Pure Strategy Equilibria of Multidimensional and Non-Monotonic Auctions*

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Abstract

We give necessary and sufficient conditions for the existence of symmetric equilibrium without ties in common values auctions, with multidimensional independent types and no monotonic assumptions. When the conditions are not satisfied, we are still able to prove the existence of pure strategy equilibrium with an exogenous and explicit tie breaking mechanism. As a basis for these results, we obtain a characterization lemma that is valid under a general setting, that includes non-independent types, asymmetrical utilities and any attitude towards risk. Such characterization gives a basis for an intuitive interpretation for the behavior of the bidder: to bid in order to equalize the marginal benefit and the marginal cost of bidding.

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1 Introduction

The main contributions of this paper are the following: (1) a general lemma of characterization of the optimum bidding behavior; (2) necessary and sufficient conditions for the existence of regular equilibria in auctions, that is, equilibria without ties with positive probability or gaps in the support of the winning bids; (3) a tie-breaking rule to ensure the existence of pure strategy equilibria when these necessary and sufficient conditions do not hold, that is, when ties with positive probability are inevitable. Additionally, we provide some models to analyze auctions with multidimensional bids.

The rest of this introduction describes the results, as well as the assumptions, relevance and the method of proof. Section 2 describes the general model, while section 3 presents the characterization lemma which is valid in the context of this general model. Section 4 describes the Indirect Auction Approach, which allows us to prove the existence of equilibrium for multidimensional,

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non-monotonic auctions. Also in section 4, we present the (exogenous) tie-breaking rule, which ensures the existence of equilibrium for all auctions in our class. Section 5 exemplifies how to use the approach described in section 4 to analyze auctions with multidimensional bids. Section 6 concludes with a discussion about the limits of our results and reviews the contributions of the paper in light of the related literature.

1.1 Basic Principle of Bidding

Many experimental and empirical works suggest that the participants of auctions do (or at least may) not follow their equilibrium strategies.[1] Although there is a considerable debate about this point, it highlights the assumption that equilibrium behavior might be too strong. An alternative approach is to assume only that the players follow rationalizable strategies, instead of equilibrium strategies. Pursuing this idea, Battigalli and Siniscalchi (2003) show that some empirical and experimental findings can be explained. Nevertheless, they still assume what Harsanyi (1967-8) calls consistency of beliefs, that is, the subjective probability that players attribute to the distribution of types of the opponents is just a conditional distribution and the conditional distribution of all players comes from the same prior distribution.[2] This is almost always assumed in game theory, but does not need to be true, as Harsanyi stresses. Indeed, at the beginning of the iteration between players, they may have inconsistent beliefs. As a result, the first rounds of the game do not satisfy the consistency of beliefs and have to be discarded in order to use the received theory.

Of course, one may think that nothing can be said without this basic assumption. We show, on the contrary, that something interesting can be said. If we adhere to the even weaker assumption that bidders are rational, then we prove that they act in order to equalize their marginal utility to the marginal cost of bidding. This basic principle can provide insights for empirical and experimental studies, since every bid (even the initial or the apparently inconsistent ones) bears valuable information about the beliefs of the players. Also, the principle holds under fairly general conditions, which are given by the Characterization Lemma.

The Characterization Lemma is valid for dependent types (with arbitrary dimension), asymmetric utilities with any attitude towards risk and does not require assumptions as to monotonicity or separability of transfers. The model embraces all kind of sealed-bid auctions where each player is interested in just one object (to buy or sell).

When one introduces the additional hypotheses of risk neutrality, symmetry and monotonicity of the utility function, the characterization provided by the Lemma reduces to the first-order conditions obtained by Milgrom and Weber (1982) for first- and second-price auctions, by Krishna and Morgan (1997) for the all-pay auction and war of attrition, and by Williams (1991) for buyers'-bids double auctions.

1.2 Multidimensional and Non-Monotonic Equilibria Existence

Two long-standing assumptions in auction theory are that types (private information) are unidimensional and that utilities (the value of the object being auctioned) are monotonic with types. Although there have been recent efforts to generalize the equilibria existence for multidimensional types (e.g. McAdams (2003a)), we are not aware of any theoretical construction for non-monotonic auctions.[3]

[1] For a survey of experimental works, see Kagel (1995) and for the empirical literature on auction data, see Laffont (1997).

[2] This is also called common prior assumption.

[3] Zheng (2001) analyzes a model where private information is the budget constraint, and the bidding behavior can be non-monotonic. Nevertheless, there is also a monotonic equilibrium. McAdams (2003b) gives an example with three bidders and affiliated types, where a non-monotonic equilibrium can exist. Athey and Levin (2001) and Ewerhart and Fieseler (2003) also give examples of non-monotonic bidding functions, but where the bids are multidimensional — a setting that we consider in section 5. None of these papers sets out to build a theory for non-monotonic equilibria.

This is probably due to the understanding that monotonicity seems a reasonable assumption when dealing with unidimensional types. Nevertheless, it is clearly less appealing when multidimensional types are considered.[4] So, it is desirable that a theory of multidimensional auctions should deal with non-monotonic assumptions.

Nevertheless, even for the unidimensional setting, things are not simple. An example provided by Jacskon, Simon, Swinkels and Zame (2002) (henceforth JSSZ) illustrates the difficulties.[5] The example is a first-price auction with two bidders, whose signals are uniformly distributed in $[0, 1]$ and utilities are given by $u_i(t) = v(t_i, t_{-i}) = 5 + t_i - 4t_{-i}$. They show that such an example does not have an equilibrium without a special tie-breaking rule.

While developing the theory for general multidimensional and non-monotonic auctions, we find the reason why this example does not possess equilibrium. Indeed, it does not satisfy the necessary and sufficient conditions for the existence of pure strategy equilibrium that we present in our Theorem 3. Theorem 3 applies to a class of symmetric interdependent values auctions (where the separable utilities case above is a particular example) with independent types.[6] Contrary to what one would expect, the conditions are in general easy to check.

If the necessary and sufficient conditions of Theorem 3 are not satisfied, the received literature can ensure the existence of equilibrium just through the result of JSSZ. This result gives the existence in mixed strategies and accepts an endogenous tie-breaking rule as the solution concept. In Theorem 4, we prove that if the conditions of Theorem 3 are not met, there exists a *fixed and previously known* tie-breaking rule that it is capable of implying the equilibrium existence in pure strategies. We call this the modified second-price auction tie-breaking rule and it applies to all kinds of auctions considered.

Our approach also provides expressions for the multidimensional strategies (even under the occurrence of ties), for auction formats that include, as special cases, the first-price, second-price, (first-price) all-pay auctions and war of attrition (second-price all-pay auction). The expressions are simple, due to the method that we follow to simplify the problem. We call it the Indirect Auction Approach and describe it below. We emphasize that it is of interest even to unidimensional auctions, as we discuss in section 6.

1.3 Detailed Description of the Indirect Auction Approach

Under the standard rule of auctions, higher bids correspond to higher probability of winning. If a bidding function $b(\cdot)$ is fixed and followed by all participants in a symmetric auction, we can associate to each bid (and so, to each type), a probability of winning. All types that bid the same bid under $b(\cdot)$ have the same probability of winning. This allows us to introduce the concept of conjugation. If $b(t) = b(s)$, and hence, t and s have the same probability of winning, we say that t and s are conjugated.[7]

The use of the probability of winning as analytical tool is not new in auction theory. Sometimes in the literature, what we call conjugation is named "reduced form": "The function relating a bidder's type to his probability of winning is the reduced form of the auction." (Border, 1991, p. 1175). See also Matthews (1984) and Chen (1986). So, what we will call "indirect auction" can be also called "reduced form auction". These papers analyze problems related to the characterization and existence of optimal auctions. So, the auction is treated, as do Myerson (1981), only by considering the probability of winning and the payments. In turn, our problem is that of finding the equilibrium for fixed auction rules. It is in light of these differences and in the attempt to do not confuse terms that we decided to maintain our original terminology. A further reason for the adoption of a different terminology comes from the fact that the indirect auction is not

[4] See examples 8, 9 and 10 in section 4.

[5] Their example was developed from Example 3 of Maskin and Riley (2000).

[6] The assumptions of Theorem 3 are related to a condition of Dasgupta and Maskin (2000). Theorems 1 and 2 hold under more general conditions.

[7] The use of conjugations is an idea borrowed from Araujo and Moreira (2000).

"equivalent" to the direct one. So, it is not a merely "reduced form" of the auction. (See Remark 1 on subsection 4.3).

Returning to the description of the method, the main idea is to reparametrize the types and to associate to all conjugated types $s \in S$, the probability of winning the auction. As stated, this idea should seem unpromising since the probability of winning will be different for each different bidding function that we begin with. Moreover, we cannot talk about conjugation if we do not previously fix a bidding function.

To overcome these problems, we define conjugations without needing to mention bidding functions. The definition comes from an insight acquired from the above notion of conjugation. We define conjugations as a suitable reparametrization of the types. Once we have defined conjugations in subsection 4.1, we can define in subsection 4.2 the Indirect Auction.[8] For this, we simply integrate the utilities of the direct auction for all types that are conjugated. From our definition of conjugation, the indirect auction is now an auction (of the same format as the direct auction, that is, a first-price auction if the original auction is a first-price auction), between two players with independent signals, uniformly distributed in $[0, 1]$. This makes the analysis of equilibrium existence easier. An important result of the subsection 4.2 is the relationship between the payoffs of direct and indirect auctions, which is made in Proposition 2.

With these preparatory results, we can finally deal with the problem of equilibrium existence in subsection 4.3. First, we prove that the existence of a regular equilibrium implies nice properties for the conjugation that it defines. This is the content of Theorem 1. These properties are almost sufficient for the existence of the equilibrium, which is proved in Theorem 2: since we have defined the conjugation without mentioning a bidding function, then whenever we can find a conjugation that meets the conditions of Theorem 2, there exists a regular equilibrium of the direct auction. These conditions are just slightly stronger than the necessary conditions given by Theorem 1. Thus, Theorem 2 turns the problem of equilibrium existence into that of finding a conjugation that meets its conditions. If we manage to find the correct conjugation, we are done. We show how to perform this task in two examples (6 and 7) at the end of subsection 4.3.

In subsection 4.4, we treat a case of utilities that include the separable utilities as a special case, that is, $v(t_i, t_{-i}) = v^1(t_i) + v^2(t_{-i})$. For the setting defined there, we are able to give necessary and sufficient conditions for the existence of regular equilibrium (Theorem 3). This is very useful, because it explain why Example 1 of JSSZ fails. But it raises the question: what can be done if the necessary and sufficient conditions of Theorem 3 are not met?

Theorem 4 provides the answer. If we conduct a modified second-price auction, the equilibrium exists in pure strategies with ties of positive probability. This last result has advantages over the result of JSSZ: it is in pure strategies, the tie-breaking rule is exogenously given, it is valid for all kind of auctions, it is fairly simple and it does not require the announcement of types. Concluding section 4, we show in subsection 4.5 that our approach can be extended to the case of risk aversion.

1.4 Multidimensional Bids

As an illustration of the Indirect Auction Approach, in section 5 we generalize two models of procurement auctions with multidimensional bids, as proposed by Che (1993) and by Ewerhart and Fieseler (2003). The analysis can also be adapted to timber auctions, described by Athey and Levin (2001). Nevertheless, this paper does not treat multi-unit auctions.

[8] This terminology comes from the "Taxation Principle" which allows us to implement the optimal direct truthful mechanism through some convenient indirect one. In this case, we are implementing the equilibrium in the auction using an indirect auction obtained from reparametrizing types through the probability of winning.

2 The Model

There are N players.[9] Player i ($i = 1, ..., N$) receives a private information, t_i, and chooses an action that is a real number (i.e., he submits a bid b_i). The "auction house" computes the bids and determines who "wins" and who "looses". If player i wins, he receives $\overline{u}_i(t, b)$ and if she looses, she receives $\underline{u}_i(t, b)$, where $t = (t_i, t_{-i})$ is the profile of all signals and $b = (b_i, b_{-i})$ is the profile of bids submitted.[10]

Information

We assume that the private signal of each player, t_i, lives in an arbitrary probabilistic space, $(T_i, \Im_i, \tau_i)$. We assume that the product space, $(T, \Im, \tau)$, is such that τ is absolutely continuous with respect to the product $\times_{i=1}^{N}\tau_i$ of its marginals.

Bidding

After receiving the private information, each player submits a sealed proposal, that is, a bid (or offer) that is a real number. A negative bid is equivalent to the non-participation decision (in which case the payoff is normalized to zero).[11] We assume that the maximum permitted bid is M, to rule out behaviors (equilibria) in which one bidder bids arbitrarily high and the others bid zero. This is a weak assumption, although it imposes some restriction on certain auction formats such as third-price auctions.

Mechanism of allocation

We suppose that each bidder sees a number that depends only on the bids submitted by the opponents and that determines the threshold of the winning and losing events. We denote such number as $b_{(-i)}$. For instance, if the auction is a one-object auction where all players are buyers, $b_{(-i)}$ is the maximum bid of the opponents, that is, $b_{(-i)} \equiv \max_{j \neq i} b_j$. If there are K objects for selling and a reserve price $b_0 > 0$, then $b_{(-i)} \equiv \max\left\{b_0, b_{(K)}^{-i}\right\}$, where $b_{(m)}^{-i}$ is the m-th order statistic of $(b_1, ..., b_{i-1}, b_{i+1}, ..., b_N)$, that is, $b_{(1)}^{-i} \geqslant b_{(2)}^{-i} \geqslant ... \geqslant b_{(N-1)}^{-i}$.[12]

Double auctions among S sellers (players $1, 2, ..., S$) and $N - S$ buyers (players $S + 1, ..., N$) can be described similarly. There are two cases to consider. If $S > N - S$, then buyers get the object as long as they bid at least the minimum bid of the sellers. So, we put $b_{(-i)} \equiv \min_{j=1,...,S} b_j$ if player i is a buyer. There is competition among sellers. Then, a player $i \leqslant S$ sells the object if and only if b_i is below the minimum bid of buyers and if it is among the $N - S$ lowest bids of sellers. In other words, $b_{(-i)} \equiv \min\left\{b_{(N-S)}^{-i,s}, \min_{j=S+1,...,N} b_j\right\}$, where $b_{(m)}^{-i,s}$ is m-th order statistic among all sellers but player i. The case when there are more buyers (that is, $S \leqslant N - S$) is similar. If the player i is a buyer, $b_{(-i)} \equiv \max\left\{b_{(S)}^{-i,b}, \min_{j=S+1,...,N} b_j\right\}$, where $b_{(m)}^{-i,b}$ is m-th order statistic among all buyers but player i. If the player i is a seller, $b_{(-i)} \equiv \min_{j=S+1,...,N} b_j$.

A third part (the auction house) computes the bids and determines what is the result of the game for each player. If $b_i < 0$ (that is, player i does not participate), the payoff is 0. If $b_i > b_{(-i)}$, the auction house declares player i "holder of an object" (and he has an ex-post payoff $\overline{u}_i(t, b)$ in this situation). If $0 \leqslant b_i < b_{(-i)}$, the auction house declares that player i is not a holder of an object (and he receives $\underline{u}_i(t, b)$).[13]

[9] Our model is inspired in auction games, although it can encompass a general class of discontinuous games. For convenience and easy understanding, we will use the terminology of auction theory, such as "bidding functions" and "bids" for strategies and actions, respectively.

[10] We consider the dependence on b instead of b_i because we want to include in our results auctions where the payoff depends on bids of the opponents, such as the second-price auction, for instance. Also, this allows the study of "exotic" auctions, i.e., auctions where the payment is an arbitrary function of all bids.

[11] We are implicitly assuming a reserve price of at least zero. This is not essential, but multiple equilibria may exist without it, as pointed out by Milgrom and Weber (1982).

[12] If there is no reserve price, simply omit b_0.

[13] In most auctions, $\underline{u}_i$ is normalized as 0. However, in double and all-pay auctions or if there is an entry fee, this is not the case.

Observe that if player i is a seller, he begins with an object and if $b_i < b_{(-i)}$, he sells his object. If he is a buyer, the situation $b_i < b_{(-i)}$ corresponds to maintaining his previous situation: without the object. We can therefore treat buyers and sellers with the same formulation. It is possible to distinguish them, through the following conditions: if player i is a buyer, then $\partial_{b_i}\overline{u}_i(t,b)$, $\partial_{b_i}\underline{u}_i(t,b) \leqslant 0$ for all (t,b). If player i is a seller, we can assume that $\partial_{b_i}\overline{u}_i(t,b)$, $\partial_{b_i}\underline{u}_i(t,b) \geqslant 0$ for all (t,b). The motivation for this definition is clear. Without changing the event of winning or losing, a higher bid may (weakly) benefit the seller and hurt a buyer. In auctions, this is a very natural discrimination, although in more general games it can be less appealing. We emphasize that these assumptions are only for purposes of interpretation. We do not use them in the results below.

If $b_i = b_{(-i)}$, there is a tie, and a specific rule — which may include a random device and/or the requirement of a further action a_i — determines if the player is a winner or a loser.[14] We model this by saying that the player receives $u_i^T(t,b,a)$, a value between $\overline{u}_i(t,b)$ and $\underline{u}_i(t,b)$.[15] We do not need to specify $u_i^T(t,b,a)$ for the first two results.

This setting is very general and applies to a broad class of discontinuous games. For example, $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = 0$ correspond to a first-price auction with risk neutrality.[16] If $\overline{u}_i(t,b) = v_i(t) - b_i$ and $\underline{u}_i(t,b) = -b_i$ we have the all-pay auction. If $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$, this is the war of attrition. As pointed out by Lizzeri and Persico (2000), we can have also combinations of these games. For example, $\overline{u}_i(t,b) = v_i(t) - \alpha b_i - (1-\alpha)b_{(-i)}$ and $\underline{u}_i(t,b) = 0$, with $\alpha \in (0,1)$, gives a combination of the first- and second-price auctions. Another possibility is the third-price auction or an auction where the payment is a general function of the others' bids. It is also useful to consider K-unit auctions with unitary demand, among N buyers, $1 < K < N$. Then, $b_{(-i)} = b^{-i}_{(K)}$. Then, a pay-your-bid auction is given by $\overline{u}_i(t,b) = v_i(t_i) - b_i$ and $\underline{u}_i(t,b) = 0$. If it is a uniform price with the price determined by the highest looser's bid, $\overline{u}_i(t,b) = v_i(t_i) - b_{(-i)}$ and $\underline{u}_i(t,b) = 0$. If it is a uniform price with the price determined by the lowest winner's bid, $\underline{u}_i(t,b) = 0$, $\overline{u}_i(t,b) = v_i(t_i) - b_{(-i)}$ if $b_i > b^{-i}_{(K-1)}$ and $\overline{u}_i(t,b) = v_i(t_i) - b_i$ otherwise. Observe that even in this last case, $\overline{u}_i(t,b)$ is continuous if $v_i(t_i)$ is.

Notation

In order to avoid confusion, we will use bold letters to denote bidding functions, i.e., $\mathbf{b} = (\mathbf{b}_i)_{i\in I} \in \times_{i\in I}\mathbb{L}^1(T_i,[-1,M])$. If we fix the other's strategies, $\mathbf{b}_{-i}$, let $F_{b_{(-i)}}(b_i|t_i) \equiv \tau_{-i}(\{t_{-i} : \mathbf{b}_{-i}(t_{-i}) < b_i\} \mid t_i)$ and $f_{b_{(-i)}}(\cdot|t_i)$ be its Radon-Nykodim derivative with respect to the Lebesgue measure, i.e., the density function.[17] We use the notation $F^{\perp}_{b_{(-i)}}(\cdot|t_i)$ for the distribution function of the singular part of the measure $F_{b_{(-i)}}(\cdot|t_i)$, that is, the part that assigns positive measure to sets of bids with zero Lebesgue measure.

If the profile $\mathbf{b}_{-i}$ is fixed, the expected payoff of bidder i of type t_i, when bidding b_i, is:

$$\begin{aligned}\Pi_i(t_i,b_i,\mathbf{b}_{-i}) \equiv \int \Big[& \overline{u}_i(t,b_i,\mathbf{b}_{-i}(t_{-i}))\, 1_{[b_i > \mathbf{b}_{(-i)}(t_{-i})]} \\ & + u_i^T(t,b_i,\mathbf{b}_{-i}(t_{-i}),a)\, 1_{[b_i = \mathbf{b}_{(-i)}(t_{-i})]} \\ & + \underline{u}_i(t,b_i,\mathbf{b}_{-i}(t_{-i}))\, 1_{[b_i < \mathbf{b}_{(-i)}(t_{-i})]} \Big] \tau_{-i}(dt_{-i}|t_i).\end{aligned} \tag{1}$$

[14] The required action can be the submission of another bid for a Vickrey auction (as in Maskin and Riley (2000)) or the announcement of the type (as in JSSZ). Since the only revealed information in the case of a tie is its occurrence, the action can be required together with the submission of the bid.

