Bargaining Equilibrium in a Non-Stationary Environment

by

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Abstract. In this paper we study an alternating-offers bargaining model in which the set of possible utility pairs evolves through time in a non-stationary, but smooth manner. In general there exists a multiplicity of subgame perfect equilibria. However, we show that in the limit as the time interval between two consecutive offers becomes arbitrarily small, there exists a unique subgame perfect equilibrium. Furthermore, and more importantly, we derive a powerful characterization of the unique (limiting) subgame perfect equilibrium payoffs, which should prove especially useful in applications. We then explore the circumstances under which Nash's bargaining solution implements this bargaining equilibrium. Finally, we extend our results to the case when the players have time-varying inside options.

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1 Introduction

There is a large literature on decentralised trade in frictional markets where the terms of trade are determined by bargaining. Typically the matching literature adopts the static Nash bargaining approach to determine the terms of trade; see, for example, Pissarides (1990). This is clearly reasonable in steady state situations where payoffs do not change over time. But the matching literature frequently considers non-steady state equilibria. For example, it might assess the transitional impact of some change in government policy on market behaviour. In such dynamical equilibria, the terms of trade will change over time. But unless those price changes are dynamically consistent, the predicted transitional equilibrium dynamics will make little sense.

The critical advantage of the Rubinstein approach (cf., Rubinstein (1982)) over the static Nash bargaining approach is that by explicitly describing a bargaining game, it can identify dynamically consistent trading prices outside of steady state. This paper formally reconsiders the original Rubinstein alternating offers bargaining game, but without the stationarity assumption. Instead payoffs are assumed to evolve in a reasonably arbitrary, though deterministic, manner. The focus is on considering the limiting properties of these equilibria as Δ, the time interval between two consecutive offers, becomes small. It is shown that this approach is surprisingly tractable and should be adopted in matching equilibria where agents face a truly dynamic bargaining problem.

Throughout we retain the standard assumptions made in the bargaining literature — that of concave, shrinking and vanishing Pareto frontiers — but allow the set of possible utility pairs to evolve through time in a *non-stationary* manner. Binmore (1987, Section 6) also considers this case and shows through an example that for any $\Delta > 0$, a continuum of subgame perfect equilibria (SPE) is possible. His example implies that once the stationarity assumption is dropped, the indeterminacy of the bargaining problem may re-appear. It also suggests that the set of equilibria may not converge to anything useful as $\Delta \to 0$.

Although we drop the stationarity assumption, we also require that the Pareto frontiers evolve *smoothly* through time. As in Binmore (1987), multiple SPE are possible for any given $\Delta > 0$, but we show that as $\Delta \to 0$, the set of SPE necessarily converges to a unique (limiting) SPE. This is quite nice, because — for well-known reasons (cf., for example, Binmore (1987, Section 8) and Muthoo (1999, Section 3.2.4)) — this limiting case is perhaps the most persuasive one to focus attention upon. Further, the limiting outcome is described by a simple differential equation which has convenient geometric properties.

Of course, as using the static Nash bargaining approach is simpler, one might prefer using it in a dynamic framework. But as Coles and Wright (1998) demonstrate, using the Rubinstein-type approach to determine the terms of trade is not only tractable for dynamic applications, it also supports qualitatively different equilibrium behaviour. They consider a monetary economy where should a buyer and seller meet, they negotiate over some 'pie'. By agreeing some partition (x_b, x_s), each obtains instantaneous payoff $u(x_i)$, $i = b, s$. But these traders do not then exit the market. Instead they exchange money and return to the market. If $v_i(t)$ denotes the expected payoff to an unmatched agent i $(i = b, s)$ at time t in a market equilibrium, then the payoff to reaching agreement x_i at time t (discounted back to time zero) is $e^{-rt}[u(x_i) + v_{-i}(t)]$. The critical feature is that non-steady state implies that the *shape* of the pie is time-varying; i.e., the Pareto frontier describing the set of efficient agreements evolves non-homothetically over time — the bargaining problem is non-stationary. Coles and Wright (1998) demonstrate that in the continuous time limit, monetary equilibria with strategic bargaining can exhibit trading cycles. In contrast, trading cycles are ruled out by the static Nash bargaining approach. This is worrisome as the primary insight is that monetary trade with market frictions is potentially destabilising — the value of money depends on what you expect others will trade for it in the future. Strategic bargaining generates dynamically consistent trading prices — the equilibrium partition at time t depends on future v_i. This is not the case with a myopic Nash bargaining rule.

Clearly such dynamic consistency issues will arise in any matching equilibria where agents do not leave the market after trade. For example, consider the real estate market where the government increases interest rates. In a non-competitive market, this might affect the market equilibrium — say fewer will choose to move house. If the house seller is also trying to buy a house elsewhere, the fall in turnover may change this seller's trading options and so change the value of the sale to the seller. Such equilibrium effects change the 'shape' of the pie and outside of a steady state, agents should anticipate how the market might change over time and use that information when bargaining over price.

This paper essentially clarifies and extends the arguments of Coles and Wright (1998) in three important ways. First, Coles and Wright (1998) did not establish that their 'differential equation' describes the limiting equilibrium to the bargaining game as

$\Delta \to 0$. They simply assumed it to be the case. This is not obvious given Binmore's continuum example. We establish a formal Convergence Theorem assuming the Pareto frontiers evolve *smoothly* through time.

Second, Coles and Wright (1998) provide a uniqueness argument which applies only to a special case; that the payoffs are additively separable (i.e., are of the form $u_i(x) + v_i(t)$) and that the $v_i(t)$ converge to some limit as $t \to \infty$. We do not impose these restrictions, especially as the latter would require that the underlying market equilibrium converges to a steady state (which is formally inconsistent with their limit cycle example, and to extended models which allow for say endogenous growth and/or technology shocks). In particular, we believe our uniqueness proof is quite powerful and expect it will generalise to other more complicated cases. The argument is based on a Liapunov-type function whose structure is closely related to the Nash bargaining product.

Finally, this paper provides a nice geometric interpretation for the limiting equilibrium which shows how the strategic bargaining approach and the Nash bargaining approach are properly related.

Other related work includes Merlo and Wilson (1995) and Cripps (1998). Those papers assume two agents negotiate over some pie (x_1, x_2) satisfying $x_1 + x_2 \leq M_t$, where if agreement is reached at time t, agent i's payoff is $e^{-rt}u_i(x_i)$. If M_t evolves deterministically, then that preference structure is a special case of those considered in this paper — it describes a one-shot bargaining game where given agreement, the traders then exit the market for good. In essence, those papers describe an optimal tree-felling problem where M_t evolves over time according to an (exogenous, stationary) Markov process. The frameworks are related but the issues are quite distinct. Indeed, Merlo and Wilson (1995) establish uniqueness of equilibrium for any $\Delta > 0$, which is not the case with non-stationary bargaining.

The rest of this paper is organized as follows. In Section 2 we lay down the model, and in Section 3 we analyze its SPE. Then, in Section 4 we study the relationship between the unique (limiting) SPE payoff pair and Nash's bargaining solution. An application of our results to a bargaining situation in which the players have time-varying discount rates is provided in Section 5. In Section 6 we extend our results to the case when the players have time-varying inside options. We conclude in Section 7.

2 The Model

Two players, A and B, bargain according to an alternating-offers procedure, where the set $\Omega(t)$ of possible utility pairs available at time t is a non-empty subset of $\Re^2$. Bargaining begins at time s, where the players negotiate according to the following procedure. At time $s + n\Delta$ (where $n \in \mathbb{N} \equiv \{0, 1, 2, \dots\}$, and $\Delta > 0$), player i makes an offer to player j ($j \neq i$), where $i = A$ if n is even (i.e., $n = 0, 2, 4, 6, \dots$) and $i = B$ if n is odd (i.e., $n = 1, 3, 5, \dots$). An offer at time $s+n\Delta$ is a utility pair (u_A, u_B) from the set $\Omega(s + n\Delta)$. Player j then decides whether to accept or reject the proposed offer. If she accepts the offer, then the bargaining game ends. Otherwise, Δ time units later, at time $s + (n + 1)\Delta$, player j makes a counteroffer to player i. This process of making offers and counteroffers continues until an offer is accepted, at which point the game ends with agreement being secured on some utility pair.

