Indivisible Goods and Fiat Money*

Michael Florig† Jorge Rivera Cayupi‡

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Abstract

In spite of fiat money is useless in a standard Arrow-Debreu model, in this paper we will show that this does not hold true anymore when goods are indivisible. In our setting, although fiat money yields no utility, its price will always be positive and the set of equilibrium allocations changes with the distribution of fiat money. Its role lies in the fact that it could be used to facilitate exchange. Since a Walras equilibrium does not always exist when goods are indivisible, a new equilibrium concept - called rationing equilibrium - is introduced and its existence is proven under weak assumptions on the economy. A Walras equilibrium exists generically on the distribution of fiat money.

Keywords: competitive equilibrium, indivisible goods, fiat money.

JEL Classification: C62, D50

1 Introduction

Most economic models assume that goods in the economy are perfectly divisible. The rational behind this assumption is that the commodities one usually considers are *almost perfectly divisible* in the sense that the minimal unit of the good is insignificant enough so that its indivisibility can be neglected. So one should be able to approximate an economy, with a *small enough level* of indivisibility of goods, by some idealized economy where goods are perfectly divisible. A competitive equilibrium of this idealized economy should thus be an approximation of some competitive outcome of the economy with indivisible goods.

The question arises what the Walras equilibrium with perfectly divisible goods is supposed to approximate - simply a Walras equilibrium of an economy with indivisible goods? Surely not, since is well known that a Walras equilibrium may fail to exist in the absence of perfectly divisible goods, and even the core may be empty (see Henry (1970) and Shapley and Scarf (1974) respectively). These facts are certainly due to some economic phenomena which cannot be modelled with the standard approach. Consequently, a richer notion of competitive equilibrium is needed, which exists even when goods are indivisible. This new notion will be called *rationing equilibrium.*

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†CERMSEM, Université de Paris 1 Panthéon-Sorbonne, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13, France, florig@univ-paris1.fr.

‡Departamento de Economía, Universidad de Chile, Diagonal Paraguay 257, Torre 26, Santiago, Chile, jrivera@econ.uchile.cl.

In order to define the rationing equilibrium, we will develop a model where (i) goods are indivisible at the individual level but perfectly divisible at the aggregate level of the economy; (ii) fiat money is used only to facilitate the exchange among consumers; and (iii) we introduce a regularized notion of demand which will be an upper semi-continuous correspondence in our framework.

With respect to (i), we proceed considering a model where there are a finite number of types of consumers, and for each type there are a continuum of individuals. The justification for this hypothesis comes from the fact that if some consumer would own a commodity which may not be considered negligible at the level of the entire economy, it would be hard to justify that this consumer acts as a price taker.

With respect to (ii), is clear that in presence of divisible goods, it could be difficult for agents to execute net-exchanges worth exactly zero. Already, Adam Smith (1776) considered the possibility for fiat money to facilitate exchange of indivisible goods as one of its crucial roles. Similarly to Drèze and Müller (1980) we introduce a slack parameter in the economy. In our case, this parameter can be identified as fiat money, whose unique role will be to facilitate the exchange of goods among individuals. Indeed, fiat money has no intrinsic value whatsoever, since in our model it does not enter in consumers' preferences. The last fact is a crucial difference with several contributions on indivisible goods, as we will see later on.

Finally, related with (iii), we point out that in presence of indivisible goods, the Walrasian demand is, in general, not an upper semi-continuous correspondence. Therefore Walras equilibria do not always exist. We introduce a regularized notion of demand, that will be a building block to define the rationing equilibrium notion.

The main results of this paper is the demonstration of the existence of a rationing equilibrium with a strictly positive price of fiat money. Fiat money having a positive price is a nice by-product of our approach which may look surprising (cf. Hahn (1965)). In fact, in the literature we can find several approaches that setup different models in order to guarantee the positiveness of price of fiat money. For example, the infinite horizon model with overlapping generations (Samuelson (1958), Balasko, Cass and Shell (1980), Balasko and Shell (1981)) or with infinitely lived agents (e.g. Bewley (1980, 1983), Gale and Hellwig (1984)). In a static or finite horizon model, one may consider money lump-sum taxation with a zero total money supply (Lerner (1947), Balasko and Shell (1986)). Finally, Clower (1967) proposed a cash in advance constraint to study similar problems (complementarily, see Dubey and Geanakoplos (1992)).

Now on, the introduction of fiat money into the Arrow-Debreu model may be necessary in a much simpler setting as the aforementioned. For example, if the non-satiation assumption does not hold, for any given price, some consumer may wish to consume a commodity bundle in the interior of his budget set. Therefore a Walras equilibrium may fail to exist. Without the non-satiation assumption, one may establish existence of an equilibrium by allowing for the possibility that some agents spend more than the value of their initial endowment. This generalization of the Walras equilibrium is called *dividend equilibrium* or *equilibrium with slack* (see Makarov (1981), Balasko (1982), Aumann and Drèze (1986) and Mas-Colell (1992) among others). This concept was first introduced in a fixed price setting by Drèze and Müller (1980). Indeed, Kajii (1996) shows that this dividend approach is equivalent to considering Walras equilibria with an additional commodity called fiat money. In his setting, fiat money can be consumed in positive quantities, but preferences are independent of the consumption of it. Thus, if local non-satiation holds, fiat money has price zero and we are back in the Arrow-Debreu setting. However, if satiation problems occur, an equilibrium with price zero of fiat money may fail to exist. Then, fiat money must have a positive price in equilibrium. In fact, if a consumer does not want to spend his entire income on consumption goods, he can satisfy his budget constraint as an equality by buying fiat money, if this fiat money has a positive price.

In our approach, neither do we use a cash in advance constraint nor do we consider an infinite horizon, nor do we consider money lump-sum taxation with a zero total money supply. The positiveness of fiat money price is only due to the indivisibility of goods and the role that this parameter plays in our model.

There also remains some questions related with the properties of the rationing equilibrium when keeping the level of indivisibility fixed. In parallel papers, we demonstrate a First and Second Welfare theorems and core equivalence for our equilibrium concept, and we prove that, under suitable conditions on the economy, a rationing equilibrium converges to a Walras equilibrium when the level of indivisibilities became small (Florig and Rivera (2004a, 2004b)).

So far, we did not yet allude to the relationship to the rather large literature on indivisible goods. One could roughly divide it into two approaches. Firstly, following Shapley and Scarf (1974) markets without a perfectly divisible good, but considering only one commodity per agent, e.g. houses. Secondly, following Henry (1970), numerous authors (including Broome (1972), Mas-Colell (1977), Kahn and Yamazaki (1981), Quinzii (1984), see Bobzin (1998) for a survey) consider economies with indivisible commodities and one perfectly divisible commodity called *money*. This should however not be confused with fiat money since it is a crucial consumption good. All these contributions suppose that the divisible commodity satisfies overriding desirability, i.e. it is so desirable by the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good in the sense that no bundle of indivisible goods can yield as much utility as consuming his initial endowment of the divisible good and nothing of the indivisible one. Then, non-emptiness of the core and existence of a Walras equilibrium can be established.