[15] The specification of a tie-breaking rule is important for the existence of equilibria, as shown by Jackson *et al.* (2002). With this terminology, the proposal of an "endogenous tie-breaking rule" of Simon and Zame (1990) corresponds to specifying endogenously u_i^T in order to ensure the equilibrium existence.

[16] If we put $\overline{u}_i(t,b) = U_i(v_i(t) - b_i)$ we can have any attitude towards risk.

[17] Note that, with such convention, the cumulative distribution functions - c.d.f.'s - are left continuous.

if $b_i \in [0, M]$ and $\Pi_i(t_i, b_i, \mathbf{b}_{-i}) = 0$ if $b_i < 0$. It is worth observing that if the probability of bid b_i being equal to $\mathbf{b}_{(-i)}$, conditional on t_i, is zero, the tie-breaking rule is not important and the second term in the integral may be omitted.

Again, when there is no possibility of confusion, we will write $\Pi_i(t_i, b_i)$ for $\Pi_i(t_i, b_i, \mathbf{b}_{-i})$ and omit the arguments and the measure. So, we have

$$
\begin{aligned}
&\Pi_i(t_i, b_i) \\
&= \int \left\{ \overline{u}_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + u_i^T 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \underline{u}_i \left(1 - 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} - 1_{\left[b_{(-i)} = \mathbf{b}_i\right]}\right)\right\} \\
&= \int \left\{ u_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \left(u_i^T - \underline{u}_i\right) 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \underline{u}_i \right\} \\
&= \int u_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \int \left(u_i^T - \underline{u}_i\right) 1_{\left[b_{(-i)} = \mathbf{b}_i\right]} + \int \underline{u}_i .
\end{aligned}
$$

where $u_i \equiv \overline{u}_i - \underline{u}_i$ is the net payoff.

3 The Basic Principle of Bidding

Our first result is a characterization of the payoff through its derivative with respect to the bid given by an integral expression, i.e., a kind of fundamental theorem of calculus. For this, we will need the following assumption:

(H) *$\overline{u}_i$ and $\underline{u}_i$ are absolutely continuous on b_i and $\partial_{b_i}\overline{u}_i$ and $\partial_{b_i}\underline{u}_i$ are essentially bounded.*

Lemma 1 (Payoff Characterization) — Assume (H). Fix a profile of bidding functions $\mathbf{b}_{-i}$. The payoff can be expressed by

$$
\begin{aligned}
\Pi_i(t_i, b_i) = E&\left[\left(u_i^T - \underline{u}_i\right)(t_i, b_i, \cdot)\, 1_{\left[\mathbf{b}_{(-i)} = b_i\right]} | t_i\right] \\
&+ \int_{[0, b_i)} E[u_i\,(t_i, b_i, \cdot)\,| t_i, \mathbf{b}_{(-i)} = \beta] dF^{\perp}_{b_{(-i)}}(\beta | t_i) + \int_{[0, b_i)} \partial_{b_i}\Pi_i(t_i, \beta) d\beta .
\end{aligned}
$$

where $\partial_{b_i}\Pi_i(t_i, \beta)$ exists for almost all β and in this case it is given by

$$
\begin{aligned}
\partial_{b_i}\Pi_i(t_i, \beta) = E&\left[\partial_{b_i}\overline{u}_i\,(t_i, \beta, \cdot)\, 1_{\left[\beta > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i\,(t_i, \beta, \cdot)\, 1_{\left[\beta < \mathbf{b}_{(-i)}\right]} | t_i\right] \\
&+ E[u_i\,(t_i, \beta, \cdot)\,| t_i, \mathbf{b}_{(-i)} = \beta] f_{b_{(-i)}}(\beta | t_i) .
\end{aligned}
\tag{2}
$$

Proof. The proof follows the demonstration of the Leibiniz rule. The main point is the use of a theorem of Rudin (1966) on the derivatives of measures and its integral expression. See the details in the Appendix A.■

The most important part of Lemma 1 is the expression of $\partial_{b_i}\Pi_i(t_i, \beta)$. One of the best ways to understand Lemma 1 is through the following:

Corollary 2 (The Basic Principle of Bidding) — Under regularity assumptions, the optimum bid is such that the marginal cost of bidding is equal to the marginal utility from bidding. More formally: if $\Pi_i(t_i, \cdot)$ is differentiable at $b_i \in \arg\max_\beta \Pi_i(t_i, \beta)$ and there is no tie with positive probability at b_i, then

$$
E[u_i | t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}(b_i | t_i) = -E\left[\partial_{b_i}\overline{u}_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i 1_{\left[b_i < \mathbf{b}_{(-i)}\right]} | t_i\right] . \tag{3}
$$

Obverse that $E[u_i|t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}(b_i|t_i)$ represents the marginal benefit of bidding, that is, the marginal utility that a bidder has from changing from losing to winning events. On the other hand, $E\left[-\partial_{b_i}\overline{u}_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]}|t_i\right]$ represents the marginal cost of changing the bid in all the events where a bidder is already winning. In the same manner, $E\left[-\partial_{b_i}\underline{u}_i 1_{\left[b_i < \mathbf{b}_{(-i)}\right]}|t_i\right]$ represents the marginal cost of changing the bid in the events where he is loosing. Thus, we can read the above condition in an intuitive and simple manner: at the optimum of the best-reply problem, the marginal benefit of bidding, $E[u_i|t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}(b_i|t_i)$, must be equal to its marginal cost, $-E\left[\partial_{b_i}\overline{u}_i 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \partial_{b_i}\underline{u}_i 1_{\left[b_i < \mathbf{b}_{(-i)}\right]}|t_i\right]$. Note that we do not require separability in the monetary transfer (risk neutrality) to reach such an interpretation.

This interpretation is useful for understanding the bidding behavior. In first-price auctions, the marginal cost of bidding is what implies a decreasing in the way bidders bid. In second-price auctions, the marginal cost of bidding is zero (because $\partial_{b_i}\overline{u}_i = 0$), so that each bidder bids until its marginal utility of bidding became zero.

Corollary 1 is a generalization of the necessary conditions first-order for the first- and second-price auctions presented in Milgrom and Weber (1982), for the war of attrition and all-pay auctions presented in Krishna and Morgan (1997), as we show in Examples 1- 4 below. Example 5 shows how the Basic Principle of Bidding is concise. Such an example is the application of Corollary 1 for double auctions and it presents a comparison with the equivalent expression obtained by Williams (1991).

Example 1 — First-price auction

When we restrict ourselves to the case of the first-price auction with risk neutrality (i.e., $\underline{u}_i = 0$ and $\overline{u}_i = v_i - b_i$), then $\partial_{b_i}\overline{u}_i = -1$ and $\partial_{b_i}\underline{u}_i = 0$. The condition (3) becomes:

$$b_i = E[v_i|t_i, \mathbf{b}_{(-i)} = b_i] - \frac{F_{b_{(-i)}}(b_i|t_i)}{f_{b_{(-i)}}(b_i|t_i)}. \tag{4}$$

This (necessary) first-order condition provides a useful way to determine best-reply bids. Note that this expression admits non-monotonic bidding functions $\mathbf{b}_{(-i)}$, contrary to Milgrom and Weber's model. It also encompasses asymmetries in utilities and distribution of types. Assuming affiliation and monotonic utilities, Milgrom and Weber (1982) can restrict themselves to the space of non-decreasing symmetric bidding functions (i.e., $\mathbf{b}_i = \mathbf{b}^*$, for all $i \in I$). Thus,

$$\mathbf{b}_{(-i)}(t_{-i}) = \max_{j \neq i} \mathbf{b}^*(t_j) = x \Longleftrightarrow t_{(-i)} \equiv \max_{j \neq i} t_j = (\mathbf{b}^*)^{-1}(x),$$

i.e., conditioning on $\mathbf{b}_{(-i)} = b_i$ is the same to conditioning on $t_{(-i)} = t_i$. Also, $f_{b_{(-i)}}(b_i|t_i) = f_{t_{(-i)}}(t_i|t_i) / (\mathbf{b}^*(t_i))'$ and $F_{b_{(-i)}}(b_i|t_i) = F_{t_{(-i)}}(t_i|t_i)$. With this, (4) becomes

$$\mathbf{b}^{*\prime}(s) = \left\{E\left[v|t_i = s, t_{(-i)} = s\right] - \mathbf{b}^*(s)\right\} \frac{f_{t_{(-i)}}(s|s)}{F_{t_{(-i)}}(s|s)}$$

whose solution is shown to be an equilibrium under affiliation.

Example 2 — Second price auction

In the second price auction, Milgrom and Weber's model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = 0$. Then, $\partial_{b_i}\overline{u}_i = \partial_{b_i}\underline{u}_i = 0$ and (3) reduces to $E[v_i - b_i | t_i, \mathbf{b}_{(-i)} = b_i]\, f_{b_{(-i)}}(b_i|t_i) = 0$ which can be simplified to

$$b_i = E[v_i | t_i, \mathbf{b}_{(-i)} = b_i].$$

Again with monotonicity and symmetry assumptions, Milgrom and Weber's expression for the equilibrium bid function can be obtained:

$$\mathbf{b}^*(s) = E\left[v | t_i = s, t_{(-i)} = s\right] \equiv \bar{v}(s,s).$$

Example 3 — All-pay auction

Krishna and Morgan (1997) extend the method of Milgrom and Weber (1982) to the cases of war of attrition and all-pay auctions. In the all-pay auction, their model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$. Then, $\partial_{b_i}\overline{u}_i = 0$ and $\partial_{b_i}\underline{u}_i = -1$. So, (3) reduces to

$$E[v_i(t) | t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}(b_i|t_i) = 1.$$

Under the same hypothesis of monotonicity and symmetry, they find the following differential equation:

$$\mathbf{b}^{*\prime}(s) = E\left[v | t_i = s, t_{(-i)} = s\right] f_{t_{(-i)}}(s|s),$$

whose solution they show to be an equilibrium under affiliation.

Example 4 — War of attrition

In the war of attrition, Krishna and Morgan (1997) model is equivalent to $\overline{u}_i(t,b) = v_i(t) - b_{(-i)}$ and $\underline{u}_i(t,b) = -b_i$. Then, $\partial_{b_i}\overline{u}_i = 0$ and $\partial_{b_i}\underline{u}_i = -1$. So, (3) reduces to

$$E[v_i(t) | t_i, \mathbf{b}_{(-i)} = b_i] f_{b_{(-i)}}(b_i|t_i) = 1 - F_{b_{(-i)}}(b_i|t_i).$$

Again, with monotonicity and symmetry, they derive the equation

$$\mathbf{b}^{*\prime}(s) = E\left[v | t_i = s, t_{(-i)} = s\right] \frac{1 - F_{t_{(-i)}}(s|s)}{f_{t_{(-i)}}(s|s)},$$

and the equilibrium is shown to exist under affiliation.

Example 5 — Double auction

In the analysis of a double auction with private values, risk neutrality, independent types and symmetry among buyers and sellers, Williams (1991) assumes that the payment is determined by the buyer's bid. So, it is optimum for the seller to bid her value. To analyze the behavior of the *buyer* i, Williams (1991) reaches the following expression:

$$\begin{aligned} \partial_{b_i}\Pi_i(t_i,\beta) = \Big[& n f_1(\beta)\, K_{n,m}\left(\mathbf{b}^{-1}(\beta), \beta\right) \\ & + (m-1)\frac{f_2(v_b)}{b'(\beta)} L_{n,m}\left(\mathbf{b}^{-1}(\beta), \beta\right)\Big](v - \beta) \\ & - M_{n,m}\left(\mathbf{b}^{-1}(\beta), \beta\right) \end{aligned} \tag{5}$$

where f_1 is the common density function of sellers, f_2 is the common density function of buyers, n is the number of sellers and m is the number of buyers. We will reproduce only the $M_{n,m}(\cdot,\cdot)$:[18]

$$M_{n,m}(v,\beta) \equiv \sum_{\substack{i+j=m,\\ 0\leqslant i\leqslant m-1}} \binom{n}{j}\binom{m-1}{i} F_1(\beta)^j F_2(v)^i (1-F_1(b))^{n-j} (1-F_2(v))^{m-1-i}.$$

The expression (5) is just a special case of (3). In fact, the expression in brackets in (5) is just $f_{b_{(-i)}}(\beta)$ and $M_{n,m}(v_b, b)$ is $F_{b_{(-i)}}(\beta)$.[19]

An important application of the Characterization Lemma will be given in the next section where we give necessary and sufficient conditions to the existence of equilibrium in common-value auctions with multidimensional independent types and non-monotonic utilities.

Another possibility is the investigation of how far auction theory can lead us under a weaker hypothesis. For instance, the Characterization Lemma can be understood as a general condition for bidding behavior, able to describe the behavior of rational bidders without assuming that bidders follow their equilibrium strategies. We have exposed such a possibility in the introduction (subsection 1.1).

4 The Indirect Auction Approach

Now we turn to the problem of the existence of equilibrium. We will consider a particularization of the model in the previous section, that is, we will work according to the following setting:

$$\begin{aligned}
\overline{u}_i(t,b) &= v(t_i,t_{-i}) - p^W\left(b_i, b_{(-i)}\right);\\
\underline{u}_i(t,b) &= -p^L\left(b_i, b_{(-i)}\right);\\
u_i^T(t,b) &= \frac{v(t_i,t_{-i}) - b_i}{m},
\end{aligned}$$

where m is the number of bidders tying and functions p^W and p^L are the payments made in the events of winning and losing, respectively. The specification of u_i^T comes from the standard solutions to ties: the payment $b_i = b_{(-i)}$ is required for any bidder that receives the object, and the object is split with equal probability among the bidders that are tying.

For further reference, we define

$$v^1(s) \equiv E\left[v(t_i,t_{-i})\,|t_i = s\right].$$

We consider an auction with a reserve price of zero.[20] We will assume the following natural conditions for p^W and p^L:

(H0) Over the domain $\mathbb{R}_+ \times \mathbb{R}$, p^W and p^L are non-negative, differentiable, and at least one of them is non-constant. If $b_i < 0$, then $p^W\left(b_i, b_{(-i)}\right) = p^L\left(b_i, b_{(-i)}\right) = 0$.

Observe that assumption (H0) is rather weak. It is possible, for instance, for the payment to be non-monotonic with the bid. Nevertheless, we are mainly interested in the following four kinds of auctions:

[18] Indeed, the other expressions are similar. To obtain $K_{n,m}(\cdot,\cdot)$ just substitute $n-1$ for n where it occurs in $M_{n,m}(\cdot,\cdot)$. To obtain $L_{n,m}(\cdot,\cdot)$, substitute $m-2$ for $m-1$ where it occurs in $M_{n,m}(\cdot,\cdot)$.

[19] Remember that the independency implies $f_{b_{(-i)}}(\beta|t_i) = f_{b_{(-i)}}(\beta)$.

[20] We can relax this assumption. Indeed, most of our results hold for any positive reserve prices, but the expressions may need some modifications.

(F) First-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.
(S) Second-price auctions: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = 0$.
(A) All-pay auctions: $p^W\left(b_i, b_{(-i)}\right) = b_i$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.
(W) War of attrition: $p^W\left(b_i, b_{(-i)}\right) = b_{(-i)}$ and $p^L\left(b_i, b_{(-i)}\right) = b_i$.

In addition to (H0), we will assume the following hypotheses:

(H1) The types are independent and identically distributed, so that $T_1 = ... = T_N = S$ and $\tau_1 = ...= \tau_N = \sigma$, with S a compact set and σ a probability measure.

(H2) v is non-negative, continuous and symmetric in its last $N-1$ arguments, that is, if t'_{-i} is a permutation of t_{-i}, $v\left(t_i, t'_{-i}\right) = v\left(t_i, t_{-i}\right)$.

Observe that we are considering a symmetric auction. Thus, throughout this section, when we talk about a strategy, we always mean a symmetric one. For instance, Theorem 3 states that the equilibrium is unique, although it is well known that second-price auctions have multiple *asymmetric* equilibria.

We denote the auction above by (S, σ, v). Note that we are still considering multidimensional types and non-monotonic utilities, with N bidders. Under these assumptions we will introduce a new approach to prove existence of equilibria in auctions. We call it the "Indirect Auction Approach". This is the subject of the following subsections.

4.1 Conjugations

We will be interested in regular bidding functions as defined below:

Definition 1 — A bounded measurable function $b : S \to \mathbb{R}$ is regular if the c.d.f.

$$F_b(c) \equiv \Pr\{s \in S : b(s) < c\}$$

is absolutely continuous and strictly increasing in its support, $[b_*, b^*]$.[21]

From the fact that $F_b(\cdot)$ is absolutely continuous, we conclude that $F_b(c) = \Pr\{s \in S : b(s) \leqslant c\}$. Let $\mathcal{S}$ denote the set of regular functions. Observe that $\mathcal{S}$ contains non-monotonic bidding functions. It is formed by functions b that do not induce ties with positive probability (because F_b is absolutely continuous) and that do not have gaps in the support of the bids (because F_b is increasing).

If a bidding function $b \in \mathcal{S}$ is fixed, let us call the c.d.f. of the maximum bid of the opponents $\tilde{P}^b$. That is, we define the transformation $\tilde{P}^b : \mathbb{R}_+ \to [0,1]$ by:

$$\begin{aligned} \tilde{P}^b(c) &= \left(\Pr\{t_i \in S : b(t_i) < c\}\right)^{N-1} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) < c, j \neq i\right\} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b(t_j) \leqslant c, j \neq i\right\}. \end{aligned} \tag{6}$$

By the definition of $\mathcal{S}$, $\tilde{P}^b$ is strictly increasing and its image is the whole interval $[0,1]$.

Now, we will denote by $P^b : S \to [0,1]$ the composition $P^b = \tilde{P}^b \circ b$. So, for a fixed $b \in \mathcal{S}$, followed by all players, $P^b(t_i)$ is the probability of player i of type t_i winning the auction:

$$\begin{aligned} P^b(t_i) &= \Pr\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\right\} \\ &= \Pr\left\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) \leqslant b(t_i)\right\}. \end{aligned} \tag{7}$$

[21] The reader should note that we are changing our notation from the previous sections. Since we are now dealing with the symmetric case, we will note use subscripts. Also, we are not using bold letters to denote functions.

The following observation is important: from H1, the above function does not depend on i and $P^b(t_i) \lesseqqgtr P^b(t_j)$ if and only if $b(t_i) \lesseqqgtr b(t_j)$. Obviously, two players have the same probability of winning if and only if they play the same bids. So, we have the following:

$$\{t_{-i} \in S^{N-1} : b_{(-i)}(t_{-i}) < b(t_i)\} = \left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\},$$

where, as natural, $P^b_{(-i)}(t_{-i}) \equiv \max_{j \neq i} P^b(t_j)$. The equality of these events implies that

$$P^b(t_i) = \Pr\left\{t_{-i} \in S^{N-1} : P^b_{(-i)}(t_{-i}) < P^b(t_i)\right\}.$$

This observation is what will allow us to define conjugations without mentioning bidding functions. This will be very important in order to state our results. We have the following:

Definition 2 — A conjugation for the auction (S, σ, v) is a measurable and surjective function $P : S \to [0,1]$ such that for each $i = 1, \ldots N$,

$$P(t_i) = \Pr\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) \leqslant P(t_i)\} = [\Pr\{t_j \in S : P(t_j) < P(t_i)\}]^{N-1}. \tag{8}$$

Observe that in the above definition, we do not need to mention the strategy $b \in \mathcal{S}$. It is also clear, from the previous discussion, that the definition is not empty, that is, for any regular function $b \in \mathcal{S}$ there exists a conjugation defined by (7) that satisfies the above definition.

Observe also that, since the range of P is $[0,1]$, we have, for all $c \in [0,1]$,

$$\Pr\left\{t_{-i} \in S^{N-1} : P_{(-i)}(t_{-i}) < c\right\} = c. \tag{9}$$

The above equation will be important in the sequel. It simply means that the distribution of $P_{(-i)}(t_{-i})$ is uniform in $[0,1]$.