The payoffs are as follows. If the players reach agreement at time $s+n\Delta$ (where $n \in \mathbb{N}$) on $(u_A, u_B) \in \Omega(s + n\Delta)$, then player i's ($i = A, B$) payoff is u_i. On the other hand, if the players perpetually disagree (i.e., each player always rejects any offer made to her), then each player obtains a payoff of zero.

Let $\Omega^P(t)$ denote the Pareto frontier at time t — that is, the set of Pareto efficient utility pairs available at time t.[1] We assume that $\Omega^P(t)$ is a connected set. Furthermore, there exists $\bar{u}_A^t > 0$ and $\bar{u}_B^t > 0$ such that $(0, \bar{u}_B^t) \in \Omega^P(t)$ and $(\bar{u}_A^t, 0) \in \Omega^P(t)$. For convenience, we describe this frontier by a function ϕ where $u_B = \phi(u_A, t)$ if and only if $(u_A, u_B) \in \Omega^P(t)$. Notice that (by the definition of Pareto efficiency) ϕ is strictly decreasing in u_A for all $u_A \in [0, \bar{u}_A^t]$. The following two assumptions are standard in the literature:

Assumption 1 (Concave Pareto Frontiers). For each $t \geq 0$, $\phi(., t)$ is concave in u_A on the interval $[0, \bar{u}_A^t]$.

Assumption 2 (Shrinking and Vanishing Pareto Frontiers). (i) For any $t \geq 0$ and $u_A \in [0, \bar{u}_A^t]$, $\phi(u_A, t) < \phi(u_A, t')$ for all $t' < t$, and (ii) for any $\epsilon > 0$ there exists a $T > 0$ such that $\bar{u}_A^t < \epsilon$ and $\bar{u}_B^t < \epsilon$ for all $t > T$.

Our third assumption replaces the (standard) stationarity assumption — we only

[1] By definition, a utility pair $(u_A, u_B) \in \Omega^P(t)$ if and only if $(u_A, u_B) \in \Omega(t)$ and there does not exist an alternative utility pair $(u'_A, u'_B) \in \Omega(t)$ such that for each $i = A, B$, $u'_i \geq u_i$, and for some i ($i = A$ or $i = B$), $u'_i > u_i$.

require that the Pareto frontier evolves smoothly over time:

Assumption 3 (Smoothly Evolving Pareto Frontiers). ϕ is continuously differentiable in t and u_A.

As Assumption 1 implies that (for any t) ϕ is differentiable in u_A *almost everywhere*, the main role of Assumption 3 is that it ensures that the time derivative exists — the Pareto frontier evolves smoothly over time. This plays no role when $\Delta > 0$, but implies "asymptotic smoothness" in the limit as $\Delta \to 0$.

3 Characterizing Equilibria

Given $\Delta > 0$, Binmore (1987, Section 6) constructs an example which demonstrates that a continuum of SPE are possible. Here, using Assumptions 1-3, we focus on characterizing the set of equilibria in the limit as $\Delta \to 0$. By restricting attention to Markov SPE, we first establish that as $\Delta \to 0$, all Markov SPE converge to the same limiting SPE, and provide a complete characterization of that limiting SPE. It is then established that all non-Markov SPE must also converge to the same limiting SPE. Hence we can conclude that the limiting SPE exists and is unique.

Before restricting attention to Markov SPE, note that in any SPE of any subgame beginning at any time t, player i's ($i = A, B$) equilibrium payoff lies between zero and $\bar{u}_i^t$.

3.1 Markov Equilibria: Characterization and Existence

For any $\Delta > 0$, we first characterize the set of all Markov SPE. As the argument is well-known (see Binmore (1987)), we quickly sketch the appropriate details.

Let Γ_i ($i = A, B$) denote the set of times at which player i has to make an offer. That is,

$$\Gamma_A = \{s, s + 2\Delta, s + 4\Delta, \dots\} \quad \text{and} \quad \Gamma_B = \{s + \Delta, s + 3\Delta, s + 5\Delta, \dots\},$$

and define $\Gamma = \Gamma_A \cup \Gamma_B$.

Now consider an arbitrary Markov SPE. For each $t \in \Gamma$, let $v(t) = (v_A(t), v_B(t))$ (where

$v(t) \in \Omega(t)$) denote the equilibrium offer made at time t. It is straightforward to show that for any $t \in \Gamma$, the equilibrium offer $v(t)$ is accepted.[2] This implies that at any time $t \in \Gamma_i$ ($i = A, B$), in equilibrium player j ($j \neq i$) accepts an offer $(u_A, u_B) \in \Omega(t)$ if and only if $u_j \geq v_j(t + \Delta)$. It thus follows that the equilibrium offer $v(t)$ at time $t \in \Gamma_i$ satisfies two standard properties, which are formally stated below in equations 1 and 2. Equation 1 states that in equilibrium player j is indifferent between accepting and rejecting player i's equilibrium offer $v(t)$ made at time $t \in \Gamma_i$, and equation 2 states that the equilibrium offer $v(t)$ lies on the Pareto frontier.

$$v_j(t) = v_j(t + \Delta) \quad \text{for} \quad t \in \Gamma_i \ (j \neq i) \tag{1}$$

$$v_B(t) = \phi(v_A(t), t). \tag{2}$$

For $t \in \Gamma_A$, these equations imply that the sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ must satisfy the following recursive equation:

$$\phi(v_A(t), t) = \phi(v_A(t + 2\Delta), t + \Delta). \tag{3}$$

Furthermore, as was noted above, it must also satisfy the following condition:

$$v_A(t) \in [0, \bar{u}_A^t] \qquad \text{for all } t \in \Gamma_A. \tag{4}$$

This argument implies the following Proposition.

Proposition 1 (Characterization of Markov SPE). *Fix $\Delta > 0$. Given any sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ satisfying (3) and (4), there corresponds a unique Markov SPE, with the following pair of strategies:*

- *At time $t \in \Gamma_A$ player A offers $(v_A(t), \phi(v_A(t), t))$, and at times $t \in \Gamma_B$ she accepts an offer $u \in \Omega(t)$ if and only if $u_A \geq v_A(t + \Delta)$.*
- *At times $t \in \Gamma_B$ player B offers $(v_A(t + \Delta), \phi(v_A(t + \Delta), t))$, and at times $t \in \Gamma_A$ she accepts an offer $u \in \Omega(t)$ if and only if $u_B \geq \phi(v_A(t + 2\Delta), t + \Delta)$.*

There exists no other Markov SPE.

This proposition implies that a Markov SPE exists if and only if a sequence $\langle v_A(t) \rangle_{t \in \Gamma_A}$ satisfying (3) and (4) exists. By slightly amending the arguments used in Binmore

[2] Assumption 2(i) and the restriction to Markov strategies imply no delay in equilibrium.

(1987), it is straightforward to establish that such a sequence always exists, and hence a Markov SPE exists.[3]

Proposition 2 (Existence of Markov SPE). *For any* $\Delta > 0$ *there exists a Markov SPE.*

As Binmore (1987) demonstrates, multiple solutions to (3) and (4) may exist as (3) does not contain an unstable forward looking root. However, we now focus on the set of Markov SPE in the limit as $\Delta \to 0$.