The paper which our approach is closest to is Dierker (1971). He proposed a quasi-equilibrium for exchange economies existing without a perfectly divisible consumption good. However, at such an equilibrium agents do not necessarily receive an individually rational commodity bundle.

2 Motivation and examples

Previous to enter in specific details on the model, in this section we will emphasize three aspects related with fiat money and Walras equilibria that have importance in both the model and definitions we are going do in next sections.

(i.) *Fiat money may change the set of Walrasian equilibria*

Suppose $I = \{1, 2, 3\}$ and let $u_i(x, y) = x \cdot y$ be the utility function for individual $i \in I$. Let $e_1 = (7, 0), e_2 = (0, 3), e_3 = (0, 4) \in I\!R^2$ be the initial endowment for them. In this case, there exists a unique Walras equilibrium price $p = (1, 1)$ with the equilibrium allocations $x_1 = (4, 3), x_2 = (1, 2), x_3 = (2, 2)$ and $x'_1 = (3, 4),\ x'_2 = (2, 1),\ x'_3 = (2, 2)$. Suppose now we endow each consumer with an initial amount of fiat money, let say, $0 < m_1 < 1/8$, $m_2 = 1$ and $0 < m_3 < 1/2$. Given that, it is possible to check that $p^* = (1, \frac{9}{8}) \in I\!R^2_{++}$, $q^* = 1$, $x^*_1 = (3, 3)$, $x^*_2 = (2, 2)$ and $x^*_3 = (2, 2)$ is a Walras equilibrium (with money) for this economy. In this example, the introduction of fiat money in the economy changes the set of equilibria, even this parameter does not enter in consumers' preferences.

(ii.) *Without fiat money markets may be non viable*

Consider an exchange economy with three types of consumers ($I = \{1,2,3\}$) and for each type we have a continuum of them, indexed by compacts and disjoint intervals $T_i \subseteq I\!R, i \in I$, all of them with identical Lebesgue measure. Suppose that each consumer choose their consumption bundle on a discrete consumption set $X_i = \{0,1,2\}$ and that the utility functions for each type are $u_1(x) = -x$, $u_2(x) = u_3(x) = x$. Finally, let $e_1 = 2$ and $e_2 = e_3 = 0$ be the initial endowment for each type of individual. Given previous definitions, we may check that there is no a Walrasian equilibrium in the economy: if $p < 0$ then Walrasian demand will be above the total initial endowment of the economy; if $p > 0$, the total initial endowment is above demand. However, if we endow consumers' type 2 and 3 with an initial amount of fiat money, let say, $m_2 = m_3 > 0$, then it can be proved that prices $p = m_2,\ q = 1$, and demands for each type of individual given by $x_1 = 0,\ x_2 = x_3 = 1$, conforms a *weak equilibrium* for this economy, and it is the only one with $p \neq 0$[1]. By other hand, if for instance, $m_2 = 3,\ m_3 > 0$, then $p = m_2/2,\ q = 1$, $x_1 = 0, x_2 = 2, x_3 = 0$ is the unique weak equilibrium with $p \neq 0$. This example stressed that in absence of fiat money non necessarily exists a Walras equilibrium in the economy and the introduction of this parameter could implies the existence of a new equilibrium concept. Unfortunately, the weak equilibrium notion presents serious inconveniences that oblige us to consider a refinement of it[2]. This refinement will be called *rationing equilibrium*, concept that we will introduce in Section 3.

(iii). *A Walras equilibrium may not exists but a rationing equilibrium*

Consider an exchange economy with three consumer indexed by $I = \{1,2,3\}$ and two goods. Let $u_1(x,y) = x + 2y$ and $u_2(x,y) = u_3(x,y) = 2x + y$ be the utility functions for each individual and let $e_1 = (0,1), e_2 = (1,0), e_3 = (1,0) \in I\!R^2$ be the initial endowment for them. In this case, there exists no Walras equilibrium in the economy and it is easy to check that $p = (1,1) \in I\!R^2$, $K = \{\mu(1,-1)|\ \mu \geq 0\}$ and demands given by the initial endowment is a rationing equilibrium (without fiat money) for the economy. Thus, this example show us that in some cases may not exist a Walras equilibrium but a rationing equilibrium. Indeed, the main result of this work will say that under very weak assumptions on the economy, provided that for each consumer the initial endowment of fiat money is strictly positive, then a rationing equilibrium will exists, with price of fiat money strictly positive.

3 The model

In this section we introduce definitions that will play a relevant role in the rest of the paper. We begin with basic concepts and continue introducing some auxiliary notions that help us

[1] This auxiliary notion will be introduced in Section 3.

[2] Consider an exchange economy with three types of consumers ($I = \{1,2,3\}$) and for each type there are a continuum of them, indexed by compact and disjoint intervals $T_i \subseteq I\!R, i \in I$, with identical Lebesgue measure. Suppose there are two commodities and for all $i \in I$, $X_i = \{0,1,2\}^2$, $u_1(x) = -x^1 - x^2$, $u_2(x) = 2x^1 + x^2$, $u_3(x) = x^1 + 2x^2$, $e_1 = (1,1), e_2 = e_3 = (0,0)$ (cf. Konovalov 1998). If $m_1 = m_2 = m_3 = 1$, then (x,p,q) with $x_1 = (0,0)$, $x_2 = (0,1)$, $x_3 = (1,0)$ the demand for each type $i \in I$ consumer, and $p = (1,1)$, $q = 1$, is a weak equilibrium. However, once the allocation is realized, consumers of type two and three wish to swap their allocations leading to $\xi_1 = (0,0)$, $\xi_2 = (1,0)$, $\xi_3 = (0,1)$ as their final demand.

to establish our main result.

3.1 Basic concepts

We set $L \equiv \{1, \ldots, L\}$ to denote the finite set of commodities. Let $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ be finite sets of types of identical consumers and producers respectively.

We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals represented by a set $T_k \subset I\!R$ of finite Lebesgue measure[3]. We set $\mathcal{I} = \cup_{i \in I} T_i$ and $\mathcal{J} = \cup_{j \in J} T_j$. Of course, $T_t \cap T_{t'} = \emptyset$ if type t and t' are different. Given $t \in \mathcal{I}$ $(\mathcal{J})$, let $i(t) \in I$ $(j(t) \in J)$ be the index such that $t \in T_{i(t)}$ $(t \in T_{j(t)})$.