A natural question that arises is which consistency is necessary between the bidding function $b \in \mathcal{S}$ and a conjugation P in order that they become compatible. Equation (7) gives the condition in one direction. The other direction is very simple, requiring only that the bidding function be an increasing function of the conjugation. That is, we have the following:

Proposition 1 — Given a conjugation $P : S \to [0,1]$, for any increasing function $h : [0,1] \to \mathbb{R}_+$, the function given by $b(t_i) = h(P(t_i))$ is consistent with P, that is,

$$P(t_i) = \Pr\{t_{-i} : b(t_j) < b(t_i), \forall j \neq i\}.$$

Proof. For an increasing function $h : [0,1] \to \mathbb{R}_+$, the function $b(t_i) = h(P(t_i))$ is such that $b(t_i) \lesseqqgtr b(t_j)$ if and only if $P(t_i) \lesseqqgtr P(t_j)$. Then, $\{t_{-i} : b(t_j) < b(t_i), \forall j \neq i\} = \{t_{-i} : P(t_j) \leqslant P(t_i), \forall j \neq i\}$ and the result follows.■

Proposition 1 says that given a conjugation P, there are many bidding functions that are consistent with it. In particular, $b = P$ is a bidding function consistent with P. On the other hand, given a bidding function, there is just a conjugation P^b that is consistent with it.

4.2 Indirect Auctions

We proceed to define the indirect auction. We begin by the definition of (indirect) strategies. To justify the definitions, remember that, given a direct strategy b, we have defined P^b as $\tilde{P}^b \circ b$. We want the indirect strategy, when composed with the reparametrized type, given by the conjugation P^b, to lead to the same bid. That is, if $\tilde{b}$ is the indirect strategy, we would like to have $b(s) = \tilde{b}(P^b(s)) = \tilde{b} \circ P^b(s) = \tilde{b} \circ \tilde{P}^b \circ b(s)$. Then, since $\tilde{P}^b$ is increasing, we must have $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$, which will also be increasing. On the other hand, given a conjugation P and an indirect strategy $\tilde{b}$, we should define the related direct strategy as $b = \tilde{b} \circ P$. So the following definitions are the natural ones.

Definition 3 — (i) An indirect bidding function is a bounded increasing function $\tilde{b} : [0,1] \to \mathbb{R}_+$.

(ii) Given a (direct) bidding function $b : S \to \mathbb{R}_+$, the indirect bidding function $\tilde{b} : [0,1] \to \mathbb{R}_+$ associated to b is

$$\tilde{b}(\phi) = \left(\tilde{P}^b\right)^{-1}(\phi), \tag{10}$$

where $\tilde{P}^b$ is given by (6).

(iii) Conversely, given an indirect bidding function $\tilde{b} : [0,1] \to \mathbb{R}_+$ and a conjugation P of (S, σ, v), the direct bidding function associated to $\tilde{b}$ and P is $b : S \to \mathbb{R}_+$ given by

$$b(s) = \tilde{b} \circ P(s). \tag{11}$$

Now, we define the indirect utility function.

Definition 4 — Fix a conjugation P for an auction (S, σ, v). The indirect utility function of bidder i associated to this conjugation is $\tilde{v} : [0,1]^2 \to \mathbb{R}$, given by

$$\tilde{v}(x, y) \equiv E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = y]. \tag{12}$$

Our method is to construct an auction that will have the above (indirect) utility function. The reader should keep in mind that the indirect auction is just an auxiliary and fictitious auction that will help in the analysis of the direct one.

Fix a conjugation P and define the following function:

$$\tilde{\Pi}(x, c) \equiv E\left[\Pi(t_i, c) | P(t_i) = x\right], \tag{13}$$

where, $\Pi(t_i, c)$ is the interim payoff of the direct auction, given by (1) in section 2. The notation should suggest to the reader that $\tilde{\Pi}_i(x, c)$ will be the payoff function of the indirect auction. Indeed, the indirect auction will be defined in the sequel in such a way for this to become true. In Appendix A we prove the following crucial result for our approach:

Proposition 2 — Given $b \in \mathcal{S}$, consider the corresponding conjugation $P = P^b$ (as defined by (7)) and the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$. Alternatively, given a conjugation P and an indirect bidding function $\tilde{b}$, let $b = \tilde{b} \circ P$ be the corresponding direct bidding function. In any case, we have the following:

(i)

$$\tilde{\Pi}(x, c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(x, \alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha - \int_{\tilde{b}^{-1}(c)}^1 p^L\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{14}$$

(ii) Assume that P is such that for all s with $P(s) = x$, and for all $x, y \in [0,1]$,

$$\tilde{v}(x, y) = E[v(t) | P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t) | t_i = s, P_{(-1)}(t_{-i}) = y]. \tag{15}$$

Then, for all t_i such that $P(t_i) = x$ and for all $c \in \mathbb{R}$, we have:

$$\tilde{\Pi}(x, c) = \Pi(t_i, c). \tag{16}$$

Observe that, because (14), $\tilde{\Pi}(x, c)$ is formally equivalent to the interim payoff of an auction between two bidders, with signals uniformly distributed in $[0, 1]$, where the opponent is following the strategy $\tilde{b}(\cdot)$ and the (common-value) utility function is given by $\tilde{v}(x, \alpha)$. So, we define the indirect auction as follows:

Definition 5 — Given an auction (S, σ, v) and a conjugation P for it, the associated indirect auction is an auction between two players with independent types uniformly distributed in $[0, 1]$ and where the utility function is $\tilde{v}$ defined by (12). The indirect auction is denoted by $\left(\tilde{S}, \tilde{\sigma}, \tilde{v}\right)$ where $\tilde{\sigma}$ is the Lebesgue measure in $\tilde{S} = [0, 1]$.

It is clear through definitions 1-4 how a conjugation relates the direct auction and the indirect auction. Obviously, a function $\hat{b} : [0, 1] \to \mathbb{R}_+$ is equilibrium of the indirect auction if for almost all $x \in [0, 1]$, $\tilde{\Pi}\left(x, \hat{b}(x)\right) \geqslant \tilde{\Pi}(x, c)$, $\forall c \in \mathbb{R}_+$. Equivalently, $\hat{b} : [0, 1] \to \mathbb{R}_+$ is equilibrium of the indirect auction if for almost all x, $y \in [0, 1]$, $\tilde{\Pi}\left(x, \hat{b}(x)\right) \geqslant \tilde{\Pi}\left(x, \hat{b}(y)\right)$. Indeed, to bid above the support of $\hat{b}$ cannot improve the probability of winning and to bid below leads to a zero payoff.[22]

4.3 Characterization and Sufficient Conditions for Regular Equilibria

The results and definitions of the two previous subsections allow us to show that the existence of a direct equilibrium implies the existence of the indirect one (Theorem 1, below). Conversely, (with an extra relatively weak assumption of consistency of payoffs), the existence of equilibrium in indirect auctions allows us to prove the existence in direct ones (Theorem 2).

Theorem 1 — Assume (H0), (H1) and (H2). If there is a pure strategy equilibrium $b \in \mathcal{S}$ for the direct auction (S, σ, v) and there exists $\partial_b \Pi(s, b(s))$ for all s, then:

(i) the associated conjugation $P = P^b$ (given by (7)) satisfies the following property: if $s \in S$ is such that $P(s) = x$, then:[23]

$$E[v(t_i, t_{-i}) | P(t_i) = x, P_{(-i)}(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P_{(-i)}(t_{-i}) = x]; \tag{17}$$

(ii) the indirect bidding function $\tilde{b} = \left(\tilde{P}^b\right)^{-1}$, where $\tilde{P}^b$ is given by (6), is the increasing equilibrium of the indirect auction. Moreover, if it is differentiable at x, it satisfies the following:

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) > \tilde{b}(\alpha)\right]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{\left[\tilde{b}(x) < \tilde{b}(\alpha)\right]}\right]}; \tag{18}$$

(iii) for all x and $y \in [0, 1]$,

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)] \, d\alpha \geqslant 0; \tag{19}$$

[22] We are using the implicit assumption that the reserve price is weakly above the minimum utility. See appendix B for details.

[23] This condition is related to a condition of Araujo and Moreira (2001).

(iv) the expected payment of a bidder of type t_i is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha, \alpha)\, d\alpha.$$

Remark 1 — One can understand Theorem 1 as saying that if a multidimensional auction has a regular equilibrium, then it can be reduced to a unidimensional auction (the indirect one). However, the reader should note that such reduction is non-trivial and that the indirect auction is not equivalent to the direct one. The indirect auction is a "fictitious" game, where each bidder is facing up a "fictitious" player, the "opponent". The "opponent" does not correspond to a real player. So, the dimension reduction is meant in this particular sense and it is valid even when bids are multidimensional, but there are just two situations considered for each bidder: to receive or not the object. This is shown in section 5, where we analyze multidimensional bids.

Observe that the expression in condition (iv) does not depend on the specific format of the payment rules, p^W and p^L. This is interesting, and implies a kind of Revenue Equivalence Theorem. Nevertheless, the payment still depends on the conjugation. So, it can be different for different auction formats, if the conjugation is different. Fortunately, we can prove that for a still general class of auction the conjugation is unique and the Revenue Equivalence Theorem holds. Although of some importance on its own, this result is a natural generalization. What is less expected is the result presented in Theorem 2. There we prove that condition (iv) (which is, in fact, equivalent to the Revenue Equivalence Theorem) is an essential part to prove the existence of equilibria.

Theorem 2 is a kind of converse of Theorem 1. The assumptions are exactly the conclusions of Theorem 1, but for condition (i): we need the slightly stronger condition (i)$'$. This is the only reason for not stating an "if and only if" theorem. Fortunately, as we will show in the next section, there are still interesting cases that permit us to state a simple necessary and sufficient condition for the existence of equilibria.

Theorem 2 — Consider a direct auction (S, σ, v) and a conjugation P for an indirect auction $\left(\tilde{S}, \tilde{\sigma}, \tilde{v}\right)$. Assume that

(i)$'$ for all $s \in S$ such that $P(s) = x$, and all $y \in [0, 1]$,

$$\tilde{v}(x, y) = E[v(t)|P(t_i) = x, P_{(-i)}(t_{-i}) = y] = E[v(t)|t_i = s, P_{(-1)}(t_{-i}) = y]; \tag{20}$$

(ii) there is an increasing function $\tilde{b}$, solution of the differential equation:

$$\tilde{b}'(x) = \frac{\tilde{v}(x, x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]}\right]}.$$

(iii) for all x and $y \in [0, 1]$,

$$\int_y^x [\tilde{v}(x, \alpha) - \tilde{v}(\alpha, \alpha)]\, d\alpha \geqslant 0.$$

(iv) The expected payment of a bidder of type t_i is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha, \alpha)\, d\alpha.$$

Then, there exists an equilibrium of the direct auction, given by $b = \tilde{b} \circ P$ and $\tilde{b}$ is the equilibrium of the indirect auction. Moreover, if $\tilde{v}$ is continuous, there exists $\partial_b \Pi(s, b(s))$ for all s.

Proof. See Appendix C.■

Remark 2 — For the four specific formats, namely, the first-price auction (F), second-price auction (S), all-pay auction (A) and war of attrition (W), the function $\tilde{b}$ is given, respectively, by

$$\text{(F)} \quad \tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha \tag{21}$$

$$\text{(S)} \quad \tilde{b}(x) = \tilde{v}(x,x) \tag{22}$$

$$\text{(A)} \quad \tilde{b}(x) = \int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha \tag{23}$$

$$\text{(W)} \quad \tilde{b}(x) = \int_0^x \frac{\tilde{v}(\alpha,\alpha)}{1-\alpha}d\alpha \tag{24}$$

Condition (ii) reduces to the requirement that the function $\tilde{b}$ above is increasing. Observe also that is possible that the equilibrium exists for an all-pay auction, for instance, but not for a first-price auction.

Remark 3 — Although natural, condition (i)' can be still too restrictive. We need it in order to apply Proposition 2 and reach the conclusion that for all t_i such that $P(t_i) = x$ and for all $c \in R$, we have: $\tilde{\Pi}(x,c) = \Pi(t_i,c)$ (see (16) in Proposition 2). In turn, this implies that the equilibrium of the indirect auction is equilibrium of the direct auction. So, instead of assuming condition (i)' above, it would be sufficient to require the (necessary) condition (i) of Theorem 1 and that it is valid (16).

Theorem 2 simplifies the problem of existence of equilibrium to find a conjugation that meets requirements (i)$'$ and (ii)-(iv). In the next subsection we treat a still general case where such conjugation can be easily defined. Nevertheless, before we deal with that case, we would like to give two examples where the assumptions of the next subsection are not satisfied, but where we still can prove the existence of equilibrium. This is worthwhile, since it provides a kind of heuristics for the existence problem. The heuristics is based in condition (i)$'$ and is illustrated in Appendix D for the examples 6 and 7, below.

Example 6 — Consider a symmetric first-price auction with two bidders, types uniformly distributed on $[0,1]$ and utility function given by:

$$v(t_i,t_{-i}) = t_i + \left(3 - 4t_i + 2t_i^2\right)t_{-i}.$$

Observe that $\partial_i v(t_i,t_{-i}) = 1 - 4t_{-i} + 4t_i t_{-i}$ can be negative. Thus, the received theory cannot be applied. Nevertheless, there exists a monotonic equilibrium. Indeed, in this case, the conjugation will be given by $P(t_i) = t_i$ and we obtain

$$\tilde{v}(x,y) = x + \left(3 - 4x + 2x^2\right)y.$$

This clearly satisfies condition (i)'. Condition (iii) follows from the fact that $x > y$ implies

$$\int_y^x [\tilde{v}(x,z) - \tilde{v}(z,z)]\,dz = \frac{(x-y)^2}{6}\left[3 + 3x^2 - 8y + 3y^2 + x(-4+6y)\right] \geqslant 0.$$

Condition (ii) is also satisfied, because the function

$$\tilde{b}(x) = \frac{1}{x}\int_0^x v(z,z)\,dz = \frac{x\left(24 - 16x + 3x^2\right)}{12}$$

is increasing. Clearly, the above function implies condition (iv). Thus, there exists a monotonic equilibrium by Theorem 2.

Nevertheless, this is not the unique equilibrium. If we assume that there exists a U-shaped equilibrium, the conjugation can be expressed by $P(t_i) = |c(t_i) - t_i|$, where $c(t_i)$ is the type that bid the same as t_i (see Figure 1). Observe that $c \circ c(t_i) = t_i$. Condition (i) of Theorem 1 requires that

$$s + \left(3 - 4s + 2s^2\right) \frac{s + c(s)}{2} = c(s) + \left(3 - 4c(s) + 2c(s)^2\right) \frac{s + c(s)}{2},$$

that is,

$$s - c(s) = [s - c(s)]\,[4 - 2c(s) - 2s] \frac{s + c(s)}{2},$$

which simplifies to $[s + c(s)]\,[2 - s - c(s)] = 1 \Rightarrow s + c(s) = 1$. Then, $c(s) = 1 - s$ and $P(s) = |1 - 2s|$. This gives the expression:

$$\tilde{v}(x, y) = \frac{1}{2} + \frac{1}{4}\left[3 - 4\left(\frac{1-x}{2}\right) + 2\left(\frac{1-x}{2}\right)^2 + 3 - 4\left(\frac{1+x}{2}\right) + 2\left(\frac{1+x}{2}\right)^2\right]$$

which simplifies to $\tilde{v}(x, y) = \left(5 + x^2\right)/4$ and condition (i)' and (iii) are easily seen to be satisfied. Also, condition (ii) and (iv) are satisfied, since

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(z, z)\, dz = \frac{5}{4} + \frac{x^2}{12}$$

is increasing. Then, $b(s) = \frac{5}{4} + \frac{(1-2s)^2}{12}$ is a direct equilibrium, and it is plotted in Figure 1.

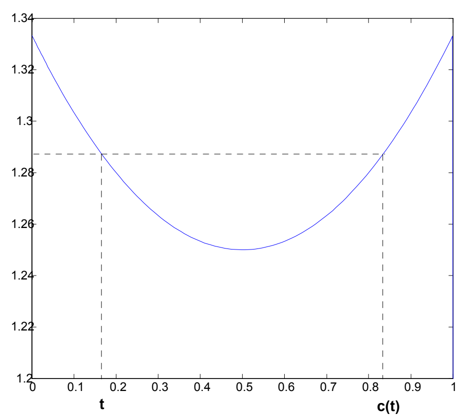

Figure 1: Equilibrium bidding function in Example 6.

Observe that no tie rules are needed in this case, because ties occur with zero probability. However, for each equilibrium bid, exactly two types pool and have the same probability of winning.

Example 6 has a monotonic equilibrium, as is usual in auction theory, but there is another non-monotonic equilibrium. Example 7 below shows a case where there is no monotonic equilibrium, but there is a bell-shaped equilibrium, showed in Figure 2.

Example 7 — Consider again a symmetric first-price auction with two bidders and signals uniformly distributed in $[1.5, 3]$, such that the value of the object is given by $v(t_i, t_{-i}) = t_i\left(t_{-i} - \frac{t_i}{2}\right)$.

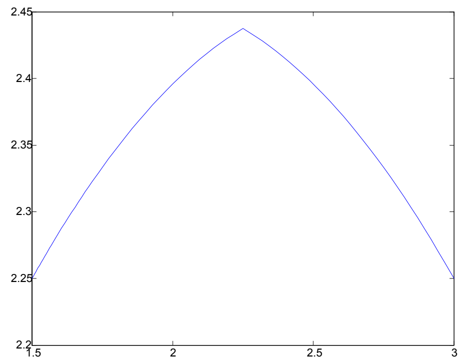

Figure 2: Equilibrium bidding function in Example 7.

In Appendix D, we show that this auction does not have monotonic regular equilibria, but there is a bell-shaped equilibria as shown in Figure 2.

Example 6 shows that it is possible for a standard auction to have multiplie equilibria. Example 7 suggests that the correct conjugation can fail to exist — at least with a fixed shape (that we begin by assuming). Thus, one would be interested in cases where it is possible to ensure the uniqueness of the equilibrium and where it is possible to find explicitly the conjugation. We do this under the context of assumption H3, to be presented in the next subsection.

4.4 Necessary and Sufficient Conditions for the Equilibrium Existence of Regular Auctions

Theorem 2 teaches us that the question of equilibrium existence is solved if we are able to find the proper conjugation. In examples 6 and 7 of the previous subsection we have shown situations where the conjugations could be obtained. However, there we assumed some features of the conjugation that are not necessary and were able to find the correct conjugation for those settings. Now we want to specify a setting where a conjugation always exists. The setting is that of auctions that satisfy assumptions (H1), (H2) and

(H3) $v(t_i, t_{-i})$ is such that if $v(t_i, t_{-i}) < v(t'_i, t_{-i})$ for some t_{-i} then $v\left(t_i, t'_{-i}\right) < v\left(t'_i, t'_{-i}\right)$ for all t'_{-i}. Moreover, if $C \subset \mathbb{R}$ has zero Lebesgue measure, then $\sigma\{s \in S : v^1(s) \in C\} = 0$.

The reader should remember that we defined

$$v^1(s) \equiv E\left[v(t_i, t_{-i}) \,|t_i = s\right].$$

Assumption (H3) is restrictive, but it is the natural context of many economic meaningful cases. For instance, for separable utilities such as $v(t_i, t_{-i}) = u^1(t_i) + u^2(t_{-i})$, it requires only that $u^1(t_i)$ does not assume any value with positive probability. The same requirement is sufficient for utilities like $v(t_i, t_{-i}) = \{\left[u^1(t_i)\right]^\alpha + \left[u^2(t_{-i})\right]^\beta\}^\gamma$, or $v(t_i, t_{-i}) = \gamma\left[u^1(t_i)\right]^\alpha \left[u^2(t_{-i})\right]^\beta$, with α, β, $\gamma > 0$. Of course, private values are included in the separable utilities case. It seems that the majority of utility functions considered in applications satisfy (H3). Of course, there are cases that do not satisfy it, such as the (mathematical) examples 6 and 7 above. It is also clear that (H3) can deal with even more complicated dependences, as example 8 below illustrates.

