The different consumption functions of products and product differentiation

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Abstract: This paper develops a monopolistic competition model to study the characteristics of products, such as the different consumption functions of products, vertical differentiation (quality) and horizontal differentiation (individuality), and the division of labor in production. Our model contributes in several aspects: first, in contrast to many vertical and horizontal product differentiation models which assume that each consumer demands one unit of the product, the quantity demanded in our model is determined by the market. Second, models of general purpose products suggest that, as an extension of Hotelling's model, products with more functions will have higher production costs but lower per-unit-distance transaction costs, but no clear explanation is given. In our model, the consumption functions served by a product are represented by a functional interval, with the usage cost and production cost both variable. Third, we gave a simple production function which includes these product's characteristics and has the economies of functional specialization. Last, these factors are analyzed systematically in a general equilibrium model. Moreover, through comparative static analysis, we examine the correlations between the products' characteristics and the exogenous variables such as population and people's preferences, transaction costs and management efficiency, and production technology.

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1. Introduction

In reality, markets not only determine the quantity and price of products, but also the characteristics of products. One might summarize these characteristics into three aspects. The first is the different consumption functions of products. For example, as functional generalization, computer provides more and more functions of media play (such as TV, CD, VCD and DVD) and communication (such as telephone, fax and Email). On the other hand, thirty years ago, people used to wear a kind of canvas shoes for many sports items. As athletic shoes becoming more and more functionally specialized, you can buy any specialized athletic shoes for most sports items. The second is horizontal differentiation (Lancaster, 1979). Different people will prefer different products' characteristics, such as style, design, size, color, and others which make no obvious cost difference to the producers. The third is vertical differentiation (quality). Products with higher quality usually need more production costs.

In order to model the variation (the variety of differentiated products), two prevalent research streams evolved. The first is the class of spatial models in the spirit of Hotelling (1929) and Lancaster (1979). They use an interval or a circle to represent the characteristics space of products in which firms can compete in price, location (horizontal differentiation), and quality (vertical differentiation). It has been extensively treated in a series of papers, such as D′Aspremont, Gabszewicz and Thisse (1979), Gabszewicz and Thisse (1979 and 1986), Shaked and Sutton (1982, 1983 and 1987), Ferreira and Thisse (1996), De Frutos, Hamoudi and Jarque (1999) and others.

The second is the class of non-spatial models in the spirit of Chamberlin (1933), Spence (1976), and Dixit and Stiglitz (1977), Krugman (1980, 1981), Yang and Shi (1992) and others. They have endogenized the differentiation by formalizing a trade off between the economies of scale and the preference for diverse consumption.

Comparing the spatial with the non-spatial models, each framework has strong points. First, in the spatial models, consumers and products can be described intuitively by the characteristics space, while the degree of differentiation the non-spatial models is represented exogenously by the elasticity coefficient in a CES utility function. As a result of this difference, a consumer in spatial models will choose the best variety in characteristic space; while in non-spatial models, consumer may choose all the varieties. Second, for the convenience of computation, Liner and Singer (1937) assume that the demand function in spatial models is rectangular: each consumer demands one or zero unit of the product. And it has become a default assumption in many spatial models, while there is no such a constraint in non-spatial models.

As an extension of Hotelling's model, Von Ungern-Sternberg (1988) and Weitzman (1994) study the general purpose products. They assume that the products with more general purpose will have higher production costs but less per-unit-distance transaction costs, the trade off between these two types of costs determines the degree of specialization of the product. However, no explanation is given on the reasons why there is a tradeoff between these two types of costs.

The purpose of this paper is to formalize these characteristics into a monopolistic competition model. Clearly, the explanation is not new as it comes from the Hotelling-Lancaster spatial model. What we do is to make some improvements and extensions, and synthesize them into a general equilibrium framework. First, in contrast to the rectangular demand function assumption in many of the spatial models which misses the result that people's trade off between the variety and quantity of products, the quantity demanded in our model is determined by the market. Second, to keep the intuition in Hotteling's model, we use another circle with unit circumference to represent the products' functions. The consumption functions served by a product are represented by a functional interval, with the usage costs and production costs both variable. Products with a wider range of functional interval will need more usage costs and variable costs but less fixed costs, and vice versa. This balance decides the boundary of products. Third, we gave a simple production function which includes these characteristics of products, and reveals the economies of functional specialization and economies of scale. Last, these factors are analyzed systematically in a general equilibrium model. Moreover, through comparative static analysis, we examine the effects of changes in the exogenous variables such as population size, preferences, transaction costs and management efficiency, and production technology on the characteristics of products.

This paper is organized as follows. In section 2, we study the model and its main results. Five steps were taken in the analysis of the model. First, we introduce the utility function. Then, we introduce the production function. Third, through general equilibrium analysis, we find the equilibrium characteristics of products. Fourth, comparative-static analysis is undertaken. Section 3 is the discussion of results. This paper concludes in section 4, and section 5 contains all the proofs of propositions as an appendix.

2. The Model

Suppose in an economy there are M identical individuals each with one unit of time endowment, and each kind of product has three characteristic indexes defined below. Suppose that there are two circles, C_1 and C_2, where C_1 is unit circumference and C_2 is l_2 in circumference; each kind can be represented as $(s, s']\times t\times x$, with $(s, s']\subset C_1$, $t\in C_2$ and $x\in R^+=[0, +\infty)$. Where C_1 is the **function space** and each element of C_1 is the **index of the product function**, and $(s, s']$ is the **functional interval**; C_2 is the **individuality space** and t the **index of individuality**. l_2 represents the extent of the diversity of individuality in the market; R^+ is the **quality space** and x is the **index of quality**. $(s, s']\times t\times x$ represents the product which provides the function of $(s, s']$, individuality index t and quality index x. For simplicity we suppose that different persons have different t values which uniformly distribute in C_2. For convenience, we use t to represent individuals later. Denote m $(=M/l_2)$ as the **density of population in individuality space**.

There are some differences between the product of this paper with that of the others. In most economics model, the characteristics of a product is (partially) exogenous; while in our model, the characteristics of a product is endogenous by some characteristics variables, such as the function interval, individuality and quality indexes. Moreover, because these three indexes are represented by continuums, that will be more appropriate to analyse the variability of product. Furthermore, we use different function intervals in C_1 to represent different **kinds** of products (differing in functions), and use the same function interval but different individuality or quality index in C_2 to represent different **varieties** of products (differing in individuality).

2.1 Utility functions

Suppose that there are I kinds of product and J_i varieties for each kind i, for $1\le i\le I$ and $1\le j\le J_i$, product ij represented as $(s_{i-1}, s_i]\times t_{ij}\times x_{ij}$, where $(s_{i-1}, s_i]$ is product ij's functional interval; t_{ij} is its individuality index and x_{ij} is its quality index. For simplicity, we also suppose that $\bigcup_{i=1}^{N_s}(s_{i-1}, s_i]=C_1$ and $(s_{i-1}, s_i]\cap(s_{h-1}, s_h]=\Phi$, for $i\neq h$. That means all products consumed will cover the whole function space, and different products will not overlap in function.

For any $t\in C_2$, suppose **person t's utility function** is

$$u(q_{ij},1\le i\le I,1\le j\le J_i)=\prod_{i=1}^{I}\left(\sum_{j=1}^{J_i} e^{-\rho_J\left|t_{ij}-t\right|}x_{ij}^{\rho_x}q_{ij}\right)^{s_i-s_{i-1}} \tag{2.1}$$

Where q_{ij} is the quantity of product $(s_{i-1}, s_i]\times t_{ij}\times x_{ij}$, ρ_J ($\rho_J>0$) is the **individuality coefficient**, and ρ_x ($\rho_x>0$) is the **quality elasticity coefficient**.

Suppose person t's **budget constraint** is

$$\Sigma_{i=1}^{I}\Sigma_{j=1}^{J_i} p_{ij} k e^{c_I(s_i - s_{i-1})} q_{ij} = r_t \tag{2.2}$$

Where p_{ij} is the mill price of product $(s_{i-1}, s_i]\times t_{ij}\times x_{ij}$ which excludes transaction cost; there are transaction costs in the market, where k (k>1) is the **transaction cost coefficient** (when there is no transaction costs, k=1) and $p_{ij}k$ is the price of product for which consumers need to pay; there are also usage costs when people consume products, where c_I (c_I>0) is the **functional interval coefficient** and $e^{c_I(s'_i - s_i)}$ represents the usage costs. That means that consumers need to pay more usage costs for a product with a longer length of functional interval.

