Lemma 3. If $\gamma=0$, or if $\gamma\neq 0$ and $\lambda=\lambda_0$, then $T^{-2}\tilde{S}_{-1}M\tilde{S}_{-1}\Rightarrow\sigma^2\int_0^1\underline{V}(r,\lambda)^2dr$ where $\underline{V}(r,\lambda)$ denotes the residuals from the projection of the process $V(r,\lambda)$ onto the subspace generated by the functions $\{1,du(r,\lambda)\}$ with $du(r,\lambda)=1(r>\lambda)$.

Proof of Lemma 3. The result follows directly by (A.26) in Lemma 1. □

Lemma 4. If $\gamma\neq 0$ and $\lambda\neq\lambda_0$ then $T^{-3}\tilde{S}_{-1}M\tilde{S}_{-1}\xrightarrow{p}\gamma^2 h(\lambda,\lambda_0)$ where

$$h(\lambda,\lambda_0)=\begin{cases} \frac{(\lambda_0-\lambda)^2(1-\lambda_0)^2}{12(1-\lambda)} & \text{if} \quad \lambda<\lambda_0, \\ \frac{(\lambda-\lambda_0)^2\lambda_0^2}{12\lambda} & \text{if} \quad \lambda>\lambda_0. \end{cases} \tag{A.31}$$

Proof of Lemma 4. $\tilde{S}_{-1}M\tilde{S}_{-1}$ represents the sum of the squared residuals from the regression of $\tilde{S}_{t-1}$ on 1 and DU_t. This is equivalent to a regression of $\tilde{S}_{t-1}$ on $1-DU_t$ and DU_t. Therefore, by (A.29) in Lemma 2 it follows that:

$$\begin{aligned} T^{-3}\tilde{S}_{-1}M\tilde{S}_{-1} \quad \xrightarrow{p} \quad & \int_0^\lambda\left[\gamma f(r,\lambda)-\frac{1}{\lambda}\int_0^\lambda\gamma f(s,\lambda)ds\right]^2dr \\ & +\int_\lambda^1\left[\gamma f(r,\lambda)-\frac{1}{1-\lambda}\int_\lambda^1\gamma f(s,\lambda)ds\right]^2dr. \end{aligned}$$

By computing the integrals we arrive at the desired result. □

Lemma 5. If $\gamma \neq 0$ then $T^{-1}\Delta\tilde{S}'\Delta\tilde{S} \xrightarrow{p} \sigma_\epsilon^2 + \gamma^2 g(\lambda, \lambda_0)$ where

$$g(\lambda,\lambda_0) = \begin{cases} \frac{(1-\lambda_0)(\lambda_0-\lambda)}{1-\lambda} & \text{if} \quad \lambda < \lambda_0, \\ 0 & \text{if} \quad \lambda = \lambda_0, \\ \frac{\lambda_0(\lambda-\lambda_0)}{\lambda} & \text{if} \quad \lambda > \lambda_0. \end{cases}$$

If $\gamma = 0$ then $T^{-1}\Delta\tilde{S}'\Delta\tilde{S} \xrightarrow{p} \sigma_\epsilon^2$.

Proof of Lemma 5. From (A.23) we have:

$$T^{-1}\Delta\tilde{S}'\Delta\tilde{S} = T^{-1}\sum_{t=2}^{T}\left(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1) - \tilde{\gamma}DU_t + \gamma DU_t^0\right)^2. \tag{A.32}$$

Consider first the case $\gamma \neq 0$. When $\lambda = \lambda_0$ we have that $DU_t = DU_t^0$ and, from Lemma 1, $\tilde{\gamma}_1 \xrightarrow{p} \gamma_1$ and $\tilde{\gamma} \xrightarrow{p} \gamma$. It then follows that $T^{-1}\Delta\tilde{S}'\Delta\tilde{S} \xrightarrow{p} \sigma_\epsilon^2$. When $\lambda < \lambda_0$ we rewrite (A.32) as

$$T^{-1}\Delta\tilde{S}'\Delta\tilde{S} = T^{-1}\sum_{t=2}^{T}\left(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1) - \tilde{\gamma}(DU_t - DU_t^0) - (\tilde{\gamma} - \gamma)DU_t^0\right)^2. \tag{A.33}$$

