Information Flow, Social Interactions and the Fluctuations of Prices in Financial Markets

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Abstract

We model how excess demand or excess supply can be generated in the presence of a social network of interactions, where agents are subject to external information and individual incentives. In this context we study price fluctuations in financial markets under equilibrium. In particular, we isolate the role of these different factors in the determination of price fluctuations and describe non trivial sensitivities to changes in equilibrium due to the existence of social interactions. We characterize equilibrium and distinguish between stable and unstable equilibrium. Crashes or bubbles are seen as out-of-equilibrium situations, preceeded by unstable equilibrium. Fluctuations under unstable equilibrium are shown to be abnormal and particulary large. Also, we show how fluctuations of the external information flows affect the fluctuations of the return process. In all cases we explain the well-known phenomena that prices do not fluctuate upwards in the same way as they fluctuate downwards. This asymmetry of price fluctuations is due to asymmetries in the price elasticity of demand and supply curves at the level defining equilibrium.

1 Introduction

This paper builds a model of individual decision making in the context of equilibrium financial markets with social interaction. The notion of social interaction takes into account the fact that individual choices are affected by the others' decisions and characteristics. In any model in Economics this happens somehow. Typically, agents interact indirectly through the prices, these prices reflecting the participation of each individual in the market place. In contrast, the type of interdependencies that we model directly link the individuals. With this type of models we can see how the behavior of one agent may affect the preferences of the others without being mediated through the equilibrium price.

Among the economists there has been an increasing recognition of the importance of social interactions in economic behavior. A very early study of the role of social interactions in binary choice is Schelling (1973). The intuition that individuals seek to conform to the behavior of reference groups has found many applications. An important case is to consider the interactions within a close neighborhood as in Loury (1977), Bénabou (1993) and Durlauf (1996a,b).

This concept of behavior driven by social interactions, although recent in economics, has been theorized by sociologists. Early examples can be found on the literature about ghetto poverty, such as Lewis (1966) and Liebow (1967), for example. More recent treatments such as Wilson (1987) emphasize the social multiplier that converts changes in private utility to changes into community-wide behavior. This tradeoff between private behavior and pression to conform will drive our theoretical work.

The potential role of social interactions has been demonstrated in contexts that are not restricted to neighborhoods. Brock (1993) shows how these effects may help explaining asset market volatility and Brock and Hommes (1997) show further how they can produce complex aggregate price dynamics. This paper is very close in spirit to this line of research. More recent papers in this line linking directly these models to their Statistical Mechanics counterpart are the works of Durlauf (1999) and Brock and Durlauf (2001).

We apply this model specifically to the financial markets, assuming a population that either buy or sell a given financial asset. The buying/selling behavior of the others affects one's utility and, therefore, one's decision. The utility function is also considered as depending on external information and on individual incentives. Individuals being different from each other have different private incentives. In a very large population it is not feasible to describe the different incentives one by one. Rather, the distribution of such private incentives is modeled by a probability density. Thus, private incentives are described as a random variable with different realizations for each of the market participants. We refer to such random variable as a *random field*. The recent papers of Durlauf (1999) and Brock and Durlauf (2001) develop a particular treatment of individual preferences that incorporate individual incentives in an explicit way that maps the problem into the Ising Model with random fields. The randomness of such individual incentives generate fluctuations in equilibrium prices. The presence of social interactions are shown to affect the nature of prices even in the absence of external information flow. Such fluctuations are shown to be particularly sensitive to change in individual incentives for some critical values of the model's parameters. Moreover, we show how the presence of fluctuations of external information is propagated to the market price fluctuations.

This paper is organized as follows. In Section 2 we introduce the model. Section 3 characterizes the equilibrium values of the average trading attitude (buying or selling) and distinguishes between stable and unstable equilibria. Under the latter, deviations from equilibrium are much more likely to occur. Section 4 describes to type of price fluctuations: the first due to the randomness of individual incentives, characterizing the finite population equilibrium convergence to an infinite population equilibrium, and the second due to changes in the external information. Section 5 concludes.

2 The Model

Consider a system of N investors in a given stock. At each point in time some of these investors are willing to buy, and the others are willing to sell. This willingness is defined as a trading attitude. For simplicity, we assume that each investor buys or sells the same amount of shares, normalized to 1.

Let the trading attitude of individual i at time t be modelled as a binary variable $s_i(t) = \pm 1, i = 1, \ldots, N$. These variables are interpreted to have the following meaning. If $s_i = +1$, individual i is willing to buy, if $s_i = -1$, individual i is willing to sell. We say that a configuration of attitudes at time t is *stable* if no investor feels pressure to change his/her attitude. As usual, we also say that markets are in *equilibrium* at time t if we have $\sum s_i(t) = 0$, in which case attitudes may be transformed into buying and selling actions. An *equilibrium configuration* is a stable configuration satisfying the equilibrium condition above.

