Private Strategy and E¢ciency: Repeated Partnership Games Revisited[¤]

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First version: March, 1999
This version: January, 2000

Abstract

This paper studies repeated partnership games with only two public signals. It is well known that the public perfect equilibrium payo¤ set is bounded away from the e¢cient frontier of the stage game in this class of game. In this paper, I construct a strongly symmetric sequential equilibrium whose equilibrium payo¤ dominates the best symmetric payo¤ by PPE. The strategy used to construct the equilibrium depends not only on the public signal but also on the realization of one's own past action. I call this class of strategy private strategy. I also provide an example where this private strategy sequential equilibrium approximates the e¢cient outcome, but the PPE payo¤ set is contained in an arbitrary small neighborhood of the stage game Nash equilibrium payo¤. This example suggests that the di¤erence between a PPE payo¤ set and a sequential equilibriujm payo¤ set can be potentially signi…cant.

[¤]I am grateful to George Mailath and Andrew Postlewaite for their encouragement and valuable suggestions. I would like to thank to Je¤rey Ely, Johannes Horner, Michihiro Kandori, Steven Matthews, Rafael Rob, and Tadashi Sekiguchi for their helpful comments. All remaining errors are my responsibility.

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1 Introduction

Since many economic problems are naturally described in the framework of repeated games with imperfect monitoring, and since partnership game is surely one of them, the example in Radner, Myerson and Maskin[11] had a big impact on the research of repeated games. Their example shows that the folk theorem can fail for repeated games with discounting and imperfect monitoring. In precise, they show that the payo¤ set of public perfect equilibria is bounded away from the e¢cient frontier independent of the discount factor less than 1. This example is important for two reason. First, it is well known that the e¢cient outcome can be sustained in repeated partnership game without discounting (Radner [10]). So, this example illustrates whether to discount or not to discount really make a di¤erence for the outcome of repeated games with imperfect monitoring. Secondly, this example shows that there exists an e¢ciency loss which is purely associated with the monitoring structure. This anti-folk theorem example motivated researches on the condition under which e¢ciency or folk theorem can be recovered when players are impatient. These researches lead to papers such as Abreu, Pearce, and Stachetti[2] and Fudenberg, Levin, and Maskin [5]. Abreu, Pearce, and Stachetti[2] invents the way to characterize PPE payo¤ set for repeated games with discounting. Fudenberg, Levin, and Maskin[5] shows that a folk theorem still obtains in this class of repeated games generically if the space of public signal is rich enough such that each player's deviation can be statistically detected separately. Needless to say, this is a weaker condition than perfect information.

So far most of the research has focused on public strategy, strategy which only depends on the history of public signals. This is mainly because one can exploit a nice recursive structure introduced by this restriction on strategy. The original game and the continuation game become isomorphic with public strategies. However, there are not convincing arguments to justify such a restriction on strategy. Such a restriction may not be a problem in the environment where the folk theorem obtains, that is, the environment with a rich public signal space and very patient players because almost everything can be achieved only with public strategies. Otherwise, it is not reasonable to impose such a restriction.

So, an obviously interesting question is the following: what a general strategy can do for repeated games with discounting and imperfect monitoring? This paper attempts to address this question. The purpose of this paper is to show that this restriction to public strategy is sometimes really restrictive in terms of efficiency. This paper sticks to the original formulation by Radner, Myerson, and Maskin in the sense that there are only two public signals, which is a ideal situation to address this question because we already know that the PPE payoff is strictly smaller than the individually rational payoff set and, moreover, the bound of PPE can be characterized to some extent. This paper explicitly consider strategies which depend on a player's own past action rather than restricting attentions to public strategies. For some parameter values, a strongly symmetric sequential equilibrium is explicitly constructed whose equilibrium payoff locates outside of the bound of PPE payoff set.[1]

In order to show how non-public strategy can be used to improve efficiency, I explain briefly the cause of inefficiency in repeated partnership games and suggest a way to circumvent that inefficiency. What causes inefficiency in partnership games with two public signal is following: since there are only two public signals available, the only way to deter deviation is to "punish" both players at the same time when a "bad" signal is observed. Since this punishment happens with positive and nonnegligible probability every period on the equilibrium path, inefficiency arises independent of the level of the discount factor.

The following public strategy achieves the upper bound of the strongly symmetric PPE payoff and often the upper bound of the symmetric PPE payoff:

(#)
- (1): Play the cooperative profile in the stage game.
- (2): If the signal is "good",go back to (1)
  If the signal is "bad", rangomize between going back to (1) and playing the Nash equilibrium forever using some randomization device.

The equilibrium payoff is given by the following formula, which is first derived in [1]:

$$\text{cooperative payoff} - \frac{\text{deviation gain}}{\text{likelihood ratio} - 1}$$

[1] In the following, strongly symmetric sequential equilibrium means a sequential equilibrium supported by strongly symmetric strategies. Strategies are strongly symmetric if players' behavior strategies is equivalent.

Symmetric squential equilibirum is a sequential equilibrum with the equilibirum payoff on the 45° degree line, but the strategy which generates that payoff can be asymmetric.

Likelihood ratio here is about a "bad" signal with respect to the deviation from the cooperative action profile. The second term measures the inefficiency in repeated partnership games with two public signals. This implies that the upper bound of (strongly) symmetric PPE payoff is larger if the deviation gain is smaller or the likelihood ratio is higher.