The result follows easily from (A.33) since by Lemma 2 we have that $\tilde{\gamma}_1 - \gamma_1 \xrightarrow{p} 0$ and $\tilde{\gamma} - \gamma \xrightarrow{p} -\gamma\frac{\lambda_0-\lambda}{1-\lambda}$. Finally, when $\lambda > \lambda_0$, we rewrite (A.32) as

$$\begin{aligned} T^{-1}\Delta\tilde{S}'\Delta\tilde{S} &= T^{-1}\sum_{t=2}^{T}\Big(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1)(1 - DU_t^0) \\ &\quad -(\tilde{\gamma}_1 - \gamma_1 - \gamma)(DU_t^0 - DU_t) - (\tilde{\gamma}_1 - \gamma_1 + \tilde{\gamma} - \gamma)DU_t\Big)^2. \end{aligned} \tag{A.34}$$

The result follows again easily from (A.34) since by Lemma 2 we have that $\tilde{\gamma}_1 - \gamma_1 \xrightarrow{p} \gamma\frac{\lambda-\lambda_0}{\lambda}$ and $\tilde{\gamma} - \gamma \xrightarrow{p} \gamma\frac{\lambda_0-\lambda}{\lambda}$.

For $\gamma = 0$ we have from (A.23) that

$$T^{-1}\Delta\tilde{S}'\Delta\tilde{S} = T^{-1}\sum_{t=2}^{T}\left(\epsilon_t - (\tilde{\gamma}_1 - \gamma_1) - \tilde{\gamma}DU_t\right)^2.$$

Since by Lemma 1 we have that $\tilde{\gamma}_1 - \gamma_1 \xrightarrow{p} 0$ and $\tilde{\gamma} \xrightarrow{p} 0$, it follows that $T^{-1}\Delta\tilde{S}'\Delta\tilde{S} \xrightarrow{p} \sigma_\epsilon^2$ proving the result. □

Lemma 6. If $\gamma = 0$, or if $\gamma \neq 0$ and $\lambda = \lambda_0$, then $\hat{\phi} = O_p(T^{-1})$. If $\gamma \neq 0$ and $\lambda \neq \lambda_0$ then $\hat{\phi} = O_p(T^{-2})$.

Proof of Lemma 6. Using (A.6) the result follows from Lemmas 3, 4 and 5. □

Lemma 7. We have that $s^2 \xrightarrow{p} \sigma_\epsilon^2 + \gamma^2 g(\lambda, \lambda_0)$.

Proof of Lemma 7. Using (A.4) and (A.5) in (A.8), we can write s^2 as

$$\begin{aligned} s^2 &= \frac{1}{T-1}(\Delta\tilde{S}'M\Delta\tilde{S} - 2\hat{\phi}\tilde{S}'_{-1}M\Delta\tilde{S} + \hat{\phi}^2\tilde{S}'_{-1}M\tilde{S}_{-1}) \\ &= \frac{1}{T-1}((1+\hat{\phi})\Delta\tilde{S}'\Delta\tilde{S} + \hat{\phi}^2\tilde{S}'_{-1}M\tilde{S}_{-1}). \end{aligned}$$

The result then follows easily from Lemmas 3, 4, 5 and 6. □

Lemma 8. If $\gamma = 0$, or if $\gamma \neq 0$ and $\lambda = \lambda_0$, then $t_\phi(3, [\lambda T]) \Rightarrow R(\lambda)$ with $R(\lambda)$ as defined in (15).

