Unique Stability in Simple Coalition Formation Games

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Abstract

We investigate the uniqueness of stable coalition structures, when the value of a coalition to a member depends solely on the identity of the other members of the coalition. We search for collections of admissible coalitions that induce uniquely stable coalition structures, that is, ensure that there is a unique stable coalition structure at every preference profile when only admissible coalitions may form. A collection of coalitions satisfies the *single-lapping property* if (a) no two coalitions have more than one member in common, and (b) in a cycle formed by coalitions with a non-empty intersection all the coalitions have the same member in common. We prove that a collection of coalitions induces a unique stable coalition structure if and only if it satisfies the single-lapping property. We also provide an alternative characterization based on a graph representation of collections of coalitions that satisfy the single-lapping property. This alternative characterization is used to explore the implications for matching problems, such as the marriage and roommate problems. Finally, we examine the relationship of our results with the existence of strategyproof rules of coalition formation.

Keywords: coalition formation, stable matching, core, strategyproofness

1 Introduction

We study the uniqueness of stable coalition structures in a simple coalition formation game, where the society is partitioned into coalitions, that is, a player may not be a member of more than one coalition. The value of a coalition to a member depends solely on the identity of the other members of the coalition, the "hedonic aspect" (Drèze and Greenberg (1980)), and players only care about the coalition they join.

In the previous literature personal preferences for membership in specific coalitions are usually not taken into account. Instead, an objective worth of each coalition is specified, which may be split among the members of a coalition; or each coalition chooses an alternative on its own for its members. The current model may be viewed, however, as a simplified version of earlier models. In particular, if the payoffs within each coalition are a priori determined (e.g., by some solution concept), or may be predicted in advance, then we can simply specify players' preferences over coalitions, without making explicit either the potential alternatives for the coalitions or the payoff divisions among the members of each coalition. Finally, we note that our model generalizes the much studied *matching models*,[1] since matchings are based on purely hedonic preferences.

This simple hedonic model of coalition formation is examined by two recent papers, Banerjee, Konishi, and Sönmez (1999), and Bogomolnaia and Jackson (1998). Both of these papers deal with the existence of stable coalition structures, and provide, among other results, restrictions on preference profiles that ensure the non-emptiness of the core (in this model the core and stable outcomes are the same, since any coalitional deviations are ruled out by both concepts). Another paper that considers a similar model is Cechlárová and Romero-Medina (1998). In their paper preferences over coalitions are a priori assumed to be based on the first-ranked or last-ranked member of the coalition.

Our study differs from the above mentioned papers in that it allows for the possibility that not all coalitions may form, since some groups may not actually be able to cooperate (if the members don't know or like each other, or for reasons of efficiency, size, costs of cooperation, etc.). Thus, instead of imposing restrictions on preference profiles, we impose

[1]See Roth and Sotomayor (1990) for an introduction.

conditions on admissible coalitions, which has a very natural interpretation. This approach is taken by a number of papers, with various goals and concerns.[2] Greenberg and Weber (1986) define *consecutive games*, where players are ordered on a line and only connected coalitions may form. The restriction on the formation of coalitions is based on a graph structure in Myerson (1977) and Owen (1986), where cooperation between two players is only possible if there exists a communication link between them. The a priori restrictions on admissible coalitions are much related, and are sometimes equivalent, to the restrictions imposed on the preferences of players over coalitions. For example, Demange (1994) does not restrict the coalitions that may be formed a priori. Instead, she uses the restrictions on preferences to show that only some coalitions matter. She also employs a graph-theoretic structure and focuses on connected coalitions that are represented by connected subgraphs.

Kaneko and Wooders (1982) introduce *partitioning games* in which the set of essential coalitions (coalitions with power) is a priori given, and they examine the non-emptiness of the core of partitioning games (see also Quint (1991) on partitioning games). Le Breton, Owen, and Weber (1992) characterize, by normal hypergraphs, the strong balancedness condition derived by Kaneko and Wooders (1982). Families of coalitions that guarantee the non-emptiness of the appropriately defined core are also characterized by Boros, Gurvich, and Vasin (1997) for a wide range of games.

The focus of this study is the uniqueness rather than the existence of stable collections of coalitions. We search for collections of admissible coalitions that induce uniquely stable coalition structures. A coalition structure is *stable* at a particular preference profile with respect to the admissible coalitions if there is no admissible coalition, which is not part of the coalition structure, such that all its members prefer to join this coalition. We provide a complete characterization of collections of admissible coalitions that guarantee the existence of a unique stable coalition structure with respect to the admissible coalitions for every preference profile.

We derive a simple property of collections of coalitions, the *single-lapping property*, as a necessary and sufficient condition for unique stability (Section 3). This property is easy to

[2]See Kalai, Postlewaite, and Roberts (1979) for one of the first discussions of this approach.

understand, and it has an intuitively clear graph representation (Section 4). Implications of our characterization results for matching problems, in particular, for the marriage and roommate problems are also explored (Section 5). Furthermore, we show that the uniqueness of stable coalition structures is very closely linked to the existence of strategyproof coalition formation rules, and this relationship, which is examined in a more general framework by Sönmez (1999), is also investigated (Section 6).

2 Definitions

There is a **finite set of players** $N = \{1, \ldots, n\}$, and a **set of preferences** $\mathcal{R}_i$ for each player $i \in N$. Player i's preferences $R_i \in \mathcal{R}_i$ strictly order all nonempty subsets of N containing i, that is, R_i is a complete, reflexive, transitive, and antisymmetric binary relation. We will refer to each nonempty subset of N as a **coalition**. Note that we assume that preferences are strict over all the coalitions that a player is a member of, and that players only care about the coalition they join. We denote the strict component corresponding to R_i by P_i. A **coalition formation problem** is defined by a pair (N, R), where $R = (R_1, \ldots, R_n)$ is a **preference profile** in $\mathcal{R}$ and $\mathcal{R} = \times_{i \in N} \mathcal{R}_i$. Throughout this paper N is fixed, and thus a coalition formation problem is simply defined by a preference profile $R \in \mathcal{R}$. We will use the notation top (R_i) to indicate the top-ranked coalition according to (R_i) : top $(R_i) = S$ if $S \subseteq N$, $i \in S$, and for all $T \subseteq N$ with $i \in T$, SR_iT. Given $R \in \mathcal{R}$ and $S \subseteq N$, we denote $(R_i)_{i \in S}$ by R_S. We also write $R_{-i} = R_{N-\{i\}}$ and $R_{-S} = R_{N-S}$.