Person t will choose the products through:

$$\begin{aligned} &\max \prod_{i=1}^{I}(\Sigma_{j=1}^{J_i} e^{-\rho_J|t_{ij}-t|} x_{ij}^{\rho_x} q_{ij})^{s_i - s_{i-1}} \\ &st: \Sigma_{i=1}^{I}\Sigma_{j=1}^{J_i} p_{ij} k e^{c_I(s_i - s_{i-1})} q_{ij} = r_t \end{aligned} \tag{2.3}$$

(2.3) implies several properties.

1. The function space represents all the necessary function in consumption. From (2.1) we know that, as a basic property of CD utility function, if the union of function intervals of product from which we consume doesn't cover the whole function space, no mater how much we consume, the utility level is always zero. That means in this case we miss out on some essential products.

2. In each kind of product, consumers (except marginal consumers) will choose only one variety of products according to their preference. From (2.3) we know that for each functional interval, consumer t will choose these J_i varieties by

$$\max_{1\le j\le J_i} \frac{e^{-\rho_J|t_{ij}-t|} x_{ij}^{\rho_x}}{p_{ij}} \tag{2.4}$$

It implies that consumer t will just choose one variety of products unless more than one variety of products gets the maximum at the same time, a case where consumer t is a marginal consumer.

3. It is assumed that the consumer is indifferent to how the whole function space is covered by different combinations of different kinds of products, provided that the individuality and quality indexes are the same. For example, suppose that consumer t has two products, $(s_1, s_2]\times t_1\times x_1$ and $(s_2, s_3]\times t_1\times x_1$. They have the same individuality and quality indexes but different function intervals. The combination of these two kinds is $(s_1, s_3]\times t_1\times x_1$. Because $(e^{-\rho_J|t_1-t|}x_1^{\rho_x}q_1)^{s_2-s_1}(e^{-\rho_J|t_1-t|}x_1^{\rho_x}q_1)^{s_3-s_2} = (e^{-\rho_J|t_1-t|}x_1^{\rho_x}q_1)^{s_3-s_1}$, From utility function (2.1) we know that Consumer t will get the same utility.

4. The usage costs are economies of functional specialization. From budget line (2.2) we know that the usage cost in consuming q piece of products with functional interval $(s_1, s_3]$ is $(e^{c_I(s_3-s_1)}-1)q$. Suppose that $s_1<s_2<s_3$, from $e^{c_I(s_2-s_1)}+e^{c_I(s_3-s_2)}<e^{c_I(s_3-s_1)}+1$, we have $(e^{c_I(s_2-s_1)}-1)q+(e^{c_I(s_3-s_2)}-1)q<(e^{c_I(s_3-s_1)}-1)q$. That means the usage costs are economies of functional specialization.

5. Each person has a particular preference for the style or individuality (which could be colour, size, etc.) of a product. Since ρ_J>0, the lower is $|t_i - t|$, the distance of individuality index between the product and consumer, the higher the utility level. For example, suppose someone wears L size shirt, so he/she will get more satisfaction if the shirt he/she wears has the size closer to L.

6. Since ρ_x>0, from (2.1) we know that a product with higher quality will provide more utility.

2.2 production function

In this model, we suppose that the factories are privately own firms. The owner's goal is maximum profit. For the convenience of computation, we assume that labor time is the only input; the owner's and all workers' time is used in production; and the management cost is assumed to be proportional to the labor time.

Suppose that a factory which produces product $(s, s']\times t\times x$ has a pair of output, quantity and quality, and its production functions is

$$c_q(s'-s)q + c_x(s'-s)x + a_1 + a_2(s'-s) = eL \tag{2.5}$$

where q and x are the quantity and quality of the product. The marginal cost of quantity and quality, $c_q(s'-s)/e$ and $c_x(s'-s)/e$, are assumed to vary in accordance to the functional interval, and c_q and c_x are called as **marginal cost coefficients** of quantity and quality, respectively; is the entry cost for each kind of product is taken to be $[a_1+a_2(s'-s)]/e$, where positive numbers a_1 and a_2 are the **entry cost coefficients**; labor time L is the only input in production, and e is the **management efficiency coefficient**, where 0<e<1. For L units of labor time, the factory need (1-e)L units of management time.

The definition of production function (2.5) implies two properties.

1. The production functions reveal economies of functional specialization. From (2.5) we know that $a_1+a_2(s'-s)$ is the entry cost in producing quantity and quality. A product with less length of functional interval $(s'-s)$ will have less entry cost, so that functional specialization can improve productivity.

2. The production functions display economies of scale. From (2.5) we know that when $k\geq 1$, the output $(c_q q+c_x x)$ with kL input is more than the k times of output with L.

2.3 Equilibrium analysis

The goal of this section is to find the characteristics of products, such as quality, the range of functional interval and the distribution of individuality indexes, in a symmetric equilibrium state.

Although we use static equilibrium analysis to deal with our model, strategies are arranged in three steps. First, each individual decides to become the owner of a factory or an employee. Second, each employee's strategy is to work with one unit time for income, while an owner has four strategy variables to maximize profit: price (or equivalently the quantity) and the quality of the product; the individuality index (location) in individuality space C_2; the last strategy variable is the functional interval. Third, individuals trade, distribute and consume.

For computational simplicity, our model only deals with the special case in which there is no overlapping of functional interval. Next, we suppose that each factory produces only one kind of products, $(s, s']\times t\times x$, that means each factory chooses only one functional interval $(s, s']$, one individuality index t, and one quality index x.

Because all individuals are assumed identical and are free to decide to become the owner of a firm or become an employee, and production functions are unique, without loosing any generality, we assume that the wage rate is one, so that each employee will get one unit of income, and each owner will get one unit of profit in equilibrium states.

Because all owners and employees have the same income in the equilibrium state, lemma 1 reveals a necessary condition of equilibrium solution.

Lemma 1 Suppose that there are J factories in an industry, say industry i, and their positions in C_2 is $\{t_{i1}, t_{i2}\dots t_{iJ} \mid t_{iJ}=t_{i0}=0\}$. For $1\leq j\leq J$, suppose that factory ij maximizes profit in (p_{ij}, x_{ij}, t_{ij}), where p_{ij} is the price and x_{ij} is the quality of product. For $1\leq j\leq J$, if t_{ij+1}-t_{ij-1}don't have the same value, then (p_{ij}, x_{ij}) will not be the same, and the maximized profit π (p_{ij}, x_{ij}, t_{ij}) will not have the same value.

Actually, there are infinite symmetric solutions in our model, but all these solutions have the same structure. For this reason, two restrictions are added to our model so that we have only to consider one symmetric solution.

In the symmetric state where there are I kinds of goods and there are J varieties in each kind of goods, we suppose that, for $1\leq i\leq I$, and $0\leq j\leq J-1$, the kind of products produced by factory ij can be described as $(\frac{i-1}{I},\frac{i}{I}]\times\frac{j}{J}l_2\times x_{ij}$. This assumption adds two restrictions to the symmetric state. First, the functional interval of the first kinds of products begins at point 0 in C_1. Next, in each C_2, the first factory always stays at point 0 in C_2.

If we relax these two restrictions, we will have infinite symmetric solutions. For example, for $1\leq i\leq I$, and $0\leq j\leq J-1$, we assume that the kind of products produced by factory ij can be described as $(\frac{i-1}{I}+f,\frac{i}{I}+f]\times\frac{(j+h_i)l_2}{J}\times x_{ij}$. That means that the functional interval of the first kinds of products begins at point f in C_1, and in industry i, the first factory stays at point h_i in C_2. It's easy to see that all these symmetric solutions will have the same structure.