3.2 A Candidate Limiting Equilibrium

To emphasize the dependence of the set of Markov SPE on Δ, it is helpful to define the following sets. For each $\Delta > 0$, let $\mathcal{F}(\Delta)$ denote the set of all sequences $\langle v_A(t) \rangle_{t \in \Gamma_A}$ which satisfy (3) and (4).[4] Moreover, for each $\Delta > 0$, let $\mathcal{G}(\Delta)$ denote the set of all Markov SPE *payoffs* to player A. Formally,

$$\mathcal{G}(\Delta) = \{u_A : \text{ there exists a sequence } \langle v_A(t) \rangle_{t \in \Gamma_A} \in \mathcal{F}(\Delta) \text{ s.t. } v_A(s) = u_A\}.$$

Of course, as Δ changes, the set $\mathcal{G}(\Delta)$ changes. In Section 3.3 below we provide a formal convergence theorem: in the limit as $\Delta \to 0$, the set $\mathcal{G}(\Delta)$ converges to a single point, denoted by $v_A^*(s)$. We first describe $v_A^*(s)$.

Fix $\Delta > 0$ and an arbitrary Markov SPE, as characterized by an element of the set $\mathcal{F}(\Delta)$. Using Assumption 3, a first-order Taylor expansion of equation 3 implies

$$\begin{aligned} \phi(v_A(t+2\Delta), t+\Delta) &= \phi(v_A(t), t) + \big[v_A(t+2\Delta) - v_A(t)\big]\phi_u(v_A(t), t) \\ &\quad + \Delta\phi_t(v_A(t), t) + R, \end{aligned} \tag{5}$$

where ϕ_u and ϕ_t denote the first-order derivatives of ϕ *w.r.t.* u_A and t, respectively, and R is the remainder term. Using (3) to substitute for $\phi(v_A(t+2\Delta), t+\Delta)$ in (5),

[3] A proof is available upon request.

[4] Notice that it follows from Proposition 1 that, for any $\Delta > 0$, the set of Markov SPE are essentially defined by the set $\mathcal{F}(\Delta)$. Furthermore, notice that Proposition 2 implies that this set is non-empty.

rearranging and dividing by 2Δ, it follows that

$$\frac{v_A(t+2\Delta)-v_A(t)}{2\Delta} = -\frac{1}{2}\frac{\phi_t(v_A(t),t)}{\phi_u(v_A(t),t)} + \frac{R}{2\Delta}. \tag{6}$$

If we could argue that the ratio of the remainder term to Δ disappears in the limit as $\Delta \to 0$, we might interpret (6) as a differential equation describing how player A's equilibrium payoff changes over time in the limiting equilibrium. We define such a solution as our candidate limiting equilibrium:

Definition 1 (CLE). A *candidate limiting equilibrium* (CLE) is a pair of functions $(v_A^*(.), v_B^*(.))$ such that

$$\text{for all } s \geq 0, \quad v_B^*(s) = \phi(v_A^*(s), s), \text{ where} \tag{7}$$

$v_A^*(.)$ is a solution to the differential equation

$$\frac{dv_A}{ds} = -\frac{1}{2}\frac{\phi_t(v_A,s)}{\phi_u(v_A,s)} \tag{8}$$

$$\text{subject to} \quad v_A(s) \in [0, \bar{u}_A^s] \quad \text{for all } s \geq 0. \tag{9}$$

Notice that the CLE describes a path $(v_A^*(s), v_B^*(s))$ for all s, while in the previous section s was fixed, but arbitrary. We now establish two results: (i) a CLE exists, and (ii) the CLE is unique.

Lemma 1 (Existence). *A CLE exists.*

Proof. In the Appendix. □

Establishing uniqueness is much less straightforward. As the underlying difference equation (3) does not contain an unstable forward looking root, it should be no surprise that the corresponding differential equation (8) does not contain an unstable forward looking root. Nonetheless, the CLE is unique (given Assumptions 1–3). Furthermore, an interesting feature of the proof is that it relies on constructing a Liapunov-type function whose structure is closely related to that of the Nash-product (which, recall, is a key object in the definition of the Nash bargaining solution). Indeed, the proof of the Convergence Theorem stated below relies on the same construction.

Lemma 2 (Uniqueness). *The CLE is unique.*

Proof. Suppose, to the contrary, that there exists two or more solutions which satisfy the differential equation in (8) subject to (9). Let $x_1^*(.)$ and $x_2^*(.)$ denote two arbitrary such solutions such that for some $s' \geq 0$, $x_1^*(s') \neq x_2^*(s')$. For each $s \geq 0$, define

$$\Psi(s) = -[x_1^*(s) - x_2^*(s)][y_1^*(s) - y_2^*(s)],$$

where $y_i^*(s) = \phi(x_i^*(s), s)$ $(i = 1, 2)$.[5] Differentiating Ψ with respect to s and using (8), we obtain that

$$\Psi'(s) = \frac{1}{2}\frac{[x_1^*(s) - x_2^*(s)]\phi_t(x_1^*(s), s)}{\phi_u(x_1^*(s), s)}\left[\frac{y_1^*(s) - y_2^*(s)}{x_1^*(s) - x_2^*(s)} - \phi_u(x_1^*(s), s)\right] - \frac{1}{2}\frac{[x_1^*(s) - x_2^*(s)]\phi_t(x_2^*(s), s)}{\phi_u(x_2^*(s), s)}\left[\frac{y_1^*(s) - y_2^*(s)}{x_1^*(s) - x_2^*(s)} - \phi_u(x_2^*(s), s)\right].$$

Hence, by concavity of ϕ (Assumption 1) and ϕ decreasing in t (Assumption 2), it follows that for any $s \geq 0$, $\Psi'(s) \geq 0$. Since $\Psi(s') > 0$, this implies that $\lim_{s\to\infty} \Psi(s) > 0$. Using Assumption 2, the condition in (9) implies that for each $i = 1, 2$, $x_i^*(s) \to 0$ as $s \to \infty$. This implies that $\lim_{s\to\infty} \Psi(s) = 0$. Hence, we have a contradiction. □

3.3 The Unique Limiting Subgame Perfect Equilibrium

The aim now is to show that in the limit as $\Delta \to 0$, any Markov SPE converges to the CLE. We then establish that in this limit, any non-Markov SPE also converges to the CLE.

Theorem 1 (The Convergence Theorem). *Fix an arbitrary s. For any $\epsilon > 0$ there exists $\bar{\Delta}$ such that for all $\Delta < \bar{\Delta}$*

$$\max_{u_A \in \mathcal{G}(\Delta)} \mid u_A - v_A^*(s) \mid < \epsilon.$$

Proof. In the Appendix. □

[5] It should be noted that since $x_i^*(s)$ satisfies (9), $\phi(x_i^*(s), s)$ is well-defined.

Theorem 1 implies that the Hausdorff distance between the set $\mathcal{G}(\Delta)$ and $\{v_A^*(s)\}$ converges to zero as $\Delta \to 0$. Hence in this limit, all Markov SPE imply agreement occurs immediately, and the terms of trade are $(v_A^*(s), v_B^*(s))$. The final step is to show that in this limit any non-Markov SPE also converges to the CLE.

Theorem 2 (Unique Limiting SPE). *In the limit as $\Delta \to 0$, any SPE converges to the CLE.*

Proof. In the Appendix. □

In summary, we have established that in the limit as $\Delta \to 0$, our bargaining game possesses a unique SPE. In this limiting SPE, agreement is struck immediately (without any delay), at the time s when the negotiations begin. Player A's equilibrium payoff is $v_A^*(s)$, where $v_A^*(.)$ is the unique solution of the differential equation (8) subject to (9), and player B's equilibrium payoff is $v_B^*(s) = \phi(v_A^*(s), s)$.