We assume that the configuration of attitudes may evolve from an equilibrium configuration due not only to the arrival of information flows from the market place but also also due to the influence that investors exert over each other. In that case, the arrival of new information takes the system from equilibrium, generating a new stable configuration of attitudes. Within this new configuration there will be either an excess demand or an excess supply for the stock, generating price pressure. Prices change in such a way that this stable configuration becomes unstable and evolves until a new equilibrium is reached. In what follows we characterize the different elements that may affect the investors' trading attitudes.

2.1 The Effect of Social Norms

Under no interaction between individuals, we assume that an isolated individual will not discriminate between a buying and a selling attitude, and will decide with equal probability in favor of $s_i = +1$ or $s_i = -1$. Given no social interaction these choices are independent of other individuals' choices. Let the mean choice in the set of the N individuals be denoted by

$$m_N = \frac{\sum s_i}{N}. \tag{1}$$

Notice that the fraction of investors willing to buy is

$$f = \frac{m_N + 1}{2}. \tag{2}$$

Clearly, if individuals decide with equal probability between a buying and a selling attitude, the average value of m_N is zero.

Still in the setting of no direct interaction between individuals, let us now consider that the individuals are in a coercive situation, where a social pressure to conform exists. This coercion may be exerted by a majority, the existence of leaders, communication media and/or other factors, like social norms, that may induce the direction of everybody's trading attitude. A typical example is the release of public news about the stock. In the case of good news, such as an increase in dividends, a strong pressure to buy will appear. The opposite case of bad news will lead to a natural pressure to sell the stock. Let h_c denote the intensity of this coercion. Its sign just define whether this coercion is in the direction to induce to buy, in which case $h_c > 0$, or in the direction to induce people to sell, in which case $h_c < 0$. We assume that isolated individuals conform with these social forces from the external environment and each of them will choose the attitude that maximizes

$$u_{1i} = h_c s_i. \tag{3}$$

No matter how small h_c is, a small degree of coercion induces a stable configuration of either massive buying attitudes or massive selling attitudes and no equilibrium is attainable.

2.2 The Effect of Social Interaction

We now turn to the effect of interactions, exchanges and contacts between individuals, abstracting from the effect of external coercion. Considering a pair of individuals i and j, they can either agree with respect to the trading attitude, in which case $s_i s_j = +1$, or disagree, in which case $s_i s_j = -1$. We introduce $J > 0$ as a measure, in utility terms, of the degree of interaction or exchange. The level of agreement for a given pair (i, j) is thus measured, in utility units, by

$$J s_i s_j, \tag{4}$$

being $+J$ in case of agreement and $-J$ in case of disagreement. A given individual i interacts with, say, n other individuals, labeled $k_1, k_2, \ldots, k_n$, with a set of given attitudes $\{s_j\}_{j\in\mathcal{I}_i}$, where $\mathcal{I}_i = \{k_1, k_2, \ldots, k_n\}$. We assume that, in the absence of external coercion, investor i chooses his/her attitude such as to maximize what is perceived as his/her total degree of agreement (the attitude being kept constant over all social contacts). Attitudes, however, are not observable. What one observes is the buying/selling behavior. Letting $\mathbb{E}_i(x)$ denote the subjective expected value of the random variable x for agent i, we assume that agent i chooses s_i so as to maximize

$$u_{2i} = J \sum_{j\in\mathcal{I}_i} s_i \mathbb{E}_i(s_j). \tag{5}$$

Let us now determine what happens when both effects, social external coercion and interactions between individuals, occur simultaneously. In that case, it is obvious that every agent will choose the attitude that is aligned with h_c, that is to say, choose the attitude with the same signal as h_c. In that way, each individual i maximizes the sum

$$G_i = u_{1i} + u_{2i} = J \sum_{j\in\mathcal{I}_i} s_i \mathbb{E}_i(s_j) + h_c s_i \tag{6}$$

and, at the same time, maximizes each of its components. Again, no matter how small h_c is, a small degree of coercion induces a stable configuration of either massive buying attitudes or massive selling attitudes and no equilibrium is attainable.

2.3 The Effect of Personal Values

Until now we have discussed two effects: the tendency to conform with social external norms and the interaction with other individuals. We now turn to a third relevant factor, namely the fact that each person, in her or his capacity as a group member, is *a priori* bound to a certain attitude by his/her idiosyncratic preferences. An additional factor is then required in order to convey all that is incultated in each person by the culture in which he or she lives, leading the person to be 'personally' inclined to opt, for example, for a positive rather than a negative attitude. This factor should act on each individual like the external coercion factor, except that it is person specific. If, for individual i, the intensity of this factor is α_i, the isolated influence of this additional factor leads him or her to maximize

$$u_{3i} = \alpha_i s_i. \tag{7}$$

Here, α_i may vary in sign and intensity from one individual to another. Depending on the nature of the model to be implemented, one may use either a configuration of known $\{\alpha_i\}$ or else, assume a probability distribution $p\{\alpha_i\}$. Together with the other factors, we assume that individuals choose their attitudes so as to maximize

$$H_i = u_{1i} + u_{2i} + u_{3i} = J \sum_{j \in \mathcal{I}_i} s_i \mathbb{E}_i(s_j) + h_i s_i, \tag{8}$$

where $h_i = h_c + \alpha_i$ is the *effective* field acting over agent i.