Proposition 1 There is a unique solution in symmetric equilibrium state of monopolistic competition, which is decided by (2.6), (2.7), (2.8), (2.9) and (2.10).

$$t_{ij}=\frac{jl_2}{J}, where\ t_{i0}=t_{iJ} \tag{2.6}$$

$$d_I=\frac{1}{2}[-\frac{a_1}{a_2}+\sqrt{(\frac{a_1}{a_2})^2+4\frac{a_1}{a_2c_I}}] \tag{2.7}$$

$$d_J=\frac{1}{2}[(\frac{\rho_x}{\rho_J}+n)+\sqrt{(\frac{\rho_x}{\rho_J}+n)^2+\frac{4n}{\rho_J}}],\ n=\frac{a_1e^{c_Id_I}k}{c_Id_I^2me} \tag{2.8}$$

$$p=\frac{c_qd_I(1+\rho_Jd_J)}{e} \tag{2.9}$$

$$x=\frac{\rho_xmed_J}{c_xke^{c_Id_I}(1+\rho_Jd_J)} \tag{2.10}$$

$$q_f=\frac{med_J}{c_qke^{c_Id_I}(1+\rho_Jd_J)} \tag{2.11}$$

$$q_c=\frac{Jq_f}{M}=\frac{e}{c_qke^{c_Id_I}(1+\rho_Jd_J)} \tag{2.12}$$

where d_I (=1/I) is the length of functional interval and d_J (=l_2/J) is the distance of two neighbor indexes of individuality in C_2, and q_f and q_c is respectively **the quantity produced by each factory** and **the quantity consumed by each individual**.

Proposition 2 shows that, although there are infinite asymmetric solutions for the equilibrium states, from the standard of social welfare, the symmetric solution is better than the asymmetric solutions mentioned.

Proposition 2 when J is an even number, there is infinite asymmetric solutions. For example, for any -1<h<1, (2.13), (2.7), (2.8), (2.9) and (2.10) decide a solution of monopolistic competition. When h=0, it's the symmetric solution of proposition 1; while h≠0, it's the asymmetric solution. Moreover, the social average utility Eu(h) is a decreasing function of $|h|$.

$$\begin{cases} t_{ij} = \dfrac{2kl_2}{J} & \text{when } j = 2k, \text{where } t_{i0} = t_{iJ} \\ t_{ij} = \dfrac{(2k+1+h)l_2}{J} & \text{when } j = 2k+1, \ -1 < h < 1 \end{cases} \tag{2.13}$$

2.4 Comparative static analysis

Through comparative static analysis, we will find the correlations between the products' characteristics and the exogenous variables such as population size and individual preferences, transaction cost and management efficiency, and production technology.

To reduce complexity, we use some simple notes to represent the correlations. For example, if z is an increasing in x but decreasing in y, we will represent these relation with formula z=(x ↑ , y ↓). Next example, in equation F(x, y)=0, if function F(x, y) is increasing in x but decreasing in y, we will denote as F(x ↑ , y ↓)=0. This basic technique is often used in the proof of the propositions below.

Lemma 2 (1) if F(z ↓ , x ↑ , y ↓)=0, then z=(x ↑ , y ↓) (2.14)
(2) if F(z ↑ , x ↑ , y ↓)=0, then z=(x ↓ , y ↑)

Corollary 1 the equilibrium length of functional interval products satisfies:

$$d_I = \frac{1}{I} = (a_1 \uparrow, a_2 \downarrow, c_I \downarrow) \tag{2.15}$$

$$c_I d_I = (a_1 \uparrow, a_2 \downarrow, c_I \uparrow) \tag{2.16}$$

Corollary 2 the equilibrium varieties satisfy:

$$d_J = \frac{l_2}{J} = (m \downarrow, e \downarrow, k \uparrow, c_I \uparrow, \rho_J \downarrow, \rho_x \uparrow, a_1 \uparrow, a_2 \uparrow) \tag{2.17}$$

$$\rho_J d_J = (m \downarrow, e \downarrow, k \uparrow, c_I \uparrow, \rho_J \uparrow, \rho_x \uparrow, a_1 \uparrow, a_2 \uparrow) \tag{2.18}$$

Corollary 3 the equilibrium quality satisfies:

$$x = (m \uparrow, e \uparrow, k \downarrow, c_I \downarrow, \rho_J \downarrow, \rho_x \uparrow, a_1 \downarrow, a_2 \uparrow, c_x \downarrow) \tag{2.19}$$

Corollary 4 the equilibrium quantity produced by each factory (q_f) satisfies:

$$q_f = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\uparrow, a_1\downarrow, a_2\uparrow, c_q\downarrow) \tag{2.20}$$

Corollary 5 the equilibrium quantity consumed by each consumer (q_c) satisfies:

(1) $$q_f = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\downarrow, a_1\downarrow, c_q\downarrow) \tag{2.21}$$

(2) $$q_c = (a_2\uparrow)(\downarrow) \Leftrightarrow \sqrt{(1+\frac{\rho_x}{n\rho_J})^2 + \frac{4}{n\rho_J}} > (<) \sqrt{1+\frac{4a_2}{a_1 c_I}} \tag{2.22}$$

We have $q_c = (a_2\uparrow)$ when $\frac{me}{k}$ is large enough, while ρ_J is large enough we have $q_c = (a_2\downarrow)$.

The main conclusions from corollary 1 to 5 are summarized in table 1, where endogenous variables are set in column one while exogenous variables are set in row one. In table 1, sign "+" means the relevant variables have positive correlation, while "–" means negative correlation; sign "0" means no correlation between them and sign "+/-" means there are both positive and negative correlations between them.

Table 1: correlation summary

	m	k	e	c_I	ρ_J	ρ_x	a_1	a_2	c_x	c_q
d_I	0	0	0	-	0	0	+	-	0	0
J/l_2	+	-	+	-	+	-	-	-	0	0
x	+	-	+	-	-	+	-	+	-	0
q_c	+	-	+	-	-	-	-	+/-	0	-
q_f	+	-	+	-	-	+	-	+	0	-

3 Discussion of results

The advantage of the synthetic analysis in our model is that it reveals how people tradeoff between the quantity and the characteristics of product, such as the range functional interval, the index of individuality and quality. In this section, we will explain the results in table 1 intuitively and also give examples in our daily life.

For endogenous variables, the range of functional interval, the variety, the quality and the quantity consumed by each person (d_I, J/l_2, x, q_c) are four choice variables of consumption, which are directly affected by the factors from the demand side; while the former three variables and the quantity produced by each factory (d_I, J/l_2, x, q_f) are four choice variables of production, which are directly affected by the factors of the supply side.

The exogenous variables are divided into two groups. The first four variables are (m, k, e, c_I). Because they induce the entire endogenous variables to change in the same direction, we call these effects as growth effects. On the other hand, the last six variables (ρ_J, ρ_x, a_1, a_2, c_x, c_q) will bring about mixed effects, the factor may be positively correlated with some variables but negatively correlated with the others.

3.1 Growth effects

Column m, k, e, and c_I show the 'growth effects'. The increase of density of population and management efficiency, and the decrease of transaction cost and usage cost will increase the quantity and improve all three characteristics indexes. Intuitively, with the condition of increase returns in production, the increase in the density of population, the decrease in transaction costs or usage costs will increase the demand and thus improve the production efficiency, while the increase of management efficiency will directly improve the production efficiency. Consequently, positive profit will attracts more producers to enter into the market and induce them to increase output and improve the characteristics indexes.

This conclusion throws some light on the intra-industry trade theory. Krugman (1980, 1981) concludes that, with the condition of the economies of scale and identical endowment, there is no inter-industry trade, but lots of intra-industry trade which bring more variety of goods and benefit all income-earners. Nevertheless, our model points out that there is another factor, the individuality space, which will distort Krugman's conclusion. For example, suppose that there are two identity countries, say A and C; there is only one kind of products, say clothes; they have the same production function which needs one kind of input, say labor and exhibit increasing returns; the only difference in these two countries is that the people of A have bigger stature, they wear clothes of XXL size; while the people of C wear M size. According Krugman's conclusions, the common market of these two countries benefit all people with two varieties of clothes. But in reality, the people of A don't need size M and the people of C don't need size XXL, so the common market will not change the economic structures. In our model, it's the special case where two countries have the same size but completely different individuality space. The common market will double both population and circumference and hence doesn't change the density of population. As a result, there is no change in economic structures. More generally, if the common market is formed between two similar countries but with partially different individuality space, the common market will increase both the population and circumference which has the opposite effects to the density of population, and thus changes the effects of the common market.

3.2 Other effects

While the variables above bring the 'growth effects', the three pairs of variables below have mixed effects on the quantity and the three characteristics indexes.