A **coalition structure** $\sigma = \{S_1, \ldots, S_k\}$, with $n \geq k \geq 1$, is a partition of N, i.e., $\bigcup_{t=1}^{k} S_t = N$, where all S_t are pairwise disjoint. For all $i \in N$, σ_i is the coalition in σ that contains i. Let Σ denote the set of all coalition structures. Furthermore, for all coalitions $S \subseteq N$, let $[S] = \{\{i\} : i \in S\}$ denote the set of singletons for the members of S. Given a preference profile $R \in \mathcal{R}$, a coalition $S \subseteq N$ **blocks** $\sigma \in \Sigma$ if for all $i \in S$, $SP_i\sigma_i$. A coalition structure σ is **stable at $\boldsymbol{R}$** if there is no coalition that blocks σ, given R. Alternatively, we will say that a coalition formation problem (N, R) has a stable coalition structure $\sigma \in \Sigma$ if σ is stable at R. Stable coalition structures can also be referred to as core coalition structures since the two notions are identical in this context.[3]

Let $\Pi = \{S \ : S \subseteq N,\ S \neq \emptyset\}$ denote the set of all coalitions in N. A **collection of coalitions** $\Pi^* \subseteq \Pi$ is a subset of Π such that $[N] \subseteq \Pi^*$. Thus, any collection of coalitions Π^* contains all singletons. It is natural to restrict our attention to these collections of coalitions, since in most contexts individuals cannot be coerced to join any coalition. For $\Pi^* \subseteq \Pi$ and $R_i \in \mathcal{R}_i$, let $R_i|\Pi^*$ denote the restriction to Π^*, i.e., for all $i \in N$, $R_i|\Pi^*$ strictly orders the coalitions in Π^* that contain i such that the ordering of these coalitions by R_i is preserved. A coalition structure $\sigma \in \Sigma$ is **stable at $\boldsymbol{R}$ with respect to a collection of coalitions $\boldsymbol{\Pi^*}$** if $\sigma \subseteq \Pi^*$ and no coalition $S \in \Pi^*$ blocks σ, given R. Hence, a coalition structure that is stable with respect to some collection of coalitions is individually rational, since no singleton blocks it, which means that none of the players prefer to stay on their own. A collection of coalitions Π^* **induces a unique stable coalition structure** if for all $R \in \mathcal{R}$ there exists a coalition structure $\sigma \in \Sigma$ which is the unique stable coalition structure at R with respect to Π^*.

3 The Single-Lapping Property: A Characterization of Unique Stability

Now we can state the property of collections of coalitions that characterizes unique stability. A **collection of coalitions $\boldsymbol{\Pi^*} \subseteq \boldsymbol{\Pi}$ satisfies the single-lapping property** if the following two conditions hold.

Condition (a): For all $S, T \in \Pi^*$ such that $S \neq T$, $|S \bigcap T| \leq 1$.

Condition (b): For all $\{S_1, \ldots, S_m\} \subseteq \Pi^*$ such that $m \geq 3$ and for all $t = 1, \ldots, m$, $|S_t \bigcap S_{t+1}| \geq 1$, where we let $S_{m+1} = S_1$, there exists $i \in N$ such that $S_t \bigcap S_{t+1} = \{i\}$ for all $t = 1, \ldots, m$.

Condition (a) says that if there is an overlap between any two coalitions, there may be at most a single player who is a member of both coalitions, and hence any two coalitions may

[3] The core terminology is used by Banerjee, Konishi, and Sönmez (1999) and Bogomolnaia and Jackson (1998).

be at most *single-lapping.* Condition (b) is a cyclical single-lapping property: it requires that if a set of coalitions form a cycle in which any two neighbors have a common member, then all these coalitions have the same single member in common. Note that these two conditions can be combined by allowing $m = 2$ in Condition (b).

Theorem 1 (Single-lapping property) *A collection of coalitions induces a unique stable coalition structure if and only if it satisfies the single-lapping property.*

Proof:

Part 1: *If a collection of coalitions satisfies the single-lapping property then it induces a unique stable coalition structure.*

First we specify an algorithm that leads to the stable coalition structure for each preference profile $R \in \mathcal{R}$, given a collection of coalitions that satisfies the single-lapping property. Then we verify that the algorithm indeed always selects the unique stable coalition structure.

An algorithm to select the unique stable coalition structure.[4] Let Π^* be a collection of coalitions that satisfies the single-lapping property. Fix $R \in \mathcal{R}$. We will identify a coalition structure $\sigma^*(R) \subseteq \Pi^*$.

First we show that there exists $S \in \Pi^*$ such that for all $i \in S$, top $(R_i|\Pi^*) = S$. Suppose, by contradiction, that such an S does not exist. Fix $i \in N$. Then there exists $j \in$ top $(R_i|\Pi^*)$ such that top $(R_j|\Pi^*)\, P_j$ top $(R_i|\Pi^*)$. Thus, there exists $l \in$ top $(R_j|\Pi^*)$ such that top $(R_l|\Pi^*)\, P_l$ top $(R_j|\Pi^*)$. Note that $l \neq i$, by Condition (a). Then there exists $h \in$ top $(R_l|\Pi^*)$ such that top $(R_h|\Pi^*)\, P_h$ top $(R_l|\Pi^*)$. Then $h \neq j$, by Condition (a), and thus $h \neq i$, by Condition (b). Continuing similarly, we get to a contradiction, since there is a finite number of players.

Note that there may be several coalitions S such that for all $i \in S$, top $(R_i|\Pi^*) = S$, and that all these coalitions are disjoint. Let

$$M_1^* = \{S \in \Pi^* : \text{ for all } i \in S, \text{ top } (R_i|\Pi^*) = S\},$$

[4] This algorithm exhibits some similarity to the *top trading cycle* algorithm due to David Gale, which selects the unique core allocation in indivisible goods markets in which each individual owns one good (Shapley and Scarf (1974), Roth and Postlewaite (1977)).

and let $T_1^* = \bigcup_{S \in M_1^*} S$. Furthermore, let $\Pi_2^* = \{S \in \Pi^* : S \bigcap T_1^* = \emptyset\}$. It is easy to verify that Π_2^* satisfies the single-lapping property. Hence, we can apply the above argument to find

$$M_2^* = \{S \in \Pi_2^* : \text{ for all } i \in S, \text{ top } (R_i|\Pi_2^*) = S\}.$$

Repeating this procedure iteratively we can identify a partition $(M_1^* \bigcup \ldots \bigcup M_m^*) \subseteq \Pi^*$, where $n \geq m \geq 1$. Let $\sigma^*(R) = M_1^* \bigcup \ldots \bigcup M_m^*$.

