On the Failure of Core Convergence in Economies with Asymmetric Information*

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Abstract

In interim economies with asymmetric information, we show that the coarse core of Wilson (1978) does not converge to price equilibrium allocations as the economy is replicated. This failure of core convergence is a basic consequence of asymmetric information and extends to any reasonable notion of either (interim) core or price equilibrium.

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1 Introduction

Among the more enduring results in game theory and economics are those which draw upon the close relationship between Walrasian equilibria and core-like cooperative concepts in large economies. This connection dates back to Edgeworth's (1881) conjecture on the convergence of the core to the set of Walrasian allocations, which was established by Debreu and Scarf (1963) in the context of replica economies. Modeling the economy as an atomless measure space of consumers, Aumann (1964) established an equivalence between core and competitive allocations.[1] While there do exist exceptions to the convergence or equivalence principle[2] the broad scope of the convergence result is indeed remarkable.

The aim of the present paper is to study core convergence in economies with asymmetric information. However, as there are various natural core concepts as well as price equilibrium concepts in the general context of asymmetric information, one needs to be somewhat more precise about the solution concepts under consideration.

A basic distinction which needs to be made in the context of asymmetric information concerns the stage at which decisions are made, namely ex-ante or interim.[3]

Consider first the case in which decisions are made at the ex-ante stage, i.e., before information is revealed to any agent. A well-known price equilibrium concept for such a model is the notion of an Arrow-Debreu equilibrium in (complete) markets with contingent commodities. The fact that in such markets an Arrow-Debreu equilibrium is identical to a Walrasian equilibrium with an appropriate indexing of commodities by time and state immediately implies the standard relationship between such allocations and a corresponding notion of the ex-ante core. (Coalitions in this core notion use ex-ante utility computations to judge whether or not a feasible allocation is an objection to a status-quo allocation of contingent commodities). The standard argument for complete information economies implies not only

[1] We refer the reader to Anderson (1992) for a survey of results in this area. See also the section on the Equivalence Principle in Aumann (1987).

[2] See, for example, Anderson and Zame (1997), Anderson, Trockel and Zhou (1997), Hart (1974), Hart and Mas-Colell (1996), Manelli (1991).

[3] The analysis of ex-post notions of the core or Walrasian equilibria is no different from that in the case of complete information.

that Arrow-Debreu allocations are ex-ante efficient but also that they belong to the ex-ante core. Furthermore, the Debreu-Scarf argument can be applied (with no more than a re-interpretation) to assert that any ex-ante core allocation which survives replication is an Arrow-Debreu allocation.

While the Arrow-Debreu model involves incomplete information, it is essentially one of symmetric uncertainty.[4] Asymmetry of information can be incorporated into this ex-ante framework by postulating that consumers differ in their ex-post information. One such approach is the one introduced by Radner (1968), which imposes the requirement that an agent's trades be measurable with respect to her private information. Equilibrium allocations so defined (Radner allocations) bear the standard relationship with an ex-ante core concept which similarly imposes such measurability restrictions (as in Allen (1991) and Yannelis (1991)); see Einy, Moreno and Shitovitz (1998). Another approach for dealing with ex-post asymmetry of information (based on mechanism design) is to directly impose incentive compatibility on agents' trades. A corresponding price equilibrium notion is the one used in Prescott and Townsend (1984). Here, even in the ex-ante case, matters are no longer so simple. As Forges, Heifetz and Minelli (1999) show, core equivalence does not generally hold, although a positive result can be established under certain conditions.

In light of the above discussion, we shall concentrate on the interim stage, i.e., the stage when agents have received their private information. In this context, too, the existing literature (Goenka and Shell (1997), Kobayashi (1980) and Yannelis (1991)) seems to point to the validity of the convergence principle. However, we will begin by showing that the coarse core of Wilson (1978) does not converge to *any* set of price equilibrium allocations considered in the literature. To prove our main point we construct a simple example of a replicated sunspot economy with strictly convex and monotonic preferences.[5] Thus, even when uncertainty does not affect the fundamentals of the economy, core convergence fails. We also show that the coarse core may not satisfy the equal treatment property (even with strictly

[4]See Chapter 19 of Mas-Colell, Whinston and Green (1995) for an excellent presentation of this material.

[5]The example we construct for this purpose also shows that core equivalence need not hold in an economy with an atomless measure space of consumers. This refutes the conjecture on core equivalence in Kobayashi (1980), page 1647. We also refute (in Section 4) the conjecture in Yannelis (1991, Remark 6.5).

convex, monotonic preferences). Moreover, coarse core allocations satisfying the equal treatment property may not converge to market allocations. The underlying reason for our negative results can be traced to an important implication of cooperation in the presence of asymmetric information. Suppose there are two states of the world, s and t, and two agents: one is informed and the other is uninformed. A coalitional improvement will typically require (for the usual reasons related to adverse selection) that the informed consumer be made better-off in *both* states of the world. This translates into a restriction on allowable coalitions, which, as we shall see, can be enough for a failure of the standard Debreu-Scarf argument.

Are these negative results driven by the fact that, despite replication, agents in our model do not become informationally small?[6] This is not so, at least in terms of the notion of informational smallness recently formalized in McLean and Postlewaite (1999). Whether there is a different sense in which agents in our model are not informationally small is, we believe, an important question that needs to be examined more carefully in future research.

It is also natural to ask whether convergence may obtain if one considers other interim core notions. In Section 4 we establish that this is not the case – our non-convergence result is robust to many reasonable modifications of either the (interim) core or the (interim) price equilibrium concept. We conclude that the convergence/equivalence principle does not hold in the presence of informational asymmetries (at the interim stage).