1. For variables ρ_J and ρ_x, a larger ρ_J means that individuals have a higher degree of preference on individual style of the product; a larger ρ_x means that their preference for higher quality increases, As shown in Table 1, either under the same production capability of the economy (general equilibrium analysis) or the same income level of the individuals (partial equilibrium analysis), they will accommodate their higher preference by reducing the other requirements on products.

2. As c_q and c_x represent respectively the marginal cost coefficient of quantity and quality, and hence the decrease of c_q and c_x, say as the dynamic effect of learning by doing, will increase the quantity and quality respectively.

3. As a_1+ $a_2(s'$-s) represent the entry cost of products with functional range (s, s′], a change in $a_1$and a_2 has several effects.

First, formula (2.7) shows that the increase of ratio a_1/a_2 will enlarge the range of functional interval. Consumers need to buy all kinds of products. In they choose the situation of more kinds of products but less length of functional interval in each kind of products, they will pay more times of a_1. Hence, when a_1 is larger, people will prefer less kinds of goods but a larger length of functional interval. Intuitively, a_1 represents the common component and a_2 represent the different components in different products. The conclusion shows that different products which have larger portion of common component seem easier to be unified into one product, while a product with two independent components to a certain extent seems easier to be specialized into different products.

For example, as functional generalization, a computer provides more and more functions of media play (such as TV, CD, VCD and DVD) and communication (such as telephone and fax). It's due to the fact that computers already have some equipment of media play and communication. To provide these functions, it just needs to add some software and improve the equipment. On the other hand, if computers keep in the narrow function range, more basic equipment will be used in different products. This tradeoff decides the boundary of the function of the computer.

Next example, athletic shoes is becoming more and more functional specialized. Thirty years ago, people used to wear the general athletic shoes, the canvas shoes, for many sports items. Nevertheless, each sports item emphasizes different part of the shoes. For example, because basketball players often jump, so the basketball shoes have thick and soft bottom with gap in it; football players often run in wet ground and control the ball with the bottom of shoes, so that the football shoes have a hard bottom with rubber nails on it. This difference, along with others, make it difficulty to produce a kind of general shoes which keep the advantage of both basketball and football, and hence make the general athletic shoes easier to be specialized. Now you can buy any specialized athletic shoes for most sports items.

Second, when a_1 and a_2 change in opposite direction, there is a tradeoff between the range of functional interval and the quality of products. As mentioned above, say when a_1 increase and a_2 decrease, people will prefer more range of functional interval, for cost saving, they accept the lower quality. A large number of examples show that the tradeoff between the ranges of function intervals and the quality levels is ubiquitous. Here are several examples in our daily life. As mentioned above, the specialization of athletic shoes will improve their quality; the rice cooker offers people much more convenience in cooking rice than the general purpose cooker; there are many varieties of specialized automobiles, such as truck, bus, sedan and others, which perform better then the general purpose automobile. Conversely, although computer may offer the function as a media player, but its effect is not as good as the special audio and video equipment; today the mobile phone can take photo but its effect is not as good as the camera; supermarkets like Wal-Mart sell almost everything at low prices, but provides less service than the specialized stores; comparing with playing in outdoor court, people can play basketball, volleyball, badminton and other games in an indoor hall, but the overlapping area distorts players' utility.

Third, as shown in column a_1 and a_2, the decrease of entry cost will increase the variety of products, and vice versa. The decrease of entry cost will increase the profit and hence induce more producers to enter the market, so that more varieties of products will be provided. This means the cheaper products will have more varieties. For example, we have a lot of varieties of clothes or shoes, but comparatively, we have fewer varieties of cars or airplanes.

3.3 Consistent tendency rule

Besides the opposite tendency of functional range and quality, rows x and q_f in Tables 1 show that producers tend to change their quality and quantity in the same direction, except for factors c_x and c_q which affect quality and quantity differently.

From the analysis above, we know that growth effects make producers change the quantity and quality in the same direction. For other effects, let's see the endogenous variables one by one. First, because ρ_J is positively correlated only with the variety of products, so it has the same qualitative effects on quantity and quality. Second, although ρ_x is negatively correlated with the quantity consumed by each person (q_c) and the variety of products (J), but from the formulas (2.8) and (2.12) we know that the effect of J is dominating. From the formula q_f=Mq_c/J we know that the quantity produced by each factory is positively correlated to ρ_x and hence tend to change in the same direction as quality. Third, because the range of functional interval, quantity (q_f) and quality are three substitutable variables within a factory, the change of a_1 and a_2 which enlarges the range of functional interval will reduce the quantity and quality. Last, it's easy to understand why c_x and c_q will affect the quantity and quality respectively. Form the analysis of income and other effects, we conclude that the quantity and quality one of a producer tend to change in the same direction.

4. Conclusion

This paper shows the correlations between the quantity and the characteristics of products, such as the functional range, the quality and variety of products, and the exogenous variables such as population and people's preferences, transaction costs and management efficiency, and production technology. As its applications, we reveal how the market decides the boundary of functional range of the products, how the market decides the variety of differentiated products, and how differences in the preference for individual styles affect the results of a common market. Besides, benefiting from the multifactor analysis, this paper also offers a map to describe how people tradeoff among the quantity and the characteristics of products. For example, our model shows that the functional range and quality tend to move in opposite direction as exogenous variables change, and the quantity and quality of each producer tend to change in the same direction.

Nevertheless, as the cost of an integrative model, we have to simplify some assumptions which induce inevitably the gap between the model and reality. Here are two aspects.

First, in our model, there is only one kind of goods for any given function, but in reality, people may use different kinds of goods for the same function. For example, people may use mobile phone to take photo for convenience or use camera for higher quality. For more realism, one may wish to relax the assumption that the functional interval of different kinds of goods do not interact. Next, one may wish to relax the symmetry assumption in our model, or to check whether there is any asymmetric solution in our model. There may be asymmetric solutions even in a symmetric system. In our model we just have proved that there is only one symmetric solution, but have not shown whether there are asymmetric solutions.

The second aspect is the multi-products problem in Hotelling's framework. Our model assumes that each factory produces only one variety of products, while in reality each factory may produce several varieties of products.

As our model shows, there is a tradeoff between functional generalization and the quality of products. Similarly, the effort to more realism is blocked by computation complexity; model specialization is an easier way out.

5. Appendixes

Lemma 1 Suppose that there are J factories/producers in an industry, say industry i, their position in C_2 is $\{t_{i1}, t_{i2} \ldots t_{iJ} \mid t_{iJ}=t_{i0}=0\}$. For $1 \le j \le J$, suppose that factory ij maximizes profit in (p_{ij}, x_{ij}, t_{ij}), where p_{ij} is the price and x_{ij} is the quality of product. For $1 \le j \le J$, if t_{ij+1}-t_{ij-1}don't have the same value, then (p_{ij}, x_{ij}) will not be the same, and the maximized profit π (p_{ij}, x_{ij}, t_{ij}) will not have the same value.

Proof: From (2.4) we know that consumer t chooses product (s, s′]$\times t_{ij} \times x_{ij}$ iff

$$\frac{e^{-\rho_J \left|\mathrm{t}_{ij} - \mathrm{t}\right|} x_{ij}^{\rho_x}}{p_{ij}} \ge \frac{e^{-\rho_J \left|\frac{j'}{J} l_2 - \mathrm{t}\right|} x^{\rho_x}}{p}, \text{for } 1 \le j' \le J \tag{5.1}$$

It reveals that t is belonging to an interval $(t_{ij} - r_1, t_{ij} + r_2)$ and the radius satisfies:

$$\frac{e^{-\rho_J \left|\mathrm{t}_{ij} - \mathrm{t}\right|} x_{ij}^{\rho_x}}{p_{ij}} \ge \frac{e^{-\rho_J \left|\mathrm{t}_{ij+1} - \mathrm{t}\right|} x_{ij+1}^{\rho_x}}{p_{ij+1}} \tag{5.2}$$

$$\rho_J(\mathrm{t}_{ij} + r_2) - \ln\left(\frac{x_{ij}^{\rho_x}}{p_{ij}}\right) = \rho_J(\mathrm{t}_{ij+1} - r_2) - \ln\left(\frac{x_{ij+1}^{\rho_x}}{p_{ij+1}}\right) \tag{5.3}$$

$$2\rho_J r_2 = \rho_J(t_{ij+1} - t_{ij}) + \rho_x[\ln(x_{ij}) - \ln(x_{ij+1})] - [\ln(p_{ij}) - \ln(p_{ij+1})] \tag{5.4}$$

$$r_2 = \frac{1}{2}(t_{ij+1} - t_{ij}) + \frac{\rho_x}{2\rho_J}[\ln(x_{ij}) - \ln(x_{ij+1})] - \frac{1}{2\rho_J}[\ln(p_{ij}) - \ln(p_{ij+1})] \tag{5.5}$$