Proof of Lemma 8. Using (A.7), the result follows from Lemmas 3, 5 and 7. □

Proof of (17) in the text. We show that in the absence of a break, $\gamma = 0$, then (17) holds. The t-statistic for testing that the coefficient of DU_t in the first step regression (A.19) is equal to zero when $T_B = [\lambda T]$ is given by $t_{\Delta Z}(\lambda) = \tilde{\gamma}/\sqrt{\hat{\sigma}^2 m_{22}}$ where m_{22} denotes the row 2, column 2 element of $(\Delta Z'\Delta Z)^{-1}$ and $\hat{\sigma}^2 = \Delta\tilde{S}'\Delta\tilde{S}/(T-1)$ is the estimated variance of the errors in (A.19). It is easy to see that $Tm_{22} \xrightarrow{p} \frac{1}{\lambda(1-\lambda)}$. By Lemma 5 we have that $\hat{\sigma}^2 \xrightarrow{p} \sigma_\epsilon^2$. By applying Lemma 1 we have that $t_{\Delta Z}(\lambda) \Rightarrow \frac{1}{\omega}\sqrt{\lambda(1-\lambda)}\left(\frac{W(1)-W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda}\right)$. Since $T_B(t_{\Delta Z})$ is the break date that maximizes $t_{\Delta Z}(\lambda)$, by using Lemma 8 and the CMT as in Zivot and Andrews (1992), we have that $t_\phi(3, T_B(t_{\Delta Z})) \Rightarrow R(\tilde{\lambda})$ where $\tilde{\lambda} = \arg\max_{\lambda\in\Lambda} \sqrt{\lambda(1-\lambda)}\left(\frac{W(1)-W(\lambda)}{1-\lambda} - \frac{W(\lambda)}{\lambda}\right)$, proving the result. □

Proof of (21) in the text. We show that if there is a break, $\gamma \neq 0$, whose date is misplaced, $\lambda \neq \lambda_0$, then (21) holds. By using Lemmas 4, 5 and 7 in (A.7) it follows that

$$T^{1/2}t_\phi(3, [\lambda T]) \xrightarrow{p} -\frac{1}{2}\left(\gamma^2 h(\lambda, \lambda_0)\right)^{-1/2}\left(\sigma_\epsilon^2 + \gamma^2 g(\lambda, \lambda_0)\right)^{1/2},$$

proving the result. □

Proof of (23) in the text. We show that in the presence of a break, $\gamma \neq 0$, (23) holds. We first derive the limiting behavior of $t_{\Delta Z}(\lambda) = \tilde{\gamma}/\sqrt{\hat{\sigma}^2 m_{22}}$ when $\gamma \neq 0$. By Lemmas 1 and 2 we have that $\tilde{\gamma} \xrightarrow{p} \gamma k(\lambda, \lambda_0)$ where

$$k(\lambda, \lambda_0) = \begin{cases} \frac{1-\lambda_0}{1-\lambda} & \text{if} \quad \lambda < \lambda_0, \\ 1 & \text{if} \quad \lambda = \lambda_0, \\ \frac{\lambda_0}{\lambda} & \text{if} \quad \lambda > \lambda_0. \end{cases}$$

By Lemma 5 we have that $\hat{\sigma}^2 \xrightarrow{p} \sigma_\epsilon^2 + \gamma^2 g(\lambda, \lambda_0)$. We also have that $Tm_{22} \xrightarrow{p} \frac{1}{\lambda(1-\lambda)}$.

Combining these results we get

$$T^{-1/2} t_{\Delta Z}(\lambda) \xrightarrow{p} \gamma k(\lambda, \lambda_0) \sqrt{\lambda(1-\lambda)} \left(\sigma_\epsilon^2 + \gamma^2 g(\lambda, \lambda_0)\right)^{-1/2}.$$

It follows that the limiting behavior of $t_{\Delta Z}(\lambda)$ depends on λ_0. For any given λ_0 it is easy to see that this limiting function of λ attains a maximum at λ_0 when $\gamma > 0$ (and a minimum at λ_0 when $\gamma < 0$). It follows that λ_0 is chosen asymptotically which, together with Lemma 8, proves the result. $\square$

Proof of (18) and (24) in the text. Let $RSS(T_B)$ denote the residual sum of squares for the first step regression (A.19) when the break date used equals T_B. Since $\hat{T}_B = \arg\max_{T_B} RSS(T_B) = \arg\max_{T_B} t^2_{\Delta DT}(T_B)$, (18) and (24) follow by the same arguments used to prove (17) and (23) respectively. $\square$

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Table 1. Asymptotic Critical Values for $t_{\phi}(j, T_B)$ (j=2,3) for Fixed $T_B = [\lambda T]$