2 An Interim Economy with Asymmetric Information

Consider an exchange economy with a finite set of consumers, N, and a finite set of states of the world, Ω. There are a finite number of commodities, and the consumption set of each consumer is $\mathbb{R}^l_+$ in each state. A consumption plan of consumer i is a function $x_i : \Omega \mapsto \mathbb{R}^l_+$. Let X_i denote the set of all consumption plans for consumer i. For $A \subseteq \Omega$, $X_i(A)$ denotes the set of all $x_i(A) \equiv (x_i(\omega))_{\omega \in A})$ where $x_i(\omega) \in \mathbb{R}^l_+$ for all $\omega \in A$. The endowment of i is denoted $e_i \in X_i$. Consumer i has a Bernoulli utility function $u_i : \mathbb{R}^l_+ \times \Omega \mapsto \mathbb{R}$; for a consumption plan x_i, $u_i(x_i(\omega), \omega)$ denotes the utility of i in state

[6] They do, of course, become small in the traditional sense of "endowment smallness".

ω. We shall assume that for all $i \in N$ and all $\omega \in \Omega$, $u_i(.,\omega)$ is continuous, monotonic and concave.

The private information of consumer i is represented by $\mathcal{P}_i$, a partition of Ω. For a state $\omega \in \Omega$, let $\mathcal{P}_i(\omega)$ be the element of $\mathcal{P}_i$ which contains ω. Thus, when the state is ω, consumer i knows that the true state lies in $\mathcal{P}_i(\omega)$.[7] Each consumer i is assumed to have a probability measure μ_i on Ω which represents i's prior beliefs regarding the states. We assume that for each $A \in \mathcal{P}_i$, $\mu_i(A) > 0$. For $\omega \in \Omega$ we denote by $\mu_i(\omega \mid \mathcal{P}_i(\omega))$, the conditional probability assigned by consumer i to state ω. For a consumption plan x_i and $A \in \mathcal{P}_i$, consumer i's conditional expected utility is denoted $U_i(x_i \mid A)$, where

$$U_i(x_i \mid A) = \sum_{\omega \in A} \mu_i(\omega \mid A) u_i(x_i(\omega), \omega).$$

Consumer i prefers consumption plan x_i to consumption plan y_i at state ω whenever $U_i(x_i \mid \mathcal{P}_i(\omega)) > U_i(y_i \mid \mathcal{P}_i(\omega))$.

An economy is defined as $\mathcal{E} = \langle \Omega, N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i \in N} \rangle$.

An *allocation* for an economy is $(x_i)_{i \in N} \in \prod_i X_i$ such that

$$\sum_{i \in N} x_i(\omega) = \sum_{i \in N} e_i(\omega) \text{ for all } \omega \in \Omega.$$

Thus an allocation can be viewed as a state-contingent contract which is feasible in each state. Let $\mathcal{A}_N$ denote the set of allocations. A coalition S is a non-empty subset of N. An allocation x is said to be feasible for coalition S if

$$\sum_{i \in S} x_i(\omega) = \sum_{i \in S} e_i(\omega) \text{ for all } \omega \in \Omega.$$

Let $\mathcal{A}_S$ denote the set of allocations feasible for S.

Coalition formation in our model takes place at the interim stage. More precisely, if the true state is ω, each consumer observes the event $\mathcal{P}_i(\omega)$ and at this stage consumers may form coalitions and agree to an allocation which is feasible for the coalition. We shall assume that when a contract has to be enforced, the true state is publicly verifiable. This obviates the need for imposing incentive compatibility constraints on allowable allocations. Our

[7] This formulation is equivalent to one in which the private information of each consumer is described by the consumer's type and an information state for the economy refers to a profile of consumers' types. In particular, each element of the partition $\mathcal{P}_i$ refers to a particular type of consumer i.

main results continue to hold even without this simplifying assumption; see Section 4 below. A notion of the core suitable for the present context is the coarse core of Wilson (1978) which is based on the idea that a coalition, in designing a potential objection, can only consider those events which are commonly known to consumers in the coalition. To describe such events we need some additional notation. Let $(\mathcal{P}_i)_{i\in S}$ be an information structure for S. The *meet* of the partitions $(\mathcal{P}_i)_{i\in S}$ is the finest partition of Ω that is coarser than each $\mathcal{P}_i$, $i \in S$, and it is denoted by $\mathcal{P}_S = \wedge_{i\in S}\mathcal{P}_i$. An event E is said to be *common knowledge among the members of S at ω* if $\mathcal{P}_S(\omega) \subseteq E$. We can now say that coalition S has an objection to allocation x if there is another allocation $y \in \mathcal{A}_S$, and a state $\omega \in \Omega$ at which it is common knowledge among the members of S that each $i \in S$ prefers y_i to x_i. Equivalently, coalition S is said to have a *coarse objection* to an allocation $x \in \mathcal{A}_N$ if there exists $y \in \mathcal{A}_S$ and an event $E \subseteq \mathcal{P}_S$ such that

$$U_i(y_i \mid A) > U_i(x_i \mid A) \text{ for all } i \in S, \text{ for all } A \in \mathcal{P}_i \text{ such that } A \subseteq E.$$

The *coarse core* is the set of all $x \in \mathcal{A}_N$ to which there does not exist a coarse objection.