We refer to condition (8) as the *fundamental bargaining equation* (FBE). In general, finding the equilibrium payoffs $(v_A^*(s), v_B^*(s))$ will be a non-trivial problem as it is described by a non-linear differential equation. But it has a simple property. It follows from (7) that (8) can be rewritten as

$$\frac{dv_B^*(s)/ds}{dv_A^*(s)/ds} = -\phi_u(v_A^*(s), s). \tag{10}$$

The right-hand side of (10) is the marginal rate of utility substitution along the Pareto frontier at the equilibrium outcome. The left-hand side describes the marginal rate of utility loss by delay at the equilibrium outcome. Strategic bargaining implies these two trade-offs are equalized. Geometrically, it implies that the slope of the CLE $(v_A^*(s), v_B^*(s))$ at time s *equals* the absolute value of the slope of the Pareto frontier $\Omega^P(s)$ at that point.

4 The Relationship with Nash's Bargaining Solution

As is well known, the unique SPE of Rubinstein's (1982) bargaining model can be described by the Nash bargaining solution of an appropriately defined bargaining problem (cf., for example, Osborne and Rubinstein (1990) and Muthoo (1999)). Here our objective is to extend this result (where possible) to the non-stationary bargaining environments.

The Nash bargaining solution (NBS) is

$$(v_A^N(s), v_B^N(s)) = \arg \max_{(u_A, u_B) \in \Omega(s)} (u_A - d_A(s))(u_B - d_B(s)),$$

where $(d_A(s), d_B(s))$ is the as yet unspecified *disagreement point.* If the disagreement point $(d_A(s), d_B(s)) = (0, 0)$ then the NBS $(v_A^N(s), v_B^N(s))$ is the unique solution of the following pair of equations:

$$v_B = \phi(v_A, s) \tag{11}$$

$$\frac{v_B}{v_A} = -\phi_u(v_A, s). \tag{12}$$

In contrast to (10), the NBS picks a point on the Pareto frontier where the absolute value of the slope of the frontier at that point equals the slope of the line joining the disagreement point $(0, 0)$ and the NBS. The following lemma establishes conditions under which the NBS and the limiting SPE payoff pair coincide for all s.

Lemma 3. *The NBS $(v_A^N(s), v_B^N(s))$ with disagreement point (0,0) is identical to the limiting SPE payoff pair $(v_A^*(s), v_B^*(s))$ for all s if and only if $\phi_u(v_A^*(s), s)$ is constant for all s.*

Proof. In the Appendix. $\square$

If $\phi_u(v_A^*(s), s)$ is constant for all s, it follows from (10) that the locus $(v_A^*(s), v_B^*(s))$ describes a straight line, while (9) implies that line passes through the origin. This of course then corresponds to the Nash bargaining solution — a ray out of the origin with slope equal to the absolute value of the Pareto frontier.

The condition which guarantees that the strategic bargaining solution describes a ray out of the origin is that the Pareto frontier shrinks homothetically. If the Pareto frontier is now described by the implicit function

$$\widehat{\phi}(u_A, u_B, t) = 1,$$

then homotheticity requires that $\widehat{\phi}$ is separable in t and homogeneous in u_A and u_B;

i.e., $\widehat{\phi} = \gamma(u_A, u_B)\widehat{\gamma}(t)$ and γ is homogeneous. Thus we obtain a Pareto frontier of the form

$$\gamma(u_A, u_B) = \alpha(t), \tag{13}$$

for some $\alpha(t)$. Obviously, we must assume that γ and α are consistent with Assumptions 1–3. Homotheticity now requires that the time component also affects the players equally over time. As the example below demonstrates, they must have the same discount rate.

Proposition 3 (Nash Equivalence under Homotheticity). *If the Pareto frontier shrinks homothetically, then the NBS with disagreement point (0,0) and the unique limiting SPE payoff pair coincide for all s.*

Proof. In the Appendix. □

If homotheticity is satisfied, then the limiting SPE payoff pair and the NBS coincide. Further, as the payoffs coincide with a static optimisation problem, the limiting SPE payoff pair is "myopic" — it does not depend on the rate at which the Pareto frontier shrinks.

5 A Worked Example

To illustrate the Fundamental Bargaining Equation, we quickly consider a simple, non-homothetic example. Suppose players A and B are bargaining over the partition of a unit size cake, where negotiations begin at time $s = 0$. Player i's payoff from obtaining $x_i \in [0, 1]$ units of the cake at time $t \geq 0$ is $u_i = x_i\delta_i(t)$, where

$$\delta_i(t) = \exp\left[-\int_0^t r_i(z)dz\right]$$

and $r_i(z) > 0$ denotes i's instantaneous rate of time preference at time z. As $x_A + x_B = 1$, this implies that the Pareto frontier is defined by the implicit function

$$\frac{u_A}{\delta_A(t)} + \frac{u_B}{\delta_B(t)} = 1.$$

Assuming r_i finite and bounded away from zero, Assumptions 1–3 are satisfied and hence Theorems 1 and 2 apply. Notice that unless $r_A = r_B$ almost everywhere, the Pareto Frontier does not shrink homothetically and so a Nash Bargaining solution cannot be applied. Instead we have to solve directly the FBE. As the Pareto Frontier implies $\phi = \delta_B(t)[1 - u_A/\delta_A(t)]$, the FBE implies $v_A^*(s)$ satisfies

$$\frac{dv_A}{ds} = -\frac{1}{2}\bigg[v_A[r_A(s) - r_B(s)] + r_B(s)\delta_A(s)\bigg].$$

subject to the boundary condition (9). It is straightforward to verify that

$$v_A^*(s) = \delta_A(s)\left[\frac{1}{2} + \frac{1}{4}\int_s^\infty \left[\frac{\delta_A(t)\delta_B(t)}{\delta_A(s)\delta_B(s)}\right]^{1/2}[r_B(t) - r_A(t)]dt\right]$$

satisfies the FBE and (9). Hence, putting $s = 0$, the unique limiting (as $\Delta \to 0$) equilibrium share of the unit size cake to player A is

$$v_A^*(0) = \frac{1}{2} + \frac{1}{4}\int_0^\infty [\delta_A(t)\delta_B(t)]^{1/2}[r_B(t) - r_A(t)]dt.$$

The unique (limiting) equilibrium share to player A is a discounted weighting of the difference between the players' discount rates in the entire future. The more impatient player B is, the higher the payoff to player A. Of course, if they have equal discount rates then the bargaining game is perfectly symmetric and they split the cake.

6 An Extension to Time Varying Inside Options

The previous sections have assumed that the pie evolves over time in a non-stationary way. But a different class of problems arise if the agents' inside options are time varying. For example, when bargaining with a striking union, the firm might sell out of its inventory of finished goods where such sales reduce the cost of the strike to the firm; see, for example, Coles and Hildreth (2000). A different example is an unemployed worker who is bargaining with a firm for a job and who receives duration dependent unemployment insurance payments. The purpose of this section is to extend the previous results for time varying inside options and so demonstrate the robustness of this approach.

Two players, A and B, bargain according to the alternating-offers procedure as previously described. An offer at time t is a utility pair (u_A, u_B) from the set $\Omega(t)$, where $u_B = \phi(u_A, t)$ describes the Pareto frontier. If the offer is rejected then over the intervening period, player i obtains flow payoff $f_i(t)\Delta \geq 0$ (which is measured in period zero utils; i.e., it is discounted back to time zero).[6] Define $d_i(t) = \int_t^\infty f_i(z)dz \geq 0$, which is player i's discounted payoff at time t should they never reach agreement.

Assumption 1′ (Concave Pareto Frontiers). For each $t \geq 0$, $\phi(., t)$ is concave in u_A on the interval $[d_A(t), \bar{u}_A^t]$.

Assumption 2′ (Positive, Shrinking and Vanishing Pareto Frontiers). (i) For any $t \geq 0$, $d_B(t) < \phi(d_A(t), t)$, (ii) for any $t \geq 0$ and $u_A \in [d_A(t), \bar{u}_A^t]$, $\phi_t(u_A, t) + f_B(t) - \phi_u(u_A, t) f_A(t) < 0$, and (iii) for any $\epsilon > 0$ there exists a $T > 0$ such that $\bar{u}_A^t < \epsilon$ and $\bar{u}_B^t < \epsilon$ for all $t > T$.