Each firm of type $j \in J$ is characterized by a *finite* production set $Y_j \subset I\!R^L$ and the aggregate production set of the firms of type j is the convex hull of $\lambda(T_j)Y_j$, which is denoted by co $\lambda(T_j)Y_j$[4].

Every consumer of type $i \in I$ is characterized by a *finite* consumption set $X_i \subset R^L$, an initial endowment $e_i \in I\!R^L$ and a *preference correspondence* $P_i : X_i \to X_i$[5]. Let $e = \sum_{i \in I} \lambda(T_i) e_i$ be the aggregate initial endowment of the economy. For $(i, j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type i consumers in type j firms. For all $j \in J$, $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$.

The initial endowment of fiat money for an individual $t \in \mathcal{I}$ is defined by $m(t)$, where $m(\cdot)$ is a Lebesgue-measurable and bounded mapping from $\mathcal{I}$ to $I\!R_+$. Without loss of generality we may assume that $m(\cdot)$ is a continuous mapping.

In the rest of this work, we note by $L^1(A, B)$ the Lebesgue integrable functions from $A \subset I\!R$ to $B \subset I\!R^L$.

Given all foregoing, an *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = \Big((X_i, P_i, e_i, m)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J} \Big),$$

an *allocation* (or consumption plan) is an element of

$$X = \Big\{ x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x_t \in X_{i(t)} \; for \, a.e. \, t \in \mathcal{I} \Big\}$$

and a *production plan* is an element of

$$Y = \Big\{ y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y_t \in Y_{j(t)} \; for \, a.e. \, t \in \mathcal{J} \Big\}.$$

Finally, the *feasible consumption-production plans* are elements of

$$A(\mathcal{E}) = \left\{ (x, y) \in X \times Y \mid \int_{\mathcal{I}} x_t = \int_{\mathcal{J}} y_t + e \right\}.$$

3.2 Equilibria concepts

Given $p \in I\!R^L$, the *weak supply* of a firm of type $j \in J$ and their *aggregate profit* are, respectively,

$$S_j(p) = \text{argmax}_{y \in Y_j} \; p \cdot y \qquad\qquad \pi_j(p) = \lambda(T_j) \text{sup}_{y \in Y_j} \; p \cdot y.$$

[3]Without loss of generality we may assume that T_k is a compact interval of $I\!R$. In the following, we note by $\lambda(T_k)$ the Lebesgue measure of set $T_k \subseteq I\!R$.

[4]That is, $\text{co}\lambda(T_j)Y_j = \left\{ \sum_{r=0}^{n} \lambda_r y_r \mid y_r \in (T_j)Y_j, \; \lambda_r \geq 0, \sum_{r=0}^{n} \lambda_r = 1, \; n \in I\!N \right\}$.

[5]That is, $x' \in P_i(x)$ if x' is strictly preferred to x by a type $i \in I$ consumer.

Given $(p,q) \in I\!R^L \times I\!R_+$, we denote the *budget set* of a consumer $t \in \mathcal{I}$ by

$$B_t(p,q) = \Big\{x \in X_{i(t)} \,|\, p \cdot x \leq w_t(p,q)\Big\}$$

where $w_t(p,q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j}\pi_j(p)$ is the wealth of individual $t \in \mathcal{I}$. The set of *maximal elements* in the budgetary set for consumer $t \in \mathcal{I}$ is denoted by

$$d_t(p,q) = \Big\{x \in B_t(p,q) \,|\, B_t(p,q) \cap P_{i(t)}(x) = \emptyset\Big\}$$

and a collection $(x,y,p,q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ is a *Walras equilibrium* (with fiat money) of $\mathcal{E}$ if

(i) for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p,q)$;

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

It is well known that in our framework a Walras equilibrium (with fiat money) may fail to exist, mainly because, in general, the correspondence $d_t(\cdot)$ is non upper semi-continuous in presence of indivisible goods[6]. This leads us to define a regularized notion of demand, called *weak demand*, which for a consumer $t \in \mathcal{I}$ is defined by[7]

$$D_t(p,q) = \limsup_{(p',q') \to (p,q)} d_t(p',q').$$

Note that, by definition, the weak demand is an upper semi-continuous correspondence. In next section we will give an economical interpretation of it.

In the following, we note by $\mathcal{C}$ the set of closed convex cones $K \subset I\!R^L$ such that $-K \cap K = \{0_{I\!R^L}\}$[8]. Thus, given $(p,q,K) \in I\!R^L \times I\!R_+ \times \mathcal{C}$, we define the *demand* of a consumer $t \in \mathcal{I}$ by

$$\delta_t(p,q,K) = \Big\{x \in D_t(p,q) \,|\, P_{i(t)}(x) - x \subset K\Big\}$$

and the *supply* of a firm $t \in \mathcal{J}$ by

$$\sigma_t(p,K) = \Big\{y \in S_{j(t)}(p) \,|\, Y_{j(t)} - y \subset -K\Big\}.$$

Definition 3.1 *A collection $(x,y,p,q,K) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+ \times \mathcal{C}$ is a* **rationing equilibrium** *of $\mathcal{E}$ if*

(i) for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p,q,K)$;

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p,K)$.

[6]For example, given an individual whose preference correspondence (two goods) is defined by the utility function $u(x,y) = 2x + y$, his initial endowment is $e = (0,1)$, the consumption set is $X = \{0,1\}^2$, then, given $p^n = (1 + 1/n, 1) \to p = (1,1)$, $q^n = 0 = q$), we obtain that $d(p^n, q^n) = (0,1)$ and $d(p,q) = (1,0)$. Thus, $d(\cdot)$ is not upper semi-continuous at $p = (1,1)$.

[7]See Rockafellar and Wets (1998), Section, 5 for the *limsup* definition of a correspondence.

[8]Those cones are called *pointed cones*. See Rockafellar and Wets (1998) for more details.

Note that for $q > 0$ the *demand for money* of consumer $t \in \mathcal{I}$ is

$$\mu_t = \frac{1}{q}\left(p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j}\pi_j(p) - p \cdot x_t\right).$$

Walras law implies that the money market is in equilibrium at an equilibrium. A Walras equilibrium with fiat money is of course a rationing equilibrium and a rationing equilibrium is a weak equilibrium. We refer to Kajii (1996) for the links among Walras equilibrium, Walras equilibrium with fiat money and the dividend equilibrium notion.

4 Demand interpretation and a characterization

As we already know, the presence of indivisible goods may implies that in our model a consumer $t \in \mathcal{I}$ might be unable to obtain a maximal element within his budget set. Should he be unable to buy $\xi \in B_{i(t)}(p,q)$ with $p \cdot \xi < w_{i(t)}(p,q)$, then he could try to pay this bundle at a higher price than the market price in order to be "served first". Thus, there is some pressure on the price of the bundle ξ and its price would rise, if a non-negligible set of consumers is rationing in this sense. So at equilibrium, no consumer obtains a bundle of goods $x \in B_{i(t)}(p,q)$ such that a strictly preferred bundle ξ with $p \cdot \xi < w_{i(t)}(p,q)$ exists.