An allocation $x \in \mathcal{A}_N$ is said to be *interim efficient* if N does not have a coarse objection to x. Similarly, $x \in \mathcal{A}_S$ is said to be *interim efficient for S* if S does not have a coarse objection to x.

Our aim is to study the relationship between the coarse core and a corresponding price equilibrium notion as an economy is replicated.

Clearly, any such exercise must involve a price equilibrium concept which captures decision making at the interim stage.[8] Moreover, for the present exercise it is reasonable to consider a price equilibrium notion such that the corresponding allocations belong to the coarse core. We now turn to a definition of such equilibrium notion.

While Wilson (1978) established the non-emptiness of the coarse core by constructing a corresponding NTU game and proving it to be balanced, he also pointed out (footnote 6, Wilson (1978)) that an alternative proof consists of showing that the coarse core contains a constrained market allocation.

[8] It is easy to see that the ex-ante core bears no logical relationship to the coarse core since the latter is based on interim considerations; see Vohra (1999) for examples. There is, therefore, no hope of establishing a core convergence/equivalence result for the coarse core and ex-ante equilibrium concepts such as the Arrow-Debreu equilibrium.

Let $p = (p(\omega))_{\omega \in \Omega}$ denote a vector of state-contingent market prices where $p(\omega) \in \mathbb{R}^l$ for $\omega \in \Omega$. Let $\triangle$ denote the unit simplex in $\mathbb{R}^{l \times |\Omega|}$. For consumer i and $A \in \mathcal{P}_i$, the budget set of consumer i corresponding to the event A, given a price vector $p \in \triangle$ is denoted

$$\gamma_i(p \mid A) = \{x_i(A) \in X_i(A) \mid \sum_{\omega \in A} p(\omega) \cdot x_i(\omega) \leq \sum_{\omega \in A} p(\omega) \cdot e_i(\omega)\}.$$

A *constrained market equilibrium* is defined as $(x, p) \in \mathcal{A}_N \times \triangle$ such that for every $i \in N$ and $A \in \mathcal{P}_i$,

$$x_i(A) \in \arg \max_{\gamma_i(p|A)} U_i(. \mid A).^9$$

It is easy to see that constrained market equilibria satisfy several properties that are analogous to those of Walrasian equilibria. In particular:

(i) An equilibrium allocation belongs to the coarse core.

(ii) A replication[10] of an equilibrium allocation is an equilibrium allocation of the corresponding replicated economy.

(iii) The converse of (ii) also holds if all consumers have strictly concave Bernoulli utility functions.

This equilibrium notion, therefore, provides a natural benchmark as a price equilibrium concept to which one might expect coarse core allocations of replicated economies to converge. In the next section we show through a simple example of a sunspot economy that the coarse core does not converge to the set of constrained market allocations.

The critical property of a price equilibrium concept which our negative result relies upon is the following:

Property P. *Suppose (x, p) is an equilibrium and there exists $\omega \in \Omega$ and a consumer i such that $\{\omega\} \in \mathcal{P}_i$. Then*

$$x_i(\omega) \in \arg \max_{\gamma_i(p|\{\omega\})} u_i(., \omega).$$

[9]While there is some abuse of notation in the use of $U_i(. \mid A)$ above, this should not cause any confusion since $U_i(x_i \mid A)$ actually depends only on $x_i(A)$.

[10]Precise definitions follow.

Clearly, Property P is satisfied by a constrained market equilibrium. In restricted market participation economies, which we consider in the next section, sunspot equilibria (as defined by conditions (7), (8) and (9) in Cass and Shell (1983)) are identical to constrained market equilibria, and, therefore, satisfy Property P. Property P is also satisfied, in the economies we consider, by Radner equilibrium (throughout this paper a Radner equilibrium will refer to an equilibrium in the sense of Radner (1968)) and by rational expectations equilibrium; see Section 4 for details.

As we shall be dealing with replica economies, we need some additional, related definitions. Given an economy $\mathcal{E} = \langle N, (\mathcal{P}_i, u_i, e_i, \mu_i)_{i\in N}\rangle$, and an allocation $x \in \mathcal{A}_N$, replicas of $\mathcal{E}$ and x are defined as follows. For every positive integer m, let $M = \{1, 2, ..., m\}$. The m-th replica of $\mathcal{E}$ is the economy $\mathcal{E}^m = \langle N\times M, (\mathcal{P}_{(i,j)}, u_{(i,j)}, e_{(i,j)}, \mu_{(i,j)})_{(i,j)\in N\times M}\rangle$, where for all $(i, j) \in N\times M$, $\mathcal{P}_{(i,j)} = \mathcal{P}_i$, $u_{(i,j)} = u_i$, $e_{(i,j)} = e_i$, and $\mu_{(i,j)} = \mu_i$. The m-th replica of x is denoted x^m where $x^m_{ij} = x_i$ for all $(i, j) \in N \times M$.[11]

An allocation x in $\mathcal{E}^m$ is said to satisfy the *equal treatment property* if

$$x_{ij}(\omega) = x_{ik}(\omega) \text{ for all } i \in N, \text{ and for all } j, k \in M.$$

The following Lemma, the proof of which is left to the reader, will be useful in various results to follow.

Lemma 1 *Suppose x is an allocation in $\mathcal{E}^m$ to which coalition S has a coarse objection. Then*

(a) S has a coarse objection y such that y is interim efficient for S.