Assumption 3′ (Smoothly Evolving Pareto Frontiers). ϕ is continuously differentiable, and f_A, f_B are continuous.

Condition (i) in Assumption 2′ implies that there is always some partition both players would prefer rather than never reach agreement — a gain to trade always exists.[7] This implies $0 \leq d_i(t) < \overline{u}_i^t$ for all t and $i = A, B$. Condition (ii) is the appropriate shrinking pie condition. To see why, suppose rather than agree some (Pareto efficient) partition (u_A, u_B) at time t, the agreement is deferred to $t + dt$. Player A is no worse off as long as the partition (u_A', u_B') at time $t + dt$ satisfies $f_A(t)dt + u_A' \geq u_A$. As player B's maximal payoff is $\phi(u_A', t + dt) + f_B(t)dt$, then the stated condition (ii) guarantees delay makes player B strictly worse off.

Again consider an arbitrary Markov SPE where $v(t) = (v_A(t), v_B(t))$ denotes the equilibrium offer made at time $t \in \Gamma$. As before shrinking pie and Markov strategies imply there is no delay in equilibrium. Hence the equilibrium offer $v(t)$ at time $t \in \Gamma_i$ satisfies

$$\begin{aligned} v_j(t) &= f_j(t)\Delta + v_j(t + \Delta) \quad \text{for} \quad t \in \Gamma_i (j \neq i) \\ v_B(t) &= \phi(v_A(t), t), \end{aligned}$$

[6]For example, if player i obtains UI payments $b(t)$, then his/her flow payoff during disagreement might be described as $f_i = e^{-rt} u_i(b(t))$.

[7]This assumption is convenient rather than critical. If it does not hold, then shrinking pie implies a (unique) T where $d_B(T) = \phi(d_A(T), T)$. A gain to trade then exists for $t < T$, but not for $t > T$. As equilibrium implies no trade for $t \geq T$, we would then use backward induction from $t = T$ with boundary condition $v_i(T) = d_i(T)$.

where the first condition says the proposer extracts maximal rents from the responder, and the second says the offer is Pareto efficient. For any $t \in \Gamma_A$, these equations imply the difference equation

$$\phi(v_A(t), t) = f_B(t)\Delta + \phi(f_A(t + \Delta)\Delta + v_A(t + 2\Delta), t + \Delta).$$

As before, our main interest is characterising the limiting equilibria as $\Delta \to 0$. A first order Taylor expansion implies

$$0 = f_B(t)\Delta + [f_A(t + \Delta)\Delta + v_A(t + 2\Delta) - v_A(t)]\phi_u + \Delta\phi_t + R.$$

Rearranging and taking the limit $\Delta \to 0$ suggests that a candidate limiting equilibrium (CLE) is a pair of functions $(v_A(.), v_B(.))$ such that for all $s \geq 0$, $v_B(s) = \phi(v_A(s), s)$, where $v_A(.)$ is a solution to the differential equation

$$\begin{aligned} \frac{dv_A}{ds} &= -\frac{1}{2}\frac{[f_B(s) + f_A(s)\phi_u(v_A, s) + \phi_t(v_A, s)]}{\phi_u(v_A, s)}, \quad (14) \\ \text{subject to} \quad v_A(s) &\in [d_A(s), \phi^{-1}(d_B(s), s)] \quad \text{for all } s \geq 0. \end{aligned}$$

There are several points. First Assumption 2′(ii) (shrinking pie) and (14) imply $dv_A/ds + f_A(s) < 0$; along the CLE, delay always makes player A worse off. Also, using $dv_B/ds = \phi_u(v_A, s)dv_A/ds + \phi_t(v_A, s)$, it follows that $dv_B/ds + f_B(s) < 0$. Delay makes both players strictly worse off.

Given the corresponding solution for dv_B/ds, (14) can be rewritten as

$$\frac{dv_B/ds + f_B(s)}{dv_A/ds + f_A(s)} = -\phi_u(v_A, s),$$

which implies the geometric interpretation obtained previously. $dv_B/ds + f_B(s)$ is the (rate of) utility gain to player B through delay (which is negative). Strategic bargaining implies the marginal rate of utility loss by delay at the equilibrium outcome equals the marginal rate of utility substitution along the Pareto frontier.

Establishing existence of a solution to (14) is straightforward. The key is to note that the previous expression can also be written as

$$\frac{\frac{d}{ds}(v_B - d_B)}{\frac{d}{ds}(v_A - d_A)} = -\phi_u(v_A, s).$$

At each point in time, strategic bargaining shares the increase in surplus by reaching agreement today rather than deferring another instant, where the ratio depends on the slope of the Pareto frontier. By defining "surplus" variables $\widehat{x} \equiv v_A - d_A, \widehat{y} \equiv v_B - d_B$, the proof of lemma 1 can be applied to establish existence of a solution where $\widehat{x}, \widehat{y} > 0$ for all s (as required).[8]

To establish uniqueness, suppose there exist (at least) two solutions to (14) which we denote $x_1(s), x_2(s)$. Further, let $y_i(s) = \phi(x_i(s), s)$ and define

$$\Psi(s) = -[x_1 - x_2][y_1 - y_2],$$

where $x_i = x_i(s), y_i = y_i(s)$. Note that $\Psi(s) > 0$ if $x_1 \neq x_2$ and vanishing pie requires $\Psi(s) \to 0$ as $s \to \infty$. But

$$\Psi'(s) = -[x_1' - x_2'][y_1 - y_2] - [x_1 - x_2][y_1' - y_2'],$$

and as the CLE implies $y_i'(s) + f_B = -\phi_u(x_i, s)[x_i'(s) + f_A]$, we can substitute out the y_i' and rearrange to get :

$$\begin{aligned} \Psi'(s) &= [x_1' + f_A] \left[y_2 - [y_1 + (x_2 - x_1)\phi_u(x_1, s)] \right] \\ &\quad + [x_2' + f_A] \left[y_1 - [y_2 + (x_1 - x_2)\phi_u(x_2, s)] \right]. \end{aligned}$$

As an equilibrium solution implies $x'(s) + f_A < 0$ (see above), then concavity of ϕ with respect to u implies $\Psi'(s) \geq 0$ which is the required contradiction. In the same way we can adapt the limiting argument demonstrated in the proof of Theorem 2 and so establish the corresponding Convergence Theorem.

6.1 An Important Special Case

There is one special case for which a simple (dynamically consistent) solution exists. It is also a case that occurs frequently in the matching literature — that all agents are assumed to be risk neutral and have common discount rate $r > 0$. Together these

[8] In particular, define $\widehat{\phi}(x, s) \equiv \phi(x + d_A(s), s) - d_B(s)$. Then Pareto efficiency implies $\widehat{y} = \widehat{\phi}(\widehat{x}, s)$, and the CLE implies $\frac{d\widehat{y}/ds}{d\widehat{x}/ds} = -\widehat{\phi}_x(\widehat{x}, s)$. Further, given ϕ, d_i satisfy Assumptions $1' - 3'$, direct inspection shows that $\widehat{\phi}$ satisfies Assumptions 1-3. Hence the proof of Lemma 1 implies a path exists where $\widehat{x}, \widehat{y} > 0$ for all s.

assumptions imply the Pareto frontier is of the form

$$u_B + \alpha u_A = \gamma(t)$$

where α is a positive constant and γ is a positive, decreasing function. As $\phi_u \equiv -\alpha$, the CLE above implies

$$\frac{dv_A}{ds} = \frac{1}{2\alpha}[f_B(s) - \alpha f_A(s) + \gamma'(s)], \tag{15}$$

which with a vanishing frontier (e.g. positive discounting and bounded payoffs) implies the (unique) bargaining solution

$$\begin{aligned} v_A(s) &= d_A(s) + \frac{1}{2\alpha}[\gamma(s) - \alpha d_A(s) - d_B(s)] \\ v_B(s) &= d_B(s) + \frac{1}{2}[\gamma(s) - \alpha d_A(s) - d_B(s)], \end{aligned}$$

where, as previously defined, the $d_i(s)$ are the player's expected discounted payoffs by never reaching agreement. Risk neutrality and common discount rates implies each player receives an equal share of the pie, net of the 'threatpoints' $d_i(s)$. Note this solution is 'static' in the sense that it is described by a Nash bargaining product, but replicates the dynamically consistent outcome to the strategic bargaining game given appropriately defined (dynamic) threatpoints.