Example 8 — Consider three bidders with bidimensional signals, each with support equal to $[0,1]^2$. The utilities are specified by the nonlinear symmetric function

$$v(t_1, t_{-1}) = 4 + 4t_1^1 - 6t_1^2 + 3\left(t_2^1 + t_3^1\right) - 5\left(t_2^2 + t_3^2\right)$$
$$+\frac{1}{2}\left[\left(t_2^1 + t_3^1\right) + \frac{1}{2} - \frac{\left(t_2^2 + t_3^2\right)}{4}\right]\left[\left(t_1^1\right)^2 - 3t_1^1 t_1^2 - \left(\frac{3}{2}t_1^2\right)^2\right]$$

It is easy to check that the above function satisfies assumption H3 (see Appendix C). Observe that the private information of any bidder cannot be simplified to a unidimensional signal because this will make the expression ot the other's utilities impossible. Another observation is that the signals cannot be reparametrized so that the function becomes monotonic in all such reparametrized signals.[24]■

Under (H3), we can define explicitly the conjugation that will work:

$$P(t_i) \equiv \Pr\left\{t_{-i} \in T_{-i} = S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i\right\}. \tag{25}$$

Then, $\tilde{v}$ is defined as before (see (12)) for such P, that is,

$$\tilde{v}(x,y) = E[v(t)|P(t_1) = x, P_{(-1)}(t_{-1}) = y].$$

Thus, we can give a *necessary and sufficient* condition for the existence of equilibrium of the direct auction: merely that the solution $\tilde{b}$ to the first-order condition of the indirect auction be increasing. This is the content of the following:

Theorem 3 — Consider the auction (S, σ, v) that satisfies (H0)-(H3). There exists an equilibrium $b \in \mathcal{S}$ and there exists a continuous derivative $\partial_b \Pi(s, b(s))$ *if and only if* there exists a function $\tilde{b}$ that *is increasing* and satisfies, for all $x \in [0,1]$,

$$\tilde{b}'(x) = \frac{\tilde{v}(x,x) - p^W\left(\tilde{b}(x), \tilde{b}(x)\right) + p^L\left(\tilde{b}(x), \tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{[\tilde{b}(x) > \tilde{b}(\alpha)]} + \partial_1 p^L\left(\tilde{b}(x), \tilde{b}(\alpha)\right) 1_{[\tilde{b}(x) < \tilde{b}(\alpha)]}\right]},$$

and it is compatible with $\hat{p}(0) = 0$, where $\tilde{v}$ is given as above and it is continuous. If this is the case, the equilibrium of the direct auction is given by $b = \tilde{b} \circ P$ and the expected payment of a bidder of type s is given by

$$p(s) = \int_0^{P(s)} \tilde{v}(\alpha, \alpha)\, d\alpha. \tag{26}$$

Moreover, if there is a unique $\tilde{b}$ that satisfies such properties, the equilibrium of the direct auction in regular pure strategies is also unique.[25]

Proof. See Appendix C.■

Remark 4 — As we explained in Remark 1, if a multidimensional auction has a regular equilibrium, it can always be reduced (in a non-trivial way) to a one dimension auction (the indirect auction). So, for obtaining equilibrium existence, we have to consider auctions that can

[24] Of course, if one increases the dimension of the signals to six, for instance, the task can be done. However, the signals so obtained will be concentrated in Lebesgue measure zero sets.

[25] The proof also shows the existence of equilibrium even if $\tilde{v}$ is not continuous, but the payment is given by (26).

be so "reduced". This is what assumption H3 allows us to explicitly do. It still encompasses cases where such reduction is not trivial, as we show in examples 9 and 10 below. The reduction of the dimension of the types is not a novelty in auction theory. While studying the effiency of auctions, Dasgupta and Maskin (2000) use a condition close to H3, while Jehiel, Moldovanu and Stacchetti (1996) made such reduction for the purpose of revenue maximization. Nevertheless, for the purpose of equilibrium existence in standard auctions, one cannot use only H3 or the Dasgupta and Maskin's condition, since the received theory would require the extra assumption of monotonicity of the reparametrized signals. As we show in examples 10 and 11, this is not always possible. So, an important feature of Theorem 3 is that it does not require monotonic assumptions.

Example 9 — Spectrum Auction[26]

Consider a first-price auction of a spectrum license.[27] The license covers two periods of time:

(1) In the first period, the regulator lets the winner explore its monopoly power. Let t_i^1 be the estimate of bidder i of the monopolist surplus in this first period. Of course, the true surplus will be better approximated by $\left(t_1^1 + ... + t_N^1\right)/\ N$. If the bidder i (a firm) wins the auction, it has to invest t_i^2, a privately known amount, to build the network that will support the service. So, in the first period, the license gives to the firm

$$\frac{t_1^1 + ... + t_N^1}{N} - t_i^2.$$

(2) In the second period, the regulator makes an estimate of the operational costs of the firm. The regulator cannot observe the true operational cost, t_i^3, which is a private information of the firm. Nevertheless, the regulator has a proxy that is a sufficient statistic for the mean operational cost of all participants in the auction $\left(t_1^3 + ... + t_N^3\right)/\ N$. The regulator will fix a price that will give zero profit for a firm with the mean operational costs.[28] So, in the second period, the license gives to the winner

$$\frac{t_1^3 + ... + t_N^3}{N} - t_i^3.$$

So, the value of the object is given by

$$v\left(t_i, t_{-i}\right) = \frac{t_1^1 + ... + t_N^1}{N} - t_i^2 + \frac{t_1^3 + ... + t_N^3}{N} - t_i^3.$$

Let the signals $t_i = \left(t_i^1, t_i^2, t_i^3\right)$, $i = 1, ..., N$ be independent. Observe that the problem cannot be reduced to a single dimension.[29] Also, the model cannot be reparametrized to an increasing one.[30] So, the received theory does not ensure the existence of pure strategy equilibrium for this case. Nevertheless, the assumption (H3) is trivially satisfied. In Appendix D, we assume the $t_i = \left(t_i^1, t_i^2, t_i^3\right)$ are independent and uniformly distributed in $\left[\underline{s}^1, \overline{s}^1\right] \times \left[\underline{s}^2, \overline{s}^2\right] \times \left[\underline{s}^3, \overline{s}^3\right]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3 \geqslant 0$ and we show that a sufficient condition for the existence of equilibrium in pure strategy is

$$\frac{\underline{s}^1}{N} - \overline{s}^2 - \overline{s}^3 \frac{N-1}{N} - 1 \geqslant 0.$$

[26] This example is formally similar to example 5 of Dasgupta and Makin (2000), tough it is a bit more complex.

[27] The example works also for any auctions of public concessions.

[28] We assume that the regulator is institutionally constrained to follow such a procedure, so the optimality of this regulation is not an issue.

[29] Indeed, if we summarize the private information by, say, $s_i = t_i^1/N - t_i^2 + t_i^3(1/N - 1)$, we lose the information about t_i^1 and t_i^3 that are needed for the value function of bidders $j \neq i$.

[30] If we try to put $-t_i^3$ in the place of t_i^3, then the dependence of $v\left(t_i, t_{-i}\right)$ on the signals t_j^3 will be decreasing.

The derivation in Appendix D indeed provides necessary and sufficient conditions for the existence of equilibrium. Above, we have only simplified it for a sufficient condition.■

Example 10 — Job Market

We can model the job market (for, say, a manager) as an auction between competing firms, where the object is the job contract with that manager. It is natural to assume that the manager has a multidimensional vector of characteristics, $m = (m^1, ..., m^k)$. For the sake of simplicity, we assume that the firms learn such characteristics through interviews and curriculum analysis. Each firm also has a position to be filled by the manager, with specific requirements for each dimension of the characteristics. For instance, if dimension 1 is ability to communicate and the position is to be the manager of a production section, there is level of desirability of this ability. An overly comunicative person may not be good. The same goes for the other characteristics. A bank may desire a sufficiently (but not exaggeratedly) high level of risk loving or audacity on the part of the manager, while a family business may desire a much lower level. Even efficiency or qualification can have a level of desirability. Sometimes, the rejection of a candidate is explained by over-qualification. Therefore, let $t_i = (t_i^1, ..., t_i^K)$ be the value of the characteristics desired by the firm.

Since the firms are competitors, then if one hires the employee, the other will remain with a vacant position, at least for a time.[31] In this way, the winning firm also benefits from the fact that its competitors have a vacant position — and, then, are not operating perfectly well. The higher the abilities required for the job, the more the competitor suffers.[32] So, the utility in this auction is as

$$v(t_i, t_{-i}) = \sum_{k=1}^{K} a^k m^k - \sum_{k=1}^{K} b^k \left(t_i^k - m^k\right)^2 + \sum_{j \neq i} \sum_{k=1}^{K} c^k t_j^k,$$

where a^k is the level of importance of characteristic k of the manager,.$b^k > 0$ represents how important is the remove from the desired level t_i^k of the characteristic k, and c^k is the weight of the benefit that firm i receives from the fact that the competitors are lacking $\sum_{j \neq i} t_j^k$ of the ability k. As in the previous example, we cannot simplify this model for a unidimensional monotonic model. In Appendix D we analyze the case where there is just one dimension ($K = 1$), 2 players ($N = 2$) and the types are uniformly distributed in $[0, 1]$, $b = b^1 \geqslant 0$. We show that if $m^1 = m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \equiv c^1 \geqslant \max\left\{\frac{2b(m-2)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}$$

and if $m < 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leqslant \min\left\{\frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}.$$

Observe that for both cases the value $c = 0$ ensures the existence of equilibrium. This is expected, since it corresponds to a private value auction.■

Another interesting application of Theorem 3 is the example 1 of JSSZ. This consists in a first-price auction without equilibrium (with the standard tie-breaking rule). Theorem 3 explains why

[31] Of course, this model works only for non-competitive job markets. In other words, the buyers (the contracting firms) have no access to a market with many homogenous employees to hire. This is implicit when we model it as an auction. So, this is the reason why a firm that does not contract the manager suffers — it is not possible to find a suitable substitute instantaneously. It is possible that this also occurs in other kinds of auctions.

[32] If the firms act in a oligopolistic market, it is possible to justify such externality through the fact that the vacant position influences the quality of the product delivered by the firms and, hence, the equilibrium in this market.

such an auction does not have an equilibrium in regular strategies:[33]

Example 11 (JSSZ, example 1) — Let us consider a first price auction with two bidders, independent types uniformly distributed in $[0,1]$. Let $v^1(t_i) = t_i$ and $v^2(t_{-i}) = 5 - 4t_{-i}$. It is clear that $P(t_i) = t_i$ in this case and $\tilde{v}(x,y) = 5 + x - 4y$. So, $\tilde{v}(\alpha,\alpha) = 5 - 3\alpha$ and

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\,d\alpha = \frac{1}{x}\left(5x - \frac{3x^2}{2}\right) = 5 - \frac{3}{2}x,$$

which is decreasing. There is therefore no direct equilibrium $b \in \mathcal{S}$.■

The example above is used by JSSZ to show that equilibrium may fail to exist under the standard tie-breaking rule. They then provide a general existence result based on endogenous tie-breaking rules. Unfortunately, their result has some undesirable properties. First, it is in mixed strategies. Second, the tie-breaking rule is endogenous, so it is not possible to know what rule has to be applied in order to guarantee the existence. Third, the rule requires that the players announce their types, which is theoretically convenient but it is unfeasible in the real world.

We offer, in contrast, a method to overcome these difficulties. First, the equilibrium is in pure strategies. Second, the rule is the same for all auctions, so that the players know it before the game starts. Third, it is a natural method: a modified second-price auction. Although we could implement the tie-breaking rule by requiring that types be announced, we do not need to require this formally. As a matter of fact, we require that the bidders submit bids that will be ranked in the standard manner. The payment in the case of winning is given by a function of the second highest bid. The fact that it is a function of the second highest bid (and not the bid itself) is the reason why we call it a "modified" second-price auction. The rule is as follows:

Modified Second-Price Auction Tie-Breaking Rule (MTBR)- In the case of a tie, conduct a modified second-price auction among the players involved in the tie, as follows. Each bidder submits a bid b_i^2; the bidder with the highest bid wins, receives the object and makes the payment. The payment is calculated based on the second highest bid, $b_{(-i)}^2$, and it is given by $v(z,z)$, where $z \equiv \left(v^1\right)^{-1}\left(b_{(-i)}^2\right)$.

Observe that the implementation of the above rule does not require any extra information.[34] We show now that the above rule ensures the existence of equilibrium for the auctions that we are considering. Therefore, the generality and simplicity of the rule can be counted as a last advantage of it.

Theorem 4 - Consider the auction (S,σ,v) that satisfies (H0) -(H3). Assume that MTBR is adopted. Let P be given by (25). If there is a $\tilde{b}$ that satisfies (33) and is compatible with $\hat{p}(0) = 0$ then there is an equilibrium in pure strategies. Moreover, the expected payment of a bidder of type t_i is given by

$$p(t_i) = \int_0^{P(t_i)} \tilde{v}(\alpha,\alpha)\,d\alpha,$$

Proof. See Appendix C.■

Remark 5 — The main ingredient in the proof of Theorem 4 is the payment expression. So, the special characteristic of MTBR is the fact that it allows the implementation of the revenue equivalence in a situation where the usual tie breaking rule does not.[35]

[33] Of course, it would be possible for an equilibrium to exist that it is not in $\mathcal{S}$ or it is in mixed strategy. JSSZ show that this is not the case with standard tie-breaking rule. They then proceed to show that a tie-breaking rule that depends on types is sufficient to ensure the equilibrium existence.

[34] Remember that the function v^1 is common knowledge.

[35] In example 11 (example 1 of JSSZ), the MTBR gives a greater revenue than the rule proposed by JSSZ.

The reader should note that Theorem 4 does not claim the uniqueness of the equilibrium. Indeed, if $\tilde{b}$ is not increasing, there are many equilibria. There are two sources for this multiplicity.

The first source is that under MTBR, any level of the bid in the range where $\tilde{b}$ is non-monotonic can be chosen to be the level of the tie. This is shown in the Figure 3. For instance, any a_0 can be chosen between x_0 and x_1. Once one of the three elements a_k, b_k or c_k is determined, so are the other two.

However, these possibilities lead to the same expected payment and payoff for each bidder in the auction, so that Theorem 4 remains as stated.

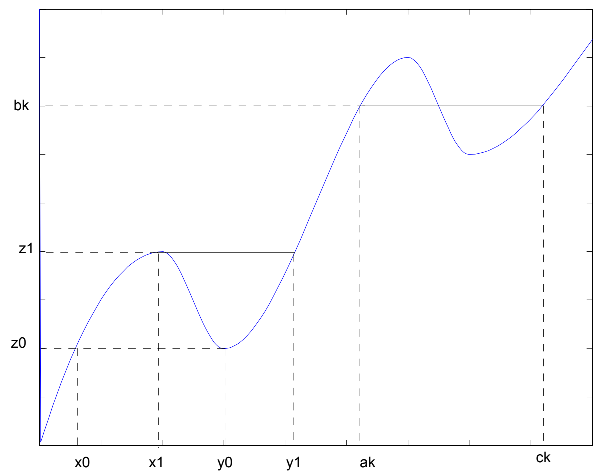

Figure 3: Possible specifications for the level of the tie.

Another point is that the tie-breaking rule is not unique, in general (although the rule just defined seems the most natural one). It can be shown, for instance, that for cases where $\tilde{b}$ is decreasing (as in example 1 of JSSZ) and for some specifications of v, there is a continuum of tie-breaking rules (like that defined by JSSZ for their example 1), which ensures the existence of equilibrium. All these tie-breaking rules nevertheless imply different revenues. In light of this observation, the existence of equilibrium with an *endogenous* tie-breaking rule seems even more problematic as a solution concept, since it can sustain very different behaviors at equilibrium.

The reader must observe that the expression of the payment in Theorem 3 depends only on the conjugation, which is fixed for all kind of auctions. Also, the payment is exactly the same under the MTBR. So, we have the following:

Theorem 5 (Revenue Equivalence Theorem) - Consider the auction (S, σ, v) that satisfies (H0) -(H3). Assume that MTBR is adopted. Then, any format of the auction gives the same revenue, provided the bidders follow a symmetric equilibrium.

4.5 Risk Aversion

Now we would like to show that the theory just developed can be extended to a setting with risk aversion. For the sake of simplicity, we will restrict ourselves to the private-value case – as Maskin and Riley (1984) do – and to the first-price auction, although it is possible to extend the analysis for the setting of the last subsection. So, we assume the following:

$$\begin{aligned}\overline{u}_i(t,b) &= U\left(v^1(t_i) - b_i\right);\\ \underline{u}_i(t,b) &= 0;\\ u_i^T(t,b) &= \frac{U(v(t_i)-b_i)}{m},\end{aligned}$$

where m is the number of tying bidders and $U : \mathbb{R} \to \mathbb{R}$ is a utility function. Also, we maintain (H1), (H3) and make the following assumption for U:

(H4) U is a strictly increasing, bounded and differentiable function, with $U(0) = 0$. Moreover, U/U' is strictly increasing.

(H4) seems a restrictive assumption, but it is implied by the assumptions of Maskin and Riley (1984). Let P be defined as in the previous section (see (25)) and

$$\tilde{v}^1(x) = E\left[v^1(t_i)\,|P^b(t_i) = x\right].$$

Let b_0 be the reserve price. We have the following:

Theorem 6 - Consider the auction described above, assume that (H1)-(H4) hold and that $\tilde{v}^1$ is continuous. Then, there exists a unique equilibrium $b \in \mathcal{S}$ given by $b = \tilde{b} \circ P$, where P is defined by (25) and $\tilde{b}$ is the solution of

$$\begin{cases} \tilde{b}'(x) = \frac{U\left(\tilde{v}^1(x) - b(x)\right)}{xU'\left(\tilde{v}^1(x) - b(x)\right)} \\ \\ b(x_0) = b_0, \text{ where } \tilde{v}^1(x_0) = b_0. \end{cases} \tag{27}$$

Proof. See Appendix C.■

5 Multidimensional Bids and Procurement Auctions

In the previous section, we extended the equilibrium existence results from unidimensional to multidimensional types. Nevertheless, we have considered only unidimensional bids. It is worth wondering what can be said about multidimensional bids.

Maybe the most obvious example of auctions with multidimensional bids are multiunit auctions. Indeed, in these auctions each bidder submits prices for each unit to be received. Our model needs important modifications to approach this case. This comes from the fact that our assumption of unitary demand allows us to consider only two situations for each bidder: to receive the object or not (ignoring the ties). Then, it is sufficient to consider just two utility functions, $\overline{u}_i$ and $\underline{u}_i$, one for each of these situations. When there are K objects in the auction, and the bidders have multiunit demand, we need to consider $K+1$ outcomes for each bidder: to end with $k = 0$, 1, ..., K objects and, for the outcome of receiving k objects, to consider the utility function u_i^k. This will require the lemma of characterization and the basic principle of bidding to be rephrased in order to take into account all these new possibilities. It seems reasonable to hope that the approach will be fruitful in this case, but, of course, careful work is needed to obtain valuable results.

Nevertheless, multiunit auctions are not the only interesting case of auctions with multidimensional bids. Indeed, many single-object auctions have multidimensional bids. For instance, in the timber auctions conducted by the U.S. Forest Service, the bidders generally are required to submit individual prices for each kind of trees to be harvested in the tract. Also, in a procurement auction for an engineering service, a buyer may request prices of the materials and of the working hours to be spent on the service. Yet another example is a procurement auction of non-homogenous products. In this case the bidders have to submit not only the price of the object but also its characteristics (quality, durability, warranty, reliability, capacity, time to delivery, etc.), that affect the utility of that product to the buyer. So, it is reasonable for the buyer to take into account such characteristics (part of the multidimensional bid) when deciding which proposal to accept.

Since the result of the auction for each bidder is only winning or losing, the seller has to specify a complete order to the multidimensional bids. We can assume that this order is given by a scoring function. For a real example, if b_i^1 and b_i^2 are the prices (bids) submitted by bidder i for the two species of trees in a tract, the U.S. Forest Service declares the winner to be the bidder with the highest expected payment $b_i^1 t_0^1 + b_i^2 t_0^2$, where t_0^1 and t_0^2 are the estimates for the quantity of each species made previously by the U.S. Forest Service. Doing so, the Forest Service is trying to maximize the expected payment that it will receive from the bidders.

In the example of procurement of non-homogenous products, the bid is (p_i, q_i), where q_i stands for the quality of the product offered and p_i, for its price. The scoring rule can be given by $U(q_i) - p_i$, where U tries to capture the value that the auctioneer attributes to quality. That is, the bid (p_i, q_i) that leads to the higher surplus $U(q_i) - p_i$ is the winning bid.

In this section, we present two models that analyze the above situations. We could try to give a general model that incorporates all the above cases, but it would be very complex and it is more instructive to treat the specific examples in turn. In subsection 5.1, we analyze a procurement auction of unit-price contracts. In section 5.2, we treat the case of non-homogenous products.