(b) Suppose all utility functions are concave. Then S has a coarse objection y with the property that for any $i \in N$:

$$\textit{if } u_i(x_{ij}(\omega)) = u_i(x_{ik}(\omega)) \textit{ for all } (i,j),(i,k) \in S \textit{ and all } \omega \in \Omega,$$

$$\textit{then } y_{ij}(\omega) = y_{ik}(\omega) \textit{ for all } j,k \in M \textit{ and all } \omega \in \Omega.$$

[11] We shall sometimes find it convenient to refer to consumer $(i, j) \in N\times M$ as consumer ij.

3 Failure of Core Convergence: A Sunspot Economy

Our main results are negative. The fact that we will derive them from examples of very simple economies makes them all the more compelling. Indeed, throughout this section, we shall consider economies in which uncertainty is extrinsic to the fundamentals of the economy. We consider sunspot economies consisting of two states and two kinds of consumers – those who are fully informed and those who cannot distinguish between either state at the interim stage. The economy can then be seen as a restricted market participation economy of Cass and Shell (1983) in which informed consumers can participate only in spot markets. For our purposes then, an economy $\mathcal{E}$ is said to be a *sunspot economy* if $\Omega = \{s, t\}$, $N = N_1 \cup N_2$, $\mathcal{P}_i = \{\Omega\}$ for all $i \in N_1$ and $\mathcal{P}_i = (\{s\}, \{t\})$ for all $i \in N_2$ and for all $i \in N$, $u_i(., s) = u_i(., t)$ and $e_i(s) = e_i(t)$. Note that for coalition S such that $S \cap N_1 \neq \emptyset$, the only common knowledge event is $\{s, t\}$, whereas for a coalition S such that $S \cap N_1 = \emptyset$, there are two common knowledge events, $\{s\}$ and $\{t\}$.

As pointed out in the previous section, in a sunspot economy, the definition of a *sunspot equilibrium* (see Cass and Shell (1983)) corresponds exactly to the definition of a constrained market equilibrium. In particular, informed consumers maximize ex-post utility subject to their ex-post budget constraint, while uninformed consumers maximize expected utility subject to a single budget constraint (involving contingent commodities). Clearly, a sunspot equilibrium satisfies property P.

Consider the following example of a two-consumer, restricted market participation economy.

Example 1.

- $N = \{1, 2\}$, $\Omega = \{s, t\}$
- $\mathcal{P}_1 = (\{s, t\})$ and $\mathcal{P}_2 = (\{s\}, \{t\})$
- $u_i((a, b), \omega) = (ab)^{1/4}$ for $i = 1, 2$ and for $\omega = s, t$
- $e_1(s) = e_1(t) = (0, 24)$ and $e_2(s) = e_2(t) = (24, 0)$
- $\mu_i(s) = \mu_i(t) = \frac{1}{2}$, for $i = 1, 2$.

This simple sunspot economy has a unique sunspot equilibrium, $(\bar{x}, \bar{p})$, where

$$\bar{x}_1(s) = \bar{x}_2(s) = \bar{x}_1(t) = \bar{x}_2(t) = (12, 12), \qquad \bar{p}(s) = \bar{p}(t) = (1/4, 1/4).$$

Thus, the unique equilibrium is actually *sunspot-free*, in the sense that for each consumer i, $\bar{x}_i(s) = \bar{x}_i(t)$. Clearly, for any integer m, $(\bar{x}^m, \bar{p})$ is the unique price equilibrium in $\mathcal{E}^m$. Of course, $\bar{x}$ belongs to the coarse core (Wilson (1978), footnote 6).

One ingredient in the Debreu-Scarf (1963) core convergence argument is the equal treatment property of core allocations (given strict convexity of preferences). In the present context, this is the property that if x belongs to the coarse core of $\mathcal{E}^m$, then x satisfies the equal treatment property. We begin by showing that, in Example 1, this property does not hold.[12]

Proposition 1 *The coarse core in a replica of size 2 of the economy in Example 1 contains an allocation which does not satisfy the equal treatment property.*

Proof. Consider a replica of size 2 of the economy described in Example 1 and the following allocation, y:

$$y_{11}(s) = y_{12}(s) = y_{11}(t) = y_{12}(t) = (12, 12).$$

$$y_{21}(s) = (10, 10), \quad y_{22}(s) = (14, 14),$$
$$y_{21}(t) = (14, 14), \quad y_{22}(t) = (10, 10).$$

Of course y does not satisfy equal treatment. We claim that y does belong to the coarse core of $\mathcal{E}^2$.

Clearly y is individually rational therefore one person coalitions cannot coarsely improve upon y. It is also easy to see that y is interim efficient. It also follows from individual rationality of y that a two-consumer coalition consisting of both informed players or of both uninformed players does not have an objection to y. Consider a two-consumer coalition containing one

[12]As we shall see, the failure of equal treatment in the present context is a result of interim restrictions on coalition formation, and is therefore quite distinct from the problem in ex-ante economies. In particular, the failure of equal treatment in Forges, Heifetz and Minelli (1999) stems from incentive compatibility constraints, and in Koutsougeras and Yannelis (1993) it is a result of measurability restrictions.

uninformed and one informed consumer, for example the coalition $\{11, 21\}$. By Lemma 1 (a) we can concentrate on potential objections which are ex-post efficient in each state. This implies that in each state consumer i's consumption of the two commodities must be the same. From the strict concavity of consumer 11's utility function it follows that it is impossible to provide him ex-ante utility of $\sqrt{12}$ while ensuring that consumer 21 receives at least a utility of $\sqrt{14}$ in state t and at least $\sqrt{10}$ in state s. Here we see why the standard proof of the equal treatment property (Debreu and Scarf (1963)) cannot be applied in the present context. The usual argument would apply if we could construct a 'coalition' consisting of consumer 11, consumer 21 *in state* s and consumer 22 *in state* t. Of course, such a 'coalition' has no meaning here. This is the fundamental reason for the failure of the equal treatment property in the presence of incomplete information. Of course, for a complete proof of the proposition we also need to verify the following Claim, whose proof is provided in the Appendix.