For simple computation below, denote $z_{ik} = \frac{1}{p_{ik}}$, where k=j, j+1, we have:

$$r_2 = \frac{1}{2}(t_{ij+1} - t_{ij}) + \frac{\rho_x}{2\rho_J}[\ln(x_{ij}) - \ln(x_{ij+1})] + \frac{1}{2\rho_J}[\ln(z_{ij}) - \ln(z_{ij+1})] \tag{5.6}$$

For the same reason, we have

$$r_1 = \frac{1}{2}(t_{ij} - t_{ij-1}) + \frac{\rho_x}{2\rho_J}[\ln(x_{ij}) - \ln(x_{ij-1})] + \frac{1}{2\rho_J}[\ln(z_{ij}) - \ln(z_{ij-1})] \tag{5.7}$$

$$r_1 + r_2 = \frac{t_{ij+1} - t_{ij-1}}{2} + \frac{\rho_x}{\rho_J}[\ln(x_{ij}) - \frac{\ln(x_{ij+1}) + \ln(x_{ij-1})}{2}] + \frac{1}{\rho_J}[\ln(z_{ij}) - \frac{\ln(z_{ij+1}) + \ln(z_{ij-1})}{2}] \tag{5.8}$$

Because everyone has one unit of income when people begin to trade in step three, from utility function (2.4) we know the demand function of factory ij is

$$D(z_{ij}, x_{ij}, t_{ij}, d_I)=\frac{(r_1+r_2)Md_I z_{ij}}{l_2 ke^{c_I d_I}}=\frac{(r_1+r_2)md_I z_{ij}}{ke^{c_I d_I}}$$

$$=\frac{md_I z_{ij}(t_{ij+1}-t_{ij-1})}{2ke^{c_I d_I}}+\frac{\rho_x md_I z_{ij}}{ke^{c_I d_I}\rho_J}[\ln(x_{ij})-\frac{\ln(x_{ij+1})+\ln(x_{ij-1})}{2}] \tag{5.9}$$

$$+\frac{md_I z_{ij}}{ke^{c_I d_I}\rho_J}[\ln(z_{ij})-\frac{\ln(z_{ij+1})+\ln(z_{ij-1})}{2}]$$

where d_I (=s′-s) is the length of the functional interval of the products, Its revenue function is

$$R(z_{ij}, x_{ij}, t_{ij}, d_I)=\frac{(r_1+r_2)Md_I}{l_2 ke^{c_I d_I}}=\frac{(r_1+r_2)md_I}{ke^{c_I d_I}}$$

$$=\frac{md_I(t_{ij+1}-t_{ij-1})}{2ke^{c_I d_I}}+\frac{\rho_x md_I}{ke^{c_I d_I}\rho_J}[\ln(x_{ij})-\frac{\ln(x_{ij+1})+\ln(x_{ij-1})}{2}] \tag{5.10}$$

$$+\frac{md_I}{ke^{c_I d_I}\rho_J}[\ln(z_{ij})-\frac{\ln(z_{ij+1})+\ln(z_{ij-1})}{2}]$$

From (2.5) we know that

$$L=\frac{c_q qd_I}{e}+\frac{c_x xd_I}{e}+\frac{a_1}{e}+\frac{a_2 d_I}{e} \tag{5.11}$$

Because the wage rate is one, so that we have:

$$C(q, x)=L-1=\frac{c_q qd_I}{e}+\frac{c_x xd_I}{e}+\frac{a_1}{e}+\frac{a_2 d_I}{e}-1 \tag{5.12}$$

$$C(z_{ij}, x_{ij}, t_{ij}, d_I)=L-1=\frac{c_q D(z_{ij}, x_{ij}, t_{ij}, d_I)d_I}{e}+\frac{c_x xd_I}{e}+\frac{a_1}{e}+\frac{a_2 d_I}{e}-1 \tag{5.13}$$

$\pi(z_{ij}, x_{ij}, t_{ij}, d_I)=R(z_{ij}, x_{ij}, t_{ij}, d_I)-C(z_{ij}, x_{ij}, t_{ij}, d_I)$

$$=\frac{md_I(t_{ij+1}-t_{ij-1})}{2ke^{c_I d_I}}(1-\frac{c_q d_I z_{ij}}{e})+\frac{\rho_x md_I}{ke^{c_I d_I}\rho_J}(1-\frac{c_q d_I z_{ij}}{e})[\ln(x_{ij})-\frac{\ln(x_{ij+1})+\ln(x_{ij-1})}{2}]$$

$$+\frac{md_I}{ke^{c_I d_I}\rho_J}(1-\frac{c_q d_I z_{ij}}{e})[\ln(z_{ij})-\frac{\ln(z_{ij+1})+\ln(z_{ij-1})}{2}]-\frac{c_x x_{ij} d_I}{e}-\frac{a_1}{e}-\frac{a_2 d_I}{e}+1 \tag{5.14}$$

For simple computation, denote

$$\hat{\pi}=\frac{ke^{c_I d_I}\rho_J\pi}{md_I}, \hat{a}=\frac{ke^{c_I d_I}\rho_J}{md_I}(\frac{a_1}{e}+\frac{a_2 d_I}{e}-1), \hat{x}_{ik}=\frac{ke^{c_I d_I}\rho_J c_x x_{ik}}{me}, \hat{z}_{ik}=\frac{c_q d_I z_{ik}}{e} \tag{5.15}$$

where k=j-1, j, j+1, then we have:

$$\hat{\pi}(\hat{z}_{ij}, \hat{x}_{ij}, t_{ij}, d_I)=\rho_x[\ln(\hat{x}_{ij})-\frac{\ln(\hat{x}_{ij+1})+\ln(\hat{x}_{ij-1})}{2}](1-\hat{z}_{ij}) \tag{5.16}$$

$$+[\ln(\hat{z}_{ij})-\frac{\ln(\hat{z}_{ij+1})+\ln(\hat{z}_{ij-1})}{2}](1-\hat{z}_{ij})+\frac{\rho_J(t_{ij+1}-t_{ij-1})(1-\hat{z}_{ij})}{2}-\hat{x}_{ij}-\hat{a}$$

For maximum profit $\pi(z_{ij}, x_{ij}, t_{ij}, d_I)$, we have the first and second-order conditions:

$$\hat{\pi}'_{\hat{x}}=\frac{\rho_x(1-\hat{z}_{ij})}{\hat{x}_{ij}}-1 \tag{5.17}$$

$$\hat{\pi}'_{\hat{z}} = -\rho_x[\ln(\hat{x}_{ij}) - \frac{\ln(\hat{x}_{ij+1}) + \ln(\hat{x}_{ij-1})}{2}] - [\ln(\hat{z}_{ij}) - \frac{\ln(\hat{z}_{ij+1}) + \ln(\hat{z}_{ij-1})}{2}] - \frac{\rho_J(t_{ij+1} - t_{ij-1})}{2} + \frac{(1-\hat{z}_{ij})}{\hat{z}_{ij}} \tag{5.18}$$

$$\hat{\pi}'_{\hat{x}} = 0 \Leftrightarrow \hat{x}_{ij} = \rho_x(1-\hat{z}_{ij}) \tag{5.19}$$

$$\hat{\pi}'_{\hat{z}} = 0 \Leftrightarrow$$

$$\frac{(1-\hat{z}_{ij})}{\hat{z}_{ij}} = \rho_x[\ln(\hat{x}_{ij}) - \frac{\ln(\hat{x}_{ij+1}) + \ln(\hat{x}_{ij-1})}{2}] + [\ln(\hat{z}_{ij}) - \frac{\ln(\hat{z}_{ij+1}) + \ln(\hat{z}_{ij-1})}{2}] + \frac{\rho_J(t_{ij+1} - t_{ij-1})}{2} \tag{5.20}$$