λ	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
.1	-3.84	-3.56	-3.30	-3.02	-2.55	-2.11	-1.71	-1.42	-1.28	-1.18	-1.06
.2	-4.06	-3.74	-3.51	-3.21	-2.76	-2.30	-1.89	-1.59	-1.43	-1.30	-1.19
.3	-4.13	-3.84	-3.59	-3.32	-2.89	-2.45	-2.06	-1.76	-1.59	-1.47	-1.35
.4	-4.15	-3.86	-3.63	-3.37	-2.97	-2.55	-2.18	-1.90	-1.75	-1.63	-1.51
.5	-4.15	-3.87	-3.66	-3.40	-2.99	-2.58	-2.22	-1.94	-1.80	-1.68	-1.57
.6	-4.12	-3.88	-3.63	-3.38	-2.98	-2.55	-2.17	-1.88	-1.73	-1.61	-1.48
.7	-4.12	-3.82	-3.58	-3.31	-2.89	-2.46	-2.05	-1.74	-1.59	-1.47	-1.33
.8	-4.03	-3.73	-3.48	-3.21	-2.76	-2.30	-1.89	-1.58	-1.42	-1.31	-1.20
.9	-3.88	-3.54	-3.29	-3.01	-2.56	-2.11	-1.72	-1.43	-1.28	-1.17	-1.06

Table 2. Critical Values for $t_\phi(1, T_B(t_\phi))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-4.46	-4.09	-3.80	-3.41	-2.87	-2.31	-1.83	-1.47	-1.31	-1.19	-1.08
	5	-5.17	-4.76	-4.43	-3.99	-3.36	-2.72	-2.20	-1.86	-1.68	-1.56	-1.44
100	0	-4.32	-3.91	-3.62	-3.26	-2.74	-2.21	-1.77	-1.45	-1.30	-1.18	-1.04
	5	-4.71	-4.33	-3.96	-3.58	-2.98	-2.42	-1.94	-1.62	-1.46	-1.36	-1.22
150	0	-4.18	-3.77	-3.47	-3.15	-2.65	-2.15	-1.73	-1.42	-1.27	-1.15	-1.04
	5	-4.37	-4.02	-3.70	-3.35	-2.82	-2.29	-1.85	-1.52	-1.37	-1.25	-1.15
∞		-3.56	-3.27	-3.02	-2.75	-2.34	-1.90	-1.54	-1.29	-1.16	-1.07	-0.97

Table 3. Critical Values for $t_\phi(2, T_B(t_\phi))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-5.46	-5.04	-4.71	-4.39	-3.87	-3.36	-2.87	-2.49	-2.30	-2.15	-2.00
	5	-6.34	-5.90	-5.59	-5.25	-4.70	-4.14	-3.63	-3.22	-3.00	-2.84	-2.66
100	0	-5.12	-4.79	-4.52	-4.21	-3.74	-3.25	-2.81	-2.45	-2.25	-2.10	-1.93
	5	-5.65	-5.30	-5.03	-4.71	-4.20	-3.65	-3.18	-2.80	-2.60	-2.45	-2.27
150	0	-5.02	-4.70	-4.44	-4.12	-3.66	-3.20	-2.77	-2.42	-2.22	-2.07	-1.90
	5	-5.41	-5.07	-4.79	-4.49	-3.99	-3.47	-3.02	-2.65	-2.46	-2.30	-2.13
∞		-4.74	-4.51	-4.27	-4.01	-3.57	-3.12	-2.70	-2.36	-2.18	-2.02	-1.87

Table 4. Critical Values for $t_\phi(3, T_B(t_\phi))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-5.32	-4.94	-4.61	-4.30	-3.78	-3.27	-2.80	-2.42	-2.24	-2.09	-1.94
	5	-6.20	-5.79	-5.48	-5.12	-4.57	-4.02	-3.49	-3.07	-2.84	-2.68	-2.47
100	0	-5.08	-4.74	-4.46	-4.16	-3.68	-3.20	-2.76	-2.40	-2.21	-2.07	-1.90
	5	-5.60	-5.23	-4.96	-4.64	-4.12	-3.57	-3.09	-2.69	-2.50	-2.34	-2.17
150	0	-4.96	-4.64	-4.37	-4.07	-3.62	-3.16	-2.73	-2.38	-2.19	-2.03	-1.88
	5	-5.34	-5.02	-4.74	-4.43	-3.92	-3.41	-2.94	-2.57	-2.38	-2.21	-2.04
∞		-4.74	-4.51	-4.27	-4.01	-3.57	-3.12	-2.70	-2.36	-2.18	-2.02	-1.87