Claim 1: No three-player coalition has an objection to y.

■

The fact that the allocation, y, constructed in the proof of the above proposition is not sunspot-free is not accidental. Suppose x is a sunspot-free allocation in the coarse core of $\mathcal{E}^m$. For $i \in N_2$, let ik^* be an informed consumer who receives, according to x, a least preferred commodity bundle (across all consumers ik, $k \in M$) in state ω, i.e.,

$$u_i(x_{ik^*}(\omega)) \leq u_i(x_{ik}(\omega)) \text{ for all } k \in M.$$

Since x is sunspot-free, this property holds for all $\omega \in \Omega$. Thus we can identify, for each $i \in N_2$, a particular consumer who (among his types in the replica) receives, at x, a least preferred commodity bundle. And this allows us to use the argument of Debreu-Scarf (1963, Theorem 2) to make the following claim:

Lemma 2 *Suppose x is sunspot-free and belongs to the coarse core of $\mathcal{E}^m$. If $u_i(., \omega)$ is strictly concave for all i and ω, then x satisfies the equal treatment property.*

Quite independently of the equal treatment property, one may still ask whether an allocation x in the economy $\mathcal{E}$ whose replica x^m is in the coarse core of $\mathcal{E}^m$ for every m, can be supported by a price equilibrium. Our next proposition provides a negative answer. As such it shows that the failure of core convergence cannot be attributed to the violation of the equal treatment property illustrated by Proposition 1.

Proposition 2 *Let $\mathcal{E}$ be the economy defined in Example 1. There exists an allocation x in $\mathcal{E}$ whose replica x^m is in the coarse core of $\mathcal{E}^m$ for all m, such that x cannot be supported as an equilibrium satisfying property P.*

Proof. Consider the following allocation x:

$$x_1(s) = (9, 9), \qquad x_1(t) = (16, 16)$$

$$x_2(s) = (15, 15), \qquad x_2(t) = (8, 8).$$

To prove the proposition, we will show that x^m belongs to the coarse core of $\mathcal{E}^m$ for all m. Clearly, x cannot be supported by an equilibrium satisfying Property P, since the equilibrium relative price in each state must be 1, and the informed consumer in state t is then trading below his budget line.

Suppose there exists m such that x^m does not belong to the coarse core of $\mathcal{E}^m$. Suppose coalition S has a coarse objection y to x^m. It is easy to see that x^m is interim individually rational. This implies that S must contain both kinds of consumers (uninformed and informed). Let k_1 and k_2 be the number of uninformed and informed consumers in S, respectively. By Lemma 1, there is no loss of generality in assuming that y satisfies equal treatment and is interim efficient for coalition S. Let y_1 denote the consumption plan of each uninformed consumer and y_2 the consumption plan of each informed consumer in coalition S. Coarse blocking implies that:

$$u_1(y_1(s)) + u_1(y_1(t)) > 7 \tag{1}$$

$$u_2(y_2(s)) > \sqrt{15} \tag{2}$$

$$u_2(y_2(t)) > \sqrt{8} \tag{3}$$

The aggregate endowment of coalition S in each state is

$$e_S(s) = e_S(t) = e_S = (24k_2, 24k_1). \tag{4}$$

By ex-post efficiency of y (for coalition S), it follows that the marginal rate of substitution of each consumer in state $\omega = s, t$ must be the same. Since the utility functions are Cobb-Douglas, this implies that there exist constants $\alpha, \beta \in (0,1)$ such that: (i) in state s, the total amount of each commodity, allocated to the uninformed consumers is the fraction α of coalition S's aggregate endowment of that commodity, and the remainder, namely the fraction $(1-\alpha)$ is allocated to the informed consumers; and (ii) in state t, the same is true with fractions β and $(1-\beta)$. Thus, the consumption plan of each uninformed consumer is $(\alpha/k_1)e_S$ in state s and $(\beta/k_1)e_S$ in state t. Similarly, the consumption plan of each informed consumer is $((1-\alpha)/k_2)e_S$ in state s and $((1-\beta)/k_2)e_S$ in state t. This implies that the utilities corresponding to allocation y are:

$$u_2(y_2(s)) = \sqrt{(1-\alpha)/k_2}[(24k_2)(24k_1)]^{1/4}.$$

$$u_2(y_2(t)) = \sqrt{(1-\beta)/k_2}[(24k_2)(24k_1)]^{1/4},$$

which implies

$$[u_2(y_2(s))]^2 k_2 = (1-\alpha)24\sqrt{k_1k_2}.$$

$$[u_2(y_2(t))]^2 k_2 = (1-\beta)24\sqrt{k_1k_2}.$$

And similarly,

$$[u_1(y_1(s))]^2 k_1 = \alpha 24\sqrt{k_1k_2}.$$

$$[u_1(y_1(t))]^2 k_1 = \beta 24\sqrt{k_1k_2}.$$

Thus

$$[u_1(y_1(\omega))]^2 k_1 + [u_2(y_2(\omega))]^2 k_2 = 24\sqrt{k_1k_2} \qquad \omega = s, t.$$