2.4 Uncertainty in the Utility

Given the impact of the social interaction in the choice of the agents' attitude, the utility of any individual will be a random variable. Its realization will depend on the realization of the others' attitudes. We assume that the optimal choice of attitude by agent i represents the solution to the maximization problem

$$\max_{s_i} H_i(s_i) + \epsilon_i(s_i).$$

Following Brock and Durlauf (2001), we make the common assumption that the difference of these two random disturbances is a random variable logistically distributed. In other words, for each i there exists $\beta_i \geq 0$ such that

$$\Pr[\epsilon_i(-1) - \epsilon_i(+1) \leq z] = \frac{1}{1 + \exp(-\beta_i z)}.$$

This is sufficient to characterize the probabilistic distributions of various choices. For example, for $i \neq j$,

$$\Pr[s_i | H_i, \mathbb{E}_i(s_j)] = \Pr[H_i(s_i) + \epsilon_i(s_i) > H_i(-s_i) + \epsilon_i(-s_i)] \propto \exp[\beta_i H_i(s_i)].$$

Independence of the random utility terms ϵ implies that the joint probability measure of a configuration $\{s_i\}$ is given by

$$\Pr\left[\{s_i\} | H_i, \mathbb{E}_i(s_j) \forall i, j \neq i\right] = \Pi_i \exp[\beta_i H_i(s_i)]. \tag{9}$$

The model is closed assuming that each agent has rational expectations, meaning that all subjective expectations $\mathbb{E}_i(s_j)$ can be replaced by the mathematical expectations $\mathbb{E}(s_j)$, where these expectations are conditioned to the different β_i and the parameters of H_i. It follows (see e.g., Brock and Durlauf, 2001) that the mathematical expectations of the individual choices are determined by the set of N coupled equations

$$\mathbb{E}(s_i) = \tanh\left\{\beta_i \left[h_i + J \sum_{j \neq i} \mathbb{E}(s_j)\right]\right\}. \tag{10}$$

3 Equilibrium Analysis

3.1 Aggregating utilities

In his seminal work, Keynes (1934) describes how professional investors behave in the market. In his view, they prefer to analyze how the crowd of investors is likely to behave in the future, rather than devoting their energy estimating fundamental values. He used the example of a beauty contest to illustrate this point. In order to predict the winner of a beauty contest, objective beauty is not as important as the knowledge (or prediction) of others' prediction of beauty.

In this paper we shall make this same important simplifying assumption in order to solve the model and understand the possible equilibria of this system. The main assumption is that individuals that take seriously into account the influence of others in the determination of their attitudes, tend to adopt the same attitude as what they predict the average buying or selling attitude to be.

Similarly, Galam and Moscovici (1991) formalize the emergence of a group as such in a social-psychological context and assume that, in equilibrium, the interaction of individual i with each of his/her neighbors with expected attitude $\mathbb{E}(s_j)$ can be replaced by the Law of Large Numbers with the interaction with an *average attitude*. That is done by replacing each $\mathbb{E}(s_j)$ by

$$\mathbb{E}(s_j) = \frac{1}{N-1} \sum_{k=1, k \neq j}^{N-1} s_k \tag{11}$$

If this is the case, the n neighbors become identical and

$$\sum_{j \in \mathcal{I}_i} \mathbb{E}(s_j) = \frac{n}{N-1} \sum_{k=1, k \neq j}^{N-1} s_k.$$

Notice that, as N increases without bound, the sum above tends to nm_N. Defining h_J as

$$h_J = J \sum_{j \in \mathcal{I}_i} \mathbb{E}(s_j) \to Jnm_N \text{ as } N \to \infty,$$

and substituting the sum above in the expression (8) for H_i we get

$$\begin{aligned} H_i &= h_J s_i + h_i s_i \\ &= \tilde{h}_i s_i, \end{aligned}$$

where

$$\tilde{h}_i = h_J + h_i.$$

In other words, the result of the assumption underlying (11) is that, when N increases without bound, the attitudes become asymptotically uncoupled. Notice, however, that this is not the same asymptotic system as if $J = 0$ from the beginning. In fact, H_i above resembles very much to u_{3i} in equation (7), with α_i replaced by $\tilde{h}_i$. But, as opposed to α_i, this last factor depends on J, the coupling constant. Hence, each agent will maximize H_i above by choosing s_i =sign $\tilde{h}_i$. Thus, when N is arbitrarily large, the aggregate utility may be written as

$$\mathcal{H}\left(\{s_i\}, J, h_c, \{\alpha_i\}\right) = \sum_{i=1}^{N} H_i = Nm_N h_J + \sum_{i=1}^{N} h_i s_i \tag{12}$$