Now we show that $\sigma^*(R)$ is the unique stable coalition structure at R with respect to Π^*.

Step 1a: $\sigma^*(R)$ *is the only conceivable stable coalition structure at* R *with respect to* Π^*. For all $\sigma \subseteq \Pi^*$ such that σ is a stable coalition structure at R with respect to Π^*, it must be the case that $M_1^* \subseteq \sigma$, since otherwise each coalition $T \in M_1^*$ blocks σ. If $m \geq 2$, this implies that for all $\sigma \subseteq \Pi^*$ such that σ is a stable coalition structure at R with respect to Π^*, we must have $M_2^* \subseteq \sigma$, since, given that $M_1^* \subseteq \sigma$, each coalition $T \in M_2^*$ would block σ otherwise. Continuing this way, we can show that the only coalition structure that is possibly stable at R with respect to Π^* is $\sigma^*(R) = M_1^* \bigcup \ldots \bigcup M_m^*$.

Step 1b: $\sigma^*(R)$ *is a stable coalition structure at* R *with respect to* Π^*.
Suppose there exists $S \in \Pi^*$ such that S blocks $\sigma^*(R)$. Then $S \bigcap T_1^* = \emptyset$, since for all $i \in T_1^*$ and for all $T \in \Pi^*$ with $i \in T$, $\sigma_i^*(R) R_i T$. Therefore, $S \in \Pi_2^*$. Let $T_t^* = \bigcup_{S \in M_t^*} S$ for all $t = 1, \ldots, m$. Then $S \bigcap T_2^* = \emptyset$, since for all $i \in T_2^*$ and for all $T \in \Pi_2^*$ with $i \in T$, $\sigma_i^*(R) R_i T$. Continuing iteratively we can show that for all $t = 1, \ldots, m$, $S \bigcap T_t^* = \emptyset$. Given that $M_1^* \bigcup \ldots \bigcup M_m^*$ is a partition of N, $\bigcup_{t=1}^m T_t^* = N$. Therefore, $S = \emptyset$, which is a contradiction. This implies that $\sigma^*(R)$ is stable at R with respect to Π^*.

Part 2: *If a collection of coalitions induces a unique stable coalition structure then it satisfies the single-lapping property.*

Let $\Pi^* \subseteq \Pi$ be a collection of coalitions that induces a unique stable coalition structure. Suppose it doesn't satisfy the single-lapping property.

Step 2a: *Suppose Condition (a) is violated.*
Then there exist $S, T \in \Pi^*$ such that $S \neq T$ and $|S \bigcap T| \geq 2$. Let $i, j \in S \bigcap T$ such that

$i \neq j$. Let $R \in \mathcal{R}$ satisfy the following.

i) For all $l \in S - (S \bigcap T)$, let R_l rank S first and $\{l\}$ second.

ii) For all $l \in T - (S \bigcap T)$, let R_l rank T first and $\{l\}$ second.

iii) Let R_i rank S first, T second, and $\{i\}$ third.

iv) For all $l \in (S \bigcap T)$ such that $l \neq i$, let R_l rank T first, S second, and $\{l\}$ third. Note, in particular, that R_j ranks T first, S second, and $\{l\}$ third.

v) Finally, for all $l \in N - (S \bigcup T)$, let R_l rank $\{l\}$ first.

Then both $\sigma = \{S\} \bigcup [N - S]$ and $\sigma' = \{T\} \bigcup [N - T]$ are stable at R with respect to Π^*, and we have a contradiction.

Step 2b: *Suppose Condition (b) is violated.*

Then there exists $\{S_1, \ldots, S_m\} \subseteq \Pi^*$ with $m \geq 3$ such that for all $t = 1, \ldots, m$, $|S_t \bigcap S_{t+1}| \geq 1$, where $S_{m+1} = S_1$, and there does not exist $i \in N$ such that $S_t \bigcap S_{t+1} = \{i\}$ for all $t = 1, \ldots, m$. Then Step 2a implies that for all $t = 1, \ldots, m$, $|S_t \bigcap S_{t+1}| = 1$, and that there exists $\{T_1, \ldots, T_k\} \subseteq \{S_1, \ldots, S_m\}$ with $3 \leq k \leq m$ such that for all $t = 1, \ldots, k$, there exists $\{i_t\} = T_t \bigcap T_{t+1}$, where we let $T_{k+1} = T_1$, and all i_t are distinct.

Let $R \in \mathcal{R}$ satisfy the following.

i) For all $t = 1, \ldots, k$ and for all $l \in T_{t+1}$ such that $l \neq i_t, i_{t+1}$, where we let $i_{k+1} = i_1$, let R_l rank T_{t+1} first and $\{l\}$ second.

ii) For all $t = 1, \ldots, k$, let R_{i_t} rank T_{t+1} first, T_t second, and $\{i_t\}$ third.

iii) Finally, for all $l \in N - \bigcup_{t=1}^{k} T_t$, let R_l rank $\{l\}$ first.

Let $\sigma \in \Sigma$ be the unique stable coalition structure at R with respect to Π^*. Then there exists $t = 1, \ldots, k$ such that $T_t \in \sigma$, say $T_1 \in \sigma$, since for all $t = 1, \ldots, k$, and for all $l \in T_t$, $T_t P_l \{l\}$, and for all $T' \in \Pi^* - \bigcup_{t=1}^{k} \{T_t\}$ and for all $l \in T'$, $\{l\} R_l T'$. Then $T_2 \notin \sigma$, since $T_1 \bigcap T_2 = \{i_1\}$. Then T_2 blocks σ unless $T_3 \in \sigma$, which in turn implies that $T_4 \notin \sigma$. Continuing this way we get a contradiction if k is odd, since in this case $T_k \in \sigma$, which implies that $T_1 \notin \sigma$, given $T_k \bigcap T_1 = \{i_k\}$. Therefore, k is even, and $\sigma \supseteq \{T_1, T_3, \ldots, T_{k-1}\}$. In this case, however, a similar argument implies that $\sigma \supseteq \{T_2, T_4, \ldots, T_k\}$, and we have a contradiction, since the stable coalition structure at R with respect to Π^* is unique. □

Next, we extend the definition of the single-lapping property to a coalition formation problem in a natural way.