Letting $z = k_2/k_1$, the above equation can now be rewritten as:

$$[u_1(y_1(\omega))]^2 + [u_2(y_2(\omega))]^2 z = 24\sqrt{z} \qquad \omega = s, t. \tag{5}$$

Using (2) and (3), this yields:

$$[u_1(y_1(s))]^2 < 24\sqrt{z} - 15z \tag{6}$$

and

$$[u_1(y_1(t))]^2 < 24\sqrt{z} - 8z. \tag{7}$$

Let

$$g_s(z) = [24\sqrt{z} - 15z]^{1/2}$$

and

$$g_t(z) = [24\sqrt{z} - 8z]^{1/2}.$$

Taking square roots on both sides of (6) and (7), and using (1), we have:

$$g_s(z) + g_t(z) > 7 \tag{8}$$

To complete the proof the reader must show that (8) cannot hold. To do this, notice that the functions $g_s(\cdot)$ and $g_t(\cdot)$ are both differentiable and concave in z. Moreover $g_s(1) + g_t(1) = 7$. It then suffices to show that the derivative of $g_s(z) + g_t(z)$ is 0 at $z = 1$, which can be easily done.

This completes the proof that x^m belongs to the coarse core of $\mathcal{E}^m$ for any m. ■

Notice that the argument we have used for showing that x^m does not belong to the coarse core of every replicated economy also applies (with obvious modifications) to an economy with an atomless measure space of consumers. In an economy in which half the consumers have the characteristics of consumer 1 and half have the characteristics of consumer 2 in Example 1, x belongs to the core where x_1 and x_2 denote the consumption of all consumers of each of the two kinds of consumers.

Recall that in any replication of the economy of Example 1 there are several agents who are completely informed. In particular, no single agent possesses information which is not available elsewhere in the economy. Replication therefore ensures that information is non-exclusive in the sense of Postlewaite and Schmeidler (1986), and agents in our model are, therefore, informationally (arbitrarily) small according to the definition introduced in McLean and Postlewaite (1999).

The proof of Proposition 2 is instructive in that it shows why the standard Debreu-Scarf argument does not apply to the coarse core. A coarse objection for a coalition consisting of informed as well as uninformed consumers must provide for an improvement in the expected utility of the uninformed, and in the interim (in the present case, ex-post) utility of *both* 'types'[13] of the informed consumers. This requirement is necessary for the potential objection to be common knowledge among all members of the coalition. As in Goenka and Shell (1997), one can transform the restricted participation

[13]Not to be confused with the common usage of 'types' in replica economies!

economy into a quasi-Walrasian economy in which an informed consumer is transformed into two consumers, one for each state of the world. Thus an informed consumer of 'type' s has endowment only in state s and consumes only in state s. In terms of the quasi-Walrasian economy, the common knowledge requirement of a coarse objection means that allowable coalitions are restricted to have the same number of informed consumers of type s as of type t. It is this restriction which accounts for the non-convergence phenomenon. Without such a restriction the usual Debreu-Scarf argument, applied to quasi-Walrasian consumers does yield 'convergence'. However, the corresponding notion of the core with quasi-Walrasian consumers does not have any natural interpretation in terms of a core with asymmetric information; see for example the discussion in Vohra (1999).

There is one particular case in which the core of the economy with quasi-Walrasian consumers coincides with the coarse core, and this provides us with a positive convergence result, at least for certain core allocations. As the next proposition shows, in economies such as the one in Example 1, convergence does indeed hold if one restricts attention to those allocations in the coarse core which are sunspot-free. (Recall that the core allocation considered in Proposition 2 is *not* sunspot-free.)

Proposition 3 *Let $\mathcal{E}$ be a sunspot economy, and suppose that the core convergence theorem holds at the ex-post stage. If x is sunspot-free and x^m belongs to the coarse core of $\mathcal{E}^m$ for every m, then x is a sunspot equilibrium allocation.*

Proof. Suppose not, i.e., suppose x is a sunspot-free allocation such that x^m belongs to the coarse core of $\mathcal{E}^m$ for every m but x is not a sunspot equilibrium allocation. Since x is sunspot-free, this must mean that in any state ω, the projection of x onto that state, $x(\omega)$, is not a Walrasian allocation of the ex-post economy. Since the core convergence theorem holds at the ex-post stage, there exists a replica of size m of the ex-post economy in state ω and a coalition $S(\omega)$ of agents improving upon $x^m(\omega)$. Because the allocation x is sunspot-free, the same is true in every ω, where the same coalition of ex post consumers is an improving coalition. Therefore, there exists a coarse improvement upon x^m: letting $S(\omega) = S_1 \cup S_2(\omega)$, the types in the coarsely improving coalition are $S_1 \cup \bigcup_{\omega \in \Omega} S_2(\omega)$ and the coarse improvement is the allocation that uses the ex-post objection in each state.

■

4 Robustness of the Results

4.1 Modifications of the Coarse Core

In this subsection we shall consider several modifications of the coarse core, already suggested in the literature,[14] and show that our results in the previous section are robust to each of these.

A model in which private information is not publicly revealed even after exchange takes place motivates the introduction of incentive compatibility constraints. Analogous to the efficiency notions incorporating incentive compatibility, as introduced in Holmstrom and Myerson (1983), one can consider a corresponding notion of the incentive compatible coarse core, as in Vohra (1999). Recall that in our model replication renders information nonexclusive in the sense of Postlewaite and Schmeidler (1986). It then follows from Proposition 3.1 in Vohra (1999) that in every replication of the economy, the allocation, x^m, constructed in the proof of Proposition 2 belongs to the incentive compatible core of $\mathcal{E}^m$. Thus, in the economy of Example 1, the incentive compatible core does not converge to the equilibrium allocation (which is also incentive compatible).