Previous fact could be explained, for instance, if the agents have more information than their own characteristics and the market price. To eliminate this "instability" it is however not necessary that the agents have a precise information on their trading partners. It is enough that they know which kind of net-trades are difficult to realize on the market (which is the "short" side of the market) when formulating their demand. This short side of the market could be modelled using a cone $K \subseteq \mathbb{R}^L$ which do not contain straight lines, i.e. if a direction of net-trade is difficult to realize, the opposite direction is easy to realize. One could think of the new demand as follows. Agents perceive the market price and the cone K (information) and then they compute their budget set. Given that, they try to find out for which type of allocations they could find a counterpart. So an allocation is not acceptable, if there exists a preferred one in the budget set which costs less than their total wealth. Moreover, they do not accept an allocation x, if a preferred allocation x' exists which is contained in the budget set and such that $x' - x \notin K$. In fact, it should not be difficult to find a counterpart for the net-exchange $x' - x$. Alternatively think that they first accept the allocation x, but then they make another net-exchange $x' - x$ leading to x' and so on, until they are at an allocation ξ such that $P_i(\xi) - \xi \subset K$. At this stage, obtaining a preferred allocation would require a net-exchange of a direction for which it is difficult to find a counterpart.

Finally, the following proposition give us an economic interpretation of weak demand. The most relevant case is when the value of fiat money is strictly positive. In such case, we will prove that for given prices $(p,q) \in \mathbb{R}^L \times \mathbb{R}_+$, the weak demand corresponds to those allocations that can be affordable by the consumer, such that the budgetary set and the convex hull of the strictly preferred points to these allocations can be strictly separated by an hyperplane and any other consumption bundle that is strictly preferred to them is costly. We recall that in absence of indivisible goods, this characterization coincide with the standard demand definition. The proof of the Proposition 4.1 is given in the Appendix.

Proposition 4.1 *Given $t \in \mathcal{I}$, we have that:*

(a) if $qm(t) > 0$ then

$$D_t(p,q) = \left\{ x \in B_t(p,q) |\ p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \mathrm{co}P_{i(t)}(x) \right\},$$

(b) if $m(t) > 0$ then

$$D_t(p,q) = \left\{ x \in B_t(p,q) \left| \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p,q), \\ \mathrm{co}P_{i(t)}(x) \cap \mathrm{co}\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) Y_j\} = \emptyset \end{array} \right. \right\},$$

(c) if $m(t) = 0$ then

$$D_t(p,q) = \left\{ x \in B_t(p,q) | p \cdot P_{i(t)}(x) \geq w_t(p,q),\ \mathrm{co}P_{i(t)}(x) \cap C(p,x) = \emptyset \right\}$$

where

$$C(p,x) = \mathrm{co} \left\{ \theta x + (1-\theta) \left[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right] | \theta \geq 0 \right\}.$$

To end this section, we point out that the condition $x \notin \mathrm{co}P_{i(t)}(x)$ in Proposition 4.1 (i) is redundant if one considers the demand as defined for the rationing equilibrium.

5 Existence of equilibrium

The strongest condition we use to ensure existence of equilibrium is the finiteness of the consumption and production sets. The rest of our assumptions are quite weak. In particular, we do not need a strong survival assumption, that is, our consumers may not own initially a strictly positive quantity of every good and the interior of the convex hull of the consumption sets may be empty (cf. Arrow and Debreu (1954)).

Assumption C. For all $i \in I$, P_i is irreflexive and transitive.

Assumption S. (Weak survival assumption). For all $i \in I$,

$$0 \in \mathrm{co}X_i - \{e_i\} - \sum_{j \in J} \theta_{ij} \lambda(T_j) \mathrm{co}Y_j.$$

Following lemma will be very relevant to demonstrate our main theorem. To present this result, we must introduce an auxiliary concept, called weak equilibrium. Thus, we say that a collection $(x, y, p, q) \in A(\mathcal{E}) \times I\!R^L \times I\!R_+$ is a *weak equilibrium* of $\mathcal{E}$ if

(i) for a.e. $t \in \mathcal{I}$, $x_t \in D_t(p,q)$;

(ii) for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

Lemma 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S, there exists a weak equilibrium with price of fiat money strictly positive.*

So now we are in conditions to enunciate our main result.

Theorem 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S and $m(t) > 0$ for all $t \in \mathcal{I}$, there exists a rationing equilibrium with price of fiat money strictly positive.*

As a consequence of Theorem 5.1 we can deduce following corollary, which establish that under the same hypotheses made on to prove Theorem 5.1, it is possible to conclude the existence of a Walras equilibrium on $\mathcal{E}$, generically on the distribution of fiat money.

Corollary 5.1 *For every economy $\mathcal{E}$ satisfying Assumptions C, S, $m(t) > 0$ for all $t \in \mathcal{I}$ and for all $M > 0$, $\lambda(\{t \in \mathcal{I} \mid m(t) = M\}) = 0$ there exists a Walras equilibrium with price of fiat money strictly positive.*

6 Appendix

6.1 Proof of Proposition 4.1.

Part (a). Given $t \in \mathcal{I}$, let

$$a(p,q) = \left\{x \in B_t(p,q) \mid p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \mathrm{co}P_{i(t)}(x)\right\}.$$

First of all, note that by definition $D_t(p,q) \subset a(p,q)$. Let $x \in a(p,q)$. If $p \cdot x < w_t(p,q)$, then for all small enough $\varepsilon > 0$, $x \in d_t(p, q-\varepsilon)$ and hence $x \in D_t(p,q)$. Otherwise, note that there exists p' such that $p' \cdot P_{i(t)}(x) > p' \cdot x$. For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$ and let

$$q^\varepsilon = \left[\frac{p^\varepsilon \cdot (x - e_{i(t)}) - \sum_{j \in J} \theta_{i(t)j} \pi_j(p^\varepsilon)}{m(t)}\right].$$

Note that $\lim_{\varepsilon \to 0}(p^\varepsilon, q^\varepsilon) = (p,q)$. Moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > p^\varepsilon \cdot x = w_t(p^\varepsilon, q^\varepsilon).$$

Since for $\varepsilon > 0$ small enough, $q^\varepsilon > 0$, we have $x \in D_t(p,q)$. Thus $a(p,q) \subset D_t(p,q)$.