Now suppose that the likelihood ratio when one player does not cooperate is much higher than when both players cooperate. In other words, it is much easier to detect the other player's non-cooperative behavior when one plays non-cooperative action. Although it might be the case that some asymmetric strategy can generate the best symmetric payoff, just focus on strongly symmetric strategies such as described above for now. Players face a serious dilemma in terms of efficiency with PPE. To get closer to the cooperative outcome, players have to use the cooperative action profile frequently, but then they cannot use the profile with high likelihood ratio to detect the other player's deviation. If they try to use this action profile with the high likelihood ratio, then the strategies are likely to be asymmetric. It might be a strategy such as alternating between asymmetric profiles, which may give players the payoff far below the efficient level. As is shown in section 3, mixing helps to increase the likelihood ratio even within the class of strongly symmetric strategy, but with the cost of reducing the stage game payoff.

There is a way to resolve this conflict. Consider the following strategy: Mix between the cooperative action and the noncooperative action, but put most of the probability on the cooperative one, and punish the other player in some way only if you play the noncooperative action and observe a bad signal. Then, the stage game payoff is close to the cooperative one and only the action profile with the high likelihood ratio is used for punishment by this strategy pair.

Note that this strategy is not a public strategy, but a private strategy because player's continuation strategy does depend on one's own past action in addition to the public signal. Moreover, players' continuation strategy pair is not common knowledge after one period and a recursive structure is lost because players cannot observe the realization of behavior strategy by the other players. This actually explains why this kind of strategy has been very difficult to analyze within the current theoretical framework. The main contribution of this paper is to succeed in constructing equilibria using private strategy such as one described above and to show that private strategy sometimes works significantly better than any public strategy by using a signal structure in a more efficient way.

Section 2 describes the details of the model. In section 3, the upper bound of all PPE payoff, including mixed strategy PPE, is derived. In the section 4, a strongly symmetric sequential equilibrium is constructed with a private strategy, which I call private sequential equilibrium, and one sufficient condition is given where the private sequential equilibrium (PSE) payoff dominates the best symmetric payoff by PPE. Section 5 gives some example which illustrates a clear difference between PPE and PSE. In that example, the PSE payoff dominates not only the maximum symmetric PPE payoff, but also the whole PPE payoff set. In precise, a sequence of stage game is constructed in such a way that the PSE payoff converges to the efficient frontier, while the whole PPE payoff set shrinks to the stage game Nash equilibrium payoff. Section 6 discusses related literature and Section 7 concludes the paper.

2 The Model

Two players are taking part in a joint production. Players choose an effort level: $a_i \in A_i = \{H, L\}$ simultaneously. H and L can be regarded as a high effort and a low effort respectively. After they choose actions; they observe a public signal or outcome $\omega \in \Omega = \{g, b\}$. The public signal and player i's action determine player i's payoff at that period. Distribution of ω depends on how many players put effort in production. $0 < \pi^j < 1$ is a probability to observe b when j players choose L. It is assumed that $\pi^0 \leq \pi^1 \leq \pi^2$. This implies that the signal structure satisfies Monotone Likelihood Ratio Property(MLRP)[2].

Let $\Delta\pi^0 = \pi^1 - \pi^0$; $\Delta\pi^1 = \pi^2 - \pi^1$; and let $L^p = \frac{(1-p)\pi^1 + p\pi^2}{(1-p)\pi^0 + p\pi^1}$ be the likelihood ratio of the signal b with respect to the effort level when the other player is randomizing H and L with probability $1-p$ and p. This gives the ratio of how the signal b is likely to realize when a player plays L instead of H in such a situation.

It is assumed that $(1, 1), (1+\alpha, -\beta), (-\beta, 1+\alpha), (0, 0)$ is the payoff profile corresponding to the action profile $(H, H), (L, H), (H, L), (L, L)$ respectively.

Since I am interested in the situation where (1): the cooperative outcome (H, H) is not a stage game Nash equilibrium, (2): (L, L) is a Nash Equi-

[2] This assumption is made for simplicity. As long as the assumption is the paper is satisfied, the order of π^0, π^1, π^2 is not essential.

librium (3): (H, H) is efficient, it is assumed that α and β satisfies $\alpha > 0$, $\beta > 0$, and $\frac{1+\alpha-\beta}{2} < 1$.

For $t \geq 2$, $h^t = (\omega_1, \ldots, \omega_{t-1}) \in H^t = \{g, b\}^{t-1}$ is a $t-1$ period public history and $h_i^t = (a_{i,1}, \ldots, a_{i,t-1}) \in H_i^t = \{H, L\}^{t-1}$ is a private history. Hence the space of player i's history is $H_i = \bigcup_{t=1}^{\infty} (H_i^t \times H^t)$ with $H_i^1 \times H^1 = \emptyset$. Player i's (behavior) strategy σ_i is a mapping from H_i to ΔA_i.

I call σ_i a private strategy if there exists some history $(h_i'^t, h^t)$, $(h_i''^t, h^t)$, $(h_i'^t \neq h_i''^t)$ such that $\sigma_i(h_i'^t, h^t) \neq \sigma_i(h_i''^t, h^t)$. This is the complement of the set of public strategies in the whole space of behavior strategies. From now on, I call sequential equilibrium with private strategy private sequential equilibrium, denoted by PSE.

Let me note that restricting attention to public strategy is not so restrictive as it seems. As [2] noted, for any pure strategy sequential equilibrium, it is possible to construct a corresponding outcome equivalent pubic perfect equilibrium. So, as long as pure strategy sequential equilibrium is concerned, one can restrict attention to PPE without loss of generality. Private strategy may matter only when player's strategy depends on the past realization of one's own randomization on $A_i = \{H, L\}$.

3 The Upper Bound of PPE

In this section, I derive an analogue of the inefficiency result in Radner, Myerson, and Maskin[11] for this discrete version of partnership game.[3]

The upper bound of the pure strategy strongly symmetric PPE payoff is easy to obtain. Let $\overline{V}_{ps}$ be the best pure strategy symmetric PPE payoff. Since there are only two signals available, it is not possible to "reward" one player when the other player is "punished". Both player has to be punished at the same time when signal b is observed. So it is efficient to set the punishment level as small as the level exactly where players are indifferent between H and L. Of course, when signal g is observed, it is efficient to use $\overline{V}_{ps}$ again. These observations lead to an equation :[4]

[3] In [11], the action space is continuum.