Refinements of the coarse core, which allow for some pooling of private information, such as the coarse+ core introduced by Lee and Volij (1997)[15] and the core with endogenous communication of Volij (1997) do not help in terms of convergence either. In fact it can be shown that in the economy $\mathcal{E}$ of Example 1, in every replication, m, the allocation x^m (constructed in the proof of Proposition 2) belongs to these cores in $\mathcal{E}^m$.

Going further in the direction of sharing information, one can consider arbitrary forms of information pooling, corresponding to Wilson's (1978) fine core. As Wilson showed, the fine core may be empty. For this reason alone, the fine core does not converge to a price allocation.

The model in Goenka and Shell (1997) can be viewed as one of asymmetric information. In their definition 5.6, the authors consider a variant of the coarse core where objections are defined without reference to a common knowledge event.[16] For our purposes, it will be enough to concentrate on

[14]See Forges (1998) for a survey.

[15]See also Lee (1998).

[16]Note that for the particular case of sunspot economies, the core (in the pooling case) used in Ichiishi and Idzik (1996) is the same as the one in definition 5.6 of Goenka and

sunspot economies as defined in Section 3. In particular, $\Omega = \{s, t\}$ and the randomizing device is based on the σ-algebra generated by the fine partition $(\{s\}, \{t\}))$. In this setting, the essential difference[17] between their core notion and the coarse core concerns *only* the case in which an objecting coalition, S, contains no uninformed consumers. In such a case, they require all the (informed) members of the S to be better-off in *both* states. In contrast, recall that for a coarse objection from S (containing no uninformed consumers) it suffices that there is *some* state in which all its members are better-off. Therefore, their core contains the coarse core, and the conclusions of Propositions 1 and 2 extend to it. In light of these remarks, Lemma 7.1 and Theorem 7.3 in Goenka and Shell (1997) should be seen as applying to the core of an economy with quasi-Walrasian consumers rather than to a notion of the core in asymmetric information economies.[18]

4.2 Other Notions of Price Equilibria

Proposition 2 applies to any price equilibrium notion satisfying Property P.[19] As we have already observed, constrained market equilibria and sunspot equilibria (in a restricted market participation economy) satisfy this property. So do Radner equilibria and rational expectations equilibria of an economy in which trade takes place only in spot markets. It is easy to see that these equilibrium concepts also yield $\bar{x}$ as the unique equilibrium allocation in the economy described in Example 1.

One may also consider Radner equilibria or rational expectations equilibria in an economy in which trade, at the interim stage, is in contingent commodities. In the economy described in Example 1, this means that four contingent commodities are traded at the interim stage. Since the informed consumers trade after receiving their signal, the market prices for these four commodities, in general, depend on the signal received by the informed. Let

Shell (1997).

[17]There are two other differences: (i) they require allocations to be measurable with respect to the σ-algebra used for defining randomizing devices. However, this measurability restriction is void if one considers, as we do, the fine σ-algebra. (ii) they define objections using weak inequalities (and some strict inequality), but this does not affect our arguments.

[18]We thank Karl Shell for clarifying this point.

[19]Also implicit in our argument is the linearity of the price functional. The possibility of examining this issue in the context of non-linear prices, as in Bisin and Gottardi (1998), remains open.

$p^s = (p^s(s), p^s(t))$ denote the market prices when the signal received by the informed consumer is s. Similarly, let p^t denote the market prices when the signal received by the informed consumer is t. Since informed consumers are allowed to trade in contingent commodities, in general, it is possible that Property P is not satisfied in equilibrium. However, it can be shown that in the economy of Example 1, these equilibria do satisfy property P. It is easy to see that the equilibrium prices are $p^s(s) = (1/2, 1/2)$, $p^s(t) = (0, 0)$ and $p^t(s) = (0, 0)$, $p^t(t) = (1/2, 1/2)$. A rational expectations equilibrium, therefore, results in both consumers consuming $(12, 12)$ in each state. (There is no non-revealing rational expectations equilibrium). Since this allocation is the same in both states, it is also the unique outcome of a Radner equilibrium.

A skeptical reader may still wonder whether non-convergence can be shown for a core allocation which is measurable with respect to the private information of the uninformed consumers. In Example 1, this measurability restriction is the same as requiring the coarse allocation to be sunspot-free. And as we have seen in Proposition 3, it was critical for the proof of Proposition 2 that x was *not* sunspot-free. However, in the next subsection we will show, by considering an economy which is not a sunspot economy, that measurability restrictions are not enough to restore core convergence.

4.3 Measurability Considerations

We shall now construct a non-sunspot economy in which there exists an allocation, x, which is constant across states such that its replication belongs to the coarse core in the corresponding replicated economy and x is not a price equilibrium allocation. This allocation also belongs to the core studied in Yannelis (1991).[20]

Example 2.

- $\Omega = \{s, t\}$, $N = \{1, 2\}$
- $\mathcal{P}_1 = (\{s, t\})$ and $\mathcal{P}_2 = (\{s\}, \{t\})$
- $u_1((a, b), s) = a^{1/4}b^{3/4}$, $u_2((a, b), s) = 2a^{1/4}b^{3/4}$

[20] In this core notion, allocations are required to be measurable with respect to each consumer's private information and, as in Goenka and Shell (1997) and Ichiishi and Idzik (1996), informed consumers in an objecting coalition must be made better-off in each state.

and $u_i((a,b),t) = a^{3/4}b^{1/4}$ for $i = 1,2$

- $e_1(s) = e_1(t) = (0, 24)$ and $e_2(s) = e_2(t) = (24, 0)$
- $\mu_i(s) = \mu_i(t) = \frac{1}{2}$, $i = 1, 2$.