Using the fact that $m_N = (\sum_{i=1}^N s_i)/N$ and that h_J tends to Jnm_N as N tends to infinity, for large enough N we may write

$$\mathcal{H}\left(\{s_i\}, J, h_c, \{\alpha_i\}\right) = \frac{Jn}{N} \left(\sum_{i=1}^{N} s_i\right)^2 + \sum_{i=1}^{N} h_i s_i. \tag{13}$$

Thus, given $\{s_i\}, J, h_c$ and $\{\alpha_i\}$, the value of aggregated utility $\mathcal{H}$ is a function only of the mean attitude m_N. Clearly, the optimal value of $\mathcal{H}$ must be associated with a unique value of m_N. A value of m_N, however, is not associated with a unique configuration of attitudes $\{s_i\}$. Several different configurations may lead to the same value of m_N. In that sense, equilibrium is not unique. Also, because the solution of our model comes only within the decoupling context, and this requires an asymptotic system, we shall refer to the stable (and equilibrium) values of m_N as m. The number of possible configurations compatible with one given value of m may thus become very large. In order to have the relative weight of different values of m, a probability measure describing its asymptotic distribution is required.

3.2 The Stable Values of m

For the special case of deterministic private incentives ($h_i = h, \forall i$) where the distribution of random terms are identical across agents ($\beta_i = \beta, \forall i$), use of equation (9) allows to write the probability distribution of m in equilibrium as

$$\Pr(\{s_i\}) = \frac{1}{Z_N} \exp\{\beta\mathcal{H}(\{s_i\})\}.$$

We define $B = \beta h$. Thus, the proportionality constant Z_N should correspond to the sum over all states (all possible configurations)

$$Z_N = \sum_{\{s_i\}} \exp\{\beta\mathcal{H}(\{s_i\})\}.$$

With the help of equation (13), Z_N may be rewritten as

$$Z_N = \sum_{\{s_i\}} \exp\left\{\frac{\beta J n}{N}\left(\sum_{i=1}^{N} s_i\right)^2 + B\sum_{i=1}^{N} s_i\right\}. \tag{14}$$

It then follows that the expected value of m can be written as

$$\mathbb{E}(m) = \frac{1}{Z_N} \sum_{\{s_i\}} \left(\frac{\sum_{i=1}^{N} s_i}{N} \right) \exp\{\beta \mathcal{H} (\{s_i\})\},$$

or still,

$$\mathbb{E}(m) = \frac{1}{N} \frac{\partial}{\partial B} \ln Z_N \tag{15}$$

Hence the expected mean trading attitude m can be directly obtained from Z_N. In the appendix we show that $1/N \ln Z_N$ can be written asymptotically as

$$\lim_{N \to \infty} \frac{1}{N} \ln Z_N = \min_{\eta} f(\eta)$$

where

$$f(\eta) = \eta^2 \frac{\nu}{2} - \ln \cosh(\eta \nu + B)$$

with $\nu = \beta 2Jn$, implying that all the probabilistic mass is concentrated at the minima of the above function satisfying the first-order conditions

$$\eta = \tanh(\eta \nu + B).$$

The value of η that solves this equation is thus a function of B, to be denoted by $\eta(B)$. From equation (15) we can now write

$$\begin{aligned}
\mathbb{E}(m) &= \frac{\partial}{\partial B} [\max_{\eta} f(\eta)] \\
&= \frac{\partial}{\partial B} \left\{ -\eta(B)^2 \frac{\nu}{2} + \ln \cosh[\eta(B)\nu + B] \right\} \\
&= \eta(B).
\end{aligned}$$

Since by the Law of Large Numbers the probability distribution of m is degenerated in the considered limit, it follows that $\mathbb{E}(m) = m$ with probability 1 and thus m must satisfy

$$m = \tanh\left[\beta\left(nJm + h\right)\right],$$

that can be seen as the particular version of equation (10) for this case.

In order to consider the contribution of the idiosyncratic influences h_i, we should replace the expression $nJm + h$, reflecting the limit value of $h_J + h$, by the effective total random influence $nJm + h_i$. Under the integrability condition $\mathbb{E}(|h|) < \infty$, Amaro de Matos and Perez (1991) show that the average attitude minimizes the function

$$f^*(\eta) = \eta^2\frac{\nu}{2} - \int p(h_i)\ln\cosh(\eta\nu + B_i)dh_i,$$

where $B_i = \beta h_i$. The solution for the stable mean attitude is shown to read

$$m = \int p\left(h_i\right)\tanh\left[\beta\left(nJm + h_i\right)\right]dh_i \tag{16}$$

This equation generalizes the above equation for m and gives the implicit stable values of m for arbitrary random fields h_i.