We have the second-order derivation matrix:

$$\hat{\pi}'' = \begin{pmatrix} \hat{\pi}''_{\hat{x}\hat{x}} & \hat{\pi}''_{\hat{x}\hat{z}} \\ \hat{\pi}''_{\hat{x}\hat{z}} & \hat{\pi}''_{\hat{z}\hat{z}} \end{pmatrix} = -\begin{pmatrix} \frac{\rho_x(1-\hat{z}_{ij})}{\hat{x}_{ij}^2} & \frac{\rho_x}{\hat{x}_{ij}} \\ \frac{\rho_x}{\hat{x}_{ij}} & \frac{(1+\hat{z}_{ij})}{\hat{z}_{ij}^2} \end{pmatrix} \tag{5.21}$$

$$\hat{\pi}'' < 0 \Leftrightarrow \hat{z}_{ij}^2 < \frac{1}{1+\rho_x} \tag{5.22}$$

That means if $\hat{z}_{ij}^2 < \frac{1}{1+\rho_x}$, the solution $(\hat{z}_{ij}, \hat{x}_{ij})$ from (5.19) and (5.20) will maximize profit; while $\hat{z}_{ij}^2 > \frac{1}{1+\rho_x}$, the $(\hat{z}_{ij}, \hat{x}_{ij})$ will not maximize profit.

From (5.16) and (5.19) we have

$$\hat{\pi}(\hat{z}_{ij}) = \hat{\pi}(\hat{z}_{ij}, \hat{x}_{ij}, t_{ij}, d_I) = \frac{(1-\hat{z}_{ij})^2}{\hat{z}_{ij}} - \rho_x(1-\hat{z}_{ij}) - \hat{a} \tag{5.23}$$

From (5.23) we know that $\hat{\pi}(\hat{z}_{ij})$ is a decreasing function when $\hat{z}_{ij}^2 < \frac{1}{1+\rho_x}$, it covers the range of the solution $(\hat{z}_{ij}, \hat{x}_{ij})$ which satisfies the first and second-order conditions

So we can conclude that: for $1 \le j \le J$, if t_{ij+1}-t_{ij-1}don't have the same value, then the maximization solutions $(\hat{z}_{ij}, \hat{x}_{ij})$ will not have the same value. It's easy to prove. For $1 \le j \le J$, suppose $(\hat{z}_{ij}, \hat{x}_{ij})$ are the same, then form (5.19) and (5.20) we have all t_{ij+1}-t_{ij-1} are the same. From (5.23), because $\hat{\pi}(\hat{z}_{ij})$ is a decreasing function, so that different $(\hat{z}_{ij}, \hat{x}_{ij})$ implies different $\hat{\pi}(\hat{z}_{ij})$.

Translate $(\hat{z}_{ij}, \hat{x}_{ij})$ to (p_{ij}, x_{ij}), and $\hat{\pi}$ to profit π, we get the Lemma.

Proposition 1 There is a unique solution in symmetric equilibrium state of monopolistic competition, which is decided by (2.6), (2.7), (2.8), (2.9) and (2.10).

$$t_{ij} = \frac{jl_2}{J}, where\ t_{i0} = t_{iJ} \tag{2.6}$$

$$d_I = \frac{1}{2}[-\frac{a_1}{a_2} + \sqrt{(\frac{a_1}{a_2})^2 + 4\frac{a_1}{a_2 c_I}}] \tag{2.7}$$

$$d_J = \frac{1}{2}[(\frac{\rho_x}{\rho_J} + n) + \sqrt{(\frac{\rho_x}{\rho_J} + n)^2 + \frac{4n}{\rho_J}}], \quad n = \frac{a_1 e^{c_I d_I} k}{c_I d_I^2 me} \tag{2.8}$$

$$p = \frac{c_q d_I (1+\rho_J d_J)}{e} \tag{2.9}$$

$$x = \frac{\rho_x med_J}{c_x ke^{c_I d_I}(1+\rho_J d_J)} \tag{2.10}$$

$$q_f = \frac{med_J}{c_q ke^{c_I d_I}(1+\rho_J d_J)} \tag{2.11}$$

$$q_c = \frac{Jq_f}{M} = \frac{e}{c_q ke^{c_I d_I}(1+\rho_J d_J)} \tag{2.12}$$

Proof: Suppose in a symmetric state, there are I kinds of products and J varieties in each kind of product. For $1 \le i \le I$, and $1 \le j \le J$, suppose that factory ij produces product $(\frac{i-1}{I}, \frac{i}{I}] \times \frac{j}{J} l_2 \times$x, and all factories have price p. The idea to prove this symmetric state is a equilibrium solution is that, for each factory ij, given the variables of other factories, when any factory ij takes the strategy variables of the assumed symmetric state, it will gets the maximum profit, and this profit level equals the income of all workers and owners.

Because of $\max_{z_{ij}, x_{ij}, t_{ij}, d_I} \pi(z_{ij}, x_{ij}, t_{ij}, d_I) = \max_{d_I}\{\max_{z_{ij}, x_{ij}, t_{ij}} \pi(z_{ij}, x_{ij}, t_{ij}, d_I)\}$, three steps will be taken. First, look for the unique solution of $\pi(d_I) = \max_{z_{ij}, x_{ij}, t_{ij}} \pi(z_{ij}, x_{ij}, t_{ij}, d_I)$. Second, look for the unique solution of $\max_{d_I} \pi(d_I)$. Third, prove that there is a unique solution of the monopolistic competition equilibrium.

First, we will prove that there is a unique solution for $\max_{z_{ij}, x_{ij}, t_{ij}} \pi(z_{ij}, x_{ij}, t_{ij}, d_I)$ with the method that there is a unique solution of the first-order conditions which has negative second-order derivation matrix.

From (5.14) we know that t_{ij} doesn't affect profit $\pi(z_{ij}, x_{ij}, t_{ij}, d_I)$, but does affect the profit $\pi(z_{ik}, x_{ik}, t_{ik}, d_I)$, where k=j-1 and j+1. As a requirement of a symmetric solution, we set

$$t_{ij} = \frac{jl_2}{J}, where\ t_{i0} = t_{iJ} \tag{5.24}$$

From the first-order condition (5.20) we have $\hat{z} = \hat{z}_{ij} = \frac{1}{1+\rho_J d_J}$, hence we have

$$p = \frac{1}{z} = \frac{c_q d_I (1+\rho_J d_J)}{e} \tag{5.25}$$

And from the first-order condition (5.19) we have $\hat{x} = \hat{x}_{ij} = \frac{\rho_x \rho_J d_J}{1+\rho_J d_J}$, hence we have:

$$x = \frac{\rho_x med_J}{c_x k e^{c_I d_I}(1+\rho_J d_J)} \tag{5.26}$$

From the second-order condition (5.22) we have:

$$\hat{\pi}'' < 0 \Leftrightarrow \hat{z}_{ij}^2 < \frac{1}{1+\rho_x} \Leftrightarrow (1+\rho_J d_J)^2 > 1+\rho_x \tag{5.27}$$

From (5.31) we know that $\rho_J d_J - \rho_x > 0$ which implies $(1+\rho_J d_J)^2 > 1+\rho_x$, so that we have negative definite second-order derivation matrix, it implies that the symmetric solution (p, x) of (5.25) and (5.26) maximizes the profit.

From (5.9) and (5.25) we have

$$q_f = D(z, x, t, d_I) = \frac{m d_I d_J z}{k e^{c_I d_I}} = \frac{med_J}{c_q k e^{c_I d_I}(1+\rho_J d_J)} \tag{5.28}$$

$$\begin{aligned} \pi(d_I) &= \max_{z_{ij}, x_{ij}, t_{ij}} \pi(z_{ij}, x_{ij}, t_{ij}, d_I) = \pi(z, x, \frac{jl_2}{J}, d_I) \\ &= \frac{m d_J d_I}{k e^{c_I d_I}}(1 - \frac{c_q d_I z}{e}) - \frac{c_x x d_I}{e} - \frac{a_1}{e} - \frac{a_2 d_I}{e} + 1 \\ &= \frac{m d_I d_J(\rho_J d_J - \rho_x)}{c_q k e^{c_I d_I}(1+\rho_J d_J)} - \frac{a_1}{e} - \frac{a_2 d_I}{e} + 1 \end{aligned} \tag{5.29}$$

Second, we also use the first and second-order conditions to look for the $\max_{d_I} \pi(d_I)$

$$\pi'_{d_I} = \frac{m(1 - c_I d_I) d_J(\rho_J d_J - \rho_x)}{c_q k e^{c_I d_I}(1+\rho_J d_J)} - \frac{a_2}{e} \tag{5.30}$$

We have the first-order condition:

$$\pi'_{d_I} = 0 \Leftrightarrow \frac{med_J(\rho_J d_J - \rho_x)}{k(1+\rho_J d_J)} = \frac{a_2 e^{c_I d_I}}{1 - c_I d_I} \tag{5.31}$$

From (5.30) we have the negative second-order condition, $\pi''_{d_I} < 0$. So that from (5.31) we get the maximize solution.