[4] It is assumed that players can access to a public randomization device. This is an innocuous assumption because I am trying to get the upper bound of PPE, while we do NOT use any public randomization device later in the construction of a private sequential equilibrium. This assumption also implies the convexity of the PPE payoff set.

$$(1-\delta)\alpha = \delta\Delta\pi^1(\bar{v}_{ps} - \theta\bar{v}_{ps}) \tag{1}$$

A recursive formula for $\bar{v}_{ps}$ is also obtained:

$$\bar{v}_{ps} = (1-\delta) + \delta\left(\left(1-\pi^0\right)\bar{v}_{ps} + \pi^0\theta\bar{v}_{ps}\right) \tag{2}$$

Solving equations (1) and (2) for $\bar{v}_{ps}$ and θ, the following well-known formula is obtained.[5]:

$$\bar{v}_{ps} = 1 - \frac{\alpha}{L^0 - 1} \tag{3}$$

The upper bound of the pure strategy symmetric PPE payoff is the efficient stage game payoff minus the deviation gain over the likelihood ratio minus 1. Note that this value is bounded away from 1 independent of the discount factor. This is basically because punishment phase can start with the probability π^0 every period.

In the appendix, it is shown that a similar formula indeed gives the upper bound of the strongly symmetric PPE payoff Moreover, the best symmetric payoff is actually generated by the symmetric PPE for some parameter values.[6] The best mixed strategy symmetric PPE is obtained just by using a mixture of H and L with probability $(1-p)$ and p instead of using the profile (H, H) in (#). The equilibrium payoff is given by

$$1 - p - p\beta - \frac{(1-p)\alpha + p\beta}{L^p - 1} \tag{4}$$

The interpretation of this formula is exactly the same as before. It is the stage game payoff minus the deviation gain over likelihood ratio minus1

[5] See [1].

[6] This contrast with a case with a rich signal space[5]. For example, if there are three public signals available, then it might help to introduce the asymmetry to players' strategies to break the symmetry of information structure, which prevents players from punishing each deviator separately. Such trick is not useful here just because there are only two signals.

when the other player is mixing H and L with probability $1 - p$ and p in the cooperative phase. Note that if $p = 0$, then this is equivalent to (3). Why can mixing help to improve the best symmetric payoff even though it reduces the stage game payoff? It is because (1): deviation gain can become small if $\alpha > \beta$ or/and (2): the likelihood ratio may increase. Now let $p^* = \arg\max_{p \in [0,1]} 1 - p - p\beta - \frac{(1-p)\alpha + p\beta}{L^p - 1}$. The following is the formal statement with the strongly symmetric strategies:

Proposition 1 The bound of the strongly symmetric PPE payoff of this repeated partnership game is given by:

$$\overline{V}_s = \max\left\{1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*} - 1}; 0\right\}.$$

Proof: See Appendix.

As it should be clear from the construction of the equilibrium strategy, this bound is a tight one. Whether the stationary strategy described above obtains $1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*} - 1}$ or no cooperation is possible.

In order to get the bound of all the symmetric PPE payoff, I have to take care of the cases where the optimal strategy pare is asymmetric. If that possibility is taken account, the upper bound has to be modified in the following way:

Proposition 2 The bound of the symmetric PPE payoff of this repeated partnership game is given by:

$$\overline{V}_s = \max\left\{1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*} - 1}; \frac{1+\alpha-\beta}{2}; 0\right\}$$ and $\overline{V}_s \geq \frac{V_1^* + V_2^*}{2}$ for any PPE payoff $(V_1^*; V_2^*)$.

Proof: See Appendix.

Interestingly, when the asymmetric strategy achieves the best symmetric payoff, at least one player has to play L with probability 1 in the first period. The payoff $\frac{1+\alpha-\beta}{2}$ is the upper bound for such a case. The equilibrium where each player uses a different degree of mixture is not an efficient one. It is easy to pick up a set of parameters where $\frac{1+\alpha-\beta}{2}$ is really the bound and it is obtained by the asymmetric PPE where players play (H; L) (L; H) alternatively. However, this bound may not be tight. When π^n is linear in n; which is the case analyzed in detail by Fudenberg and Levin [?], The bound in Proposition 2 is tight in the sense that one of the three number $\left\{1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*} - 1}; \frac{1+\alpha-\beta}{2}; 0\right\}$ is the bound and the bound is really achieved by some strategy.

4 Construction of a Private Sequential Equilibrium

In this section, a private sequential equilibrium is constructed and compared to the bound of symmetric PPE obtained in the last section. The strategy is described by a machine $M_i = \langle Q_i, q_{i0}, f_i, \mu_i \rangle$. In this quadruple, $Q_i = \{q_{i,g}, q_{i,b}\}$ is the states of the machine with $q_{i,g}$ being the initial state. The level of mixture between H and L at each state is determined by a function $f_i : Q_i \to [0,1]$. For example, $f_i(q_{i,k})$ is the probability to play L when player i is in the state k. The transition function is $\mu_i : Q_i \times A_i \times \Omega \times \Omega_i \to Q_i$. Here $\omega_i \in \Omega_i$ is an outcome of a player i's personal randomization device and used to generate a behavior strategy after a certain type of history, which is specified later. Let $\Omega_i = \{0, 1\}$ be the space of the outcome of the randomization device and $Pr(\omega_i = 0) = 1 - \rho$ and $Pr(\omega_i = 1) = \rho$ where $\rho \in [0,1]$ can be chosen arbitrary. Note that the state transition depends on one's own action. Each machine M_i induces a mixed strategy (but not a behavior strategy). We denote by $\sigma_i(M_i)$ a behavior strategy corresponding to the mixed strategy generated by the machine M_i.[7][8]