3.3 Phase Diagram

The stable values of m are solutions of the above equation (16). In the simple initial case, where the probability mass of the random field h_i is concentrated around the value h, if $\beta J > 1$, there is a critical value of βh above which there is a single solution to (16) and below which there exist three different solutions.

As pointed out in Durlauf (1999), this means that when private incentives (expressed by h) are sufficiently weak, then the desire for conformity (as measured by J) present in individual decisions may generate multiple, self-consistent stable behaviors. In turn, this means that the relationship between the individual incentives and aggregate behavior can be highly nonlinear. For example, close to criticality, a small change in h may change the number of stable values for m in the system.

For the more general general case of random h_i, we refer to Salinas and Wreszinski (1985) to characterize the possible minima of f, and thus the possible values of stable mean attitudes. Typically, and depending on the values of β, J and on the probability $p(h_i)$, there may be different types of minima for $f^*(\eta)$. If η^* is a minimum of $f^*(\eta)$, then a Taylor expansion around η^* will look like

$$f^*(\eta) = f^*(\eta^*) + \theta \frac{(\eta - \eta^*)^{2k}}{2k} + o\left[(\eta - \eta^*)^{2k}\right].$$

We call k the type of the minimum η^* and θ its strength.

The several different possibilities are

1. one global quadratic minimum ($k = 1$);
2. two global quadratic minima ($k = 1$);
3. several quadratic minima ($k = 1$);
4. one global quartic minimum ($k = 2$);
5. one global minimum ($k = 3$).

Notice that the solutions that minimize $f^*(\eta)$ correspond to *stable* values of average trading attitude, not necessarily to *equilibrium*. Equilibrium is attained only for those stable configurations of trading attitudes that imply $m = 0$. Also, criticality of the parameters occur only in the last two cases. Studying equilibrium means to study the cases where $m = 0$. If we have a case where $m = 0$ and $k > 1$, we then know that the system is under an *unstable equilibrium* in the sense that a little change of parameters (β, J or h) may take it away from equilibrium to a different stable configuration of trading attitudes ($m \neq 0$), generating an excess demand or an excess supply for the asset.

4 Price Fluctuations in Equilibrium

In this Section we are going to consider a system initially at equilibrium. This means that the average trading attitude is zero. If the system is composed of infinite agents this is indeed true. However, for any finite system, whenever an agent chooses his/her behavior based on the perceived mean attitude of the others, his/her resulting utility is random, allowing the average trading attitude to fluctuate around zero. In the first part of this Section we study the impact of such fluctuations in prices. In the second part we study how fluctuations of the random field affect the dynamics of prices, when the system evolves in equilibrium.

4.1 Static Fluctuations

4.1.1 Fluctuations of m

In what follows we assume equilibrium i.e., $m = 0$. For any real γ, consider the following random variable

$$A_N = \frac{\sum_{i=1}^{N} s_i - Nm}{N^{1-\gamma}}.$$

Hence, we can write

$$m_N = m + N^{-\gamma} A_N.$$

In the case of strict social interaction, i.e., when $h_i = 0$, Ellis and Newman (1978) have shown that, with $\gamma = \frac{1}{2k}$ and as $N \to \infty$, A_N is distributed according to a density proportional to

$$\exp\left(-u^{2k}\right).$$

In other words, away from criticality ($k = 1$), equilibrium fluctuations of the average trading attitude are Normal. At criticality, or at unstable equilibrium ($k = 2$), equilibrium fluctuations of the average trading attitude are non-Normal, with much higher variance. The latter case reflects a situation where any slight change of parameters leads the system into an excess demand or excess supply of assets, leading the system away from the original equilibrium.

Under the presence of random fields, Amaro de Matos and Perez (1991) have shown that, with $\gamma = \frac{k}{2(2k-1)}$ and as $N \to \infty$, A_N is distributed according to a density proportional to

$$\exp\left[-(u - u_\circ)^{2k}\right]$$

if $k=1$, where $u_\circ$ is a Normal random variable, with mean zero and variance depending on the distribution $p(h)$, or

$$u^{2(k-1)} \exp\left[\delta s^{2(2k-1)}\right]$$

if $k>1$.

Notice that the difference in the scaling factor γ implies that critical fluctuations under random fields are higher than under strict social interaction, implying that the unstable equilibrium is more unstable under the presence of individual incentives.

4.1.2 Price Fluctuations

Considering that m fluctuates around zero, we consider in this Section what happens to the equilibrium price under such fluctuations. Let P be the initial stock price and $\varepsilon_s(P)$ and $\varepsilon_d(P)$ denote respectively the elasticities of the supply and demand curve at that price level.