Since every individual can choose to become the worker or the owner freely, at the equilibrium state, the owner and worker will have the one unit of income each. So we have

$$\pi(d_I) = 1 \Leftrightarrow \frac{med_J(\rho_J d_J - \rho_x)}{k(1+\rho_J d_J)} = \frac{e^{c_I d_I}(a_1 + a_2 d_I)}{d_I} \tag{5.32}$$

Third, from (2.6), (2.7), (2.8), (2.9) and (2.10) we get the solution of the symmetric equilibrium, now we will prove that the solution is unique.

From (5.31) and (5.32) we have

$$\frac{a_2}{1 - c_I d_I} = \frac{a_1 + a_2 d_I}{d_I} \tag{5.33}$$

$$a_2 d_I = (a_1 + a_2 d_I)(1 - c_I d_I) \tag{5.34}$$

$$a_2 c_I d_I^2 + a_1 c_I d_I - a_1 = 0 \tag{5.35}$$

$$d_I = \frac{1}{2}[-\frac{a_1}{a_2} + \sqrt{(\frac{a_1}{a_2})^2 + 4\frac{a_1}{a_2 c_I}}] \tag{5.36}$$

We know that there is only one positive d_I. From (5.35) we know that

$$\frac{a_2}{1-c_I d_I}=\frac{a_1}{c_I d_I^2} \quad (5.37)$$

Denote $n=\dfrac{a_1 e^{c_I d_I} k}{c_I d_I^2 me}$ (5.38)

From (5.31) (5.37) and (5.38) we have:

$\rho_J d_J^2-(\rho_x+n\rho_J)d_J-n=0$ (5.39)

There is only one positive solution:

$$d_J=\frac{1}{2}[(\frac{\rho_x}{\rho_J}+n)+\sqrt{(\frac{\rho_x}{\rho_J}+n)^2+\frac{4n}{\rho_J}}] \quad (5.40)$$

After proving the uniqueness of d_I and d_J, from (5.36) and (5.40) we can get the uniqueness of p and x and other variables in the symmetric equilibrium. Moreover, we have the quantity consumed by each consumer:

$$q_c=\frac{q_f J}{M}=\frac{q_f}{md_J} \quad (5.41)$$

From (5.28) we have:

$$q_c=\frac{e}{c_q k e^{c_I d_I}(1+\rho_J d_J)} \quad (5.42)$$

Proposition 2: When J is an even number, and for any $-1<h<1$, (2.13), (2.7), (2.8), (2.9) and (2.10) decide a solution of monopolistic competition. When h=0, it's the symmetric solution of proposition 1; while $h\neq0$, it's the asymmetric solution. Moreover, the social average utility Eu(h) is a decreasing function of $|h|$.

$$\begin{cases} t_{ij}=\dfrac{2kl_2}{J} & \text{when } j=2k, \text{where } t_{i0}=t_{iJ} \\ t_{ij}=\dfrac{(2k+1+h)l_2}{J} & \text{when } j=2k+1,\ -1<h<1 \end{cases} \quad (2.13)$$

Proof: when h=0, (2.6) and (2.13) are the same, so that the solution is the symmetric solution in proposition 1; while in the case of $h\neq0$, although the set of t_{ij} from (2.13) distribute asymmetrically in C_2, but they keep t_{ij+1}-t_{ij-1} the same value for $1\le j\le J$. consequently. From the proof of proposition 1 we know that, the rest variables will be the same as that in the symmetric equilibrium state in proposition 1, so that (2.13), (2.7), (2.8), (2.9) and (2.10) will determine an asymmetric equilibrium solution.

In the equilibrium state with $-1<h<1$, t_{ij} distribute according (2.13), and hence we know that, when j=2k, factory ij locates in $\frac{2kl_2}{J}$ and its market range is $[\frac{(4k-1+h)l_2}{2J},\frac{(4k+1+h)l_2}{2J}]$; when j=2k+1, factory ij locates in $\frac{(1+h)l_2}{J}$ and its market range is $[\frac{(4k+1+h)l_2}{2J},\frac{(4k+3+h)l_2}{2J}]$. Because the market range of each factory distributes regularly on C_2, so that the social average utility equals the average utility in the market range of one factory, say factory i1.When t belong to this market range, consumer t has the utility

$$\mathrm{u(t)=e}^{\rho_J(t-\frac{(1+h)l_2}{J})}\mathrm{x}^{\rho_x}\mathrm{q_c} \tag{5.43}$$

$$\mathrm{Eu(h)}=\frac{l_2}{\mathrm{J}}\int_{\frac{(1+h)l_2}{2J}}^{\frac{(3+h)l_2}{2J}}u(t)dt=\frac{l_2}{\mathrm{J}}\int_{\frac{(1+h)l_2}{2J}}^{\frac{(3+h)l_2}{2J}}\mathrm{e}^{\rho_J(t-\frac{(1+h)l_2}{J})}\mathrm{x}^{\rho_x}\mathrm{q_c}dt \tag{5.44}$$

$$\mathrm{Eu(h)}=(\mathrm{x}^{\rho_x}\mathrm{q_c})\int_{-\frac{1+h}{2}}^{\frac{1-h}{2}}\mathrm{e}^{\rho_J s}ds \tag{5.45}$$

From (5.45) we have the result that Eu(h) is a decreasing function of $|h|$.

Lemma 2 (1) if F(z ↓ , x ↑ , y ↓)=0, then z=(x ↑ , y ↓) (2.14)

(2) if F(z ↑ , x ↑ , y ↓)=0, then z=(x ↓ , y ↑)

Proof: it's easy to prove.

Corollary 1 the equilibrium length of functional interval products satisfies:

$$d_I=\frac{1}{I}=(a_1\uparrow,a_2\downarrow,c_I\downarrow) \tag{2.15}$$

$$c_Id_I=(a_1\uparrow,a_2\downarrow,c_I\uparrow) \tag{2.16}$$

Proof: they come from (2.7).

Corollary 2 the equilibrium varieties satisfy:

$$d_J=\frac{l_2}{J}=(m\downarrow,e\downarrow,k\uparrow,c_I\uparrow,\rho_J\downarrow,\rho_x\uparrow,a_1\uparrow,a_2\uparrow) \tag{2.17}$$

$$\rho_Jd_J=(m\downarrow,e\downarrow,k\uparrow,c_I\uparrow,\rho_J\uparrow,\rho_x\uparrow,a_1\uparrow,a_2\uparrow) \tag{2.18}$$

Proof: From (2.8) we have $d_J=\frac{l_2}{J}=(\rho_J\downarrow,\rho_x\uparrow,n\uparrow)$ (5.46)

From (2.8) $n=\frac{a_1e^{c_Id_I}k}{c_Id_I^2me}=\frac{a_1c_Ie^{c_Id_I}k}{(c_Id_I)^2me}$ (5.47)

Because 0<$c_I d_I$<1, so we have $\frac{e^{c_Id_I}}{c_I^2d_I^2}=(c_Id_I\downarrow)$ (5.48)

From (2.16) we have $n=\frac{a_1c_Ie^{c_Id_I}k}{(c_Id_I)^2me}=(a_2\uparrow,\frac{me}{k}\downarrow)$ (5.49)

From (2.8) and (5.37) we have $n=\frac{a_2e^{c_Id_I}k}{(1-c_Id_I)me}=(c_Id_I\uparrow)$ (5.50)

From (2.16) and (5.50) we have $n=(c_I\uparrow,a_1\uparrow)$ (5.51)