5.1 Unit-Price Contracts

In this subsection, we present a model of procurement auction with multidimensional bids that generalizes the model of Ewerhart and Fieseler (2003). Although our model is phrased for procurement auctions, easy adaptations can be made in order to deal with the situation analyzed by Athey and Levin (2001): the timber auctions conducted by the U.S. Forest Service.

A firm (or a government) procures a service to be executed. Its engineers estimate the amount of each input to be used to execute it: materials, working hours, etc. If there are m input factors to the service, the engineers estimate the amounts t_0^1, ..., t_0^m that will be used. We denote the vector of estimates by $t_0 = (t_0^1, ..., t_0^m)$.

The potential suppliers of the service (who will be called sellers) have private information about their technologies. That is, seller i knows the quantity of inputs t_i^1, ..., t_i^m that he will need to complete the service. Let $t_i = (t_i^1, ..., t_i^m)$.

The buyer then conducts a procurement auction, and request the potential suppliers to submit multidimensional bids $b_i = \left(b_i^1, ..., b_i^m\right) \in \mathbb{R}_+^m$. The non-negative number b_i^k is the price that seller i asks for each unit of the $k - th$ input. Based on the vector of bids, the buyer decides to buy the service from the bidder with the least cost, that is, bidder i such that $b_i \cdot t_0 = \min_j b_j \cdot t_0$, where $b_j \cdot t_0$ denotes the inner product $\sum_{k=1}^m b_j^k t_0^k$. In other words, there is a scoring function that is used by the buyer to evaluate the bids. It is just a function $B : \mathbb{R}^m \to \mathbb{R}$, given by $B(b_i) = b_i \cdot t_0$. The bid with the lowest score (expected payment) is the winner.

Once the winner is chosen, say bidder i, the buyer signs a contract with him, specifying the unit price that will be charged, $p = \left(p^1, ..., p^m\right)$. The signed contract can be a lowest-score contract (corresponding to a first-price auction), where $p \cdot t_0 = \min_j B(b_j)$, or a second-score contract, in which case $p \cdot t_0 = B_{(-i)} \equiv \min_{j \neq i} B(b_j)$.[36] In the first case, $p = b_i$ is, then, the contract signed. In the second case the bidder is free to choose p in order to met the requirement $p \cdot t_0 = B_{(-i)}$.

After the contract is signed, the service is executed, the true amount of inputs used, $t_i^1, ..., t_i^m$, is revealed and the transfer (payment) $p \cdot t_i$ is made by the buyer to the seller. We assume that the buyer can observe the efforts made by the contractor so that there is no moral hazard. It would be possible to include in the model the possibility of moral hazard, but this will turn the problem much more complex. However, the reader should note that our assumption is not so restrictive.

[36] All pay auctions and war of attrition seem inadequate in this setting: the buyer pays something even to those who do not win. We will not consider these formats.

We can understand $t_i^1, ..., t_i^m$ as the optimal level of the observable variables that are chosen by the contractor i, given the technology of observation of the buyer and the technology available to the contractor. So, the unique true restriction of the model is that, at the moment of bidding at the auction, the seller has solved all uncertainties regarding its technology, so that his choices deterministically imply the outcome of the observable.

This kind of contract is called unit-price contract and it is widely used in the real world. A natural question is "why?" Indeed, one could guess that it would be better (or at least equivalent) for the buyer to ask for an unidimensional bid: the price of the whole project. Then the buyer could contract the seller with the cheapest proposal. The intuition for the use of unit-price contracts is that this enables the contractor and the buyer to share risks. With the unidimensional bid, the risk becomes entirely on the part of the contractor.

We assume that seller i faces a cost $c(t_i)$ of providing the service. The profit of seller i is, then,

$$p \cdot t_i - c(t_i).$$

So, the problem of the seller is

$$\begin{aligned}&\max_{b_i \in \mathbb{R}_+^m} E\left\{[p \cdot t_i - c(t_i)] 1_{[t_0 \cdot b_i < B_{(-i)}]}\right\} \\ &= \max_{b_i \in \mathbb{R}_+^m} [p \cdot t_i - c(t_i)] \Pr\left[t_0 \cdot b_i < B_{(-i)}\right]\end{aligned}$$

Observe that this problem can be broken into two parts. The level $\beta = t_0 \cdot b_i$ determines the probability of winning the auction. Under an optimum level β, the seller is free to choose b_i (and hence, p), which maximizes $p \cdot t_i - c(t_i)$.

So, in a first-scoring auction, where $p = b_i$, this problem is

$$\max_{b_i : b_i \cdot t_0 = \beta} b_i \cdot t_i,$$

since $-c(t_i)$ is a constant. In a second-scoring auction, the problem is

$$\max_{p \in \mathbb{R}_+^m, p \cdot t_0 = B_{(-i)}} p \cdot t_i.$$

Both problems are linear with linear restrictions and they are formally equivalent. So, the maximum is obtained by a corner solution, which is very easy to obtain. For a fixed level β or for a $B_{(-i)} = \beta$, the problem is

$$\max_{p \in \mathbb{R}_+^m, p \cdot t_0 = \beta} [p \cdot (t_i - t_0) + p \cdot t_0]$$

Let $k(t_i)$ be defined as $\arg\max_k \left(t_i^k - t_0^k\right)$. Then, the solution is, clearly,

$$b(t_i, \beta) = \left(0, ..., 0, \frac{\beta}{t_0^{k(t_i)}}, 0, ..., 0\right),$$

where all entries are zero, but that in position $k(t_i)$. With this bid, the profit is

$$\frac{\beta}{t_0^{k(t_i)}} t_i^{k(t_i)} - c(t_i) = \frac{t_i^{k(t_i)}}{t_0^{k(t_i)}} \left[\beta - c(t_i) \frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right].$$

The problem of the bidder now becomes to choose the score level β in

$$\begin{aligned}
&\arg\max_{\beta\geqslant 0} \frac{t_i^{k(t_i)}}{t_0^{k(t_i)}} \left[\beta - c(t_i)\frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right] \Pr\left[\beta < B_{(-i)}\right] \\
&= \arg\max_{\beta\geqslant 0} \left[\beta - c(t_i)\frac{t_0^{k(t_i)}}{e_i^{k(t_i)}}\right] \Pr\left[\beta < B_{(-i)}\right]
\end{aligned} \tag{28}$$

So, all types t_i that have the same $c(t_i)\, t_0^{k(t_i)}/t_i^{k(t_i)}$ must choose the same optimum bid. Then, we define the conjugation:[37]

$$P(t_i) = \Pr\left\{t_{-i} \in T_{-i} : c(t_j)\frac{t_0^{k(t_j)}}{t_j^{k(t_j)}} > c(t_i)\frac{t_0^{k(t_i)}}{t_i^{k(t_i)}},\ \forall j \neq i\right\}.$$

Also, define, for all $x = P(t_i)$,

$$\tilde{c}(x) \equiv c(t_i)\frac{t_0^{k(t_i)}}{t_i^{k(t_i)}},$$

which is well defined from the definition of the conjugation. Observe that types t_i with higher $P(t_i)$ are eager to win, because they have a lesser adjusted cost $c(t_i)\, t_0^{k(t_i)}/t_i^{k(t_i)}$. Interestingly, in the unit-price auction, it is not the seller with the lower costs that wins. Indeed, the auction favors those players who have types with high difference $t_i^{k(t_i)} - t_0^{k(t_i)}$, because the term $t_0^{k(t_i)}/t_i^{k(t_i)}$ lowers the true costs to the "virtual cost" $c(t_i)\, t_0^{k(t_i)}/t_i^{k(t_i)}$. Another important observation is that, as we have said before, the players that conjugated do not need to have the same payoff. This comes from the fact that factor $t_i^{k(t_i)}/t_0^{k(t_i)}$ adjusts the "virtual payoff", $\left[\beta - c(t_i)\, t_0^{k(t_i)}/e_i^{k(t_i)}\right]$. See (28).

Turning back to the solution of the auction, in the first-score auction the problem of the seller now simplifies to

$$\max_{\beta} \left[\beta - \tilde{c}(x)\right] \Pr\left[\beta < B_{(-i)}\right].$$

By the definition of conjugation, $F_{B_{(-i)}}(\beta(x)) = 1 - x$, so that $\Pr\left[\beta(x) < B_{(-i)}\right] = x$. The first-order condition becomes

$$\beta'(x) = \frac{\beta(x) - \tilde{c}(x)}{x},$$

which, together with the initial condition $\beta(0) = \tilde{c}(0)$, gives the symmetric equilibrium:

$$\beta^1(x) = x\left[\tilde{c}(0) - \int_0^x \frac{\tilde{c}(\alpha)}{\alpha^2} d\alpha\right].$$

For a second-score auction, the problem is

$$\max_{\beta} \left[p\left(B_{(-i)}\right) \cdot t_0 - \tilde{c}(x)\right] \Pr\left[\beta > B_{(-i)}\right].$$

Observe that the first term does not depend on β (besides the dependence on $B_{(-i)}$) and the other is increasing in β. Since the strategy is increasing in the conjugation, then the solution is given simply by that β such that $p\left(B_{(-i)}\right) \cdot t_0 - \tilde{c}(x) > 0$ if and only if $\beta > B_{(-i)}$. That is,

[37] Of course, we again work under the assumption of non-atoms in the distribution of $c(t_i)\, t_0^{k(t_i)}/t_i^{k(t_i)}$.

$$\beta^2(x) = \tilde{c}(x),$$

and, after the result of the auction, the contract $p\left(B_{(-i)}\right)$ is signed.

Ewerhart and Fieseler (2003) solve just the first-score auction for the particular case where there are two players, the types are unidimensional and the costs linear. The interpretation is that all sellers are assumed to have the same type (equal to one) for one of the inputs (materials). The cost is given by $c(t_i) = c_M + c_L t_i^L$, where c_M is the cost of materials and c_L is the cost of labor. Under these simplifications, they obtain unimodal behavior (with increasing and decreasing regions fixed). They can thus show the existence of equilibrium with the standard monotonic methods.

5.2 Non-Homogeneous Products

In this section we consider a procurement auction where the product to be delivered may have different characteristics. In other words, the products are non-homogenous. So, the buyer requires each seller to submit, together with a price b_i^0, a vector of characteristics, $b_i^c = \left(b_i^1, ..., b_i^m\right)$, of the product the seller plans to deliver. So, the whole bid is the vector $b_i = (b_i^0, b_i^1, ..., b_i^m)$.

The bids are ranked through a scoring function that we will assume to be of the form: $B(b_i) = V\left(b_i^1, ..., b_i^m\right) - b_i^0$, where V can be (or not) the utility that the buyer attributes to the good with characteristics $\left(b_i^1, ..., b_i^m\right)$. We assume this form of the scoring rule for the sake of simplicity.

Each seller has multidimensional private information t_i. The private information is related to the cost of producing the good, that is, the cost of delivering a good with characteristics $b_i^c = \left(b_i^1, ..., b_i^m\right)$ by a seller with type t_i is $c(t_i, b_i^c)$.

The payment to the seller is p_i in a first-score auction. In a second-score auction, the second highest score, $B_{(-i)} \equiv \max_{j \neq i} B(b_j)$, has to be matched, but the firm is free to choose the price and the characteristics to do so. That is, the firm chooses $\overline{b}_i$ such that $B\left(\overline{b}_i\right) = B_{(-i)}$.[38] If the contract $p = \left(p^0, p^c\right) = \left(p^0, p^1, ..., p^m\right) = \overline{b}_i$ is signed, the seller ends up with a profit of $p^0 - c(t_i, p^c)$ and the buyer a utility $U(p^c) - p^0$, where U can (or not) be equal to V. The problem of the bidder is to choose b_i in order to

$$\begin{aligned} &\max_{b_i \in \mathbb{R}_+^m} E\left\{\left[p^0 - c(t_i, p^c)\right] 1_{\left[B(b_i) > B_{(-i)}\right]}\right\} \\ &= \max_{b_i \in \mathbb{R}_+^m} \left[p^0 - c(t_i, p^c)\right] \Pr\left[B(b_i) > B_{(-i)}\right] \end{aligned}$$

Again, the problem can be broken into two parts. For each score level β, the bidder finds the contract $p = p(t_i, \beta)$ to solve

$$h(t_i, \beta) \equiv \max_{p: B(p) = \beta} p^0 - c(t_i, p^c).$$

The second problem is to choose the β in order to maximize

$$\max_{\beta \geqslant 0} h(t_i, \beta) \Pr\left[\beta > B_{(-i)}\right].$$

Let us analyze the first problem. The condition is that $B(p) = V(p^c) - p^0 = \beta$. So, the problem can be simplified to obtain p^c that solves

$$\max_{p^c \in \mathbb{R}^m} V(p^c) - c(t_i, p^c),$$

[38] Other variations are possible. For instance, the seller may be required to meet the exact bid b_j of an opponent j such that $B(b_j) = B_{(-i)}$. Another possibility is to require that the price b_j^0 of this bidder is matched and to choose a vector of characteristics $\overline{b}_i^c$ that is at least as good as that of j, that is, $V\left(\overline{b}_i^c\right) \geqslant V(b_j)$. For the sake of simplicity, we will restrict our attention to the two rules described.

since the choice $p^0 = V(p^c) - \beta$ ensures the restriction of the original problem. Suppose that there is a unique $p^c = p^c(t_i)$ that solves the above problem.

We obtain $h(t_i, \beta) = V(p^c(t_i)) - c(t_i, p^c(t_i)) - \beta$. The second problem is now

$$\max_{\beta \geqslant 0} \left[V(p^c(t_i)) - c(t_i, p^c(t_i)) - \beta\right] \Pr\left[\beta > B_{(-i)}\right].$$

It becomes clear that the types with the same level $V(p^c(t_i)) - c(t_i, p^c(t_i))$ will bid the same β. Let $v^1(t_i)$ be defined as $V(p^c(t_i)) - c(t_i, p^c(t_i))$. Then, it is natural to define the conjugation:

$$P(t_i) = \Pr\left\{t_{-i} \in T_{-i} : v^1(t_j) < v^1(t_i), \forall j \neq i\right\}.$$

Define $\tilde{v}^1(x)$ as $E\left[v^1(t_i) | P(t_i) = x\right]$. Observe that $\tilde{v}^1(x) = v^1(t_i)$ if $P(t_i) = x$ and that $v^1(t_i) = \tilde{v}^1 \circ P(t_i)$.

Then, the solution of the first-score auction is given by

$$\beta^1(x) = \frac{1}{x} \int_0^x \tilde{v}^1(\alpha)\, d\alpha.$$

For the second-score auction, the strategy is simply $\beta^2(x) = \tilde{v}(x)$.

6 Conclusion

Now we would like to highlight what in our opinion are the most important contributions of this paper and to discuss possible extensions.

6.1 The Contributions

Our contributions can be summarized as follows:

- Equilibrium Existence in the Multidimensional Setting - McAdams (2003a) generalizes Athey (2001) for multidimensional types and actions. Nevertheless, he works with discrete bids and types. Our approach offers the existence with continuum types and bids. His result is just an existence result, while ours provides the expressions of the bidding functions. His assumptions requires monotonicity, which is an undesirable restriction when one tries to work in multidimensional settings. While we do not need such an assumption, our results also do not cover multi-unit as his. JSSZ also gives the existence for multidimensional games, including cases with dependence, while we require independence. However, they need an endogenous tie breaking rule and give the existence in mixed strategies, while our results are in pure strategies.

- Equilibrium Existence in Non-Monotonic Settings - We are not aware of general non-monotonic equilibrium existence results in pure strategies. Zheng (2001), Athey and Levin (2001) and Ewerhart and Fieseler (2003) present cases where non-monotonicity arises. The cases in the last two papers seem important in practice. So, our results develop a theory to deal with the situations where the usual monotonicity is not fulfilled.

- Uniqueness of Equilibrium - We are able to ensure the uniqueness of equilibrium in the general setting analyzed (under assumption H3), extending the well known uniqueness of unidimensional and monotonic auctions.

- Necessary and Sufficient Conditions for the Existence of Equilibrium without Ties - The results of JSSZ do not allow one to distinguish when the special tie-breaking is needed or not. Our approach clarifies, under assumption H3, whether there is a need for a special tie-breaking rule.

- Exogenous Tie-Breaking Rule - Knowing exactly when there is a need for a tying with positive probability, we are able to offer an exogenous tie breaking rule, which has the advantage of being implemented through a (modified) second-price auction. Moreover, the equilibrium that the rule implements is in pure strategies.

- Revenue Equivalence Theorem - We have also generalized the Revenue Equivalence Theorem (Theorem 5). Futhermore, Theorem 2 and Appendix B show that there is a deep connection between the revenue equivalence and the existence of equilibrium. Riley and Samuelson (1981) and Myerson (1981) establish that revenue equivalence is a consequence of the equilibrium behavior. Proposition 3 in Appendix B shows that the revenue equivalence is also sufficient for the existence of equilibrium, (if another condition is satisfied). We are not aware of this connection being established previously.

So, our results have clarified some aspects of the problem of equilibrium existence in auctions. The theory shows that, under assumption H3, there is no additional difficult in working with the more general setting of multidimensional types and non-monotonic utilities besides those difficulties already possible in the unidimensional setting.[39] Moreover, this approach allows the equilibrium bidding functions to be expressed in a simple way. This is so because the equilibrium bidding function of a general auction can be expressed by the equilibrium bidding function of an auction with two bidders and uniform types in $[0, 1]$.

There are counter-examples for the existence of equilibrium with multidimensional types. See, for instance, Jackson (1999) and Fang and Morris (2003). These papers consider bidimensional types, and utilities with both private-value and common-value parts. Reading them, one might guess that the main problem with the existence is that of the multidimensionality. Our results suggest two sources for the non-existence: one is the dependence of the signals and the other is the non-monotonicity of the indirect bidding function. In the later case, it is likely that the tie-breaking proposed would solve the problem. So, the dependence can be a deeper source for non-existence.[40]

A last word about the need of tie-breaking rules is worth. Based on Theorem 4, one may conjecture that is always possible to ensure the existence of equilibrium with an exogenous tie-breaking rule. Also, it is possible to conjecture that for sufficiently regular utility functions and independent types, even discontinuous games have an equilibrium in pure strategies with an exogenous tie breaking rule. It is to be seen whether these conjectures are correct.

6.2 The Limits of the Method

Our theory makes two important assumptions: independence of the types and symmetry.

The generalization of the approach for dependent types involves some difficulties, because the conjugation would depends in a complicated manner on the types. Nevertheless, we believe that something can be done if we assume conditional independence, but little can be expected from this case.[41] It is worth remembering that the problem with dependence is not specific to our approach. Jackson (1999) gives a counter-example for the equilibrium existence of an auction with bidimensional affiliated types. Fang and Morris (2003) also obtain negative results, not only for the existence of equilibrium but also for the revenue equivalence.

On the other hand, asymmetry does not seem to impose severe restriction on the existence of equilibrium. We believe that the approach of the indirect auction can be adapted to this case, although not in a straightforward way. If this can be done, it is unlikely that we will obtain the simple expressions of this paper.

[39] Theorem 3 shows that the non-existence of the equilibrium comes from the non-monotonicity of the indirect bidding function. This can occurs also in unidimensional setting, although it can be more common in multidimensional models.

[40] For an alternative method to deal with dependence of signals, see de Castro (2004).

[41] de Castro (2004) proposes the use of conditional independence as an alternative for affiliation.

Another limitation of our theory is that it is applied to single-unit auctions. As we have discussed in section 5, it seems possible to extend the approach for multi-unit auctions.

Finally, about assumption H3, we would like to comment that, although it is not entirely general, it does seem to encompass many of the most important economical examples.

Appendix A - Proof of the Basic Results

Proof of Lemma 1. Let us first remember the expression of Π_i:

$$\Pi_i(t_i, b_i) = E\left[\underline{u}_i|t_i\right] + E\left[\left(u_i^T - \underline{u}_i\right) 1_{\left[\mathbf{b}_{(-i)}=b_i\right]}|t_i\right] + E\left[u_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}|t_i\right],$$

where $u_i = \overline{u}_i - \underline{u}_i$.

We consider each term above separately. The first one has a derivative with respect to b_i almost everywhere and is equal to $E\left[\partial_{b_i}\underline{u}_i|t_i\right]$. The derivative of the last term with respect to b_i is just $E\left[\partial_{b_i}\underline{u}_i|t_i\right]$. Also,

$$E\left[\underline{u}_i|t_i\right] = \int E\left[\partial_{b_i}\underline{u}_i|t_i\right] d\beta.$$

The second term is different from zero just where there is an atom in the distribution of $\mathbf{b}_{(-i)}$. Thus, it is equal to zero for almost all b_i, and its derivative is zero almost everywhere.