Part (b). Let

$$A(p,q) = \left\{x \in B_t(p,q) \,\middle|\, \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p,q), \\ \mathrm{co}P_{i(t)}(x) \cap \mathrm{co}\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) Y_j\} = \emptyset \end{array}\right\}.$$

Step b.1. $A(p,q) \subset D_t(p,q)$.

Let $x \in A(p,q)$. Thus, there exists p' such that

$$p' \cdot P_{i(t)}(x) > p' \cdot \left\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) Y_j\right\}.$$

For all $\varepsilon > 0$, let $p^\varepsilon = p + \varepsilon p'$. Thus, for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > p^\varepsilon \cdot \left\{x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) Y_j\right\},$$

$$p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q).$$

Let[9]

$$q^\varepsilon = q + \left[\frac{p^\varepsilon \cdot x - w_t(p^\varepsilon, q)}{m(t)}\right]_+.$$

[9] For $x \in I\!R$, we note $[x]_+ = \max\{x, 0\}$.

Note that $\lim_{\varepsilon \to 0}(p^\varepsilon, q^\varepsilon) = (p,q)$ and moreover for all $\varepsilon > 0$,

$$p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q^\varepsilon) \geq p^\varepsilon \cdot x$$

and therefore $x \in D_t(p,q)$.

Step b.2. $D_t(p,q) \subset A(p,q)$.

For all $x \in D_t(p,q)$, there exists sequences (p^n, q^n) converging to (p,q), such that for all $n \in I\!N$

$$p^n \cdot P_{i(t)}(x) > w_t(p^n, q^n) \geq p^n \cdot x.$$

Thus $p \cdot P_{i(t)}(x) \geq w_t(p,q)$ and

$$\mathrm{co}P_{i(t)}(x) \cap \mathrm{co}\left\{ x, e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) Y_j \right\} = \emptyset$$

which ends the proof of part (b).

Part (c).

Let

$$c(p) = \left\{ x \in B_t(p,q) \;\middle|\; \begin{array}{c} p \cdot P_{i(t)}(x) \geq w_t(p,q), \\ \mathrm{co}P_{i(t)}(x) \cap C(p,x) = \emptyset \end{array} \right\}.$$

Step c.1. $c(p) \subset D_t(p,q)$.

Given $x \in c(p)$ there exists p' such that

$$p' \cdot \mathrm{co}P_{i(t)}(x) > p' \cdot \left(e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right) \geq p' \cdot x.$$

Thus, for all $\varepsilon > 0$, given $p^\varepsilon = p + \varepsilon p'$ it follows that

$$\min p^\varepsilon \cdot P_{i(t)}(x) > \max p^\varepsilon \cdot \left(e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right),$$

$$\min p^\varepsilon \cdot \left(e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \mathrm{argmax}\, \pi_j(p) \right) \geq p^\varepsilon \cdot x.$$

Moreover, since Y_j is finite for all $j \in J$, we may check that for all $\varepsilon > 0$ small enough and all $j \in J$,

$$\mathrm{argmax}\, \pi_j(p^\varepsilon) \subset \mathrm{argmax}\, \pi_j(p)$$

and therefore for all small $\varepsilon > 0$,

$$\min p^\varepsilon \cdot P_{i(t)}(x) > w_t(p^\varepsilon, q) \geq p^\varepsilon \cdot x,$$

which implies that $x \in D_t(p,q)$.

Step c.2. $D_t(p,q) \subset c(p)$.

Let $x \in D_t(p,q)$. Then there exists a sequence p^n converging to p such that for all $n \in I\!N$,

$$p^n \cdot P_{i(t)}(x) > w_t(p^n, q) \geq p^n \cdot x.$$

Thus $p \cdot P_{i(t)}(x) \geq w_t(p,q)$ and p^n separates strictly co$P_{i(t)}(x)$ and

$$\text{co}\left\{\theta x + (1-\theta)[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j}\lambda(T_j)Y_j] \,|\, \theta \geq 0\right\}.$$

Since

$$C(p,x) \subset \text{co}\left\{\theta x + (1-\theta)[e_{i(t)} + \sum_{j \in J} \theta_{i(t)j}\lambda(T_j)Y_j] \,|\, \theta \geq 0\right\}$$

we can conclude that $x \in c(p)$. □

6.2 Proof of Lemma 5.1

In order to demonstrate Lemma 5.1 we use the following proposition, which is an extension of the well know Debreu-Gale-Nikaido lemma.

Proposition 6.1 *Let* $\varepsilon \in]0,1]$ *and* φ *be an upper semi continuous correspondence from* $\mathbb{B}(0,\varepsilon)$ *to* $\mathbb{R}^L$ *with nonempty, convex, compact values*[10]*. If for some* $k > 0$*,*

$$\forall p' \in \mathbb{B}(0,\varepsilon), \quad \|p'\| = \varepsilon \quad \Longrightarrow \quad \sup p' \cdot \varphi(p') \leq k(1-\varepsilon),$$

then there exists $p \in \mathbb{B}(0,\varepsilon)$ *such that, either:*

- $0 \in \varphi(p)$

or

- $\|p\| = \varepsilon$ *and* $\exists\ \xi \in \varphi(p)$ *such that* ξ *and* p *are colinear and* $\|\xi\| \leq k\frac{1-\varepsilon}{\varepsilon}$*.*

Proof of Proposition 6.1.

¿From the properties of φ, one can select a convex compact subset $K \subset \mathbb{R}^L$ such that for all $p \in \mathbb{B}(0,\varepsilon)$, $\varphi(p) \subset K$. Consider the correspondence $F : \mathbb{B}(0,\varepsilon) \times K \to \mathbb{B}(0,\varepsilon) \times K$ defined by

$$F(p,z) = \{q \in \mathbb{B}(0,\varepsilon) \mid \forall q' \in \mathbb{B}(0,\varepsilon),\ q \cdot z \geq q' \cdot z\} \times \varphi(p).$$

From Kakutani Theorem, F has a fixed point (p,ξ). If $\|p\| < \varepsilon$, then $\xi = 0$. If $\|p\| = \varepsilon$, then from the definition of F, p and ξ are colinear. Therefore, $\|\xi\| \leq k\ \frac{1-\varepsilon}{\varepsilon}$, which ends the demonstration. □

Proof of Lemma 5.1

Previous to proceed, is necessary to introduce some notations. We note by $\leq_{lex}$ the lexicographic order[11]. Given $p_0, ..., p_k \in \mathbb{R}^L$, for a $(k+1) \times L$ matrix $\mathcal{P} = [p_0, \ldots, p_k]'$ (transpose of matrix $[p_0, \ldots, p_k]$), we note for every $j \in J$,

$$S_j(\mathcal{P}) = \{y \in Y_j \mid \forall z \in Y_j,\ \mathcal{P}z \leq_{lex} \mathcal{P}y\} \qquad \pi_j(\mathcal{P}) = \lambda(T_j)\text{sup}_{lex}\{\mathcal{P}y \mid y \in Y_j\}$$

[10] $\mathbb{B}(0,\varepsilon) = \{x \in \mathbb{R}^L \mid \|x\| \leq \varepsilon\}$. The norm used here is Euclidean norm.