If $dm>0$, there will be a pressure on the demand for the stock, and the stock price should increase to restablish equilibrium. In that case the excess demand is Ndm. Then, there will be a change in price

$$dP = \frac{P}{Q}\varepsilon_s(P)\, Ndm,$$

Notice that, in equilibrium, the transacted amount Q is always $N/2$. The corresponding return on the stock is thus

$$\frac{dP}{P} = 2\varepsilon_s(P)\, dm.$$

If, on the other hand, $dm<0$, this reflects an excess supply and the stock price should decrease to restablish equilibrium. In that case the excess supply is Ndm and the change in price is

$$dP = \frac{P}{Q}\varepsilon_d\left(P\right)Ndm,$$

where $\varepsilon_d(P) < 0$ is now the elasticity of the demand. The corresponding return on the stock is

$$\frac{dP}{P} = 2\varepsilon_d\left(P\right)dm.$$

Given that we know the distribution of dm, we can characterize the distribution of the returns. Away from criticality, this model leads to Normally distributed returns if the equilibrium price is such that $|\varepsilon_d(P)| = |\varepsilon_s(P)|$. However, if these two elasticities are different, as it is likely to occur, the distribution of the returns will be Two-Piece-Normal distributed, reflecting unbalanced risk, biased towards the direction of higher elasticity.

At criticality, the distribution of returns can also be characterized although in this case there are no closed formulas for the volatility. Here, if $|\varepsilon_d(P)| > |\varepsilon_s(P)|$ there is a higher probability that excess supply dominates out-of-equilibrium deviations, leading to a crash. On the other hand, if $|\varepsilon_d(P)| < |\varepsilon_s(P)|$ there is a higher probability that excess supply dominates out-of-equilibrium deviations, leading to a bubble.

4.2 Dynamic Fluctuations

Until now, we have dealt with an effective field $h_i = h_c + \alpha_i$ decomposed as a random variable h_i plus a constant value h_c. Hence, appart from that constant value h_c, the probability density $p(h_i)$ describes the distribution of the individual incentives among the agents and can be replaced without any loss of generality by $p(\alpha_i)$ to describe the stable configurations as satisfying

$$m = \int p\left(\alpha_i\right)\tanh\left[\beta\left(nJm + \alpha_i + h_c\right)\right]d\alpha_i.$$

We now shall consider how the dynamics for the external field h_c may affect the time evolution of equilibrium prices. As before, the starting point is that for equilibrium to exist, the value of m must be zero. From equation (16), we can write

$$m = f\left[m, h_c\right],$$

where

$$f\left[m, h_c\right] = \int p\left(\alpha_i\right) \tanh\left[\beta\left(nJm + \alpha_i + h_c\right)\right] dh_i.$$

Imposing $m = 0$ as a solution implies that

$$\int p\left(\alpha_i\right) \tanh\left[\beta(\alpha_i + h_c)\right] d\alpha_i = 0.$$

Given that $p\left(\alpha_i\right)$ is given and fixed, and the integral runs over all possible values of α_i, the above equality must be seen as determining β as a function of the level of the external field h_c. Hence, in equilibrium we write

$$f\left[m, h_c\right] = \int p\left(h_i\right) \tanh\left[\beta\left(h_c\right)\left(nJm + \alpha_i + h_c\right)\right] d\alpha_i. \tag{17}$$

4.2.1 Information Flow and Price Pressures

With $m = 0$, the fact that there is a flow of information in the market about the stock is modeled in this context as a one dimensional stochastic field h_c satisfying the diffusion stochastic differential equation

$$dh_c = a_h dt + b_h dW_t.$$

A change dh_c in the external field will have as an immediate effect a change dm in the initial value $m = 0$.

As before, we have

$$\frac{dP}{P} = 2dm \times \begin{cases} \varepsilon_s\left(P\right) & \text{if } dm > 0 \\ \varepsilon_d\left(P\right) & \text{if } dm < 0 \end{cases} \tag{18}$$

The first thing to notice is that returns under good news and under bad news do not have necessarily the same behaviour. In order to study the dynamics of prices, we must therefore study the dynamics of equilibrium attitudes dm under the arrival of information flows. This is done next.

4.2.2 The Dynamics of m

Changes in h_c will generate changes in m. Given equation (17), if h_c follows a diffusion process, m will also follow a diffusion process by Itô's Lemma.