I use the following specific transition function μ_i with ρ:

$$\mu_i^{\rho}(q_{i,g}, a_i, \omega, \omega_i) = \begin{cases} q_{i,g} & \text{if } (a_i, \omega) \neq (L, b) \\ q_{i,b} & \text{if } (a_i, \omega) = (L, b) \end{cases}$$

$$\mu_i^{\rho}(q_{i,b}, a_i, \omega, \omega_i) = \begin{cases} q_{i,b} & \text{if } (a_i, \omega) \neq (L, g) \\ q_{i,g} & \text{if } (a_i, \omega, \omega_i) = (L, g, 1) \\ q_{i,b} & \text{if } (a_i, \omega, \omega_i) = (L, g, 0) \end{cases}$$

Players are using ω_i to randomize between staying $q_{i,b}$ and moving to $q_{i,g}$ when player i's action is L and a signal g is observed. Players control the level of punishment by choosing ρ. Since the strategy is strongly symmetric, the subscript i is omitted in the following.

Most important element of the strategy $\sigma(M)$ is $f(q_k)$, $k = g, b$. They are defined to be a solution to the following equations.

(*)

$$V_g = (1 - \delta)(1 - p_g - p_g\beta) + \delta\left[(1 - p_g)V_g + p_g\left\{\left(1 - \pi^1\right)V_g + \pi^1 V_b\right\}\right] \quad (5)$$

[7] Aumann(1964)[3]

[8] This automaton can be "purified" by introducing more private signal and expanding the state space using a sophisticated transition function.

$$(1-\delta)\left((1-p_g)\alpha + p_g\beta\right) = \delta p_g \Delta\pi^1 (V_g - V_b) \tag{6}$$

$$V_b = (1-\delta)(1-p_b-p_b\beta) + \delta\left[(1-p_b)V_b + p_b(1-\rho)V_b + p_b\rho\left\{\left(1-\pi^1\right)V_g + \pi^1 V_b\right\}\right] \tag{7}$$

$$(1-\delta)\left((1-p_b)\alpha + p_b\beta\right) = \delta p_b\rho \Delta\pi^1 (V_g - V_b) \tag{8}$$

If the solution $\left(p_b^*, p_g^*\right)$ of these equations are in $[0,1]$; then these numbers can be used for the function $f_i(q_{i,k})$ and generate a behavior strategy $\sigma(M)$: Then each equation has a natural interpretation. (5) is player j's continuation payoff if player $i \neq j$ is using machine M_i and in state g: The first term is a stage game payoff when player j plays H: The second term is the continuation payoff if the continuation payoff of player j is given by V_k when player i is in the state k: (6) is the indifference condition between playing H and L when he other player is in the state k. (7) and (8) can be interpreted in the same way as(5) and(6). These equations imply that whatever state the other player is in, a player is indifferent between playing H and playing L in the current period provided that one's continuation payoff is completely determined by the other player's state. Moreover, looking at these equations carefully, it can be seen that a player's continuation payoff is actually completely determined by the other player's state. So, a player cannot control one's own payoff at all. Any payoff difference one can make in the current period is offset by the difference of the continuation payoff. As a consequence, this strategy makes the other player indifferent between all the repeated game strategy and that in turn guarantees that this strategy is sequentially rational to itself.[9] Note that the logic is similar to the one for a totally mixed strategy equilibrium in a finite normal form game.

Lemma 3 If $\left(\left(p_g^*, p_b^*\right), \rho^*, V_g^*, V_b^*\right)$ solves $(*)$ and $0 \leq p_k^* \leq 1$, $k = g, b$ and $\rho^* \in [0,1]$, then the strongly symmetric strategy pair $(\sigma_1(M^*), \sigma_2(M^*))$ with

[9] The idea of strategy which makes the other player indifferent for all repeated game strategies is first found by Piccione[9] in the context of private monitoring to deal with private information. His strategy basically consists of a infinite state automaton. This paper and Ely and Välimäki[4], which also deals with private monitoring, are the first papers to show independently that the same idea can be built into a finite state automaton.

$f_i(q_{i;k}) = p_k^*$ and $\mu_i = \mu^{\rho *}$ for all $k = g; b$ and i is a sequential equilibrium with a system of belief compatible with $(\sigma_1(M^*); \sigma_2(M^*))$ and the equilibrium payoff is $(V_g^*; V_g^*)$.

Proof:

Suppose that a player, say player 1, is using this machine M^* and player 2 is playing always H after any history. If player 1 is in the state 0, player 2's expected average payoff is $V_g^* = 1 - p_g^* - p_g^{*\beta} - \frac{(1-p_g^*)\alpha + p_g^{*\beta}}{L^{\mu} - 1}$: If player 1 is in the state 1, player 2's expected average payoff is $V_b^* = 1 - p_b^* - p_b^{*\beta} + \frac{(1-p_b^*)\alpha + p_b^{*\beta}}{L^{\mu} - 1}\frac{1-\pi^{\mu}}{\pi^{\mu}}$: Now suppose that player 1 is actually in state g and there exists a pure strategy for player 2 which gives more (or less) payoff than V_g^*: Then, thanks to continuity at infinity, I can replace this strategy with another strategy which is the same as the original strategy until some period and goes back to "always H" thereafter, keeping a payoff more(or less) than V_g^*: Let the period to go back to "always H" be N: By indifference conditions (6) and (8), whatever state player 1 is going to be in the period N; players are indifferent between playing H and L in the period $N - 1$: So, I can replace this strategy with another strategy which goes back to "always H" in the period $N - 1$ with the same expected average payoff. This induction goes back to the initial period and leads to a contradiction. [10] So, any pure strategy, henceforth any mixed strategy, generates the same expected average payoff $V_g^* = 1 - p_g^* - p_g^{*\beta} - \frac{(1-p_g^*)\alpha + p_g^{*\beta}}{L^{\mu} - 1}$: Since the same result holds when player 1 is in the state b, $\sigma(M^*)$ is sequentially rational if the other player is using M^*: This implies that $(\sigma(M^*); \sigma(M^*))$ is a symmetric sequential equilibrium with the belief corresponding to the machine M^* $\blacksquare$