From (5.49) and (5.51) we have $n=(m\downarrow,e\downarrow,k\uparrow,c_I\uparrow,a_1\uparrow,a_2\uparrow)$ (5.52)

From (5.46) and (5.52) we have

$$d_J=\frac{l_2}{J}=(m\downarrow,e\downarrow,k\uparrow,c_I\uparrow,\rho_J\downarrow,\rho_x\uparrow,a_1\uparrow,a_2\uparrow) \tag{5.53}$$

From (2.8) we have $\rho_J d_J = \frac{1}{2}[(\rho_x + \rho_J n) + \sqrt{(\rho_x + \rho_J n)^2 + 4\rho_J n}]$, and it is easy to get

$$\rho_J d_J = (m \downarrow, e \downarrow, k \uparrow, c_I \uparrow, \rho_J \uparrow, \rho_x \uparrow, a_1 \uparrow, a_2 \uparrow) \tag{5.54}$$

Corollary 3 the equilibrium quality satisfies:

$$x = (m \uparrow, e \uparrow, k \downarrow, c_I \downarrow, \rho_J \downarrow, \rho_x \uparrow, a_1 \downarrow, a_2 \uparrow, c_x \downarrow) \tag{2.19}$$

Proof: From (2.10) we have $x = \frac{\rho_x med_J}{c_x k e^{c_I d_I}(1 + \rho_J d_J)}$; from (2.10) and (2.17) we know:

$$x = (\rho_J \downarrow, \rho_x \uparrow, a_2 \uparrow, c_x \downarrow) \tag{5.55}$$

From (2.10) and (5.32) we know $x = \frac{\rho_x (a_1 + a_2 d_I)}{c_x d_I (\rho_J d_J - \rho_x)}$ (5.56)

From (2.15) and (2.18) we know $x = (\frac{me}{k} \uparrow)$ (5.57)

From (2.8) we have:

$$\frac{1}{d_J} = \frac{-(n\rho_J + \rho_x) + \sqrt{(n\rho_J + \rho_x)^2 + 4n\rho_J}}{2n} \tag{5.58}$$

$$\frac{1}{d_J} + \rho_J = \frac{2\rho_J(1+\rho_x)}{(\rho_x - n\rho_J) + \sqrt{(\rho_x - n\rho_J)^2 + 4n\rho_J(1+\rho_x)}} \tag{5.59}$$

From (2.8) and (5.37) we have:

$$n = \frac{a_1 e^{c_I d_I} k}{c_I d_I^2 me} = \frac{a_2 e^{c_I d_I} k}{(1 - c_I d_I) me} \tag{5.60}$$

$$e^{c_I d_I}(\frac{1}{d_J} + \rho_J) = \frac{2\rho_J(1+\rho_x)}{(\rho_x e^{-c_I d_I} - \frac{a_1 k \rho_J}{c_I d_I^2 me}) + \sqrt{(\rho_x e^{-c_I d_I} - \frac{a_1 k \rho_J}{c_I d_I^2 me})^2 + 4\frac{a_2 e^{-c_I d_I} k(1+\rho_x)}{(1 - c_I d_I) me}}} \tag{5.61}$$

From (2.16) we have $\rho_x e^{-c_I d_I} = (a_1 \downarrow, c_I \downarrow)$ (5.62)

From (2.6) we have

$$\frac{a_1}{c_I d_I^2} = (\frac{\sqrt{a_1 c_I} + \sqrt{a_1 c_I + 4a_2}}{2})^2 \tag{5.63}$$

So that $\frac{a_1 k \rho_J}{c_I d_I^2 me} = (a_1 \uparrow, c_I \uparrow)$ (5.64)

From (2.16) we have $\frac{a_2 e^{-c_I d_I} k(1+\rho_x)}{(1 - c_I d_I) me} = (c_I d_I \downarrow) = (a_1 \downarrow, c_I \downarrow)$ (5.65)

From (5.62), (5.64) and (5.65) we have:

$$e^{c_I d_I}(\frac{1}{d_J} + \rho_J) = (a_1 \uparrow, c_I \uparrow) \tag{5.66}$$

From (2.10) and (5.66) we know that $x = (c_I \downarrow, a_1 \downarrow)$ (5.67)

From (5.55), (5.57) and (5.67) we have

$$x = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\uparrow, a_1\downarrow, a_2\uparrow, c_x\downarrow) \tag{5.68}$$

Corollary 4 the equilibrium quantity produced by each factory (q_f) satisfies:

$$q_f = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\uparrow, a_1\downarrow, a_2\uparrow, c_q\downarrow) \tag{2.20}$$

Proof: from (2.11) we know $q_f = \dfrac{med_J}{c_q k e^{c_I d_I}(1+\rho_J d_J)}$, it has the same proof and conclusion as x, except the change from c_x to c_q, so that we have:

$$q_f = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\uparrow, a_1\downarrow, a_2\uparrow, c_q\downarrow) \tag{5.69}$$

Corollary 5 the equilibrium quantity consumed by each consumer (q_c) satisfies:

(1) $q_c = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\downarrow, a_1\downarrow, c_q\downarrow)$ (2.21)

(2) $q_c = (a_2\uparrow)(\downarrow) \Leftrightarrow \sqrt{(1+\dfrac{\rho_x}{n\rho_J})^2 + \dfrac{4}{n\rho_J}} > (<) \sqrt{1+\dfrac{4a_2}{a_1 c_I}}$ (2.22)

We have $q_c=(a_2\uparrow)$ when $\dfrac{me}{k}$ is large enough; if ρ_J is large enough, we get $q_c=(a_2\downarrow)$.

Proof: (1) From (2.12) we have $q_c = \dfrac{e}{c_q k e^{c_I d_I}(1+\rho_J d_J)}$, from (2.16) and (2.18) we have:

$$q_c = (m\uparrow, e\uparrow, k\downarrow, c_I\downarrow, \rho_J\downarrow, \rho_x\downarrow, a_1\downarrow, c_q\downarrow) \tag{5.70}$$

(2) From (2.8) we have:

$$\rho_J d_J = \frac{(n\rho_J+\rho_x)+\sqrt{(n\rho_J+\rho_x)^2+4n\rho_J}}{2} \tag{5.71}$$

$$e^{c_I d_I}(\rho_J d_J + 1) = \frac{e^{c_I d_I}}{2}((n\rho_J+\rho_x)+\sqrt{(n\rho_J+\rho_x)^2+4n\rho_J}+2) \tag{5.72}$$

Denote function $f \triangleq c_I d_I + \ln((n\rho_J+\rho_x)+\sqrt{(n\rho_J+\rho_x)^2+4n\rho_J}+2)$ (5.73)

$$f'_{d_I} = c_I + \frac{\rho_J n'_{d_I}}{\sqrt{(n\rho_J+\rho_x)^2+4n\rho_J}} = c_I - \frac{c_I \rho_J n}{\sqrt{(n\rho_J+\rho_x)^2+4n\rho_J}} \frac{2-c_I d_I}{c_I d_I} \tag{5.74}$$

From (2.7) we know that $\dfrac{2-c_I d_I}{c_I d_I} = \sqrt{1+\dfrac{4a_2}{a_1 c_I}}$ (5.75)

$$f'_{d_I} > (<) 0 \Leftrightarrow \sqrt{(n\rho_J+\rho_x)^2+4n\rho_J} > (<) n\rho_J \sqrt{1+\frac{4a_2}{a_1 c_I}}$$

$$\Leftrightarrow \sqrt{(1+\frac{\rho_x}{n\rho_J})^2 + \frac{4}{n\rho_J}} > (<) \sqrt{1+\frac{4a_2}{a_1 c_I}} \tag{5.76}$$

From (2.15) we know that $d_I=(a_2\downarrow)$, so that

$q_c = (a_2\uparrow)(\downarrow) \Leftrightarrow f'_{d_I} > (<) 0 \Leftrightarrow \sqrt{(1+\dfrac{\rho_x}{n\rho_J})^2 + \dfrac{4}{n\rho_J}} > (<) \sqrt{1+\dfrac{4a_2}{a_1 c_I}}$ (5.77)

From (2.8) we know that when $\frac{me}{k}$ is large enough and hence n is small enough, we have $f'_{d_I} > 0$, so that q_c= ($a_2 \uparrow$); while when ρ_J is large enough, we have $f'_{d_I} < 0$, so that q_c= ($a_2 \downarrow$).

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