Let the diffusion process of m be given by

$$dm = a_m dt + b_m dW_t.$$

Equation (17), allows us to write $dm = df$. In order to explicitly write a_m and b_m, we use Itô's Lemma to explicitly write df and identify the drift and diffusion coefficient for both diffusions. Details for df are given in the Appendix, leading to

$$\begin{aligned} a_m &= a_h \frac{(\beta + h_c\beta')\phi(1) + \phi(\alpha)}{1 - \beta J n \phi(1)} \\ &- b_{mm}^2 \frac{J^2 n^2 \beta^2 \Delta(1)}{1 - \beta J n \phi(1)} \\ &- 2 b_m b_h \frac{Jn\beta[(\beta' h_c + \beta)\Delta(1) + \Delta(\alpha)]}{1 - \beta J n \phi(1)} \\ &- b_{hh}^2 \frac{[(\beta' h_c + \beta)\Delta(1) + \beta'\Delta(\alpha) - \beta''\phi(\alpha) - (\beta'' h_c + 2\beta')\Phi(1)]}{1 - \beta J n \phi(1)} \end{aligned}$$

where

$$\Phi(x) = \int x p(\alpha_i) d\alpha_i \frac{1}{\cosh^2[\beta(h_c)(h_c + \alpha_i)]}$$

and

$$\Delta(x) = \int xp(\alpha_i)d\alpha_i \frac{\tanh[\beta(h_c)(h_c+\alpha_i)]}{\cosh^2[\beta(h_c)(h_c+\alpha_i)]}.$$

Also,

$$b_m = b_h \frac{(\beta + h_c\beta')\phi(1) + \phi(\alpha)}{1 - \beta J n \phi(1)}.$$

These are the drift and the diffusion coefficient for the average trading attitude, generating a price process

$$\frac{dP}{P} = 2(a_m dt + b_m dW_t) \times \begin{array}{ll} \varepsilon_s(P) & \text{if } dm > 0 \\ \varepsilon_d(P) & \text{if } dm < 0 \end{array} \tag{19}$$

constrained to the values of β satisfying the equilibrium condition

$$\int p(\alpha_i) \tanh[\beta(\alpha_i + h_c)]\, d\alpha_i = 0.$$

4.2.3 Some Particular Cases

We now analyse some special cases. In the absence of social interactions ($J = 0$) we have

$$\begin{aligned} a_m &= a_h[(\beta + h_c\beta')\phi(1) + \phi(\alpha)] \\ &- b_{hh}^2[(\beta' h_c + \beta)\Delta(1) + \beta'\Delta(\alpha) - \beta''\phi(\alpha) - (\beta'' h_c + 2\beta')\Phi(1)] \end{aligned}$$

and

$$b_m = b_h[(\beta + h_c\beta')\phi(1) + \phi(\alpha)],$$

together with the equilibrium constraint

$$\int p(\alpha_i) \tanh[\beta(\alpha_i + h_c)]\, d\alpha_i = 0.$$

A more trivial solution can be found satisfying the above constraint by considering the solution for very small β. In fact, the equilibrium constraint is trivially satisfied for $\beta = 0$. In that case $\Phi(1) = 1, \Phi(\alpha) = \mathbb{E}(\alpha)$ and $\Delta(x) = 0$. We then have

$$a_m = a_h \mathbb{E}(\alpha)$$

and

$$b_m = b_h \mathbb{E}(\alpha),$$

and the price process reads

$$\frac{dP}{P} = 2\mathbb{E}(\alpha)(a_h dt + b_h dW_t) \times \begin{cases} \varepsilon_s(P) & \text{if } dm > 0 \\ \varepsilon_d(P) & \text{if } dm < 0 \end{cases} \tag{20}$$

Notice that even in this very simple case, upward fluctuations of the price process are different from downwards fluctuations.

In particular, if $h_c \to \infty$, meaning that the external information is too strong, the only equilibrium solution is

$$\beta(h_c) = o\left(h_c^{-1}\right).$$

If the above condition is not satisfied, we will have in that limit

$$\Phi(x) = \Delta(x) = 0$$

leading to $a_m = b_m = 0$.

5 Final remarks

In this work we have studied the impact of social interactions on price fluctuations in financial markets under equilibrium. Social interactions are not the only relevant ingredient determining price fluctuations. External information and individual incentives are as essential as the social pressure to conform.

We were able to understand and isolate the role of different factors in the determination of price fluctuations, and to describe non trivial sensitivities to changes in equilibrium due to the existence of social interactions.

In particular, even under the absence of information flows, fluctuations in the decision about whether or not to invest under equilibrium may lead to non-trivial price fluctuations. In equilibrium situations in the vicinity of crashes or bubbles, price fluctuations are shown to become much larger. Deviating from equilibrium, in our model, means to attain a stable configuration of trading attitudes that generate either an excess supply, in which case we would have a crash, or an excess demand, in which case we would have a bubble. Such deviations are possible from equilibrium, provided that the system passes through what we have called unstable equilibria. The point was made that such transitions are highly sensitive to variations on the individual incentives. This sensitivity is due basically to the presence of the social interactions.

We also show that upward and downward equilibrium fluctuations are different in size, basically due to the fact that, in general, the demand elasticity is different from the supply elasticity at the equilibrium price level.