What is going on here? Since the outcome of a player's randomization is a private information, a player never know what is the other player's continuation strategy or which state the other player is in after the initial period. So, in principle, players have to update the probability of the other player being in state g or b and make sure that their continuation strategy is actually sequentially rational. It is usually very difficult to guarantee sequential rationality especially off the equilibrium path and this problem makes it difficult to deal with private information in discounted repeated

[10] The argument used here is the one shot deviation principle, which is usually invalid when private information is present. Here, since the player's payoff does not depend on their belief, the accumulated private information is useless.

games. However, players do not care about their belief dynamics here because whatever state the other player is in, a player's expected payoff cannot be affected for themselves. Although whether the other player is in state g or b matters for player's expected payoff level, it does not matter in terms of player's incentive.

The following main proposition shows that for δ close to 1, I can find a solution $\left(p_g^*(\delta), p_b^*(\delta), \rho^*(\delta), V_g^*(\delta), V_b^*(\delta)\right)$ parameterized by δ for the above equations $(*)$ such that $p_g^*(\delta) \approx 0$ and $p_b^*(\delta) = 1$ with the appropriately adjusted $\rho^*(\delta) \in [0,1]$. Since I can derive $V_g^* = 1 - p_g^* - p_g^*\beta - \frac{(1-p_g^*)\alpha + p_g^*\beta}{L^1 - 1}$ as is derived in lemma 3 after some manipulation of equations $(*)$, this result implies that the payoff arbitrary close to $1 - \frac{\alpha}{L^1-1}$ is achieved as a sequential equilibrium for such δ using the private strategy generated by $M^*(\delta)$ which is based on $\left(p_g^*(\delta), p_b^*(\delta), \rho^*(\delta)\right)$. Note that this formula uses the likelihood ratio L^1 instead of L^0, while the other components are exactly the same as in the equilibrium payoff of the pure strategy strongly symmetric equilibrium. Here players spend most of their time in the state q_g playing H, but the punishment happens only after (L, b), which allows me to use L^1 instead of L^0.

Proposition 4 Suppose that M $\pi^1 > \pi^1\alpha + (1-\pi^2)\beta$[11]. Then for any $\eta > 0$, there exists a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a strongly symmetric strategy pair $(\sigma(M(\delta)), \sigma(M(\delta)))$ parameterized by δ, which is a sequential equilibrium with a compatible belief system and generates the symmetric equilibrium payoff $(V(\delta), V(\delta))$ such that $V(\delta) > 1 - \frac{\alpha}{L^1-1} - \eta$.

Proof:

Given that $0 < \delta < 1$, we can derive the following system of equations equivalent to $(*)$.

$(**)$

$$V_g = 1 - p_g - p_g\beta - \frac{(1-p_g)\alpha + p_g\beta}{L^1 - 1} \tag{9}$$

$$V_b = 1 - p_b - p_b\beta + \frac{(1-p_b)\alpha + p_b\beta}{L^1 - 1}\frac{1-\pi^1}{\pi^1} \tag{10}$$

[11] This assumption is equivalent to $V_g > V_b$ where V_g and V_b is derived as a function of parameters by solving this system of equations.

$$(1-\delta)\left((1-p_g)\alpha + p_g\beta\right) = \delta p_g \Delta\pi_1 (V_g - V_b) \tag{11}$$

$$p_b = \frac{p_g\alpha}{p_g(\alpha-\beta) + \rho\{(1-p_g)\alpha + p_g\beta\}} \tag{12}$$

Once p_g is obtained, then V_g, V_b, p_b are obtained by (9), (10) and (12) respectively. Since ρ can be an arbitrary number between 0 and 1, it is set to be $\frac{p_g}{(1-p_g)\alpha+p_g\beta}$ so that $p_b = 1$. This is actually between 0 and 1 if p_g is between 0 and 1. Substituting (9), (10) and (12) for V_g, V_b, p_b, I can get a quadratic equation, whose solution can be used for p_g:

$$F(x;\delta) = c_2(\delta)x^2 + c_1(\delta)x + c_0(\delta) = 0 \tag{13}$$

with

$$c_2(\delta) = \delta\{\pi_2(1+\beta) - \pi_1(1+\alpha)\} \tag{14}$$

$$c_1(\delta) = (1-\delta)(\beta-\alpha) + \delta\left\{\pi^1\alpha + \left(1-\pi^2\right)\beta - \Delta\pi^1\right\} \tag{15}$$

$$c_0(\delta) = (1-\delta)\alpha \tag{16}$$

$(x;\delta) = (0;1)$ is clearly a solution. Since $\frac{\partial F}{\partial x}|_{(x;\delta)=(0;1)} \neq 0$ with the assumption $\pi^1\alpha + (1-\pi^2)\beta - \Delta\pi^1$, Implicit function theorem can be applied to get a C^1 function $p_g(\delta)$ around $\delta = 1$ with
$\frac{dp_g(1)}{d\delta} = -\frac{\frac{\partial F}{\partial \delta}|_{(x;\delta)=(0;1)}}{\frac{\partial F}{\partial x}|_{(x;\delta)=(0;1)}} = \frac{\alpha}{\pi^1\alpha+(1-\pi^2)\beta-\Delta\pi^1}$