A **coalition formation problem $(\boldsymbol{N}, \boldsymbol{R})$ satisfies the single-lapping property** if the collection of coalitions

$$\{S \subseteq N \,:\, \text{there exists } i \in S \text{ such that } SR_i\{i\}\}$$

satisfies the single-lapping property.

Theorem 1 has an immediate implication for the existence of a unique stable coalition structure for a coalition formation problem, namely, that if a coalition formation problem satisfies the single-lapping property then it has a unique stable coalition structure. This follows from Part 1 of Theorem 1. The converse statement does not hold, however, since the requirement that each coalition formation problem, in the family of those problems in which only a certain restricted collection of coalitions is desired, have a unique stable coalition structure is substantially more demanding than requiring unique stability for just one such coalition formation problem. This is illustrated by the following example.

Example 1 Consider the following coalition formation problem with three players. (Preferences over coalitions are listed from the top down, omitting braces.)

R_1^*	R_2^*	R_3^*
123	123	123
12	2	3
1		

The unique stable coalition structure for this problem is $\sigma = \{\{1, 2, 3\}\}$, the joining of the grand coalition. The problem R^* does not satisfy the single-lapping property, however, since player 1 prefers both 12 and 123 to staying on her own, and Condition (a) is violated.□

Next, we show that the *top-coalition property* proposed by Banerjee, Konishi, and Sönmez (1999) is weaker than the single-lapping property when preferences are strict. Banerjee, Konishi, and Sönmez (1999) show that the top-coalition property is sufficient for a unique stable coalition structure to exist. Not surprisingly, the single-lapping property, which is necessary as well as sufficient for the existence of a unique stable coalition structure for all preference profiles, is more stringent.

Given a coalition $S \subseteq N$, a coalition $T \subseteq S$ is a **top-coalition** of S if for any $i \in T$ and any $T' \subseteq S$ with $i \in T'$, TR_iT'. A coalition formation problem satisfies the **top-coalition property** if for any coalition $S \subseteq N$ there exists a top-coalition of S.

We can verify that the single-lapping property for coalition formation problems implies the top-coalition property, based on some of the arguments in the proof of Theorem 1. Fix $R \in \mathcal{R}$ and $S \subseteq N$. Assume that R satisfies the single-lapping property. Let

$$\Pi^* = \{T \subseteq S \ : \text{there exists } j \in T \text{ such that } TR_j\{j\}\}.$$

Note that since R satisfies the single-lapping property, so does Π^*. Then we can show that there exists $T \in \Pi^*$ such that for all $i \in T$, top $(R_i|\Pi^*) = T$, using a similar argument as in the algorithm in the proof of Part 1 of Theorem 1. This means that T is a top-coalition of S, which is what we needed to show. The top-coalition property is weaker than the single-lapping property, which can be seen from Example 1. The coalition formation problem in Example 1 does not satisfy the single-lapping property, but it satisfies the top-coalition property, given that coalition $\{1, 2, 3\}$ is a top-coalition of itself, and any pair of players has a singleton as a top-coalition in this problem.

4 The Tree Structure Representation: An Alternative Characterization of Unique Stability

A collection of coalitions that satisfies the single-lapping property can be represented by a graph, in particular, a *forest*. This representation, which we call the *tree structure representation,* provides further insights about the collections of coalitions that induce unique stability.

A *graph* $G = (N, E)$ is given by a set E of unordered pairs of distinct players in N. A pair $\{i, j\} \in E$ is called an *edge* between i and j. A *path* of G is a sequence

$\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}$ of distinct edges in G, such that all players $i_1, \ldots, i_k$ are distinct, except, possibly, for i_1 and i_k. We say that this path *connects* players $i_1, \ldots, i_k$. If $i_1 = i_k$ and the path connects at least two distinct players then it is a *cycle*. A graph is *connected* if there is a path between every pair of distinct players in N. A graph that is not connected is the union of two or more connected subgraphs, each pair of which has no player in common. These disjoint subgraphs are called the *connected components* of the graph. A graph is a *tree* if any two distinct players are connected by a unique path. A graph is a *forest* if each of its connected components is a tree. It is well-known that if a graph is a tree or a forest then it contains no cycles. In fact, a graph is a forest if and only if it contains no cycles.

A set of edges $\{\{i_t, i_{t+1}\} : t = 1, \ldots, k-1\}$ in G is *consecutive* if $\{i_1, i_2\}, \{i_2, i_3\}, \ldots,$ $\{i_{t-1}, i_t\}$ is a path in G. A set of consecutive edges $\{\{i_t, i_{t+1}\} : t = 1, \ldots, k-1\}$ with $i_1 \neq i_k$ **represents** a coalition $S \subseteq N$ if $S = \{i_1, \ldots, i_k\}$. For each collection of coalitions $\Pi^* \subseteq \Pi$, we construct its **associated graph** $G = (N, E)$ as follows. If $\Pi^* = [N]$, we have $E = \emptyset$, that is, the edge set is empty in G. Otherwise, let $\Pi^* = [N] \bigcup \{S_1, \ldots, S_m\}$, without loss of generality, where $m \geq 1$, and coalitions $S_1, \ldots, S_m$ are distinct. Then the edge set E can be partitioned into $\{E_1, \ldots, E_m\}$, such that for each $t = 1, \ldots, m$, E_t is a set of consecutive edges which represents S_t. Thus, in the graph associated with Π^* there is a distinct set of consecutive edges representing each non-singleton coalition in Π^*. A graph associated with a collection of coalitions may not be unique, since it depends on which members of a coalition are connected by edges. We can ensure the uniqueness of the associated graph G for a collection of coalitions by following the convention that two players i, $j \in S$, where $i < j$, are connected by an edge in G to represent the coalition $S \in \Pi^*$ only if there is no $l \in S$ such that $i < l < j$.

We will say that a collection of coalitions has a **tree structure representation** if (any of) its associated graph(s) is a forest. An example of a tree structure representation is given in Figure 1 (which follows the above described convention). Notice that we cannot read off the admissible coalitions from the graph without specifying the partition of the edge set corresponding to the represented collection of coalitions (in Figure 1, the bold edges indicate the appropriate partition).