7 Conclusion

This paper has extended the Rubinstein bargaining model to a non-stationary environment. Although in general, multiple equilibria are possible for $\Delta > 0$, it has been established that with an appropriate continuity assumption, equilibrium is always unique in the limit as $\Delta \to 0$. Further that limiting equilibrium is described by a differential equation which has a simple geometric interpretation — at each point in time, equilibrium shares the increase in surplus by reaching agreement today rather than deferring another instant, where the ratio depends on the slope of the Pareto frontier. As Coles and Wright (1998) establish, when embedded in an extended matching framework, this property results in a tractable dynamical structure. Indeed, using this bargaining approach to extend that same monetary framework, Ennis (1999) also establishes the existence of sunspot equilibria.

Although aggregate dynamics are one source of non-stationary influences on the bargaining problem, a second source is that of time-varying inside options. We have shown that the same techniques apply, and qualitatively identical results are obtained for this case. Further, a useful simplification is obtained when all are assumed to be risk neutral and have a common discount rate. This case is standard in the non-steady-state matching literature; see, for example, Diamond and Fudenberg (1989), Boldrin, Kiyotaki and Wright (1994), Mortensen and Pissarides (1994), Lagos and Violante (1998), and Mortensen (1999). For this particular case, the equilibrium outcome to the bargaining game reduces to a Nash bargaining solution but with particular 'threatpoints' — the threatpoints are each individual's expected discounted payoff should agreement never be reached. This outcome seems particularly useful for future applications — see Cripps (1998), and Coles and Hildreth (2000) for examples where the pie evolves stochastically, or Coles and Masters (2000) for an equilibrium matching model where worker skills decline while unemployed.

Appendix

Proof of Lemma 1

Let $\widehat{\phi} : (-\infty, \bar{u}_A^0] \times [0, \infty) \to \Re$ be any function such that (i) for any $(x, s) \in [0, \bar{u}_A^0] \times [0, \infty)$, $\widehat{\phi}(x, s) = \phi(x, s)$, (ii) $\widehat{\phi}$ is continuously differentiable on its domain, and (iii) for any $(x, s) \in (-\infty, \bar{u}_A^0] \times [0, \infty)$, $\widehat{\phi}_t(x, s) < 0$ and $\widehat{\phi}_u(x, s) < 0$. Consider the following differential equation

$$\frac{dx}{ds} = -\frac{1}{2}\frac{\widehat{\phi}_t(x, s)}{\widehat{\phi}_u(x, s)} \tag{A.1}$$

$$\text{subject to} \quad x(s) \in [0, \bar{u}_A^s] \quad \text{for all } s \geq 0. \tag{A.2}$$

From the *Fundamental Theorem of Differential Equations*, we know that for any initial value $x(0) = x_0$, where $x_0 \in [0, \bar{u}_A^0]$, there exists a unique solution to the differential equation in A.1; let that solution be denoted by $\widehat{x}(s; x_0)$. To prove Lemma 1, we show an initial value $x_0 \in [0, \bar{u}_A^0]$ exists such that $\widehat{x}(s; x_0)$ satisfies A.2.

For any initial value $x_0 \in [0, \bar{u}_A^0]$, a trajectory is denoted as $\{\widehat{x}(s; x_0), \widehat{y}(s; x_0)\}$, where $\widehat{y}(s; x_0) = \widehat{\phi}(\widehat{x}(s; x_0), s)$.

Claim A.1. *[Trajectories do not cross].*
For any initial values $x_0 \in [0, \bar{u}_A^0]$ and $x_0' \in [0, \bar{u}_A^0]$ such that $x_0 > x_0'$: $\widehat{x}(s; x_0) > \widehat{x}(s; x_0')$ and $\widehat{y}(s; x_0) < \widehat{y}(s; x_0')$ for all $s \geq 0$.

This claim follows directly from the proof of Lemma 2 in the text. $\Psi(s)$ as defined in that proof is a measure of the distance between any two trajectories at any point in time. The proof of lemma 2 shows this distance increases with s and so trajectories never meet. We can see this directly by computing the slope of a trajectory in the (x, y) plane. Direct calculation of $d\widehat{y}(s; x_0)/ds$ implies this slope is

$$\frac{d\widehat{y}(s; x_0)/ds}{d\widehat{x}(s; x_0)/ds} = -\widehat{\phi}_u(\widehat{x}(s; x_0), s)$$

which is strictly positive. The slope is equal in absolute value to the slope of the graph of $\widehat{\phi}(x, s)$ at $(\widehat{x}, \widehat{y}, s)$. Concavity of ϕ implies the slope of the trajectory is small for small $\widehat{x}$, and is large for large $\widehat{x}$. As Figure 1 demonstrates, trajectories tend to diverge over time. Further, as the slope of the trajectory is always strictly positive, any trajectory $\{\widehat{x}(s; x_0), \widehat{y}(s; x_0)\}$ either (i) intersects the x-axis in finite time, or (ii) intersects the y-axis in finite time, or (iii) is always (strictly) in the positive quadrant.

Define $\Upsilon_x = \{x_0 \in [0, \bar{u}_A^0] :$ there exists an S such that $\widehat{x}(S; x_0) > 0$ and $\widehat{y}(S; x_0) = 0\}$, and $\Upsilon_y = \{x_0 \in [0, \bar{u}_A^0] :$ there exists an S such that $\widehat{x}(S; x_0) = 0$ and $\widehat{y}(S; x_0) > 0\}$.[9] Furthermore, define $\Upsilon^* = \{x_0 \in [0, \bar{u}_A^0] :$ for all $s \geq 0$, $\widehat{x}(s; x_0) > 0$ and $\widehat{y}(s; x_0) > 0\}$. Claim A.2 now shows that Υ^* is non-empty, which completes the proof of the Lemma.

Claim A.2. Υ^* *is non-empty.*

Proof of Claim A.2. By contradicton. Suppose to the contrary that Υ^* is empty. This implies that Υ_x and Υ_y form a complete partition of $[0, \bar{u}_A^0]$. Since $0 \in \Upsilon_y$ and $\bar{u}_A^0 \in \Upsilon_x$, these two sets are non-empty. Furthermore, since trajectories do not cross, the respective supports of Υ_x and Υ_y are connected. Hence, since Υ_x and Υ_y partition the interval $[0, \bar{u}_A^0]$, one of these two sets is closed. Suppose, without loss of generality, that Υ_x is closed — that is, there exists $x^c \in (0, \bar{u}_A^0)$ such that $\Upsilon_x = [x^c, \bar{u}_A^0]$. Hence, there exists a corresponding $S < \infty$ such that $\widehat{x}(S; x^c) = \bar{u}_A^S > 0$ and $\widehat{y}(S; x^c) = 0$.

Now consider $s = S + 1$ and set $x(S + 1) = \bar{u}_A^{S+1}$ and $y(S + 1) = 0$. By iterating the differential equation (A.1) backwards through time starting at $s = S + 1$ with

[9] That is, Υ_x and Υ_y are respectively the sets of initial values whose trajectories cross the x-axis and y-axis.