Since $\pi^1\alpha + (1-\pi^2)\beta - \Delta\pi^1 < 0$ by assumption, $p_g(\delta) \in (0;1)$ for some small interval $(\underline{\delta};1)$ and $p_g(\delta) \to 0$ as $\delta \to 1$. V_g and V_b can be derived from (9) and (10) with $p_g(\delta)$ and $p_b = 1$. $(p_g; p_b; \rho; V_g; V_b) = \left(p_g(\delta); 1; \frac{p_g(\delta)\beta}{(1-p_g(\delta))\alpha+p_g(\delta)\beta}; V_g(\delta); \frac{\beta(1-\cdots}{\pi_2-\cdots}\right.$ is a parametrized solution for $(**)$. Now by lemma 3, $(\sigma(M(\delta)); \sigma(M(\delta)))$ with $f(q_g) = p_g(\delta)$, $f(q_b) = 1$, and $\rho(\delta) = \frac{p_g(\delta)\beta}{(1-p_g(\delta))\alpha+p_g(\delta)\beta}$ is a sequential equilibrium with a compatible belief and the equilibrium payoff is $V_g(\delta)$, which converges to $1 - \frac{\alpha}{L^1-1}$ as $\delta \to 1$. For any $\varepsilon > 0$, we can pick $\underline{\delta}$ such that for all $\delta \in (\underline{\delta};1)$, $(\sigma(M(\delta)); \sigma(M(\delta)))$ generates the equilibrium payoff $V(\delta)$ more than $1 - \frac{\alpha}{L^1-1} - \varepsilon$. ∎

Since the pure strategy strongly symmetric PPE payoff is $1 - \frac{\alpha}{L^0 - 1}$ with high δ if $\frac{\alpha}{L^0-1} < 1$; $L^1 > L^0$ is necessary for this PSE to dominate the best symmetric PPE payoff. Another necessary condition is $\mathbf{M}\pi^1 > \pi^1\alpha + (1-\pi^2)\beta$ which is used to construct the PSE. The next theorem gives a simple sufficient condition for the PSE to dominate the best symmetric PPE payoff.

Proposition 5 If $L^1 > L^0$; $\mathbf{M}\pi^1 > \pi^1\alpha + (1-\pi^2)\beta$; $\beta > \alpha$; and $\frac{1+\beta-\alpha}{2} > \frac{\alpha}{L^1-1}$; then there exists a $\underline{\delta}$ such that for all $\delta \in (\underline{\delta}, 1)$; the equilibrium payoff generated by $(\sigma(M(\delta)), \sigma(M(\delta)))$ is larger than ∇_s:

Proof: I just need to show that $1 - \frac{\alpha}{L^1-1} > \nabla_s$:
(1): $1 - \frac{\alpha}{L^1-1} > 0$
By $\mathbf{M}\pi^1 > \pi^1\alpha + (1-\pi^2)\beta$;

$$1 - \frac{\alpha}{L^1 - 1} > \frac{(1-\pi^2)\beta}{\mathbf{M}\pi^1} > 0$$

(2): $1 - \frac{\alpha}{L^1-1} > \frac{1+\alpha-\beta}{2}$

$$1 - \frac{\alpha}{L^1 - 1} - \frac{1+\alpha-\beta}{2} = \frac{1}{2} - \frac{\alpha}{L^1-1} + \frac{\beta-\alpha}{2}$$

This is strictly positive by assumption.
(3): $1 - \frac{\alpha}{L^1-1} > 1 - p - p\beta - \frac{(1-p)\alpha + p\beta}{L^p - 1}$ for all $p \in [0,1]$

Let $M(p) = 1 - \frac{\alpha}{L^1-1} - \left\{1 - p - p\beta - \frac{(1-p)\alpha+p\beta}{L^p-1}\right\}$: Then it is easy to show that $M'(p) < 0$ for all $p \in [0,1]$ because $L^1 > L^0$ and $\beta > \alpha$; observing the fact that $L^p - 1$ is always less than $L^1 - 1$:

These imply that $1 - \frac{\alpha}{L^1-1} > \nabla_s = \max\left\{1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*}-1}; \frac{1+\alpha-\beta}{2}; 0\right\}$ ■

Although there are many restrictions on the structure of the stage game, there still exists a generic set of parameters, which satisfies all these restrictions. In the next section, I pick a example satisfying these payoff restriction, where the PSE is much more efficient than any PPE.

5 An Example.

Set $\alpha = \kappa > 0$ and $\beta = 1 - 2\kappa > 0$.

Also Set

$$\begin{cases} \pi_0 = \frac{1}{2} \\ \pi_1 = \frac{1}{2} + \varepsilon_t \\ \pi_2 = 1 - \varepsilon_t \end{cases} \text{ and } \varepsilon_t \to 0.$$

Note that these numbers guarantee that the assumption for Proposition 4 is satisfied for small ε_t. As ε_t goes to 0; it gets more difficult to detect the other players deviation if $(H;H)$ is played. At a certain level of ε_t; players have to randomize to support any strongly symmetric PPE payoff other than 0. If ε_t gets much closer to 0; then simply Nash repetition is only the feasible strongly symmetric equilibrium. This is because the stage game payoff becomes negative as players put too much weight on L for detecting deviations effectively.