Now consider the last term in its original form, $\int u_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}$. Let $a^n \to b_i^+$ (i.e., $a^n > b_i$; the other case is analogous). We have

$$\begin{aligned}
&\int \left\{u_i\left(t_i, a^n, \cdot\right) 1_{\left[a^n>\mathbf{b}_{(-i)}\right]}\right\} - \int \left\{u_i\left(t_i, b_i, \cdot\right) 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}\right\} \\
&= \int \left\{\left[u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)\right] 1_{\left[a^n>\mathbf{b}_{(-i)}\right]}\right\} + \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[a^n>\mathbf{b}_{(-i)}\geqslant b_i\right]}.
\end{aligned}$$

Since u_i has bounded derivative with respect to almost all b_i, $\frac{u_i(t_i,a^n,\cdot)-u_i(t_i,b_i,\cdot)}{a^n-b_i} \to \partial_{b_i}u_i$, for almost all b_i. Also, $1_{\left[a^n>\mathbf{b}_{(-i)}\right]} \to 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}$. These imply that

$$\frac{u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)}{a^n - b_i} 1_{\left[a^n>\mathbf{b}_{(-i)}\right]} \to \partial_{b_i}u_i 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}$$

for almost all b_i and these functions are (almost everywhere) bounded. By the Lebesgue Theorem, the integral converges, that is, there exists

$$\lim_{a^n \to b_i} \int \frac{u_i\left(t_i, a^n, \cdot\right) - u_i\left(t_i, b_i, \cdot\right)}{a^n - b_i} 1_{\left[a^n>\mathbf{b}_{(-i)}\right]}$$

and it is equal to $E\left[\partial_{b_i}u_i\left(t_i, b_i, \cdot\right) 1_{\left[b_i>\mathbf{b}_{(-i)}\right]}|t_i\right]$.

Now we want to determine the derivative of the other term. For this purpose, define for each $t_i \in T_i$ fixed, the measure ρ over $\mathbb{R}_+$ by

$$\rho(V) \equiv \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[\mathbf{b}_{(-i)}\in V\right]}.$$

We have

$$\begin{aligned}
&\lim_{a^n \to b_i} \frac{1}{a^n - b_i} \int u_i\left(t_i, b_i, \cdot\right) 1_{\left[a^n>\mathbf{b}_{(-i)}\geqslant b_i\right]} \\
&= \lim_{a^n \to b_i} \frac{1}{a^n - b_i} \rho\left(\left[b_i, a^n\right)\right) \\
&= \lim_{a^n \to b_i} \left\{\frac{\rho\left(\left[b_i, a^n\right)\right)}{m\left(\left[b_i, a^n\right)\right)}\right\} \\
&= D\rho\left(b_i\right)
\end{aligned}$$

where the existence of $\lim_{r\to 0} \frac{\rho(B(b_i,r))}{m(B(b_i,r))} = D\rho(b_i)$ is ensured by Theorem 8.6 of Rudin (1966) for almost all b_i, that is, $m\left(\left\{v : \nexists \lim_{r\to 0} \frac{\rho(B(b_i,r))}{m(B(b_i,r))}\right\}\right) = 0$. Theorem 8.6 of Rudin (1966) also says that $D\rho$ coincides almost everywhere with the Radon-Nikodym derivative $\frac{d\rho}{dm}(.)$ and that

$$\rho(V) = \rho^{\perp}(V) + \int_V \frac{d\rho}{dm}(\beta)\, d\beta.$$

where $\rho^{\perp}$ denotes the orthogonal part of ρ, and it has the property

$$\lim_{a^n \to b_i} \frac{1}{a^n - b_i} \rho^{\perp}([b_i, a^n)) = 0,$$

by the same theorem.

It is easy to see that ρ is absolutely continuous with respect to $F_{b_{(-i)}}$. The Radon-Nikodym Theorem guarantees the existence of the Radon-Nikodym derivative of ρ with respect to $F_{b_{(-i)}}$, denoted by $E[u_i(t_i, b_i, \cdot)\, ||t_i, \mathbf{b}_{(-i)}(t_{-i}) = \beta]$ such that

$$\rho(V) \equiv \int_V E[u_i(t_i, b_i, \cdot)\,|t_i, \mathbf{b}_{(-i)} = \beta] dF_{b_{(-i)}}(\beta|t_i).$$

Then, it is easy to see that the Radon-Nikodym derivative $\frac{d\rho}{dm}(b_i)$ is simply $E[u_i(t_i, b_i, \cdot)\,|t_i, \mathbf{b}_{(-i)}(t_{-i}) = b_i] f_{b_{(-i)}}(b_i|t_i)$, where $f_{b_{(-i)}}(\cdot|t_i)$ is the Radon-Nikodym derivative of $F_{b_{(-i)}}(\cdot|t_i)$. Thus,

$$\begin{aligned}\partial_{b_i}\Pi_i(t_i,\beta) = E\left[\partial_{b_i}\overline{u}_i(t_i,\beta,\cdot)\, 1_{[\beta > \mathbf{b}_{(-i)}]} + \partial_{b_i}\underline{u}_i(t_i,\beta,\cdot)\, 1_{[\beta < \mathbf{b}_{(-i)}]} |t_i\right]& \\ + E[u_i(t_i,\beta,\cdot)\,|t_i, \mathbf{b}_{(-i)} = \beta] f_{b_{(-i)}}(\beta|t_i)&,\end{aligned}$$

and, by the Lebesgue Theorem,

$$\begin{aligned}\Pi_i(t_i, b_i) = E\left[\left(u_i^T - \underline{u}_i\right)(t_i, b_i, \cdot)\, 1_{[\mathbf{b}_{(-i)} = b_i]} |t_i\right]& \\ + \int_{[0,b_i)} E[u_i(t_i, b_i, \cdot)\,|t_i, \mathbf{b}_{(-i)} = \beta] dF^{\perp}_{b_{(-i)}}(\beta|t_i) + \int_{[0,b_i)} \partial_{b_i}\Pi_i(t_i,\beta) d\beta&.\end{aligned}$$

This concludes the proof.■

Proof of Proposition 2. Let us introduce the following notation:

$$\begin{aligned}\Pi_i^+(t_i, c) &= \int \left[v(t_i, \cdot) - p^W\left(c, b_{(-i)}(\cdot)\right)\right] 1_{[c > b_{(-i)}(\cdot)]} \Pi_{j\neq i}\sigma(dt_j) \\ \Pi_i^-(t_i, c) &= \int p^L\left(c, b_{(-i)}(\cdot)\right) 1_{[c < b_{(-i)}(\cdot)]} \Pi_{j\neq i}\sigma(dt_j), \\ \tilde{\Pi}_i^{+,-}(\phi_i, c) &\equiv E\left[\Pi_i^{+,-}(t_i, c)\,|P(t_i) = \phi_i\right]\end{aligned}$$

Let us begin with the proof for $\tilde{\Pi}_i^+$ and Π_i^+. Let us denote the conditional expectation[42]

[42] We refer the reader to Lehmann (1959) p. 41-5 for a discussion of the concept of conditional expectation and its properties.

$$g^{t_i,c}(\alpha) \equiv E\left[\left(v(t_i,t_{-i}) - p^W\left(c, b_{(-i)}(t_{-i})\right)\right) | P^b_{(-i)}(t_{-i}) = \alpha\right]. \tag{29}$$

The event $\left[c > b_{(-i)}(t_{-i})\right]$ occurs if and only if $\left[\tilde{P}^b(c) > P^b_{(-i)}(t_{-i})\right]$ occurs. Then, we have

$$\Pi_i^+(t_i,c) = \int g^{t_i,c}\left(P^b_{(-i)}(t_{-i})\right) 1_{\left[\tilde{P}^b(c) > P^b_{(-i)}(t_{-i})\right]} \Pi_{j\neq i}\sigma(dt_j).$$

Now we appeal to the Lemma 2.2, p. 43, of Lehmann (1959). This lemma says the following: if R is a transformation and if $\mu^*(B) = \mu\left(R^{-1}(B)\right)$, then

$$\int_{R^{-1}(B)} g[R(t)]\,\mu(dt) = \int_B g(\alpha)\,\mu^*(d\alpha).$$

In our case, $R = P^b_{(-i)}$ and $\mu^*([0,c]) = \mu^*([0,c)) = \tau_{-i}\left(P^{-1}_{(-i)}([0,c))\right) = \Pr\{t_{-i} \in S^{N-1} : P^b(t_j) < c\} = c$, by (9). So, μ^* is exactly the Lebesgue measure, so that we have

$$\Pi_i^+(t_i,c) = \int_0^{\tilde{P}(c)} g^{t_i,c}(\alpha)\,d\alpha. \tag{30}$$

From this and the definition of $\tilde{\Pi}_i^+$, we have

$$\begin{aligned}
\tilde{\Pi}_i^+(\phi_i,c) &= E\left[\int_0^{\tilde{P}(c)} g^{t_i,c}(\alpha)\,d\alpha | P(t_i) = \phi_i\right] \\
&= \int_0^{\tilde{P}(c)} E\left[g^{t_i,c}(\alpha) | P(t_i) = \phi_i\right] d\alpha \\
&= \int_0^{\tilde{P}(c)} \left[\tilde{v}(\phi_i,\alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha,
\end{aligned}$$

where the second line comes from a interchange of integrals (Fubbini's Theorem) and the last line comes from independency and the definition of $\tilde{v}(\phi_i,\alpha)$ and $g^{t_i,c}(\alpha)$ (see (12) and (29)). Also from (10), we can substitute $\tilde{P}$, to obtain

$$\tilde{\Pi}_i^+(\phi_i,c) = \int_0^{\tilde{b}^{-1}(c)} \left[\tilde{v}(\phi_i,\alpha) - p^W\left(c, \tilde{b}(\alpha)\right)\right] d\alpha. \tag{31}$$

Now, we can repeat the above procedures with $\Pi_i^-(\phi_i,c)$ and obtain:

$$\tilde{\Pi}_i^-(\phi_i,c) = \int_{\tilde{b}^{-1}(c)}^1 p^L\left(c, \tilde{b}(\alpha)\right) d\alpha. \tag{32}$$

Adding up, that is, putting $\tilde{\Pi}_i(\phi_i,c) = \tilde{\Pi}_i^+(\phi_i,c) - \tilde{\Pi}_i^-(\phi_i,c)$, we obtain the interim payoff of the indirect auction. This concludes the proof of the first part.

For the second part, observe that the equality (15) implies that for all t_i such that $P(t_i) = P(s) = x$,

$$
\begin{aligned}
E\left[g^{t_i,c}(\alpha)\,|P(t_i)=x\right] &= E\left[\left(v(t_i,t_{-i})-p^W\left(c,b_{(-i)}(t_{-i})\right)\right)|P(t_i)=x,P^b_{(-i)}(t_{-i})=\alpha\right] \\
&= E\left[\left(v(t_i,t_{-i})-p^W\left(c,b_{(-i)}(t_{-i})\right)\right)|t_i=s,P^b_{(-i)}(t_{-i})=\alpha\right] \\
&= E\left[\left(v(s,t_{-i})-p^W\left(c,b_{(-i)}(t_{-i})\right)\right)|P^b_{(-i)}(t_{-i})=\alpha\right] \\
&= g^{s,c}(\alpha).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\tilde{\Pi}^+_i(x,c) &= E\left[\int_0^{\tilde{P}(c)} g^{t_i,c}(\alpha)\,d\alpha|P(t_i)=x\right] \\
&= \int_0^{\tilde{P}(c)} E\left[g^{t_i,c}(\alpha)\,|P(t_i)=x\right]d\alpha \\
&= \int_0^{\tilde{P}(c)} g^{s,c}(\alpha)\,d\alpha \\
\text{by (30)} &= \Pi^+_i(s,c),
\end{aligned}
$$

Obviously, the same can be shown for Π^-_i and $\tilde{\Pi}^-_i$. So, the proof is complete.■

Appendix B - Indirect Auction Equilibria

In this appendix, we will analyze auctions between two players, with independent types uniformly distributed in $[0,1]$. Since this is the setting of the indirect auction, we will use notation consistent with that, although the results of this appendix are independent from the results of section 4. For $(i,-i)=(1,2)$ or $(2,1)$ let

$$
\begin{aligned}
\overline{u}_i(t,b) &= v(t_i,t_{-i})-p^W(b_i,b_{-i})\,; \\
\underline{u}_i(t,b) &= -p^L(b_i,b_{-i})\,; \\
u^T_i(t,b) &= \frac{v(t_i,t_{-i})-b_i}{2}.
\end{aligned}
$$

If we suppose that this auction has a symmetric increasing equilibrium, the first-order condition (3) simplifies to

$$
\tilde{b}'(x)=\frac{\tilde{v}(x,x)-p^W\left(\tilde{b}(x),\tilde{b}(x)\right)+p^L\left(\tilde{b}(x),\tilde{b}(x)\right)}{E_\alpha\left[\partial_1 p^W\left(\tilde{b}(x),\tilde{b}(\alpha)\right)1_{\left[\tilde{b}(x)>\tilde{b}(\alpha)\right]}+\partial_1 p^L\left(\tilde{b}(x),\tilde{b}(\alpha)\right)1_{\left[\tilde{b}(x)<\tilde{b}(\alpha)\right]}\right]}. \tag{33}
$$

We consider an auction with a reserve price of zero. We will assume the following natural conditions for v, p^W and p^L:

Assumptions: (I) $\tilde{v}$ is measurable, non-negative and bounded above. (II) Over the domain $\mathbb{R}_+\times\mathbb{R}$, p^W and p^L are non-negative, differentiable, and at least one of them is non-constant. If $b_i<0$, then $p^W(b_i,b_{-i})=p^L(b_i,b_{-i})=0$. (III) There exists an absolutely continuous $\tilde{b}$ that satisfies (33) almost everywhere in $[0,1]$.

Observe that assumption (I) is rather weak. For instance, if $\tilde{v}$ is a conditional expectation of a measurable non-negative bounded function, it holds. Under so general $\tilde{v}$, it is not necessary for the function $\tilde{b}$ considered in assumption (III) to be increasing. So, we have to consider an modified auction, where the bidders are required to announce a type instead of submitting a bid. We then show that truth-telling is optimal for the bidders in the modified auction.

First, observe that, since $\tilde{b}$ is absolutely continuous over $[0,1]$, its image is an closed interval, which we assume to be $[b_*, b^*]$. In order not to impose restrictions on the possible bids for a bidder, we extend the domain of $\tilde{b}$ from $[0,1]$ to $\mathbb{R}$ in order to permit bids out of $[b_*, b^*]$. To submit a bid $b < b_*$, the bidder can announce a type $y = b - b_* < 0$, whereas to submit a bid $b > b^*$, it is sufficient to announce a type $y = \ b - b^* + 1$. In other words, if $y < 0$, $\tilde{b}(y) \equiv b_* + y$ and if $y > 1$, $\tilde{b}(y) \equiv b^* + y - 1$. The modified auction is described below.

Modified Auction - The bidder submits a type $y \in \mathbb{R}$. In any event, the payment is determined as if the bidder has submitted the bid $\tilde{b}(y)$. The bidder wins against opponents who announce types below y and loses to opponents who announce types above y. If there is a tie, the object is given with probability $1/2$ for each bidder.

Observe that if $\tilde{b}$ is increasing, the modified auction is simply the original auction. If $\tilde{b}$ is not increasing, the difference is that the events of winning are not determined by $\tilde{b}$ but by the announced type y. The rule of the modified auction implies the following interim payoff

$$\begin{aligned}\hat{\Pi}(x,y) = & \int_0^{\min\{y,1\}} \left[\tilde{v}(x,\alpha) - p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right)\right] d\alpha \\ & - \int_{\max\{y,0\}}^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha.\end{aligned} \tag{34}$$

We can simplify the above expression to

$$\hat{\Pi}(x,y) = \int_0^{\min\{y,1\}} \tilde{v}(x,\alpha)\, d\alpha - \hat{p}(y),$$

where

$$\hat{p}(y) \equiv \int_0^{\min\{y,1\}} p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha + \int_{\max\{y,0\}}^1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) d\alpha.$$

Now, we can see that the announcement of a type $y < 0$ or $y > 1$ is never profitable because the payment is non-negative, by assumption (II), and we can restrict attention to $y \in [0,1]$. Observe that

$$p^W\left(\tilde{b}(y), \tilde{b}(y)\right) + \int_0^y \partial_1 p^W\left(\tilde{b}(y), \tilde{b}(\alpha)\right) \tilde{b}'(y)\, d\alpha + p^L\left(\tilde{b}(y), \tilde{b}(y)\right) + \int_y^1 \partial_1 p^L\left(\tilde{b}(y), \tilde{b}(\alpha)\right) \tilde{b}'(y)\, d\alpha$$

is continuous, so that $\hat{p}$ is differentiable. So, for every y, we have

$$\hat{p}'(y) = \partial_y \left\{\int_0^y \tilde{v}(x,\alpha)\, d\alpha - \hat{\Pi}(x,y)\right\} = \tilde{v}(x,y) - \partial_y \hat{\Pi}(x,y).$$

Truth-telling is always optimal if

$$\int_y^x \partial_y \hat{\Pi}(x,\alpha)\, d\alpha = \hat{\Pi}(x,x) - \hat{\Pi}(x,y) \geqslant 0 \tag{35}$$

for any x and y. Also, in this case, if there exists $\partial_y \hat{\Pi}(x,y)\mid_{y=x}$, it must be zero, so that

$$\partial_y \hat{\Pi}(x,y)\mid_{y=x}=0 \Rightarrow \hat{p}'(x) = \tilde{v}(x,x). \tag{36}$$

Indeed, these are simply the second- and the first-order conditions, respectively. We have the following:

Proposition 3 - If truth-telling is equilibrium of the modified auction and $\hat{\Pi}(x,y)$ is differentiable w.r.t. y at $y=x$ for all $x \in [0,1]$, then

$$\hat{p}(y) = \int_0^y \tilde{v}(\alpha,\alpha)\, d\alpha, \tag{37}$$

and for all x and $y \in [0,1]$,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0. \tag{38}$$

Conversely, assume that (37) and (38) hold. Then, truth-telling is equilibrium of the modified auction.

Proof. Observe that $\hat{p}$ is non-negative by assumption (II) and its definition. Then, truth-telling implies $\hat{p}(0)=0$, otherwise $\hat{\Pi}(0,0)<0$ and bidder 0 could do better by not participating in the auction. Since there exists $\partial_y \hat{\Pi}(x,y)\mid_{y=x}$ for all $x \in [0,1]$, $\hat{p}$ is given by (37). So,

$$\hat{\Pi}(x,y) = \int_0^y [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \tag{39}$$

and (38) follows from (35), that is, the fact that truth-telling is equilibrium.

On the other hand, given (37), then (39) holds. Then, (38) implies (35), that is, truth telling is equilibrium.■

As we have said before, if $\tilde{b}$ is increasing, the modified auction is just the original (unmodified) auction. Then, we have

Corollary 4 - Let $\tilde{b}$ be such that (37) holds. If $\tilde{b}$ is increasing and (38) also holds, then $\tilde{b}$ is equilibrium of the original (unmodified) auction. In the affirmative case, we have, for all x and $y \in [0,1]$,

$$\tilde{\Pi}\left(x, \tilde{b}(y)\right) = \int_0^y [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha. \tag{40}$$

We also have the following:

Corollary 5 - Assume that $\tilde{v}$ is continuous and let $\tilde{b}$ be a solution to (33), compatible with $\hat{p}(0)=0$. Then, if $\tilde{b}$ is increasing and (38) is valid, it is the equilibrium of the original (unmodified) auction.

Proof. Since $\tilde{v}$ is continuous and $\tilde{b}$ is differentiable, there exists $\partial_y \hat{\Pi}(x,y)$ and $\partial_b \tilde{\Pi}\left(x, \tilde{b}(y)\right)$ for all $x,y \in [0,1]$. Since $\tilde{b}$ satisfies (33), then $\partial_b \tilde{\Pi}\left(x, \tilde{b}(x)\right)=0$. Since $\partial_y \hat{\Pi}(x,y)\mid_{y=x} = \partial_b \tilde{\Pi}\left(x, \tilde{b}(x)\right) \cdot \tilde{b}'(x)$, (36), together with the initial condition imply (37). So, the hypotheses of Corollary 4 are satisfied.■

Conversely, we have:

Corollary 6 - If $\tilde{b}$ is an increasing equilibrium of the original (unmodified) auction that satisfies (38) and there exists $\partial_b \tilde{\Pi}\left(x, \tilde{b}(x)\right)$ for all $x \in [0,1]$, then (37) and (38) hold.