Theorem 2 (Tree structure representation) *A collection of coalitions induces a unique stable coalition structure if and only if it has a tree structure representation.*

We will prove the following lemma, which, together with Theorem 1, yields Theorem 2.

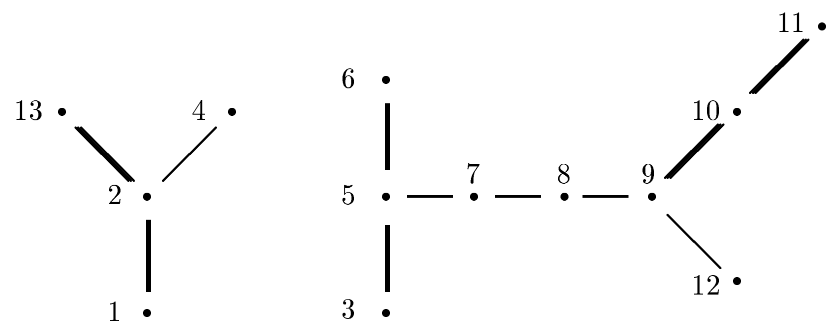

Figure 1: A tree structure representation of the collection of coalitions $\Pi^* = \{ \{1, 2, 13\}, \{2, 4\}, \{3, 5, 6\}, \{5, 7, 8, 9, 12\}, \{9, 10, 11\} \}$, which satisfies the single-lapping property

Lemma 1 *A collection of coalitions satisfies the single-lapping property if and only if it has a tree structure representation.*

Proof:

Part 1: *If a collection of coalitions satisfies the single-lapping property then it has a tree structure representation.*

Fix a collection of coalitions $\Pi^* \subseteq \Pi$ that satisfies the single-lapping property. Let $G = (N, E)$ be a graph associated with Π^*. Suppose G is not a forest. Then G contains a cycle C. Recall that E can be partitioned into $\{E_1, \ldots, E_m\}$ such that for each $z = 1, \ldots, m$, E_z is a set of consecutive edges which represents $S_z \in \Pi^*$. Then $C \subseteq E$ can be partitioned into $\{C_1, \ldots, C_k\}$ such that for all $t = 1, \ldots, k$, there exists $z_t \in \{1, \ldots, m\}$ with $C_t \subseteq E_{z_t}$. Note that $k \geq 2$, since a set of consecutive edges that constitutes a cycle cannot represent a single coalition. Suppose $k = 2$. Then there exist $i, j \in N$, $i \neq j$, such that C connects i and j and $\{i, j\} \subseteq S_{z_1} \bigcap S_{z_2}$. This contradicts Condition (a) in the definition of the single-lapping property. Hence, $k \geq 3$. Furthermore, for all $t = 1, \ldots, k-1$, $|S_{z_t} \bigcap S_{z_{t+1}}| \geq 1$, $|S_{z_1} \bigcap S_{z_k}| \geq 1$, and there does not exist $i \in N$ such that for all $t = 1, \ldots, k-1$, $S_{z_t} \bigcap S_{z_{t+1}} = \{i\}$ and $S_{z_1} \bigcap S_{z_k} = \{i\}$. This contradicts Condition (b) in the definition of the single-lapping property. Hence, G is a forest, and Π^* has a tree structure representation.

Part 2: *If a collection of coalitions has a tree structure representation then it satisfies the single-lapping property.*

Fix a collection of coalitions $\Pi^* \subseteq \Pi$ that has a tree structure representation. Let $G = (N, E)$ be a graph associated with Π^*. Then G is a forest. Suppose Π^* does not satisfy the single-lapping property. Assume first that Condition (a) does not hold. Then there exist $S, T \in \Pi^*$ such that $S \neq T$ and $|S \bigcap T| \geq 2$. This implies that there exist two sets of consecutive edges $E_S \subset E$ and $E_T \subset E$ such that $E_S \bigcap E_T = \emptyset$, E_S represents S, and E_T represents T. Then, since $|S \bigcap T| \geq 2$, G contains a cycle, which is a contradiction. Thus, Condition (a) holds, and hence Condition (b) must be violated. We can show similarly for this case that G contains a cycle, which again contradicts the fact that it is a forest. Therefore, Π^* satisfies the single-lapping property. □

A special class of collections of coalitions that satisfy the single-lapping property is one in which each member has an associated graph that is *linear* (a linear graph consists of a single path that connects all the players). The corresponding coalition formation problems are *consecutive,* a property that is defined by Bogomolnaia and Jackson (1998) as an analogue, for our simple coalition formation games, of the property given by Greenberg and Weber (1986). A coalition formation problem is **consecutive** if there exists an ordering of the players such that any coalition that is preferred to staying alone by some player is consecutive with respect to this ordering. Bogomolnaia and Jackson (1998) prove that this property is sufficient for the existence of a stable coalition structure. For coalition formation problems that satisfy the single-lapping property every coalition that is favored by some player to staying alone is consecutive on a tree rather than a line, which is more general. On the other hand, the single-lapping property is more restrictive than consecutiveness in that it does not allow the formation of every coalition that is consecutive on a tree, given that admissible coalitions cannot have more than one member in common.

The single-lapping property, or, equivalently, the existence of a tree structure representation, is extremely restrictive. Since such restrictions on admissible coalitions do not occur naturally in most contexts, it follows from our results that unique stability is not a common feature of purely hedonic coalition formation processes in various social, economic, and political settings. A normative conclusion that one may draw from our theorems is that if the coalition formation process can be affected or designed by some organization, it may nevertheless be desirable to select a collection of coalitions that meets these strict criteria, in order to ensure the existence of a unique stable coalition structure. The need for such a design may be particularly pronounced if the preferences of the players are private information, and is further underlined by the implications of unique stability for strategyproofness that we examine in Section 6. Ideally, the choice should depend on some public information relevant to the formation of coalitions, and it may take into account the preferences of the designer. The question in general is one of choosing a *a spanning tree*[5] of $G = (N, E^c)$, which will afford the greatest flexibility possible under the circumstances (where E^c denotes the *complete graph*, a graph in which any two players are connected by exactly one edge).