Finally, we considered the case where there is an information flow, leading to stochastic fluctuations of the external influence. Such fluctuations are shown to affect the equilibrium price process of assets in financial markets. The stochastic nature of the information flow influences the price process, and the equilibrium constraint implies that the probabilistic parameter regulating how the decision of an agent affects the uncertainty about his/her utility changes with the intensity of external influence. For small enough values of this probabilistic parameter, the fluctuations of prices are seen to reflect trivially the fluctuations of the external information. Another implication of the model is that for strong enough external influence, it may be extremely hard to satisfy the equilibrium condition, in which case the price process would not fluctuate.

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A Rewriting the sum over states Z_N

In order to rewrite the expression (14) for Z_N in a more suitable way, we use the trivial identity

$$\exp\left(\frac{\alpha}{2}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2} + \sqrt{\alpha}x\right) dx,$$

with

$$\frac{\alpha}{2} = \frac{\beta J n}{N} \left(\sum_{i=1}^{N} s_i\right)^2,$$

so that it follows that the sum over states Z_N can be written as

$$\begin{aligned} Z_N &= \sum_{\{s_i\}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[-\frac{x^2}{2} + \left(\sqrt{\frac{\beta 2Jn}{N}}x + B\right) \sum_{i=1}^{N} s_i\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{x^2}{2}\right) \left\{2\cosh\left[x\sqrt{\frac{\beta 2Jn}{N}} + B\right]\right\}^N . \end{aligned}$$

Making the change of variables $\nu = \beta 2Jn$ and

$$\eta = x(\nu N)^{-1/2},$$

the expression for Z_N can be rewritten as

$$Z_N = \sqrt{\frac{\nu N}{2\pi}} 2^N \int_{-\infty}^{+\infty} \exp[-Nf(\eta)] d\eta$$

where

$$f(\eta) = \eta^2 \frac{\nu}{2} - \ln\cosh(\eta\nu + B).$$

Using the Laplace asymptotic method, we can now study the asymptotic probability properties of the mean statement m. In fact,

$$\lim_{N\to\infty}\frac{1}{N}\ln\int_{-\infty}^{+\infty}\exp[Nf(\eta)]d\eta=\min_{\eta}f(\eta).$$

That is to say, asymptotically the sum over all states Z_N has its probabilistic mass fully concentrated over the configurations leading to the minimum value of f and thus satisfying the first-order conditions

$$\eta=\tanh(\eta\nu+B).$$

B Deriving the Diffusion Process for df

From Itô's Lemma,

$$df=\left\{(\nabla f)\,a+\frac{1}{2}\mathrm{tr}\left[b^{\intercal}\left(\nabla^{2}f\right)b\right]\right\}dt+(\nabla f)\,bdW_t.$$

Identification of the drift and diffusion coefficient for m leads to

$$\begin{aligned}a_m &= \frac{\partial f}{\partial m}a_m+\frac{\partial f}{\partial h}a_h+\frac{1}{2}\mathrm{tr}\left[b^{\intercal}\left(\nabla^{2}f\right)b\right]\\ &= \frac{\partial f}{\partial m}a_m+\frac{\partial f}{\partial h}a_h+\frac{1}{2}\left[b_m^2\frac{\partial^2 f}{\partial m^2}+2b_mb_h\frac{\partial^2 f}{\partial m\partial h}+b_h^2\frac{\partial^2 f}{\partial h^2}\right]\end{aligned}$$

and

$$b_m=\frac{\partial f}{\partial m}b_m+\frac{\partial f}{\partial h}b_h$$

where all the partial derivatives are calculated with $m=0$. We then get

$$a_m=\frac{1}{1-\frac{\partial f}{\partial m}}\left\{\frac{\partial f}{\partial h}a_h+\frac{1}{2}\left[b_m^2\frac{\partial^2 f}{\partial m^2}+2b_mb_h\frac{\partial^2 f}{\partial m\partial h}+b_h^2\frac{\partial^2 f}{\partial h^2}\right]\right\}$$

and

$$b_m=\frac{\frac{\partial f}{\partial h}}{1-\frac{\partial f}{\partial m}}b_h$$

Noticing that

$$\tanh x = \frac{\exp(x) - \exp(-x)}{\exp(x) + \exp(-x)}$$

and

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}$$

it follows that

$$\tanh' x = \frac{1}{\cosh^2 x}$$

and

$$\tanh'' x = \frac{\tanh x}{\cosh^2 x}.$$

We then have for the several derivatives,

$$\begin{aligned}
f_m &= \beta J n \Phi(1) \\
f_h &= (\beta + h_c \beta')\Phi(1) + \Phi(\alpha) \\
f_{mm} &= -2J^2 n^2 \beta^2 \Delta(1) \\
f_{hh} &= (2\beta' + h_c \beta'')\Phi(1) + \beta''\Phi(\alpha) - 2(\beta' h_c + \beta)\Delta(1) - 2\beta'\Delta(\alpha) \\
f_{hm} &= -2Jn\beta[(\beta' h_c + \beta)\Delta(1) + \Delta(\alpha)],
\end{aligned}$$

with $\Phi(x)$ and $\Delta(x)$ defined as in the main text.