Proof. Since $\tilde{b}$ is increasing, truth-telling is equilibrium of the modified auction. Since $\partial_y \hat{\Pi}(x,y)\mid_{y=x} = \partial_b \tilde{\Pi}\left(x, \tilde{b}(x)\right) \cdot \tilde{b}'(x)$, there exists $\partial_y \hat{\Pi}(x,y)\mid_{y=x}$ for all x. Proposition 3 gives the result.■

Observe that the four kinds of auctions that we have analyzed satisfy the previous assumptions. Indeed, their payment functions clearly satisfy assumption (II). The equilibrium in the first-price auction (F), second-price auction (S), all-pay auction (A) and war of attrition (W), with reserve price of zero, are given by (21)-(24). Those functions satisfy the first-order condition (33) and are absolutely continuous, so that assumption (III) is also satisfied. Moreover, in each auction format, it is immediate to see that these strategies lead to (37).

Appendix C - Proofs of the Theorems

Proof of Theorem 1.

(i) If $b \in \mathcal{S}$, it defines a conjugation P^b by (7). Since the bid $b(t_i) = \beta$ is optimal for bidder t_i against the strategy $b(\cdot)$ of the opponents, $\partial_b \Pi(s, b(s)) = 0$ and this implies that

$$E\left[v(t_i, \cdot) \mid t_i = s, b_{(-i)}(t_{-i}) = \beta\right] = p^W(\beta,\beta) - p^L(\beta,\beta) - \frac{E_{t_{-i}}\left[\partial_{b_i} p^W 1_{\left[b_i > \mathbf{b}_{(-i)}\right]} + \partial_{b_i} p^L 1_{\left[b_i < \mathbf{b}_{(-i)}\right]}\right]}{f_{b(-i)}(\beta)},$$

where the right-hand side does not depend on s (it depends only because $\beta = b(s)$ is the optimum bid for such bidder). Thus, the left-hand side has to be the same for all s that are bidding the same bid in equilibrium, which implies that (17) holds.

(ii) If $b(t_i)$ is the c that maximizes $\Pi(t_i, c)$ for all t_i with the same $P(t_i)$, obviously $b(t_i)$ is also the c that maximizes $\tilde{\Pi}(P(t_i), c)$. Indeed, this comes from the definition of $\tilde{\Pi}(P(t_i), c)$ given by (13). In other words, $\tilde{b}(x) = \tilde{P}^{-1}(x) = b\left(P^{-1}(x)\right)$ is the equilibrium of the indirect auction.[43]

(iii) and (iv) Since $\tilde{b}$ is an increasing equilibrium of the indirect auction, the assumptions of Corollary 6 in the Appendix B are satisfied, which implies directly (iii) and that the payment of the indirect auction is given by

$$\tilde{p}\left(\tilde{b}(x)\right) = \int_0^x \tilde{v}(\alpha, \alpha)\, d\alpha.$$

Since b is regular, there is no tie with positive probability. So, only the bid determines the payment. If we remember that all types that are conjugated bid the same, we see that (iv) holds.■

Proof of Theorem 2. In Corollary 4 in Appendix B, we prove that conditions (ii), (iii) and (iv) are sufficient for the equilibrium existence in the indirect auction. Now, Proposition 2 proves that condition (i)$'$ implies that for all s such that $P(s) = x$, $\tilde{\Pi}(x,c) = \Pi(s,c)$ (see (16)). Now, if we put $b(s) = \tilde{b}(P(s))$, then

$$\Pi(s, b(s)) = \tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \text{ and}$$
$$\Pi(s,c) = \tilde{\Pi}(P(s), c).$$

[43] Observe that all $t \in P^{-1}(x)$ bids the same $b(t)$, by the definition of P.

But this is sufficient to show the equilibrium existence in the direct auction, since $\tilde{b}$ is the equilibrium in the indirect auction, which implies that

$$\tilde{\Pi}\left(P(s), \tilde{b}(P(s))\right) \geqslant \tilde{\Pi}(P(s), c),$$

for all $c \in \mathbb{R}$. If $\tilde{v}$ is continuous, $\tilde{\Pi}(x, c)$ is differentiable at all $c \in \mathbb{R}$. This concludes the proof.■

Through the proof of Theorem 3, we will make successive use of the following fact:

Lemma 2 - Assume (H1), (H2) and (H3). For any $\sigma-$field Σ on S^{N-1}, we have

$$\begin{aligned}
&\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\
&\quad \Leftrightarrow \forall t_{-i} : v(s', t_{-i}) > v(s, t_{-i}) \\
&\quad \Leftrightarrow E[v(t_i, t_{-i}) | t_i = s', \Sigma] > E[v(t_i, t_{-i}) | t_i = s, \Sigma], \text{ a.s.}
\end{aligned}$$

Proof. (H3) gives the first equivalence. By (H2), v is continuous over a compact. So, if $\forall t_{-i}$: $v(s', t_{-i}) > v(s, t_{-i})$, there is $\delta > 0$ so that $d(t_{-i}) \equiv v(s', t_{-i}) - v(s, t_{-i}) - \delta \geqslant 0$ for all t_{-i}. Then, for any Σ, $E[d(t_{-i}) | \Sigma] \geqslant 0$ almost surely.[44] This implies that $E[v(t_i, t_{-i}) \mid t_i = s', \Sigma] > E[v(t_i, t_{-i}) \mid t_i = s, \Sigma]$, a.s. On the other hand, $E[v(t_i, t_{-i}) \mid t_i = s', \Sigma] > E[v(t_i, t_{-i}) \mid t_i = s, \Sigma]$ a.s. implies that $\exists t_{-i} : v(s', t_{-i}) > v(s, t_{-i})$.■

Proof of Theorem 3. Let us begin with the proof of the *necessity*. According to Theorem 1, given a $b \in \mathcal{S}$, the associated conjugation P^b (given by (7)) is such that for all $s \in (P^b)^{-1}(x)$,

$$E[v(t_i, t_{-i}) | P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = x] = E[v(t_i, t_{-i}) | t_i = s, P^b_{(-i)}(t_{-i}) = x].$$

If $P^b(s) = P^b(s')$ and there is some t_{-i} such that $v(s, t_{-i}) < v(s', t_{-i})$, Lemma 2 implies that

$$E[v(t_i, t_{-i}) | t_i = s, P^b_{(-i)}(t_{-i}) = x] < E[v(t_i, t_{-i}) | t_i = s', P^b_{(-i)}(t_{-i}) = x],$$

which contradicts the previous equality between the conditional expectations. We conclude that

$$P^b(s) = P^b(s') \Rightarrow v(s, t_{-i}) = v(s', t_{-i}) \text{ for all } t_{-i}. \tag{41}$$

Let us define $\tilde{v}^1(x)$ as $E\left[v(t_i, t_{-i}) | P^b(t_i) = x\right]$ and prove that it is non-decreasing. Suppose that there exist x and y, $x > y$, such that $\tilde{v}^1(x) < \tilde{v}^1(y)$. We will reach a contradiction after a series of facts.

First, we claim that for all t_i and t'_i such that $P^b(t_i) = x$ and $P^b(t'_i) = y$, we have $v(t_i, t_{-i}) < v(t'_i, t_{-i})$ for all t_{-i}. Otherwise, $v(t_i, t_{-i}) \geqslant v(t'_i, t_{-i})$ for some t_{-i} and, by (H3), $v\left(t_i, t'_{-i}\right) \geqslant v\left(t'_i, t'_{-i}\right)$ for all t'_{-i}. Then, Lemma 2 and (41) would imply that $\tilde{v}^1(x) = E\left[v(t_i, t_{-i}) | P^b(t_i) = x\right] \geqslant E\left[v(t_i, t_{-i}) | P^b(t_i) = y\right] = \tilde{v}^1(y)$, a contradiction of our initial assumption. Thus, the claim is proved.

This claim and Lemma 2 imply that

$$\begin{aligned}
\tilde{v}(x, z) &\equiv E\left[v(t_i, t_{-i}) | P^b(t_i) = x, P^b_{(-i)}(t_{-i}) = z\right] \\
&< E\left[v(t_i, t_{-i}) | P^b(t_i) = y, P^b_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y, z),
\end{aligned}$$

[44] See, for instance, Kallenberg (2002), Theorem 6.1, p. 104.

for all $z \in [0,1]$, a.s. Thus,

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(y,\alpha)]\, d\alpha < 0.$$

By condition (iii) of Theorem 1, we also have that

$$\int_y^x [\tilde{v}(y,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \leqslant 0.$$

Summing up these two integrals, we obtain

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha < 0,$$

which contradicts condition (iii) of Theorem 1. This contradiction establishes that $x > y \Rightarrow \tilde{v}^1(x) \geqslant \tilde{v}^1(y)$.

Suppose now that there exists $x > y$ such that $\tilde{v}^1(x) = \tilde{v}^1(y)$. Then, the monotonicity of $\tilde{v}^1$ (just proved) gives

$$\forall \phi \in [y,x], \tilde{v}^1(\phi) = \tilde{v}^1(x) = \tilde{v}^1(y). \tag{42}$$

Let $S' = \left\{ s \in S : \tilde{b}(y) \leqslant b(s) < \tilde{b}(x) \right\}$. From (7), for all $s \in S'$, $P^b(s) \in [y,x]$. Then, (41) and (42) imply that $s \in S' \Rightarrow v^1(s) = \tilde{v}^1(x)$. Assumption (H3) requires that $\sigma(S') = 0$. Observe that $S' = A \backslash B$, where $A \equiv \left\{ s \in S : b(s) < \tilde{b}(x) \right\}$ and $B = \left\{ s \in S : b(s) < \tilde{b}(y) \right\}$. But then, $\sigma(A) = \sigma(B)$. However, from the definition of $\tilde{b}$ as the inverse of $\tilde{P}^b$, we have the following:

$$0 < x - y = \tilde{P}^b\left(\tilde{b}(x)\right) - \tilde{P}^b\left(\tilde{b}(y)\right) = (\sigma(A))^{N-1} - (\sigma(B))^{N-1},$$

which is a contradiction. So, we have proved that $x = P^b(s') > P^b(s) = y$ implies $v^1(s') = \tilde{v}^1(x) > \tilde{v}^1(y) = v^1(s)$ and $P^b(s') = P^b(s)$ implies $v^1(s') = v^1(s)$. In other words, $P^b(s') \lesseqqgtr P^b(s)$ if and only if $v^1(s') \lesseqqgtr v^1(s)$ which allows us to conclude that

$$P^b(t_i) = \Pr\left\{ t_{-i} \in T_{-i} = S^{N-1} : v^1(t_j) < v^1(t_i), j \neq i \right\},$$

as we have defined in (25).

Now, $\tilde{v}$ and $\tilde{b}$ in Theorem 1 are exactly those defined in the statement of Theorem 3. So, Theorem 1 implies the claims about $\tilde{b}$. Moreover, if $\tilde{b}$ is unique, the fact that the conjugation is unique proves that the equilibrium of the direct auction is unique.

Sufficiency. If we define P by (25), it is a conjugation. Let us prove that it satisfies condition (i)$'$. If for some x, y and s, such that $P(s) = x$, we have

$$\tilde{v}(x,y) = E[v(t)|P(t_i) = x, P_{(-i)}(t_{-i}) = y] < E[v(t)|t_i = s, P_{(-1)}(t_{-i}) = y],$$

then, for at least one t_{-i} and s', $P(s') = x$, $v(s, t_{-i}) > v(s', t_{-i})$. But then, by (H3), $v(s, t_{-i}) > v(s', t_{-i})$ for all t_{-i} which implies $v^1(s) > v^1(s')$ and $P(s) > P(s')$, a contradiction with the assumption that $P(s) = P(s') = x$. So, condition (i)$'$ is satisfied.

Let us prove condition (iii) of Theorem 2. If $x > y$, for all t_i and t_i' such that $P(t_i') = x$ and $P(t_i) = y$, we have $v(t_i', t_{-i}) > v(t_i, t_{-i})$ for all t_{-i}, by (H3). Then, for all $z \in [0,1]$,

$$\begin{aligned}\tilde{v}(x,z) &\equiv E\left[v(t_i, t_{-i})\,|P(t_i) = x, P_{(-i)}(t_{-i}) = z\right] \\ &> E\left[v(t_i, t_{-i})\,|P(t_i) = y, P_{(-i)}(t_{-i}) = z\right] = \tilde{v}(y,z).\end{aligned}$$

Then, if $y < \alpha < x$, $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) > 0$ and we have:

$$\int_y^x [\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha)]\, d\alpha \geqslant 0.$$

Now if $x < \alpha < y$, we have $\tilde{v}(x,\alpha) - \tilde{v}(\alpha,\alpha) < 0$ so that condition (iii) is satisfied. Condition (iv) of Theorem 2 follows from Corollary 5 in Appendix B.[45] Now, since $\tilde{b}$ satisfies the first-order condition and is increasing by assumption, Theorem 2 implies the existence of equilibrium, with the equilibrium bidding function given by $b = \tilde{b} \circ P$.

Finally, if we use the assumption that $\tilde{v}$ is continuous, there exists $\partial_b \Pi\left(x, \tilde{b}(y)\right)$ for all x, $y \in [0,1]$. Since $\Pi(s, b(x)) = \tilde{\Pi}\left(P(s), \tilde{b}(P(x))\right)$, there exists $\partial_b \Pi(s,b)$ and it is continuous on b. Observe that this assumption is used only for the proof of this last fact and for the proof of condition (iv). Then, if we assume condition (iv) instead of $\tilde{v}$ being continuous, the equilibrium existence would also follow.■

Proof of Theorem 4. If $\tilde{b}$ is strictly increasing, it is a symmetric equilibrium by Theorem 3. If it is not, let $\overline{b}(x) = \sup_{\alpha \in [0,x]} \tilde{b}(\alpha)$. As we discussed after the statement of Theorem 4, this is just one of the possible specification for the equilibrium bidding function. Remember that $\tilde{b}$ is absolutely continuous. Then, there is an enumerable set of intervals $[a_k, c_k]$ where $\overline{b}(x)$ is constant. Let $b_k \equiv \overline{b}(x)$ for $x \in [a_k, c_k]$. (See Figure 4.)

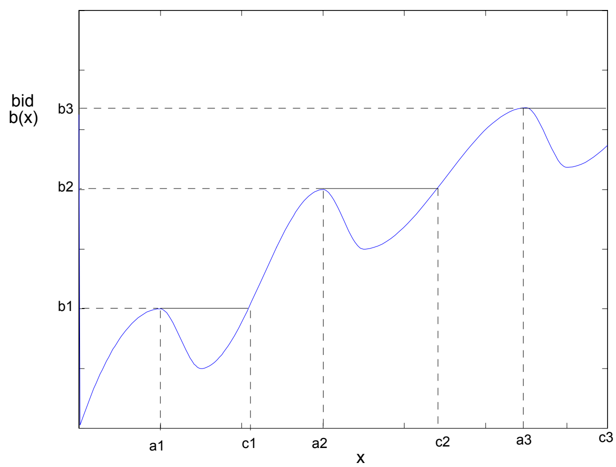

Figure 4: Indirect Equilibrium Bidding Function

Therefore, there is a tie among the indirect types in $[a_k, c_k]$. The tie is solved by the MTBR. We show that it is a dominant strategy for the modified second-price auction to bid $b_i^2 = v^1(t_i)$. Suppose that the opponent is following this strategy, that is, $b_{(-i)}^2(z) = \tilde{v}^1(z)$. Player i will receive the payoff

$$\int_{a_k}^{c_k} [\tilde{v}(x,z) - \tilde{v}(z,z)]\, 1_{[b_i^2 > \tilde{v}^1(z)]} dz$$

for bidding b_i^2. But $b_i^2 > b_{(-i)}^2(z) = \tilde{v}^1(z)$ if and only if $\left(v^1\right)^{-1}\left(b_i^2\right) > \left(v^1\right)^{-1}\left(b_{(-i)}^2(z)\right) = z$. Then, if player i bids $b_i^2 = v^1(t_i) = \tilde{v}^1(x)$ when $P(t_i) = x$, then he will win (and receive $\tilde{v}(x,z) - \tilde{v}(z,z)$)

[45]We could have established condition (iii) also from that Corollary. We preferred to establish it directly to observe that the existence of equilibrium would follow if, instead of assuming $\tilde{v}$ continuous, we assumed condition (iv) directly.

if and only if $x > z$, which is equivalent to $\tilde{v}(x,z) > \tilde{v}(z,z)$, by the proof of Theorem 3. So this is the optimum bid for him. This proves that this strategy is an equilibrium for the modified second-price auction and we assume that this is the equilibrium played in case of a tie. Then, following this strategy, each participant is, indeed, getting the payoff

$$\int_{a_k}^{x} [\tilde{v}(x,z) - \tilde{v}(z,z)]\, dz.$$

Thus, in the whole auction, the bidder who follows the strategy $\overline{b}(x)$ and, in case of a tie, the above strategy, will receive the payoff

$$\tilde{\Pi}_i\left(x, \overline{b}(x)\right) = \int_0^x [\tilde{v}(x,z) - \tilde{v}(z,z)]\, dz.$$

By deviating from $\overline{b}$, that is, bidding $\overline{b}(y) \neq \overline{b}(x)$, he will get

$$\tilde{\Pi}_i\left(x, \overline{b}(y)\right) = \int_0^y [\tilde{v}(x,z) - \tilde{v}(z,z)]\, dz,$$

if $\overline{b}(y)$ is not a bid with positive probability. So, the MTBR implements the modified auction defined in Appendix B. By Proposition 3 there, the assumptions of Theorem 4 ensure the existence of equilibrium of the modified auction, and hence of the indirect auction. Under the properties of the conjugation, all the assumptions of Proposition 2 are satisfied, so that the equilibrium of the indirect auction is also an equilibrium of the direct one.■

Proof of Theorem 6. Suppose first that there is an equilibrium $b \in \mathcal{S}$. We begin by reproducing the first argument of Theorem 1. If $b \in \mathcal{S}$ is the equilibrium, then the first-order condition (3) implies that at $\beta = b(t_i)$,

$$\begin{aligned} &E\left[U\left(v^1(t_i) - \beta\right) | t_i, b_{(-i)}(t_{-i}) = \beta\right] f_{b(-i)}(\beta) + \partial_{b_i} U\left(v^1(t_i) - \beta\right) F_{b(-i)}(\beta) = 0 \\ &\Rightarrow \frac{U\left(v^1(t_i) - \beta\right)}{U'\left(v^1(t_i) - \beta\right)} = \frac{F_{b(-i)}(\beta)}{f_{b(-i)}(\beta)} \end{aligned}$$

The right-hand side does not depend on t_i (it depends only because $\beta = b(t_i)$ is the optimum bid for such bidder). Thus, the left-hand side has to be the same for all $s = t_i$ that are bidding β in equilibrium. By (H4), this implies that all conjugated types have the same $v^1(t_i) - \beta$, and hence, the same $v^1(t_i)$. So, the conjugation is the one defined in (25) and it is unique. The indirect equilibrium has to be given by (27), which can be seen from Maskin and Riley (1984).

On the other hand, the function $\tilde{b}$, solution of the first-order condition of the indirect auction is increasing, because we are in a private-value setting. Now, we have just to check that the signal of the derivative of Π at the equilibrium:

$$\begin{aligned} &U\left(\tilde{v}^1(x) - \tilde{b}(y)\right) \frac{1}{\tilde{b}'(y)} - U'\left(v^1(x) - \tilde{b}(y)\right) x \gtreqless 0 \\ &\qquad\qquad \Leftrightarrow \frac{U\left(\tilde{v}^1(x) - \tilde{b}(y)\right)}{xU'\left(v^1(x) - \tilde{b}(y)\right)} \gtreqless \tilde{b}'(y) \end{aligned}$$

Thus, since $\tilde{b}$ is increasing, $\partial_b \tilde{\Pi}\left(x, \tilde{b}(y)\right) \gtreqless 0$ if and only if $x \gtreqless y$. This concludes proof.■

7 Appendix D - Proofs for the Examples

Proof of example 7.