“initial” value $x(S+1) = \bar{u}_A^{S+1}$, we obtain another trajectory $\{\widehat{x}(s; x_0'), \widehat{y}(s; x_0')\}$ where $x_0' \in \Upsilon_x$. But as trajectories cannot cross, this implies $x_0' < x^c$ which is the required contradiction.$\square$.

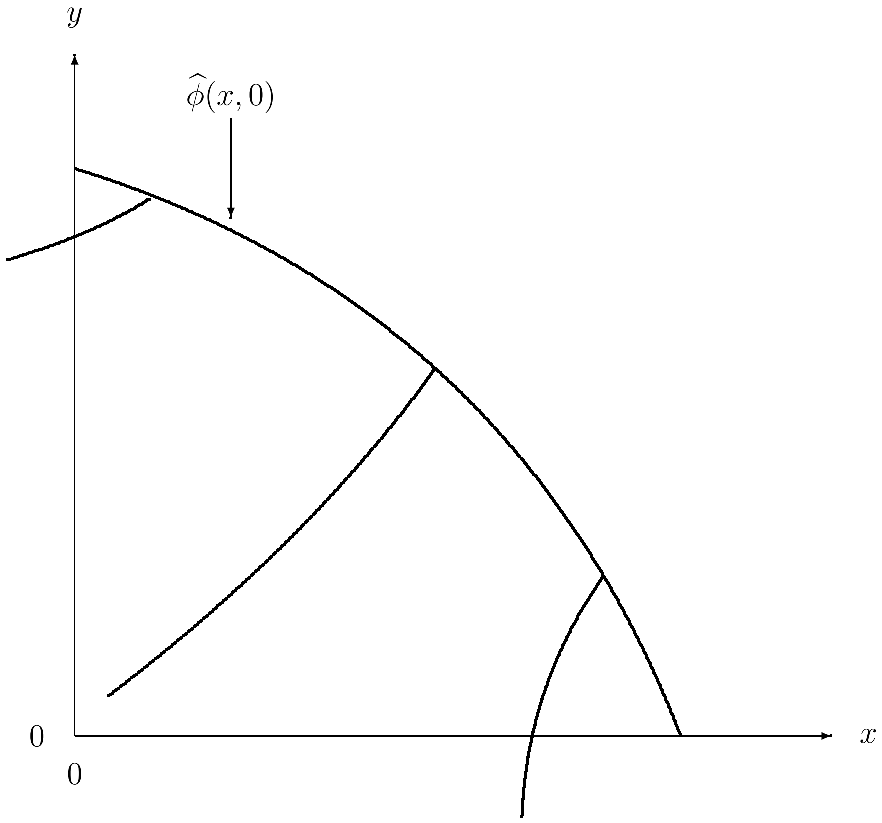

Figure 1: Three typical trajectories of the differential equation in A.1.

Proof of Theorem 1

Fix an arbitrary sequence $\langle \Delta_n \rangle$ such that $\Delta_n > 0$ (for all $n \in \mathbb{N}$) and $\Delta_n \to 0$ as $n \to \infty$. This defines a sequence $\langle \mathcal{F}_n \rangle$ where $\mathcal{F}_n \equiv \mathcal{F}(\Delta_n)$. Now define a sequence $\langle x_n \rangle$ where for each $n \in \mathbb{N}$, x_n is an arbitrary element of $\mathcal{F}_n$. That is, for each $n \in \mathbb{N}$, x_n is an arbitrary sequence $\langle x_n(t) \rangle_{t \in \Gamma_A^n}$ that satisfies

$$\phi(x_n(t), t) = \phi(x_n(t + 2\Delta_n), t + \Delta_n) \tag{A.3}$$

and $x_n(t) \in [0, \bar{u}_A^t]$ for all $t \in \Gamma_A^n$, where $\Gamma_A^n \equiv \Gamma_A(\Delta_n) = \{s, s + 2\Delta_n, s + 4\Delta_n, \ldots\}$. We have to show that the sequence $\langle x_n(s) \rangle$ converges to $v_A^*(s)$.

For each $n \in \mathbb{N}$ and $t \in \Gamma_A^n$ define

$$\Psi(n,t) = -[v_A^*(t) - x_n(t)][v_B^*(t) - y_n(t)], \tag{A.4}$$

where (v_A^*, v_B^*) is the unique CLE and $y_n(t) = \phi(x_n(t), t)$. One might interpret $\Psi(n,t)$ as a measure of the distance between the CLE $(v_A^*(t), v_B^*(t))$ and the SPE payoff pair $(x_n(t), y_n(t))$. In particular, $\Psi(n,t) = 0$ if and only if $x_n(t) = v_A^*(t)$, and $\Psi(n,t) > 0$ for $x_n(t) \neq v_A^*(t)$. Most importantly, by establishing that $\Psi(n,s) \to 0$ as $n \to \infty$ it follows that $x_n(s) \to v_A^*(s)$ in this limit. Hence we establish the Theorem by proving that for any $\varepsilon > 0$ there always exists an N such that $\Psi(n,s) < \varepsilon$ for all $n > N$.

Fix an arbitrary $\varepsilon > 0$. If $\bar{u}_A^s \bar{u}_B^s \leq \varepsilon/2$, then $\Psi(n,s) \leq \bar{u}_A^s \bar{u}_B^s \leq \varepsilon/2$ (for all $n \in \mathbb{N}$), and we are done. Now suppose that $\varepsilon < 2\bar{u}_A^s \bar{u}_B^s$. Define T such that $\bar{u}_A^T \bar{u}_B^T = \varepsilon/2$. Assumptions 2 and 3 imply T exists, is unique and is strictly greater than s. Also $\Psi(n,T) \leq \bar{u}_A^T \bar{u}_B^T = \varepsilon/2$ for all $n \in \mathbb{N}$. Furthermore, define for each $n \in \mathbb{N}$,

$$M_n = \min\{m \in \mathbb{N} : m \geq (T-s)/2\Delta_n\} \quad \text{and} \quad T_n = s + 2M_n\Delta_n.$$

Notice that $T_n \in \Gamma_A^n$ for all $n \in \mathbb{N}$. Further $T_n \geq T$, and Assumption 2 implies that $\Psi(n, T_n) \leq \bar{u}_A^{T_n} \bar{u}_B^{T_n} \leq \varepsilon/2$.

Now for any $n \in \mathbb{N}$,

$$\Psi(n,s) = \Psi(n,T_n) - \sum_{i=0}^{M_n-1} \left[\Psi(n, s + 2(i+1)\Delta_n) - \Psi(n, s + 2i\Delta_n)\right].$$

Claim A.3 — which is stated below — implies

$$\Psi(n,s) = \Psi(n,T_n) - \sum_{i=0}^{M_n-1} \left[F(n, s + 2i\Delta_n)\Delta_n + o(\Delta_n)\right],$$

where $o(\Delta_n)$ denotes a remainder term that is of order smaller than Δ_n (i.e. $o(\Delta_n)/\Delta_n$ converges to zero as $n \to \infty$). As Claim A.3 also implies $F(n,t) \geq 0$ for all $t \in \Gamma_A^n$, this now implies

$$\Psi(n,s) \leq \Psi(n,T_n) - \sum_{i=0}^{M_n-1} o(\Delta_n).$$

But $M_n = 0(1/\Delta_n)$ and so it follows that $\sum_{i=0}^{M_n-1} o(\Delta_n)$ converges to zero as $n \to \infty$. Hence there exists an N such that for any $n > N$, $\mid \sum_{i=0}^{M_n-1} o(\Delta_n) \mid < \varepsilon/2$. As

$\Psi(n, T_n) \leq \varepsilon/2$, this implies $\Psi(n, s) < \varepsilon$ for all $n > N$ (as required).