You can see why strongly symmetric strategies do not work by examining the formula of the payoff $:1 - p - p\beta - \frac{(1-p)\alpha + p\beta}{L_t^p - 1}$. This is $1 - 2p(1-\kappa) - \frac{(1-p)\kappa + p(1-2\kappa)}{p} = -\left(2p(1-\kappa) + 3\kappa - \frac{\kappa}{p}\right)$; which is clearly negative if k is small enough (if $k < \frac{8}{17}$)

An another candidate of the PPE upper bound is simply $\frac{1+\alpha-\beta}{2} = \frac{3}{2}\kappa$. So there exists a $\underline{t}(\kappa)$ such that the sum of any PPE payoff is bounded by 3κ for all $t = \underline{t}(\kappa)$.

On the other hand, the upper bound of the private sequential equilibrium payoff converges to $1 - \frac{\alpha}{L_t^1 - 1} \to 1 - \kappa$. Since κ can be an arbitrary small positive number, I can construct an example where the PSE approximates the efficient outcome arbitrarily close and the whole PPE payoff set is contained in an arbitrary small neighborhood of Nash repetition payoff $(0;0)$.

6 Related Literature and Comments

Since one of the important point in constructing PSE is to deal with private information generated endogenously by private strategies, this paper is closely related to literature on repeated games with private monitoring, where signals are private information. In particular, Ely and Välimäki[4] independently created a similar strategy which is also described by a finite automaton. The idea behind these strategies are the same as one in Piccione[9], where the strategy is basically an automaton with countably in…nite number of states.

However there is a critical di¤erence between this paper and Ely and Välimäki[4]. While players play a pure action at each state in their paper, players have to randomize at the initial state in this paper to use the private action-signal pro…le with the highest likelihood ratio, which is the main focus of this paper. This idea of e¢cient monitoring is an old and simple idea which lies at the heart of the analysis of moral hazard. This paper suggests a way to use private information to enhance informational e¢ciency in repeated games.

As for e¢cient detection based on randomization, Kandori[7] applied the same idea to an example of repeated partnership game. His example corresponds to the case where $¼^1$ is 0 and $¼^0$, $¼^2$ is between 0 and 1 in my model: With this parameter, a similar strategy is actually able to achieve e¢ciency. This is natural based on the result in this paper because it corresponds to $L^1 = 1$ in my model. There are two comments on Kandori[7] worth mentioning.

First, if $¼^1$ is 0; the timing to go to the punishment phase is common knowledge. If a player plays L and observe b; then it has to be the case that the other player also plays L and observe b: This implies that players do not have to face with serious problems associated with private information and they can use the Nash repetition as a punishment. This is why his construction requires $¼^1 = 0$: On the other hand, my private strategy works in a full support environment with some additional assumptions on the parameter.

Secondly, it is interesting to see that what is going to happen if I take a sequence of $¼^1$ converging to 0, keeping $¼^0$ and $¼^2$ constant. As long as the assumption on the parameter is satis…ed, it is easy to see that the private strategy in this paper can be constructed for each such $¼^1$:[12] Is this sequence of strategies converging to Kandori's strategy? The answer is No. Since players have to be indi¤erent between H and L even when the other player is in state b; which is not the property of his model, the continuation payo¤ V_b is bounded away from 0 independent of discount factor. This property is re‡ected in the fact that $¼^1 = 0$ is not su¢cient condition for the construction of the private strategy in this paper. It also makes a di¤erence on the degree of mixture at the state g for a …xed discount factor. However, this di¤erence

[12] As mentioned in the beginning, the assumption $¼^0$ **5** $¼^1$ **5** $¼^2$; which is violated here, is not important for the construction of the private strategy in this paper.

disappears as $\delta \to 1$ because $p_g(\delta)$ converges to 0 in both model.

7 Discussion

In this paper, since the strategy is constructed in such a way that players are always indifferent among all strategies, they can make their behavior depend on not only public information but also private information, namely the past realization of one's own action. Because of this prevailing indifference, this PSE is robust to a various sort of change in information structure. First, if parameters such as $(\alpha; \beta; \pi^0; \pi^1; \pi^2)$ change slightly, then, of course, there exists a PSE close to the original PSE. Secondly, suppose that each player can observe additional signals which is informative about the other player's state. This does not change anything because a player does not care what the other player's state is. Finally, note that this strategy works even if there is no public signal at all, in which case PPE does not have any bite at all by definition. Suppose that the stage game is perturbed in the following way: public signals follow the same distribution as before, but they are not observable to players. Players observe a public signal plus private noise. Each player observes the true public signal most of the time, but observe the wrong one with small probability. The private strategy in this paper works even in this setting. Again, this is because players do not care about the other player's state. Formally, this model belongs to repeated games with private monitoring. However, this is not a model of repeated games with almost perfect monitoring, which has been the main focus of private monitoring literature because of its tractability. This is a repeated games with almost public monitoring (Mailath and Morris[8]). Hence this strategy can serve as a good example of strategy which works with almost public monitoring without almost perfectness. Observe how this strategy is related to the conditions suggested by Mailath and Morris[8]; which plays an important for a PPE to remain an equilibrium with an almost public monitoring when a public signal structure is perturbed slightly. The private strategy satisfies the necessary condition (connectedness) but does not satisfy the sufficient condition (finite memory).

There is one important open question left. Even though it is shown that PSE is much more efficient than PPE in some case, we have no idea what is really the best symmetric sequential equilibrium payoff. More insight is needed to see whether RMM's inefficiency result extends to a class of sequential equilibrium or some efficiency result stands out surprisingly.

appendix:

Proof of Proposition 1

In this proof, it is shown that the strongly symmetric PPE achieves the best symmetric PPE payoff among a large class of asymmetric strategies, not just strong symmetric strategies. For this purpose, I do not restrict attention to strongly symmetric strategies from the beginning.