[11] For $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $x \leq_{lex} y$, if $x_r > y_r$, $r \in \{1,\ldots,n\}$ implies that $\exists \rho \in \{1,\ldots,r-1\}$ such that $x_\rho < y_\rho$. We write $x <_{lex} y$ if $x \leq_{lex} y$, but not $[y \leq_{lex} x]$. In an obvious manner we define $x \geq_{lex} y$ and $x >_{lex} y$.

where $\sup_{lex}$ is the supremum with respect to the lexicographic order. Given $\mathcal{Q} = (q_r) \in \mathbb{R}^{k+1}$, for every $t \in \mathcal{I}$ let

$$B_t(\mathcal{P}, \mathcal{Q}) = \left\{ x \in X_{i(t)} \mid \mathcal{P} \cdot (x - e_{i(t)}) \leq_{lex} m(t)\mathcal{Q} + \sum_{j \in J} \theta_{i(t)j} \pi_j(\mathcal{P}) \right\}$$

and finally, for $\varepsilon > 0$, we note $\mathcal{P}(\varepsilon) = \sum_{r=0}^{k} \varepsilon^r p_r$ and $\mathcal{Q}(\varepsilon) = \sum_{r=0}^{k} \varepsilon^r q_r$. So now we are in conditions to demonstrate the result. To do so, we proceed in nine steps.

Step 1. *Perturbed equilibria.*

For simplicity, for all $t \in \mathcal{I}$ we note $D_t(p)$ instead of $D_t(p, 1 - \|p\|)$. Given that, it is easy to check that for all $\varepsilon \in [0, 1]$, all $t \in \mathcal{I}$, and all $j \in J$ the set-valued mappings

$$D_t : \mathbb{B}(0, \varepsilon) \to \text{co}X_{i(t)} \qquad \text{co}S_j : \mathbb{B}(0, \varepsilon) \to \text{co}Y_j$$

are upper semi-continuous, nonempty and compact valued.

Now, define the excess demand mapping

$$\varphi : \mathbb{B}\,(0, 1 - 1/n) \to \sum_{i \in I} \lambda(T_i)(\text{co}X_i - e_i) - \sum_{j \in J} \lambda(T_j)\text{co}Y_j$$

by

$$\varphi(p) = \int_{t \in \mathcal{I}} (D_t(p) - e_{i(t)}) - \sum_{j \in J} \lambda(T_j)\text{co}S_j(p).$$

Obviously $\varphi(\cdot)$ is nonempty, convex, compact valued and upper semi-continuous. For each $n \in \mathbb{N}$ and each $p \in \mathbb{B}(0, 1 - 1/n)$ we have that

$$p \cdot \varphi(p) \leq (1 - \|p\|) \int_{\mathcal{I}} m(t).$$

So we may apply Proposition 6.1 to conclude that for all $n > 1$ there exists

$$(x^n, y^n, p^n, q^n) \in \prod_{i \in I} L^1(T_i, X_i) \times \prod_{j \in J} L^1(T_j, S_j(p^n)) \times \mathbb{B}\,(0, 1 - 1/n) \times \mathbb{R}_{++}$$

such that for all $t \in \mathcal{I}$, $x_t^n \in D_t(p^n, q^n)$, $q^n = 1 - \|p^n\|$, $\int_{t \in \mathcal{I}} x_t^n + \int_{t \in \mathcal{J}} y_t^n - e \in \varphi(p^n)$ and

$$\left\| \int_{t \in \mathcal{I}} x_t^n + \int_{t \in \mathcal{J}} y_t^n - e \right\| \leq \frac{1}{n-1} \int_{\mathcal{I}} m(t).$$

Step 2. *Construction of $\mathcal{P}$ and $\mathcal{Q}$.*

For the construction of a hierarchic price we will proceed as in Florig (2002). For that, our objective is to define a set of vectors $\{\psi_0, \psi_1, ..., \psi_L\} \subseteq \mathbb{R}^{L+1}$ which help us to define both $\mathcal{P}$ and $\mathcal{Q}$ as required. To do so, set $\psi^n = (p^n, q^n)$ and taking a subsequence, we may assume that ψ^n converges to $(p_0, q_0) \in \mathbb{R}^{L+1}$. Let ψ_0, ψ_0^n and $\mathcal{H}^0$ defined as follows:

$$\psi_0 = (p_0, q_0),$$
$$\psi_0^n = \psi^n,$$
$$\mathcal{H}^0 = \psi_0^{\perp} = \{x \in R^{L+1} \mid \psi_0 \cdot x = 0\}.$$

Using a recursive procedure, for every $r \in \{1, 2, \ldots, L-1\}$ we define ψ_r, ψ_r^n and $\mathcal{H}^r$ as follows:

$$\psi_r^n = \mathrm{proj}_{\mathcal{H}^{r-1}}(\psi_{r-1}^n),$$

and given that, if for all large enough $n \in I\!N$, $\psi_r^n \neq 0$, then let $\psi_r \equiv (p_r, q_r)$ be the limit of $\psi_r^n / \parallel \psi_r^n \parallel$ for some subsequence. In such case,

$$\mathcal{H}^r = \psi_r^{\perp}$$

and

$$\psi_{r+1}^n = \mathrm{proj}_{\mathcal{H}^r}(\psi_r^n).$$

We continue with previous algorithm until for all large enough $n \in I\!N$, $\psi_r^n = 0$ for some subsequence. In such case, we set $\psi_r = \ldots = \psi_L = 0$ and define

$$k = \min\{r \in \{0, \ldots, L\} \mid \psi_{r+1} = \ldots = \psi_L = 0\}.$$

Given all foregoing, we had obtained a set $\{\psi_r = (p_r, q_r),\ r = 1, \ldots, k\}$ of orthonormal vectors. Note that for all $r \in \{0, \ldots, k\}$[12],

$$\|\psi_{r+1}^n\| = \|\psi_r^n\|\, o(\|\psi_r^n\|)$$

which allow us to decompose the sequence ψ^n in the following way

$$\psi^n = \sum_{r=0}^{k} (\parallel \psi_r^n \parallel - \parallel \psi_{r+1}^n \parallel)\psi_r = \sum_{r=0}^{k} \varepsilon_r^n \psi_r,$$

with $\varepsilon_r^n = \parallel \psi_r^n \parallel - \parallel \psi_{r+1}^n \parallel$ for $r \in \{0, \ldots, k\}$. Thus, $\varepsilon_{r+1}^n = \varepsilon_r^n o(\varepsilon_r^n)$ for $r \in \{0, \ldots, k-1\}$, and ε_0^n converges to 1.