Claim A.3. *For any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$:*

$$\Psi(n, t + 2\Delta_n) - \Psi(n, t) = F(n, t)\Delta_n + o(\Delta_n),$$

where $F(n, t)$ is defined by

$$F(n,t) = \frac{[v_A^*(t) - x_n(t)]\phi_t(v_A^*(t), t)}{\phi_u(v_A^*(t), t)} \left[\frac{v_B^*(t) - y_n(t)}{v_A^*(t) - x_n(t)} - \phi_u(v_A^*(t), t)\right] - \frac{[v_A^*(t) - x_n(t)]\phi_t(x_n(t), t)}{\phi_u(x_n(t), t)} \left[\frac{v_B^*(t) - y_n(t)}{v_A^*(t) - x_n(t)} - \phi_u(x_n(t), t)\right].$$

Furthermore, for any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$: $F(n, t) \geq 0$.

Proof of Claim A.3. As $v_A^* : [0, \infty) \to \Re$ satisfies the differential equation in (8), then for any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$,

$$v_A^*(t + 2\Delta_n) - v_A^*(t) = -\frac{\phi_t(v_A^*(t), t)}{\phi_u(v_A^*(t), t)}\Delta_n + o(\Delta_n). \tag{A.5}$$

Further, Assumption 3 (differentiability) implies that we can consider a first order Taylor expansion of $\phi(v_A^*(t + 2\Delta_n), t + 2\Delta_n)$ around $\phi(v_A^*(t), t)$, and A.5 then implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$:

$$\phi(v_A^*(t + 2\Delta_n), t + 2\Delta_n) = \phi(v_A^*(t), t) + \left[v_A^*(t + 2\Delta_n) - v_A^*(t)\right]\phi_u(v_A^*(t), t) + 2\Delta_n\phi_t(v_A^*(t), t) + o(\Delta_n). \tag{A.6}$$

Recalling that $x_n(t)$ satisfies A.3, Assumption 3 (differentiability) implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$:

$$x_n(t + 2\Delta_n) - x_n(t) = -\frac{\phi_t(x_n(t), t)}{\phi_u(x_n(t), t)}\Delta_n + o(\Delta_n). \tag{A.7}$$

Now consider a first order Taylor expansion of $\phi(x_n(t+2\Delta_n), t+2\Delta_n)$ around $\phi(x_n(t), t)$.

A.7 then implies that for any $n \in \mathbb{N}$ and $t \in \Gamma_A^n$:

$$\phi(x_n(t+2\Delta_n), t+2\Delta_n) = \phi(x_n(t), t) + \Big[x_n(t+2\Delta_n) - x_n(t)\Big]\phi_u(x_n(t), t) \\ + 2\Delta_n \phi_t(x_n(t), t) + o(\Delta_n). \quad \text{(A.8)}$$

Given the definition of Ψ in (A.4), and using (A.5)–(A.8) to substitute out terms dated at time $t + 2\Delta_n$, straightforward (but messy) algebra establishes the equations stated in the Claim. $F(n, t) \geq 0$ follows from the concavity of ϕ, and from $\phi_t < 0$ and $\phi_u < 0$.

Proof of Theorem 2

Fix $\Delta > 0$. For each $i = A, B$ and $t \in \Gamma_i$, let $G_i(t)$ denote the set of SPE payoffs to player i in any subgame beginning at time t. Formally, $G_i(t) = \{g_i$: there exists an SPE in any subgame beginning at time t (when player i makes an offer) that gives player i a payoff of $g_i\}$. Since $G_i(t)$ is bounded, we denote its supremum and infimum by $M_i(t)$ and $m_i(t)$, respectively.

It follows from Claim A.4 below that both the sequence $\langle M_A(t)\rangle_{t\in\Gamma_A}$ and the sequence $\langle m_A(t)\rangle_{t\in\Gamma_A}$ are elements of the set $\mathcal{F}(\Delta)$. Theorem 1 implies that in the limit, as $\Delta \to 0$, the set $\mathcal{F}(\Delta)$ converges to a unique element. Hence, it follows (by appealing to Claim A.4) that in the limit, as $\Delta \to 0$, the set of SPE payoffs to the players in any subgame are *uniquely* defined: in the limit as $\Delta \to 0$, any SPE in any subgame gives player A a payoff of $v_A^*(s)$ and player B a payoff of $v_B^*(s)$. This implies that in any limiting (as $\Delta \to 0$) SPE, each player's offer (in any subgame when she has to make an offer) is accepted by her opponent. Hence, it immediately follows that in the limit as $\Delta \to 0$, any SPE converges to the CLE.

Claim A.4. *Fix $\Delta > 0$. $\forall t \in \Gamma_A$, $M_A(t) = \phi^{-1}(m_B(t+\Delta), t)$ and $m_A(t) = \phi^{-1}(M_B(t+\Delta), t)$, and $\forall t \in \Gamma_B$, $M_B(t) = \phi(m_A(t+\Delta), t)$ and $m_B(t) = \phi(M_A(t+\Delta), t)$.*

Proof of Claim A.4. The proof — which is available upon request — follows from a straightforward adaptation of standard arguments (which are, for example, presented in Osborne and Rubinstein (1990, chapter 3) and Muthoo (1999, chapter 3)).

Proof of Lemma 3

We first establish *sufficiency*. If $\phi_u(v_A^*(s), s)$ is constant for all s, then (10) implies that the locus $\{(v_A^*(s), v_B^*(s)) : s \geq 0\}$ is a straight line, being a ray through the origin with slope equal to the absolute value of the slope of the frontier $\Omega^P(s)$ at $(v_A^*(s), v_B^*(s))$. Hence, for all s the NBS and the limiting SPE payoff pair are identical. We now establish *necessity.* If $v_A^N(s) = v_B^*(s)$ for all s, then (10) and (12) imply

$$\frac{dv_B^*(s)/ds}{dv_A^*(s)/ds} = \frac{v_B^N(s)}{v_A^N(s)} \qquad \text{for all } s. \tag{A.9}$$

Suppose, to the contrary, that there exists $s'' > s'$ such that $\phi_u(v_A^*(s''), s'') \neq \phi_u(v_A^*(s'), s')$. Then (10) and (A.9) imply

$$\frac{v_B^N(s'')}{v_A^N(s'')} \neq \frac{v_B^N(s')}{v_A^N(s')}.$$

But this implies that there exists $s \in (s', s'')$ such that

$$\frac{dv_B^*(s)/ds}{dv_A^*(s)/ds} \neq \frac{v_B^N(s)}{v_A^N(s)},$$

which contradicts (A.9).

Proof of Proposition 3

Given (13), the Nash bargaining solution satisfies:

$$\gamma(v_A^N(t), v_B^N(t)) = \alpha(t) \tag{A.10}$$

$$\frac{v_B^N(t)}{v_A^N(t)} = \frac{\gamma_B(v_A^N(t), v_B^N(t))}{\gamma_A(v_A^N(t), v_B^N(t))} \tag{A.11}$$

where $\gamma_i \equiv \partial\gamma/\partial u_i$. Homogeneity of γ and (A.11) implies $v_B^N(t) = \lambda v_A^N(t)$, where λ is defined by

$$\lambda = \frac{\gamma_B(1, \lambda)}{\gamma_A(1, \lambda)}.$$

Assumptions 1–3 guarantee a solution exists and is unique.[10] Given that solution,

[10]Pick any point on the Pareto frontier. λ is the slope of the line from the origin to this point, while the right-hand side is the (absolute) slope of the Pareto frontier at this point. The right-hand side is positive, decreasing in λ and is continuous.

$v_A^N(t)$ is then uniquely determined by

$$\gamma(v_A^N(t), \lambda v_A^N(t)) = \alpha(t).$$

Direct inspection shows that this solution also satisfies (10) and therefore satisfies the FBE.

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