First, let us show that there is no monotonic equilibria for this auction. By contradiction, assume that there is a increasing bidding function. Then, $P(t_i) = \frac{t_i - 1.5}{1.5}$ and condition (i)' is trivial. We have

$$\begin{aligned}\tilde{v}(x, y) &= (1.5x + 1.5)\left[1.5y + 1.5 - \frac{1.5x + 1.5}{2}\right] \\ &= \frac{9(x+1)(2y - x + 1)}{8}\end{aligned}$$

Thus, the necessary condition (iii) is not satisfied, because $x > y$ implies

$$\int_y^x [\tilde{v}(x, z) - \tilde{v}(z, z)]\, dz = -\frac{3(x-y)^3}{8} < 0.$$

Now, we will show that there are multiple equilibria for this auction. Assume that there exists a bell-shaped equilibrium and that, for each x, there are two types, $f(x)$ and $g(x)$ such that $P(t_i) = x = \frac{3 - g(x) + f(x) - 1.5}{1.5}$, which implies that $g(x) = f(x) + 1.5(1 - x)$. (See Figure 5).

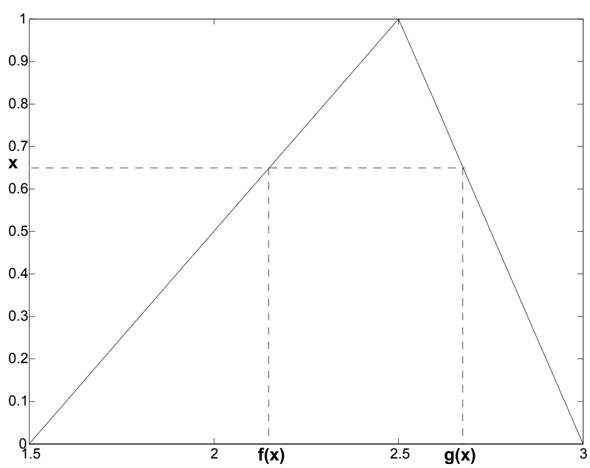

Figure 5: Functions f(x) and g(x) that define the conjugation in example 7.

Condition (i)' requires

$$\begin{aligned}f(x)\left(\frac{f(y)+g(y)}{2} - \frac{f(x)}{2}\right) &= \frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right) - \frac{f^2(x)+g^2(x)}{4} \\ &\Leftrightarrow \frac{f(y)+g(y)}{2}\left[f(x) - \frac{f(x)+g(x)}{2}\right] \\ &= \frac{f(x)^2}{2} - \frac{f^2(x)+g^2(x)}{4} \\ &= \frac{f(x)^2 - g(x)^2}{4} \\ &\Leftrightarrow \frac{f(y)+g(y)}{2} = \frac{f(x)+g(x)}{2}\end{aligned}$$

Then, $f(y)+g(y)$ is a constant, and we have $f(x)=k+3/4x$. Since $f(0)=1.5$, $k=1.5$. We obtain:

$$\begin{aligned}\tilde{v}(x,y) &= \frac{f(x)+g(x)}{2}\left(\frac{f(y)+g(y)}{2}\right)-\frac{f^2(x)+g^2(x)}{4}\\ &= \left(\frac{9}{4}\right)^2-\frac{(3/2+3/4x)^2+(3-3/4x)^2}{4}\\ &= \left(\frac{9}{4}\right)\left[1+\frac{x}{4}-\frac{x^2}{8}\right],\end{aligned}$$

which satisfies condition (iii) because it is increasing in x on $[0,1]$. Condition (ii) and (iv) are also satisfied, since

$$\tilde{b}(x)=\frac{1}{x}\int_0^x \tilde{v}(z,z)\,dz=\frac{3\left(24+3x-x^2\right)}{32}$$

is increasing on $[0,1]$.■

Proof for Example 8 - We have

$$\begin{aligned}&v\left(\tilde{t}_1,t_{-1}\right)-v\left(t_1,t_{-1}\right)\\ &=\left\{4+\left[2\left(t_2^1+t_3^1\right)-\frac{\left(t_2^2+t_3^2\right)}{2}+1\right]\left[\left(\tilde{t}_1^1-\frac{3}{2}\tilde{t}_1^2\right)+\left(t_1^1-\frac{3}{2}t_1^2\right)\right]\right\}\\ &\cdot\left[\left(\tilde{t}_1^1-\frac{3}{2}\tilde{t}_1^2\right)-\left(t_1^1-\frac{3}{2}t_1^2\right)\right].\end{aligned}$$

The term on the right in the first line is always positive, so that the signal of the difference $v\left(\tilde{t}_1,t_{-1}\right)-v\left(t_1,t_{-1}\right)$ depends only on $\tilde{t}_1$ and t_1.■

Proof for Example 9 - Spectrum Auction

Let us assume the $t_i=\left(t_i^1,t_i^2,t_i^3\right)$ are independent and uniformly distributed on $\left[\underline{s}^1,\overline{s}^1\right]\times\left[\underline{s}^2,\overline{s}^2\right]\times\left[\underline{s}^3,\overline{s}^3\right]$, with $\underline{s}^1$, $\underline{s}^2$, $\underline{s}^3\geqslant 0$. We have

$$\begin{aligned}v^1\left(t_i^1,t_i^2,t_i^3\right)&=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3\\ &+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]\end{aligned}$$

Let us denote by $\overline{v}^1$ the expression in the first line above, that is,

$$\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)=\frac{t_i^1}{N}-t_i^2-\frac{N-1}{N}t_i^3.$$

The conjugation P and the c.d.f. $\tilde{P}$ are given by:

$$P\left(t_i^1,t_i^2,t_i^3\right)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)<\overline{v}^1\left(t_i^1,t_i^2,t_i^3\right)\right\}\right]^{N-1}.$$

and

$$\tilde{P}(k)=\left[\Pr\left\{\left(s^1,s^2,s^3\right):\overline{v}^1\left(s^1,s^2,s^3\right)+\frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2-\left(\underline{s}^1\right)^2+\left(\overline{s}^3\right)^2-\left(\underline{s}^3\right)^2\right]<k\right\}\right]^{N-1}.$$

We can reparametrize the problem so that

$$\tilde{P}(k) = \left[\Pr\left\{(x,y,z) \in [0,1]^3 : ax + by + cz < l(k)\right\}\right]^{N-1},$$

where $a = \left(\overline{s}^1 - \underline{s}^1\right)/N > 0$, $b = -\left(\overline{s}^2 - \underline{s}^2\right) < 0$, $c = -(N-1)\left(\overline{s}^3 - \underline{s}^3\right)/N < 0$ and

$$l(k) = k - \frac{\underline{s}^1}{N} + \overline{s}^2 + \frac{N-1}{N}\underline{s}^3 - \frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2 - \left(\underline{s}^1\right)^2 + \left(\overline{s}^3\right)^2 - \left(\underline{s}^3\right)^2\right].$$

It is elementary to obtain that, for a uniform distribution in $[0,1]^3$ and $a > 0$, $b < 0$, $c < 0$ and $k > b + c$,

$$\Pr\left\{(x,y,z) \in [0,1]^3 : ax + by + cz < l\right\} = \frac{(l-b-c)^3}{6abc}.$$

So,

$$\tilde{P}(k) = \frac{[l(k) - b - c]^{3(N-1)}}{(6abc)^{N-1}}$$

and

$$\begin{aligned}\tilde{v}(x,y) = & \left\{\tilde{P}^{-1}(x) - \frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2 - \left(\underline{s}^1\right)^2 + \left(\overline{s}^3\right)^2 - \left(\underline{s}^3\right)^2\right]\right\} y \\ & + E\left[\frac{\sum_{j\neq i}\left(t_j^1 + t_j^3\right)}{N} | \max_{j\neq i} P(t_j) = y\right].\end{aligned}$$

The candidate for the equilibrium of the first-price indirect auction is

$$\tilde{b}(x) = \frac{1}{x}\int_0^x \tilde{v}(\alpha,\alpha)\, d\alpha,$$

which is differentiable, with $\tilde{b}'(x) = [\tilde{v}(x,x) - x]/x$. Then, Theorem 3 teaches us that there exists an equilibrium in regular pure strategies for this auction if and only if

$$\begin{aligned}\tilde{v}(x,x) - x = & \left\{\tilde{P}^{-1}(x) - \frac{N-1}{2N}\left[\left(\overline{s}^1\right)^2 - \left(\underline{s}^1\right)^2 + \left(\overline{s}^3\right)^2 - \left(\underline{s}^3\right)^2\right] - 1\right\} x \\ & + E\left[\frac{\sum_{j\neq i}\left(t_j^1 + t_j^3\right)}{N} | \max_{j\neq i} v^1(t_j) = \tilde{P}^{-1}(x)\right]\end{aligned}$$

is positive. Depending on the values of $\underline{s}^n$, $\overline{s}^n$, for $n = 1, 2, 3$, the above expression can be positive or negative. If it is always positive, $\tilde{b}$ is increasing and it is the equilibrium of the indirect auction. In the other case, there is no equilibrium without ties. For instance, a sufficient condition for the existence of equilibrium in pure strategy is

$$\frac{\underline{s}^1}{N} - \overline{s}^2 - \overline{s}^3\frac{N-1}{N} - 1 \geqslant 0,$$

since the expectation above is always positive.■

Proof for Example 10 - Job Market

We assume that there are two players with unidimensional signals uniformly distributed on $[0,1]$ and that $m \in [0,1]$, $b \geqslant 0$. Following the method given by Theorem 3, we first obtain

$$v^1(t_i) = am + \frac{c}{2} - b(t_i - m)^2.$$

We will consider two cases.

First case: $m \leqslant 1/2$. In this case, we have

$$P(t_i) = \begin{cases} 1 - 2m + 2t_i, & \text{if } 0 \leqslant t_i < m \\ 1 - 2t_i + 2m, & \text{if } m \leqslant t_i < 2m \\ 1 - t_i, & \text{if } 2m \leqslant t_i \leqslant 1 \end{cases}$$

So, we have

$$\tilde{v}(x,y) = \begin{cases} am + c(1-y) - b(1-x-m)^2, & \text{if } 0 \leqslant x, y < 1-2m \\ am + c(1-y) - \frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 1-2m \leqslant x \leqslant 1 \\ (a+c)m - b(1-x-m)^2, & \text{if } 0 \leqslant x < 1-2m \leqslant y \leqslant 1 \\ (a+c)m - \frac{b}{4}(1-x)^2, & \text{if } 1-2m \leqslant x, y \leqslant 1 \end{cases}$$

Now, it is easy to obtain, for $x < 1-2m$,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x}\int_0^x \tilde{v}(y,y)\,dy \\ &= \frac{1}{x}\int_0^x \left[am + c(1-y) - b(1-y-m)^2\right] dy, \\ &= am + c - b(1-m)^2 - x\left[\frac{c}{2} + b(m-1)\right] - \frac{b}{3}x^2, \end{aligned}$$

which is increasing if

$$x_V = \frac{\frac{c}{2} + b(m-1)}{\frac{-2b}{3}} \geqslant 1 - 2m,$$

that is, if $c \leqslant \frac{2b(m+1)}{3}$. For $x > 1-2m$,

$$\begin{aligned} \tilde{b}(x) &= \frac{1}{x}\{\frac{1}{6}(1-2m)\left[6am + 3c(1+2m) - 2b\left(1-m+m^2\right)\right] \\ &\quad + \int_{1-2m}^{x}\left[(a+c)m - \frac{b}{4}(1-y)^2\right] dy\} \\ &= \frac{6(c - 2cm + 2amx + 2cmx) - b\left(3 - 12m + 12m^2 + 3x - 3x^2 + x^3\right)}{12x} \\ &= \frac{(1-2m)\left[2c - b(1-2m)\right]}{4x} + m(a+c) - \frac{b}{4} + \frac{b\left(3x - x^2\right)}{12} \end{aligned}$$

whose derivative can be simplified to

$$\tilde{b}'(x) = -\frac{(1-2m)\left[2c - b(1-2m)\right]}{4x^2} + \frac{b(3-2x)}{12}$$

The term $x^2(3-2x)$ is increasing, so that, the bidding function will be increasing if and only if $\tilde{b}'(1-2m) \geqslant 0$, that is,

$$\begin{aligned}\frac{(1-2m)^2 b[3-2(1-2m)]}{3} &\geqslant (1-2m)[2c-b(1-2m)] \\ &\Leftrightarrow c \leqslant \frac{2b(1-2m)(1+m)}{3}\end{aligned}$$

We conclude that in the case of $m < 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \leqslant \min\left\{\frac{2b(m+1)}{3}, \frac{2b(1-2m)(1+m)}{3}\right\}.$$

Second Case: $m > 1/2$. We have

$$P(t_i) = \begin{cases} t_i, & \text{if } 0 \leqslant t_i < 2m-1, \\ 1-2m+2t_i, & \text{if } 2m-1 \leqslant t_i < m \\ 1-2t_i+2m, & \text{if } m \leqslant t_i \leqslant 1 \end{cases}$$

and

$$\tilde{v}(x,y) = \begin{cases} am+cy-b(x-m)^2, & \text{if } 0 \leqslant x,y < 2m-1 \\ am+cy-\frac{b}{4}(1-x)^2, & \text{if } 0 \leqslant y < 2m-1 \leqslant x \leqslant 1 \\ (a+c)m-b(x-m)^2, & \text{if } 0 \leqslant x < 2m-1 \leqslant y \leqslant 1 \\ (a+c)m-\frac{b}{4}(1-x)^2, & \text{if } 2m-1 \leqslant x,y \leqslant 1 \end{cases}$$

For $x < 2m-1$,

$$\begin{aligned}\tilde{b}(x) &= \frac{1}{x}\int_0^x \tilde{v}(y,y)\,dy \\ &= \frac{1}{x}\int_0^x \left[am+cy-b(y-m)^2\right]dy, \\ &= am-bm^2+x\left(\frac{c}{2}+bm\right)-\frac{b}{3}x^2,\end{aligned}$$

which is increasing in the considered interval if and only if

$$\frac{\frac{c}{2}+bm}{-2\left(-\frac{b}{3}\right)} \geqslant 2m-1,$$

that is, $c \geqslant \frac{2}{3}b(m-2)$. For $x > 2m-1$,

$$\tilde{b}(x) = \frac{-2c(2m-1)-b(2m-1)^2}{4x} + \frac{12(a+c)m-b\left(3-3x+x^2\right)}{12}$$

which gives

$$\tilde{b}'(x) = \frac{2c(2m-1)+b(2m-1)^2}{4x^2} + \frac{b(3-2x)}{12},$$

Following the same procedure of the first case, $\tilde{b}'(x) \geqslant 0, \forall x \in [2m-1,1]$ if and only if

$$\begin{aligned}\frac{(2m-1)^2 b[3-2(2m-1)]}{3} &\geqslant -(2m-1)[2c+b(2m-1)] \\ &\Leftrightarrow c \geqslant -\frac{2b(2m-1)(1+m)}{3}\end{aligned}$$

We conclude that, if $m > 1/2$, there exists a pure strategy equilibrium in regular strategies if and only if

$$c \geqslant \max\left\{\frac{2}{3}b\left(m-2\right), \frac{2b\left(1-2m\right)\left(1+m\right)}{3}\right\}.$$

■

References

[1] Araujo, A. and H. Moreira (2000), "Adverse Selection Problems without the Spence-Mirrlees Condition," working paper, IMPA.

[2] Araujo, A. and L. I. de Castro (2004): "Equilibria in Auctions: From Fundamentals to Open Problems," working paper, IMPA.

[3] Araujo, A., L. I. de Castro and H. Moreira (2004), "Pure Strategy Equilibria of Multidimensional Non-Monotonic Auctions," working paper, IMPA.

[4] Athey, S. (2001): "Single Crossing Properties and The Existence of Pure Strategy Equilibria in Games of Incomplete Information," Econometrica, 69, 861-899.

[5] Athey, S. and J. Levin (2001): "Information and Competition in U.S. Forest Service Timber Auctions," Journal of Political Economy, 109, 375-417.

[6] Battigalli P and M. Siniscalchi (2003): "Rationalizable bidding in first-price auctions," Games and Economic Behavior, 45, 38-72.

[7] Bernheim, B. D. (1984): "Rationalizable Strategic Behavior," Econometrica, 52, 1007-1028.

[8] Border, K. (1991): "Implementation of Reduced Form Auctions: A Geometric Approach," Econometrica, 59, 1175-1187.

[9] de Castro, L. I. (2004): "A Critique to Affiliation," Working paper. IMPA.

[10] Che, Y.-K. (1993): "Design Competition Through Multidimensional Auctions," Rand Journal of Economics, 24, 668-680.

[11] Chen, Y.-M. (1986): "An Extension to the Implementability of Reduced Form Auctions," Econometrica, 54, 1249-1251.

[12] Dasgupta, P. and E. Maskin (2000): "Efficient Auctions," The Quartely Journal of Economics, 115, 341-388.

[13] Ewerhart, C. and K. Fieseler (2003): "Procurement Auctions and Unit-Price Contracts," Rand Journal of Economics, 34, 569-581.

[14] Fang, H. and S. Morris (2003): "Multidimensional Private Value Auctions," Cowles Foundation Discussion Paper No. 1423.

[15] Harsanyi, J. C. (1967-8): "Games With Incomplete Information Played by Bayesian Players," Management Science, 14, 159–182, 320-334 and 486-502.

[16] Jackson, M. O. (1999): "The Non-Existence of Equilibrium in Auctions with Two Dimensional Types," working paper.

[17] Jackson, M. O., L. K. Simon, J. M. Swinkels and W. R. Zame (2002): "Communication and Equilibrium in Discontinuous Games of Incomplete Information," Econometrica, 70, 1711-1741.

[18] Jehiel, P., B. Moldovanu and E. Stacchetti. (1996): "How (Not) to Sell Nuclear Weapons," American Economic Review, 86, 814-829.

[19] Kagel, J. H. (1995): "Auctions: A Survey of Experimental Research," in: Kagel, J. Roth, A. (Eds.), The Handbook of Experimental Economics. Princeton Univ. Press, Princeton.

[20] Kallenberg, Olav (2002): "Foundations of Modern Probability," 2nd. edition, Springer-Verlag, New York.

[21] Krishna, V. and J. Morgan (1997): "An Analysis of the War of Attrition and the All-Pay Auction," Journal of Economic Theory, 72, 343-362.

[22] Laffont, J.-J. (1997): "Game Theory and Empirical Economics: The Case of Auction Data," European Economic Review, 41, 1-35.

[23] Lebrun, B. (1999): "First-Price Auction in the Asymmetric N Bidder Case," International Economic Review, 40, 125-142.

[24] Lehmann, E. L. (1959): "Testing Statiscal Hypotheses," Wiley Series in Probability and Mathematical Statistics.

[25] Lizzeri, A. and N. Persico (2000): "Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price," Games and Economic Behavior, 30, 83-114.

[26] Maskin, E. and J. Riley (1984): "Optimal Auctions with Risk Averse Buyers," Econometrica, 52, 1473-1518.

[27] Maskin, E. and J. Riley (2000): "Equilibrium in Sealed High Bid Auctions," Review of Economic Studies, 67, 439-454.

[28] Matthews, S. A. (1984): "On the Implemetability of Reduced Form Auctions," Econometrica, 52, 1519-1522.

[29] McAdams, D. (2003a): "Isotone Equilibrium in Games of Incomplete Information," Econometrica, 71, 1191–1214.

[30] McAdams, D. (2003b): "Characterizing Equilibria in Asymmetric First-Price Auctions," working paper, MIT.

[31] Milgrom, P. R. and J. Weber (1982): "A Theory of Auctions and Competitive Bidding," Econometrica, 50, 1089-1122.

[32] Myerson, R. (1981): "Optimal Auction Design," Mathematics of Operations Research, 6, 58-73.

[33] Pearce, D. G. (1984): "Rationalizable Strategic Behavior and The Problem of Perfection," Econometrica, 52: 1029-50.

[34] Reny, P. and S. Zamir (2002): "On the Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions," mimeo.

[35] Riley, J. and W. Samuelson (1981): "Optimal Auctions," American Economic Review, 71, 381-392.

[36] Rudin, W. (1966): Real and Complex Analysis. McGraw-Hill, N. York.

[37] Simon, L. K. and W. R. Zame (1990): "Discontinuous Games and Endogenous Sharing Rules," Econometrica, 58, 861-872.

[38] Williams, S. R. (1991): "Existence and convergence of equilibria in the buyer's bid double auction," Review of Economic Studies, 58, 351-374.

[39] Zheng, C. Z. (2001): "High Bids and Broke Winners," Journal of Economic Theory, 100, 129-171.