Table 5. Critical Values for $t_\phi(1, T_B(t_{\Delta Z}))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-3.77	-3.45	-3.16	-2.85	-2.40	-1.95	-1.57	-1.30	-1.18	-1.09	-1.00
	5	-4.29	-3.92	-3.60	-3.21	-2.63	-2.11	-1.69	-1.38	-1.23	-1.09	-0.89
100	0	-3.75	-3.37	-3.13	-2.84	-2.37	-1.92	-1.57	-1.32	-1.19	-1.09	-0.98
	5	-4.01	-3.63	-3.31	-2.99	-2.48	-2.00	-1.61	-1.35	-1.21	-1.11	-1.00
150	0	-3.60	-3.28	-3.04	-2.76	-2.34	-1.91	-1.55	-1.30	-1.17	-1.07	-0.99
	5	-3.78	-3.47	-3.17	-2.87	-2.42	-1.96	-1.58	-1.31	-1.19	-1.09	-0.98
∞		-3.56	-3.27	-3.02	-2.75	-2.34	-1.90	-1.54	-1.29	-1.16	-1.07	-0.97

Table 6. Critical Values for $t_\phi(2, T_B(t_{\Delta Z}))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-4.53	-4.17	-3.87	-3.49	-2.94	-2.38	-1.91	-1.57	-1.40	-1.27	-1.14
	5	-5.11	-4.67	-4.32	-3.92	-3.29	-2.63	-2.07	-1.70	-1.50	-1.35	-1.15
100	0	-4.31	-3.99	-3.70	-3.36	-2.86	-2.35	-1.89	-1.56	-1.39	-1.25	-1.11
	5	-4.64	-4.26	-3.93	-3.56	-3.02	-2.45	-1.96	-1.60	-1.42	-1.29	-1.15
150	0	-4.21	-3.90	-3.63	-3.29	-2.80	-2.31	-1.86	-1.53	-1.37	-1.24	-1.12
	5	-4.53	-4.10	-3.76	-3.42	-2.91	-2.37	-1.90	-1.55	-1.39	-1.26	-1.13
∞		-4.00	-3.72	-3.47	-3.18	-2.72	-2.23	-1.81	-1.50	-1.33	-1.21	-1.09

Table 7. Critical Values for $t_\phi(3, T_B(t_{\Delta Z}))$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-4.47	-4.11	-3.81	-3.46	-2.91	-2.37	-1.91	-1.56	-1.40	-1.26	-1.14
	5	-5.01	-4.58	-4.21	-3.83	-3.22	-2.59	-2.06	-1.69	-1.49	-1.32	-1.11
100	0	-4.27	-3.96	-3.67	-3.34	-2.85	-2.35	-1.90	-1.56	-1.39	-1.26	-1.11
	5	-4.58	-4.18	-3.88	-3.53	-2.99	-2.43	-1.95	-1.60	-1.41	-1.29	-1.14
150	0	-4.21	-3.87	-3.59	-3.27	-2.80	-2.30	-1.87	-1.54	-1.37	-1.24	-1.12
	5	-4.46	-4.07	-3.74	-3.39	-2.88	-2.36	-1.89	-1.55	-1.38	-1.26	-1.13
∞		-4.00	-3.72	-3.47	-3.18	-2.72	-2.23	-1.81	-1.50	-1.33	-1.21	-1.09

Table 8. Critical Values for $t_\phi(1, \hat{T}_B)$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-3.83	-3.47	-3.19	-2.88	-2.41	-1.95	-1.58	-1.31	-1.18	-1.09	-0.99
	5	-4.35	-3.96	-3.62	-3.24	-2.64	-2.12	-1.69	-1.39	-1.23	-1.10	-0.95
100	0	-3.75	-3.37	-3.11	-2.82	-2.37	-1.93	-1.57	-1.31	-1.19	-1.09	-0.98
	5	-3.99	-3.61	-3.30	-2.97	-2.49	-2.00	-1.61	-1.34	-1.21	-1.10	-0.99
150	0	-3.61	-3.32	-3.06	-2.77	-2.33	-1.91	-1.55	-1.29	-1.17	-1.07	-0.98
	5	-3.83	-3.49	-3.18	-2.88	-2.41	-1.96	-1.58	-1.31	-1.19	-1.08	-0.98
∞		-3.56	-3.27	-3.02	-2.75	-2.34	-1.90	-1.54	-1.29	-1.16	-1.07	-0.97