One way of constructing a spanning tree, based on a priori knowledge of the desirability of placing pairs of players in the same coalition, is given by the following procedure. Order the pairs of players according to how desirable it is to have them in the same coalition (e.g., put pairs of "friends" first and pairs of "enemies" last, or place pairs that are well-matched according to their skills, views, etc., on the top, depending on the context). Starting from the top, let an edge connect the two players in the first pair, in the second pair, and so on, until the addition of the edge corresponding to a pair would create a cycle. Skip this pair, and continue in a similar manner down the list of pairs, until the addition of any edge would yield a cycle. Clearly, we have arrived at a spanning tree. Now partition the edge set into sets of consecutive edges so that the resulting admissible coalitions are desirable according to some criteria (e.g., optimal size of coalitions). In this way we have constructed

[5]Given a connected graph G, we can choose a cycle and remove any of its edges, and the remaining graph stays connected. If we repeat this procedure until there are no cycles left, we arrive at a tree, which is called a *spanning tree of G*. More generally, when G may not be connected, we find similarly a *spanning forest of G*.

a collection of coalitions that satisfies the single-lapping property.

This procedure illustrates how one may restrict a priori the coalition formation procedure in order to guarantee unique stability for all preference profiles. Admittedly, if restrictions don't arise naturally, or are insufficient in numbers, the construction of a spanning tree (or a spanning forest) essentially determines a priori the coalitions to be formed, without much reference to the preferences. On the other hand, imposing restrictions may be quite natural in some contexts, for example, in the cases of marriage and roommate markets, which we explore next.

5 Implications for Matching Problems

Two special cases of coalition formation problems are the well-studied problems of matching in marriage and roommate markets. For these problems any collection of admissible coalitions $\Pi^* \subseteq \Pi$ is such that for all $S \in \Pi^*$, $|S| \leq 2$, that is, any non-singleton admissible coalition is of size two. In the case of the marriage problem it is also required that for all $S \in \Pi^*$ with $S = \{i, j\}$, $i \neq j$, we have $i \in W$ and $j \in M$, where $N = W \bigcup M$ and W and M are the disjoint sets of women and men. The existence of a stable marriage matching (i.e., a stable coalition structure) for every preference profile is shown by Gale and Shapley (1962) in their classical paper (for an alternative non-constructive proof see Sotomayor (1996)), and they also show in the same paper that there may not exist any stable roommate matching. It is also well-known that stable marriage matchings are typically not unique.

Alcalde (1995) gives a sufficient condition for the existence of stable roommate matchings. His sufficiency property, α-reducibility, is the two-person equivalent of the top-coalition property of Banerjee, Konishi, and Sönmez (1999), and is therefore implied by the single-lapping property for roommate problems. A complete characterization of preference profiles for which a stable roommate matching exists is provided by Tan (1991), and a stronger sufficiency condition, which allows for indifferences, the *no-odd-rings* condition, is identified by Chung (1999), both of which are implied by the single-lapping property.

Our characterization results, Theorems 1 and 2, shed some light on these matching problems as well, by providing necessary and sufficient conditions for collections of coalitions to induce a unique stable marriage matching as well as a unique stable roommate matching. Our results also apply to the more complex problem of multi-sided matching (whereas the marriage problem is a two-sided matching problem), for which the admissible coalitions are singletons and coalitions of size k ($k \geq 3$), and every coalition of size k has one player from each of the k sides of the market (see Alkan (1988) for a non-existence result for three-sided matching markets). Theorems 1 and 2 do not apply, however, to two-sided matching problems in which players on one or both sides of the market may be matched to more than one player on the other side of the market, because it is usually assumed that the players do not care about who else is matched to the same player as they are.[6]

In the case of roommate problems (and marriage problems as a special case), unique stability may be understood best by considering the tree structure characterization given in Theorem 2. In this special case there is a unique graph associated with any collection of coalitions (i.e., sets of acceptable pairs), regardless of any convention, since in this case there is an edge between two players if and only if they constitute an acceptable pair. Note also that the graph associated with a roommate problem unambiguously indicates the admissible coalitions. The only difference between the associated graphs of marriage and roommate problems is that for roommate problems there may be an edge between any two players, whereas for marriage problems the graph is a priori restricted to be *bipartite*[7] with respect to the partition of women and men, and thus an edge is ruled out a priori between players of the same sex.

Given that there is an edge between two players in a graph representation of collections of coalitions for marriage and roommate problems if and only if this pair of players is acceptable, in the following we will call a graph associated with a collection of coalitions (**collection of acceptable pairs**) for these two special problems the **associated acceptability graph**. It follows from Lemma 1 that a collection of coalitions satisfies the

[6]The best-known of these problems is the college admissions problem, or many-to-one matching problem (for an introduction, see Roth and Sotomayor (1990)). Even more complex is the many-to-many matching problem (see, for example, Alkan (1999) and Sotomayor (1999)).

[7]A graph is *bipartite* if the set of players can be split into two disjoint sets such that each edge connects a player from one set with a player in the other set. Note that if G is a bipartite graph then each cycle of G is even (i.e., consists of an even number of edges).

single-lapping property in a marriage or roommate market if and only if its associated acceptability graph has no cycles. Therefore, the following corollary is implied by Theorem 2.

Corollary 1 *In marriage and roommate problems a collection of acceptable pairs induces a unique stable matching if and only if the associated acceptability graph contains no cycles.*

It is easily seen that Theorem 1 leads to the same implication. Since the requirement that two distinct coalitions cannot have more than one member in common is trivially satisfied in these simpler cases, the definition of the single-lapping property reduces to Condition (b). Condition (b), in turn, is immediately seen to be equivalent to the requirement that there are no cycles in the associated acceptability graph. Let us also remark that for marriage and roommate problems unique stability is consistent with *any* consecutive coalition being admissible (whether consecutiveness is meant on a line or on a tree), unlike for general coalition formation problems, since Condition (a) holds vacuously in these cases. Thus, for these matching problems the single-lapping property may be regarded as a consecutiveness property on a tree.

A result that is related to Corollary 1, but deals with the existence rather than the uniqueness of stable coalition structures, is due to Abeledo and Isaak (1991). They show that, in our terminology, a collection of acceptable pairs induces *some* stable matching, which is not necessarily unique, if and only if the associated acceptability graph is bipartite. This is a finding that explains the gap between the existence result for marriage problems and the non-existence result for roommate problems, namely, that marriage problems rule out odd cycles (cycles that consist of an odd number of edges). Our condition is more demanding, since it rules out every cycle, not only odd cycles.