Let $\mathcal{P} = [p_0, \ldots, p_k]'$ (transpose of matrix $[p_0, \ldots, p_k]$), and $\mathcal{Q} = (q_0, q_1, ..., q_k) \in I\!R^{k+1}$.

Step 3. *Equilibrium allocation candidate.*

There exists by Fatou's lemma (Arstein (1979)) $(x^*, y^*) \in A(\mathcal{E})$ such that for a.e. $t \in \mathcal{I}$ and a.e. $t' \in \mathcal{J}$[13]

$$x_t^* \in \mathrm{cl}\{x_t^n\}, \quad y_{t'}^* \in \mathrm{cl}\{y_{t'}^n\}.$$

Step 4. *For all $\varepsilon > 0$ small enough and all n large enough, for a.e. $t \in \mathcal{J}$,*

$$y_t^* \in S_{j(t)}(\mathcal{P}(\varepsilon)) = S_{j(t)}(p^n) = S_{j(t)}(\mathcal{P}).$$

Since for all $j \in J$, Y_j is finite, for all $\varepsilon > 0$ small enough and for all $j \in J$ we have that $S_j(\mathcal{P}(\varepsilon)) = S_j(\mathcal{P})$ and similarly, for $n \in I\!N$ large enough, for all $j \in J$, $S_j(p^n) = S_j(\mathcal{P})$. Since for a.e. $t \in \mathcal{J}$, $y_t^n \in S_{j(t)}(p^n)$ for all $n \in I\!N$, and since $y_t^* \in \mathrm{cl}\{y_t^n\}$, y_t^n is constant and equal to y_t^* for a subsequence. Thus, $y_t^* \in S_{j(t)}(\mathcal{P})$.

Let ρ be the smallest $r \in \{0, \ldots, k\}$ such that $q_r \neq 0$. Since for all $n \in I\!N$, $q^n > 0$, $q_\rho > 0$. Let $\widetilde{\mathcal{P}} = [p_0, \ldots, p_\rho]'$ and $\widetilde{\mathcal{Q}} = (q_0, \ldots, q_\rho)$. For all $j \in J$, let $\bar{y}_j = y_t^*$, provided that $y_t^* \in S_j(\mathcal{P})$. Since that $S_j(\mathcal{P}) \subset S_j(\widetilde{\mathcal{P}})$, $\bar{y}_j \in S_j(\widetilde{\mathcal{P}})$.

[12] Throughout the paper we denote by $o : R \to R$ a function which is continuous in 0 with $o(0) = 0$.
[13] In the following, he closure of set A is denoted by $\mathrm{cl}A$.

Step 5. *For a.e.* $t \in \mathcal{I}$, $x_t^* \in B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$.

By the previous step, one may check that $B_t(p^n, q^n)$ converges in the sense of Kuratowski - Painlevé to $B_t(\mathcal{P}, \mathcal{Q})$[14]. Thus $x_t^* \in B_t(\mathcal{P}, \mathcal{Q}) \subset B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$.

Step 6. *For all* $\varepsilon > 0$ *small enough, for a.e.* $t \in \mathcal{I}$, $x_t^* \in B_t(\widetilde{\mathcal{P}}(\varepsilon), \widetilde{Q}(\varepsilon))$.

For a.e. $t \in \mathcal{I}$, we have, by the previous step, that $x_t^* \in B_t(\widetilde{\mathcal{P}}(\varepsilon), \widetilde{\mathcal{Q}}(\varepsilon))$ for all small enough $\varepsilon > 0$. Since $m : \mathcal{I} \to I\!R_+$ is bounded and since there are only finitely many values for x_t^*, there exists $\varepsilon > 0$ satisfying this property for a.e. $t \in \mathcal{I}$.

Step 7. *For a.e.* $t \in \mathcal{I}$, $x_t \in P_{i(t)}(x_t^*)$ *implies that*

$$\widetilde{\mathcal{P}} \cdot (x_t - e_{i(t)}) - m(t)\widetilde{\mathcal{Q}} - \sum_{j \in J} \theta_{i(t)j} \pi_j(\widetilde{\mathcal{P}}) \geq_{lex} 0.$$

Otherwise

$$\widetilde{\mathcal{P}} \cdot (x_t - e_{i(t)}) - m(t)\widetilde{\mathcal{Q}} - \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \widetilde{\mathcal{P}} \bar{y}_j <_{lex} 0$$

and then for all large enough $n \in I\!N$,

$$p^n \cdot (x_t - e_{i(t)}) - q^n m(t) - \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) p^n \cdot \bar{y}_j < 0.$$

By Proposition 4.1, this contradicts $x_t^* \in D_t(p^n, q^n)$ for a subsequence of (p^n, q^n).

Step 8. *For all* $\varepsilon > 0$ *small enough, for a.e* $t \in \mathcal{I}$, $x_t \in P_{i(t)}(x_t^*)$ *implies that*

$$\widetilde{\mathcal{P}}(\varepsilon) \cdot (x_t - e_{i(t)}) - \widetilde{\mathcal{Q}}(\varepsilon) m(t) - \sum_{j \in J} \theta_{i(t)j} \pi_j(\widetilde{\mathcal{P}}(\varepsilon)) \geq 0.$$

Since X_i is finite, there exists a finite partition $\{\widetilde{T}_1, \dots, \widetilde{T}_f\}$ of $\mathcal{I}$ such that the sets $B_t(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$ are constant on each of the elements of the partition. We may choose the partition such that for every $s \in \{1, \dots, f\}$, there exists $i \in I$ such that $\widetilde{T}_s \subset T_i$ and x_t^* is constant on $\widetilde{T}_s$. Let $m^s = \text{essup}\{m(t) \mid t \in \widetilde{T}_s\}$ (essential supremum) and suppose for all $\bar{\varepsilon} > 0$, there exists $\varepsilon \in]0, \bar{\varepsilon}]$ such that

$$\widetilde{\mathcal{P}}(\varepsilon) \cdot (x_t - e_{i(t)}) - m^s \widetilde{\mathcal{Q}}(\varepsilon) - \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) \widetilde{\mathcal{P}}(\varepsilon) \cdot \bar{y}_j < 0.$$

Thus there exists $\eta \in]0, m^s]$ such that for all large $n \in I\!N$,

$$p^n \cdot (x_t - e_{i(t)}) - q^n (m^s - \eta) - \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) p^n \cdot \bar{y}_j < 0.$$

Hence, for all large $n \in I\!N$ there exists $\bar{T}_s \subset \widetilde{T}_s$ with $\lambda(\bar{T}_s) > 0$ such that for a.e. $t \in \bar{T}_s$

$$p^n \cdot (x_t - e_{i(t)}) - q^n m(t) - \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) p^n \cdot \bar{y}_j < 0.$$

By Proposition 4.1, this contradicts $x_t^* \in D_t(p^n, q^n)$ for a subsequence of (p^n, q^n).