Consider the following allocation x:

$$x_1(s) = x_2(s) = (12, 12)$$

$$x_1(t) = x_2(t) = (12, 12).$$

It is easy to check that in an economy with spot markets there is a unique, fully revealing rational expectations equilibrium[21] with prices $p(s) = (1/4, 3/4)$ and $p(t) = (3/4, 1/4)$. The corresponding allocation is $\tilde{x}$, where

$$\tilde{x}_1(s) = (18, 18), \qquad \tilde{x}_2(s) = (6, 6),$$

$$\tilde{x}_1(t) = (6, 6), \qquad \tilde{x}_2(t) = (18, 18).$$

It can also be shown, as in the previous subsection, that in a market with contingent commodities, $\tilde{x}$ is the unique allocation corresponding to a rational expectations equilibrium. While $\tilde{x}$ is not the allocation corresponding to a Radner equilibrium (with spot markets or with contingent commodity markets), it can, nevertheless, be shown that in either case the equilibrium allocation is not x.

We will now show that x^m belongs to the coarse core for every replication. We argue by contradiction. Suppose there is a replica $\mathcal{E}^m$ such that x^m does not belong to its Coarse Core. Let S be a coalition which improves upon x^m with an allocation y. Since x^m is (interim) individually rational and preferences are convex, S must contain both kinds of consumers (uninformed and informed). Let k_1 and k_2 be the number of uninformed and informed consumers in S, respectively. By Lemma 1, we may assume that y satisfies equal treatment and is interim efficient for S. Following the same steps as in the proof of Proposition 2, and letting $z = k_2/k_1$, one arrives at the inequality

$$z^{3/4} + z^{1/4} - z > 1.$$

To complete the proof we must show that this inequality cannot hold. To do this, notice that the function on the LHS is differentiable and concave

[21] This is also the unique constrained market equilibrium

in z. Moreover the LHS = RHS when $z = 1$. In fact, the function on the LHS reaches a maximum at $z = 1$, and can, therefore never exceed the RHS. Therefore, x^m belongs to the coarse core of $\mathcal{E}^m$ for any m.

Appendix

Proof of Claim 1. Consider a possible objection from coalition $\{11, 21, 21\}$. Suppose this coalition has an objection, z. The aggregate endowment of this coalition is $(24, 48)$. Ex-post efficiency in this coalition then requires that for each consumer, in each state, the consumption of commodity 1 be twice as much as that of commodity 2. Thus z can be written as follows:

$$z_{21}(s) = (2a, a), \qquad z_{22}(s) = (2b, b).$$

Since $(2a^2)^{1/4} \geq \sqrt{10}$ and $(2b^2)^{1/4} \geq \sqrt{14}$, it follows that

$$a + b \geq 10/\sqrt{2} + 14/\sqrt{2} = 12\sqrt{2}.$$

But then

$$z_{11}(s) \leq (48 - 24\sqrt{2}, 24 - 12\sqrt{2}).$$

By a similar argument, the same is true in state t,

$$z_{11}(t) \leq (48 - 24\sqrt{2}, 24 - 12\sqrt{2}).$$

Since z is an objection, consumer 11 must have expected utility higher than $\sqrt{12}$. Using the above inequalities, this implies $2(24 - 12\sqrt{2})^2 \geq 144$, or

$$\sqrt{2} - 1 \geq 1/2,$$

which is clearly false. Thus there exists no objection from a coalition consisting of one uninformed consumer and the two uninformed consumers.

Finally, consider a coalition consisting of both uninformed consumers and one informed consumer, for example, $\{11, 12, 21\}$. Given Lemma 1 (b), it suffices to consider a potential objection in which both uniformed consumers are treated identically. It is easy to see that in this three-consumer coalition it is not possible to provide all consumers interim utilities corresponding to the equal-split allocation $\bar{x}$. However, consumer 21 needs to be given at least $\sqrt{10}$ and $\sqrt{14}$ in states s and t respectively. From the strict concavity of the utility functions it follows that this, too, is impossible if the uninformed are to be made better-off. The details are as follows. Suppose there exists an objection z. Then $z_{21}(s) = (a, 2a)$ and $z_{21}(t) = (b, 2b)$ such that $2a^2 > 100$ and $2b^2 > 196$, which yields, $a > 10/\sqrt{2}$ and $b > 14/\sqrt{2}$. This means that

$$z_{11}(s) = z_{12}(s) \leq (12 - 5/\sqrt{2}, 24 - 10/\sqrt{2}),$$

$$z_{11}(t) = z_{12}(s) \leq (12 - 7/\sqrt{2}, 24 - 14/\sqrt{2}).$$

Since consumers 11 and 12 receive a higher expected utility than y, we must have:

$$(2)^{1/4}[0.5\sqrt{12 - 7/\sqrt{2}} + 0.5\sqrt{12 - 5/\sqrt{2}}] > \sqrt{12}.$$

Since the function on the LHS is strictly concave, this implies that

$$(2)^{1/4}\sqrt{12 - 6/\sqrt{2}} > \sqrt{12},$$

or

$$12\sqrt{2} - 6 > 12,$$

which is false. ■

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