Table 9. Critical Values for $t_\phi(2, \hat{T}_B)$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-4.52	-4.17	-3.88	-3.54	-3.03	-2.51	-2.06	-1.72	-1.54	-1.42	-1.28
	5	-5.15	-4.71	-4.41	-4.05	-3.45	-2.83	-2.28	-1.88	-1.67	-1.50	-1.29
100	0	-4.37	-4.02	-3.73	-3.43	-2.95	-2.48	-2.05	-1.72	-1.55	-1.40	-1.24
	5	-4.71	-4.34	-4.03	-3.68	-3.14	-2.62	-2.15	-1.78	-1.59	-1.44	-1.28
150	0	-4.26	-3.93	-3.67	-3.37	-2.92	-2.46	-2.04	-1.69	-1.52	-1.39	-1.25
	5	-4.47	-4.15	-3.85	-3.55	-3.05	-2.54	-2.09	-1.72	-1.55	-1.41	-1.26
∞		-4.07	-3.75	-3.50	-3.22	-2.78	-2.32	-1.90	-1.60	-1.44	-1.32	-1.22

Table 10. Critical Values for $t_\phi(3, \hat{T}_B)$

T	$kmax$	1%	2.5%	5%	10%	25%	50%	75%	90%	95%	97.5%	99%
50	0	-4.59	-4.22	-3.93	-3.57	-3.04	-2.52	-2.06	-1.72	-1.54	-1.43	-1.28
	5	-5.04	-4.64	-4.31	-3.95	-3.36	-2.74	-2.20	-1.82	-1.62	-1.45	-1.27
100	0	-4.35	-4.04	-3.78	-3.46	-2.96	-2.47	-2.03	-1.69	-1.53	-1.40	-1.26
	5	-4.62	-4.27	-3.98	-3.64	-3.10	-2.55	-2.09	-1.74	-1.55	-1.41	-1.27
150	0	-4.29	-3.97	-3.68	-3.37	-2.90	-2.43	-1.99	-1.67	-1.50	-1.38	-1.25
	5	-4.45	-4.14	-3.83	-3.51	-2.99	-2.47	-2.02	-1.68	-1.51	-1.37	-1.25
∞		-4.07	-3.75	-3.50	-3.22	-2.78	-2.32	-1.90	-1.60	-1.44	-1.32	-1.22

Table 11. Frequency of null rejections (Rej.) and correct break date selection (T_B^0) for LM-type tests: $t_\phi(j,\cdot)$ $(j=1,2,3)$

DGP: $y_t = \delta DU_t^0 + \gamma DT_t^0 + x_t$, $x_t = \alpha x_{t-1} + e_t$, e_t i.i.d. $N(0,1)$