Step 9. *For all* $\varepsilon > 0$ *small enough, for a.e.* $t \in \mathcal{I}$, $x_t^* \in D_t((\mathcal{P}(\varepsilon), (\mathcal{Q}(\varepsilon))$.

Let $\bar{\varepsilon} > 0$ small enough satisfying the previous steps. Let $(p^*, q^*) = \sum_{r=0}^{\rho} \bar{\varepsilon}^r (p_r, q_r)$.

By Proposition 4.1, $x_t^* \notin \text{co} P_{i(t)}(x_t^*)$. Then, since $q^* > 0$ and for a.e. $t \in \mathcal{I}$, $m(t) > 0$, we can deduce by Proposition 4.1 that $x_t^* \in D_t((\mathcal{P}(\varepsilon), (\mathcal{Q}(\varepsilon))$.

Thus, (x^*, y^*, p^*, q^*) is a weak equilibrium and $q^* > 0$. □

[14] This concept is widely used to define set - convergence. See Rockafellar and Wets (1998), Section 4, for more details.

6.3 Proof of Theorem 5.1

Let $m^1 : \mathcal{I} \to I\!R_{++}$ be a mapping strictly increasing and bounded and let (x^0, y^0, p^0, q^0) be a weak equilibrium of $\mathcal{E}$. Let $\mathcal{E}^1$ be an economy defined as follows. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \dots, A\}$ satisfying the following:

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t^0 = x_a$.

Set $X_a^1 = (P_a(x_a) \cup x_a) \cap (x_a + (p^0)^\perp)$ and $e_a^1 = x_a$, with P_a^1 the restriction of P_a to X_a^1.

Since there is also a finite number of types of producers and production sets are finite, we can define a finite set of producer types $B \equiv \{1, \dots, B\}$ satisfying the following:

(i) $(T_b)_{b \in B}$ is a finer partition of $\mathcal{J}$ than $(T_j)_{j \in J}$,

(ii) for every $b \in B$, there exists y_b such that for every $t \in T_b$, $y_t^0 = y_b$.

Given $Y_b^1 = ((Y_b - y_b) \cap (p^0)^\perp)$, define the economy by $\mathcal{E}^1$ as

$$\mathcal{E}^1 = \left((X_a^1, P_a^1, e_a^1, m^1)_{a \in A}, (Y_b^1)_{b \in B}, (\theta_{ab})_{(a,b) \in A \times B} \right),$$

where m^1 defines the initial endowments of fiat money. The economy $\mathcal{E}^1$ satisfies Assumptions C, S. So by the Lemma 5.1 there exists a weak equilibrium with $q^1 > 0$ and therefore a Walras equilibrium (with fiat money) for the economy $\mathcal{E}^1$, which is denoted by (x^1, y^1, p^1, q^1), with $q^1 > 0$. Set $\mathcal{P} = [p^0, p^1]'$.

Claim 6.1 *For a.e. $t \in \mathcal{I}$, $\mathcal{P}x_t^1 \leq_{lex} w_t$ with $w_t = (w_t^0, w_t^1) \in I\!R^2$ such that*

$$w_t^0 = p^0 \cdot e_{i(t)} + q^0 m(t) + \sum_{j \in J} \theta_{i(t)j} \lambda(T_j) p^0 \cdot y_j^0$$

$$w_t^1 = p^1 \cdot e_{i(t)}^1 + q^1 m^1(t) + \sum_{b \in B} \theta_{i(t)b} \lambda(T_b) p^1 \cdot y_b^1.$$

Note that by the construction of X_t^1, we have for a.e. $t \in \mathcal{I}$, $p^0 \cdot x_t^0 = p^0 \cdot x_t^1$. Since for every $r \in \{0, 1\}$, $p^r \cdot x_t^r \leq w_t^r$ we have for a.e. $t \in \mathcal{I}$, $\mathcal{P}x_t^1 \leq_{lex} w_t$. By other hand, note that for all $t \in \mathcal{J}$, $y_t = y_t^0 + y_t^1 \in S_{j(t)}(\mathcal{P})$.

Claim 6.2 *For a.e. $t \in \mathcal{I}$, $\xi_t \in P_{i(t)}(x_t^1)$ implies $\mathcal{P}x_t^1 <_{lex} \mathcal{P}\xi_t$.*

By transitivity of the preferences, $\xi_t \in P_{i(t)}(x_t^1)$ implies that $\xi_t \in P_{i(t)}(x_t^0)$. Thus, $p^0 \cdot x_t^1 = p^0 \cdot x_t^0 \leq p^1 \cdot \xi_t$. Since (x^1, y^1, p^1, q^1) is a Walras equilibrium of $\mathcal{E}^1$, $p^1 \cdot x_t^1 < p^0 \cdot \xi_t$ for a.e. $t \in \mathcal{I}$.

Set $(\bar{x}, \bar{y}, \bar{p}, \bar{q}) = (x^1, y, p^1, q^1)$, with y_t as in Claim 6.1. Let $K' = \{x \in I\!R^L \mid (0,0) <_{lex} \mathcal{P}x\} \cup \{0\}$. Clearly this is a convex and pointed cone (that is, $-K' \cap K' = \{0\}$). Since for all $t \in \mathcal{J}$, $y_t \in S_{j(t)}(\mathcal{P})$, we have for all $t \in \mathcal{J}$, $Y_{j(t)} - y_t \subset -K'$. For all $t \in \mathcal{J}$, let K_t be the positive hull of $K' \cap (y_t - Y_{j(t)})$. Note that for all $t \in \mathcal{I}$, if $x_t \in P_{i(t)}(\bar{x}_t)$, then $(0,0) <_{lex} \mathcal{P}(x_t - \bar{x}_t)$. For all $t \in \mathcal{I}$, let K_t be the positive hull of $K' \cap (P_{i(t)}(\bar{x}_t) - \bar{x}_t)$. Let $K = \text{cl}\{\text{co} \cup_{t \in \mathcal{I} \cup \mathcal{J}} K_t\}$. Of course K is a convex cone and by the finiteness of the consumption and production sets $K \subset K'$. Thus, $-K \cap K = \{0\}$. For all $t \in \mathcal{I}$, $P_{i(t)}(\bar{x}_t) - \bar{x}_t \subset K$, for all $t \in \mathcal{J}$, $Y_{j(t)} - \bar{y}_t \subset -K$, which ends the proof. □

6.4 Proof of Corollary 5.1

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