We conclude this section by giving some examples of acceptability graphs without cycles that may arise naturally or may be constructed easily, based on a priori information. For example, if it is publicly known that some mates are not acceptable to some players (the most obvious example of this is players of the same sex for each player in marriage markets), then it is natural to restrict E accordingly, and then remove further edges from the graph, if necessary, in order to arrive at a spanning forest.

Our first example is for marriage markets. Suppose it is desirable to match men and women closely according to a one-dimensional criterion such as height. Suppose, furthermore, that it is required that a woman should never be taller than her mate. Then, assuming that there are no two women or two men who are exactly of the same height, one may use the following acceptability graph that has no cycles. A pair of a woman and a man is an acceptable pair (i.e., there is an edge connecting them in the acceptability graph), whenever the woman is not taller than the man, and there is no other man who is shorter than the man in question, but not shorter than the woman.

The acceptability graph consists of *stars* (one player is connected to all the other players), as connected components, for roommate problems, if the players are a priori partitioned into small groups of "friends" (or "acquaintances"), so that it would be undesirable to match players from different groups, and if each group of friends that consists of more than two players has a "center," a person who is the only acceptable roommate to all the others in her group of friends. Finally, the acceptability graph for roommate problems is a linear graph if we order the players according to their level of tidiness (or some other one-dimensional criterion), and, assuming that no two players have exactly the same level of tidiness, any pair of players is acceptable as roommates if they are adjacent in this ordering. Hence, this acceptability graph is based on the assumption that roommates are best matched if their levels of tidiness are matched.[8]

6 Strategyproofness and Unique Stability

The problem we investigate in this study, the existence of a unique stable coalition structure, is an interesting problem in its own right. Our characterization results, Theorems 1 and 2, are given further importance, in addition, by the fact that unique stability is closely related to the existence of strategyproof coalition formation rules. A rule is strategyproof if players cannot misrepresent their preferences profitably, regardless of what preferences the others have. We will show that if there exists a collection of coalitions Π^* such that every preference profile satisfies the single-lapping property with respect to Π^*, then the coalition formation rule which selects the unique stable coalition structure at every pref-

[8]The example of tidiness as a criterion for choosing roommates is due to Chung (1999).

erence profile is strategyproof, Pareto efficient, and individually rational. That is, if there is a collection of coalitions that satisfies the single-lapping property, and if it is known a priori that players prefer staying on their own to joining any of the coalitions not in this collection, then not only the existence of a unique stable coalition structure is guaranteed for every preference profile, but also the coalition formation rule that chooses the unique stable coalition structure at each profile possesses desirable properties, namely, stragyproofness, Pareto efficiency, and individual rationality. Furthermore, Sönmez (1999) proved for a general indivisible goods allocation model, which includes our model as a special case, that if there is a strategyproof, Pareto efficient, and individually rational allocation rule, whenever the core is nonempty, the allocation rule must select an allocation in the core, and the core must be a singleton when preferences are strict.[9] Therefore, the coalition formation rule that chooses the unique stable coalition structure at each preference profile is the only rule that has these three desirable properties. While this follows from Sönmez (1999), we provide an alternative direct proof of uniqueness for our model.

Fix a collection of coalitions $\Pi^* \subseteq \Pi$. Let $\mathcal{R}^{\Pi^*} \subseteq \mathcal{R}$ denote the set of all preference profiles $R \in \mathcal{R}$ such that for all $i \in N$ and $S \subseteq N$, $SR_i\{i\}$ implies that $S \in \Pi^*$. Note that $\mathcal{R}^{\Pi} = \mathcal{R}$. A **coalition formation rule** $f : \mathcal{R}^{\Pi^*} \mapsto \Sigma$ selects for each preference profile $R \in \mathcal{R}^{\Pi^*}$ a coalition structure $\sigma \in \Sigma$. We use the notation $f_i(R) = \sigma_i$, where $f(R) = \sigma$. A coalition formation rule f is **strategyproof** if for all $R \in \mathcal{R}^{\Pi^*}$, $i \in N$, and $\tilde{R}_i \in \mathcal{R}_i^{\Pi^*}$, $f_i(R) R_i f_i(\tilde{R}_i, R_{-i})$. A coalition formation rule f is **Pareto efficient** if for all $R \in \mathcal{R}^{\Pi^*}$, there does not exist $\sigma \in \Sigma$ such that for all $i \in N$, $\sigma_i R_i f_i(R)$, and for some $j \in N$, $\sigma_j P_j f_j(R)$. A coalition formation rule f is **individually rational** if for all $R \in \mathcal{R}^{\Pi^*}$ and for all $i \in N$, $f_i(R) R_i \{i\}$.

Theorem 3 *Let $\Pi^* \subseteq \Pi$ be a collection of coalitions that satisfies the single-lapping property. Then there exists a unique coalition formation rule $f : \mathcal{R}^{\Pi^*} \mapsto \Sigma$ that is strategyproof, Pareto efficient, and individually rational, namely, the rule that selects the unique stable coalition structure with respect to Π^* at every preference profile.*

[9] Another related paper is Ledyard (1977), which studies strategyproof rules that select a core allocation, and finds that the existence of a best unblocked allocation for all the participants is a necessary and sufficient condition for the existence of a strategyproof rule in core selecting organizations.

Proof: Fix a collection of coalitions $\Pi^* \subseteq \Pi$ that satisfies the single-lapping property.

Part 1: Existence. Let $f : \mathcal{R}^{\Pi^*} \mapsto \Sigma$ be the coalition formation rule that selects the unique stable coalition structure with respect to Π^* at every preference profile. We will show that f is strategyproof, Pareto efficient, and individually rational.

First, suppose that f is not individually rational. Then there exist $i \in N$ and $R \in \mathcal{R}^{\Pi^*}$ such that $\{i\} P_i f_i(R)$. This means that $\{i\}$ blocks $f(R)$, given R, and $f(R)$ is not stable at R with respect to Π^*, given that $\{i\} \in \Pi^*$. Hence, f is individually rational.

Next, suppose f is not Pareto efficient. Then there exist $R \in \mathcal{R}^{\Pi^*}$ and $\sigma \in \Sigma$ such that for all $i \in N$, $\sigma_i R_i f_i(R)$, and for some $j \in N$, $\sigma_j P_j f_j(R)$. Let $S = \sigma_j$. Then for all $i \in S$, $S P_i f_i(R)$. This means that S blocks $f(R)$. Furthermore, for all $i \in S$, $S P_i f_i(R) R_i \{i\}$, by individual rationality. This implies that $S \in \Pi^*$, which contradicts the fact that σ is stable at R with respect to Π^*. Hence, f is Pareto efficient.