$T=100$; $T_B^0 = 50$; 5% nominal size; $kmax = 0$

			t_ϕ	$t_\phi(1,T_B(t_\phi))$		$t_\phi(1,T_B(t_{\Delta Z}))$		$t_\phi(1,\hat{T}_B)$		$t_\phi(2,T_B(t_\phi))$		$t_\phi(2,T_B(t_{\Delta Z}))$		$t_\phi(2,\hat{T}_B)$		$t_\phi(3,T_B(t_\phi))$		$t_\phi(3,T_B(t_{\Delta Z}))$		$t_\phi(3,\hat{T}_B)$	
α	δ	γ	Rej.	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0	Rej.	T_B^0
1	5	0	.049	.047	.358	.051	.987	.052	.979	.044	.152	.043	.000	.061	.973	.041	.129	.043	.038	.042	.024
1	10	0	.034	.036	.477	.050	1.00	.052	1.00	.026	.282	.022	.007	.061	1.00	.021	.184	.021	.084	.018	.056
1	0	1	.000	.000	.000	.000	.019	.000	.011	.019	.085	.071	.210	.059	.131	.021	.101	.072	.264	.057	.264
1	0	2	.000	.000	.001	.000	.021	.000	.011	.017	.142	.067	.424	.060	.345	.019	.204	.068	.631	.052	.631
1	5	1	.000	.000	.001	.000	.992	.000	.986	.036	.072	.071	.576	.061	.996	.039	.097	.068	.542	.053	.542
1	10	2	.000	.000	.000	.000	1.00	.000	1.00	.078	.044	.066	1.00	.061	1.00	.084	.019	.045	.827	.033	.827
.8	0	0	.762	.750	–	.598	–	.573	–	.461	–	.475	–	.421	–	.447	–	.438	–	.310	–
.8	5	0	.356	.555	.671	.730	.976	.739	.964	.282	.408	.162	.000	.518	.958	.203	.216	.161	.025	.111	.008
.8	10	0	.027	.452	.915	.743	1.00	.756	1.00	.167	.721	.012	.000	.536	1.00	.030	.257	.012	.075	.006	.038
.8	0	1	.000	.000	.000	.000	.020	.000	.011	.267	.125	.537	.257	.454	.131	.291	.157	.541	.306	.476	.306
.8	0	2	.000	.000	.000	.000	.021	.000	.011	.249	.213	.546	.430	.510	.331	.268	.309	.555	.629	.490	.629
.8	5	1	.000	.000	.000	.000	.985	.000	.975	.333	.178	.493	.521	.535	.993	.341	.066	.371	.554	.312	.554
.8	10	2	.000	.000	.000	.000	1.00	.000	1.00	.380	.195	.558	1.00	.536	1.00	.376	.003	.124	.817	.095	.817

Table 12. Frequency of null rejections for LM-type tests for Model 2: $t_\phi(2,\cdot)$

DGP: $y_t = \delta DU_t^0 + \gamma DT_t^0 + x_t$, $x_t = \alpha x_{t-1} + \rho \Delta x_{t-1} + e_t + \psi e_{t-1}$, e_t i.i.d. $N(0,1)$

$T = 100$; $T_B^0 = 50$; 5% nominal size; $kmax = 5$

			Size ($\alpha = 1$)					Power ($\alpha = 0.8$)				
			δ, γ	$\delta(\gamma = 0)$		$\gamma(\delta = 0)$		δ, γ	$\delta(\gamma = 0)$		$\gamma(\delta = 0)$	
ρ	ψ	T_B	0.0	5.0	10.0	1.0	2.0	0.0	5.0	10.0	1.0	2.0
0.0	0.0	$T_B(t_\phi)$	.050	.043	.024	.028	.026	.332	.073	.168	.167	.156
		$T_B(t_{\Delta Z})$	.050	.045	.022	.076	.079	.348	.102	.008	.398	.414
		$\hat{T}_B$	.050	.063	.064	.058	.065	.291	.364	.377	.321	.358
0.6	0.0	$T_B(t_\phi)$	.055	.049	.036	.049	.042	.881	.750	.599	.751	.708
		$T_B(t_{\Delta Z})$	.048	.043	.034	.081	.083	.697	.476	.162	.726	.768
		$\hat{T}_B$	.066	.072	.072	.072	.068	.701	.733	.780	.680	.748
-0.6	0.0	$T_B(t_\phi)$	.072	.050	.017	.034	.032	.224	.076	.025	.089	.077
		$T_B(t_{\Delta Z})$	.053	.031	.013	.078	.078	.156	.030	.005	.186	.193
		$\hat{T}_B$	.054	.058	.064	.050	.056	.143	.131	.164	.103	.141
0.0	0.5	$T_B(t_\phi)$	.073	.067	.046	.048	.042	.359	.263	.140	.221	.204
		$T_B(t_{\Delta Z})$	.061	.062	.040	.095	.100	.283	.187	.053	.333	.361
		$\hat{T}_B$	.077	.086	.087	.082	.086	.284	.326	.336	.297	.327
0.0	-0.5	$T_B(t_\phi)$	.430	.297	.118	.255	.236	.859	.565	.358	.635	.594
		$T_B(t_{\Delta Z})$	.158	.089	.022	.244	.250	.561	.083	.017	.576	.595
		$\hat{T}_B$	.176	.220	.229	.196	.223	.496	.526	.577	.433	.537