Let $\bar{v}_s$ be the best symmetric PPE payoff. First, at least one player has to play H with positive probability in the initial period. Otherwise, $\bar{v}_s$ is 0; the payoff by Nash repetition. Suppose that both player plays H with positive probability in the initial period for now, which is a necessary condition for the strongly symmetric PPE to achieve any outcome other than the stage game Nash. Let p_i be the probability for player i to play L in the initial period. Let $\bar{v}_1$ and $\bar{v}_2$ be the equilibrium payoff such that $\bar{v}_s = \frac{\bar{v}_1+\bar{v}_2}{2}$[13] and $(\sigma_1^*; \sigma_2^*)$ be a strategy profile supporting that payoff profile.:

$\bar{v}_1$ and p_2 satisfy the following inequalities derived from the recursive equation:

$$\bar{v}_1 \geq (1-\delta)(1-p_2-p_2\beta) + \delta\left[(1-p_2)\left\{\left(1-\pi^0\right)v_1^* + \pi^0\theta_1 v_1^*\right\} + p_2\left\{\left(1-\pi^1\right)v_1^* + \pi^1\theta_1 v_1^*\right\}\right] \tag{17}$$

and the incentive constraint:

$$(1-\delta)((1-p_2)\alpha + p_2\beta) = \delta\left\{(1-p_2)\Delta\pi^0 + p_2\Delta\pi^1\right\}(1-\theta_1)v_1^* \tag{18}$$

v_1^* is the payoff generated by the continuation strategy of $(\sigma_1^*; \sigma_2^*)$ after the signal g. $\theta_1 v_1^*$ might be higher than the true continuation payoff generated by $(\sigma_1^*; \sigma_2^*)$ after the signal b because it is set as large as possible to satisfy the incentive constraint to let player 1 to play H in the initial period

Similar inequality and equation holds for player 2:

$$\bar{v}_2 \geq (1-\delta)(1-p_1-p_1\beta) + \delta\left[(1-p_1)\left\{\left(1-\pi^0\right)v_2^* + \pi^0\theta_2 v_2^*\right\} + p_1\left\{\left(1-\pi^1\right)v_2^* + \pi^1\theta_2 v_2^*\right\}\right] \tag{19}$$

[13] Here I take account of the possibility that the best symmetric payoff is achieved by randomizing between asymmetric equilibriua using a public randomization device.

$$(1-\delta)\{(1-p_1)\alpha + p_1\beta\} = \delta\left[(1-p_1)\cdot\pi^0 + p_1\cdot\pi^1\right](1-\mu_2)v_2^* \quad (20)$$

Adding (17) and (19) and using $v_1^* + v_2^* \le \bar{v}_1 + \bar{v}_2$; we can get

$$\bar{v}_1 + \bar{v}_2 \le 1 - p_i - p_i\beta - \frac{\delta\{(1-p_i)\pi_0 + p_i\pi_1\}(1-\mu_j)v_j^*}{1-\delta} + 1 - p_j - p_j\beta - \frac{\delta\{(1-p_j)\pi_0 + p_j\pi_1\}(1-\mu_i)v_i^*}{1-\delta}$$

Substituting (18) and (20) into this equation ,

$$\bar{v}_1 + \bar{v}_2 \le 1 - p_i - p_i\beta - \frac{(1-p_i)\alpha + p_i\beta}{L^{p_i}-1} + 1 - p_j - p_j\beta - \frac{(1-p_j)\alpha + p_j\beta}{L^{p_j}-1}$$

Note that the bound of the player 1's(2's) payoff only depends on $p_2\,(p_1)$: Then, $p_1 = p_2 = p^*$ gives the optimal bound of $\bar{v}_1 + \bar{v}_2$ and

$$\bar{v}_s = \frac{\bar{v}_1 + \bar{v}_2}{2} \le 1 - p^* - p^*\beta - \frac{(1-p^*)\alpha + p^*\beta}{L^{p^*}-1}$$

It is clear that this bound is supported by the strongly symmetric strategy PPE where randomizing H and L with $(1-p^*; p^*)$ is used instead of $(H;H)$ in (#) and that $\bar{v}_s = \bar{v}_1 = \bar{v}_2$. ■

This proof assumes that players can access to some public randomization device, but it turns out they do not really need one. Another fact which becomes clear from the proof is that if any asymmetric profile is used to support $\bar{v}_s$; then some player has to play L with probability 1 in the initial period. This fact makes the proof of the next proposition simple.

Proof of 2

Consider the case where one player play L with probability 1 in the initial period. Suppose that this player is player 2 without loss of generality. The payoff profile $(\bar{v}_1; \bar{v}_2)$ satisfies $\bar{v}_s = \frac{\bar{v}_1 + \bar{v}_2}{2}$ as before.

The recursive equations for players are:

$$\Pi_1 = -(1-\delta)(1-p_1)\beta + \delta\left[(1-p_1)\left(\left(1-\pi^1\right)v_1^{*} + \pi^1 v_1^{**}\right) + p_1\left(\left(1-\pi^2\right)v_1^{*} + \pi^2 v_1^{**}\right)\right] \tag{21}$$

$$\Pi_2 = (1-\delta)(1-p_1)(1+\alpha) + \delta\left[(1-p_1)\left(\left(1-\pi^1\right)v_2^{*} + \pi^1 v_2^{**}\right) + p_1\left(\left(1-\pi^2\right)v_2^{*} + \pi^2 v_2^{**}\right)\right] \tag{22}$$

Since $v_1^{*} + v_2^{*} \leq \Pi_1 + \Pi_2$ and $v_1^{**} + v_2^{**} \leq \Pi_1 + \Pi_2$; adding 21 and 22,

$$\Pi_1 + \Pi_2 \leq (1-p_2)(1+\alpha-\beta) \leq 1+\alpha-\beta$$

So, $\Pi_s \leq \frac{1+\alpha-\beta}{2}$

$\blacksquare$

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