It remains to show that f is strategyproof. First note that for all $R \in \mathcal{R}^{\Pi^*}$, $f(R) \subseteq \Pi^*$, by individual rationality, and thus for all $i \in N$ and $R \in \mathcal{R}^{\Pi^*}$, $f_i(R) \in \Pi^*$. Suppose f is not strategyproof. Then there exist $R \in \mathcal{R}^{\Pi^*}$, $i \in N$, and $\tilde{R}_i \in \mathcal{R}_i$ such that $f_i(\tilde{R}_i, R_{-i}) P_i f_i(R)$. Since $f_i(\tilde{R}_i, R_{-i})$ does not block $f(R)$, there exists $j \in f_i(\tilde{R}_i, R_{-i})$ such that $f_j(R) P_j f_j(\tilde{R}_i, R_{-i})$. Since both $f_j(R)$ and $f_i(\tilde{R}_i, R_{-i})$ are in Π^*, and since Π^* satisfies the single-lapping property, $i, j \in f_i(\tilde{R}_i, R_{-i})$ and $j \in f_j(R)$ imply that $i \notin f_j(R)$, otherwise Condition (a) would be violated. Hence $f_j(R) \neq f_i(R)$. Then, since $f_j(R)$ does not block $f(\tilde{R}_i, R_{-i})$, there exists $l \in f_j(R)$ such that $f_l(\tilde{R}_i, R_{-i}) P_l f_l(R)$. Note that $l \neq i$. Since both $f_l(\tilde{R}_i, R_{-i})$ and $f_j(R)$ are in Π^*, and since Π^* satisfies the single-lapping property, $j, l \in f_j(R)$ and $l \in f_l(\tilde{R}_i, R_{-i})$ imply that $j \notin f_l(\tilde{R}_i, R_{-i})$, otherwise Condition (a) would be violated. Hence $f_l(\tilde{R}_i, R_{-i}) \neq f_i(\tilde{R}_i, R_{-i})$. Then, since $f_l(\tilde{R}_i, R_{-i})$ does not block $f(R)$, there exists $h \in f_l(\tilde{R}_i, R_{-i})$ such that $f_h(R) P_h f_h(\tilde{R}_i, R_{-i})$. Note that $h \neq j$. Furthermore, Condition (b) implies in this case that $h \neq i$. Continuing similarly, we reach a contradiction, given that the set of players is finite. Therefore, f is strategyproof.

Part 2: Uniqueness. Let $f : \mathcal{R}^{\Pi^*} \mapsto \Sigma$ be a coalition formation rule that satisfies strategyproofness, Pareto efficiency, and individual rationality. We will show that f selects the unique stable coalition structure with respect to Π^* at every preference profile. Fix $R \in \mathcal{R}^{\Pi^*}$. Let $\sigma^* \in \Sigma$ denote the unique stable coalition structure at R with respect to Π^*.

We will prove that $f(R) = \sigma^*$.

Specify $\tilde{R} \in \mathcal{R}^{\Pi^*}$ as follows. For all $i \in N$, let $\tilde{R}_i$ rank σ_i^* first, and, provided $\sigma_i^* \neq \{i\}$, let $\tilde{R}_i$ rank $\{i\}$ second. Let $\sigma^* = M_1^* \bigcup \ldots \bigcup M_m^*$ be the partition of the players as defined in the algorithm described in the proof of Part 1 of Theorem 1. For all $t = 1, \ldots, m$, let $T_t^* = \bigcup_{S \in M_t^*} S$. Suppose there exist $S \in M_1^*$ and $\bar{R}_{-S} \in \mathcal{R}^{\Pi^*}_{-S}$ such that $S \notin f(R_S, \bar{R}_{-S})$. Let $S = \{1, \ldots, s\}$, without loss of generality. Note that $s \geq 2$, by individual rationality. Since for all $i \in S$, top $(R_i) = S$, strategyproofness implies that for all $i \in S$, $f_i(\tilde{R}_1, \ldots, \tilde{R}_i, R_{i+1}, \ldots, R_s, \bar{R}_{-S}) \neq S$. Hence $S \notin f(\tilde{R}_S, \bar{R}_{-S})$. Then, since $s \geq 2$, individual rationality implies that $[S] \subseteq f(\tilde{R}_S, \bar{R}_{-S})$, which contradicts Pareto efficiency. Therefore, for all $S \in M_1^*$ and $\bar{R}_{-S} \in \mathcal{R}^{\Pi^*}_{-S}$, we have $S \in f(R_S, \bar{R}_{-S})$. This implies that for all $\bar{R}_{-T_1^*} \in \mathcal{R}^{\Pi^*}_{-T_1^*}$, $M_1^* \subseteq f(R_{T_1^*}, \bar{R}_{-T_1^*})$.

Let $\Pi_2^* = \{S \in \Pi^* : S \bigcap T_1^* = \emptyset\}$. Suppose that there exist $S \in M_2^*$ and $\bar{R}_{-(T_1^* \bigcup S)}$ such that $S \notin f(R_{T_1^*}, R_S, \bar{R}_{-(T_1^* \bigcup S)})$. Then, given that for all $i \in S$, top $(R_i | \Pi_2^*) = S$, and that for all $\bar{R}_{-T_1^*} \in \mathcal{R}^{\Pi^*}_{-T_1^*}$ $M_1^* \subseteq f(R_{T_1^*}, \bar{R}_{-T_1^*})$, a similar argument to the above implies a contradiction. Therefore, for all $\bar{R}_{-(T_1^* \bigcup T_2^*)} \in \mathcal{R}^{\Pi^*}_{-(T_1^* \bigcup T_2^*)}$, $M_2^* \subseteq f(R_{T_1^*}, R_{T_2^*}, \bar{R}_{-(T_1^* \bigcup T_2^*)})$. Continuing iteratively, we can establish that for all $t = 1, \ldots, m$, $M_t^* \subseteq f(R)$. This means that $f(R) = \sigma^*$, as desired. $\square$

The coalition formation rule f which always selects the unique stable coalition structure with respect to Π^* (given Π^* that satisfies the single-lapping property) is also group-strategyproof. A coalition formation rule $f : \mathcal{R}^{\Pi^*} \mapsto \Sigma$ is **group-strategyproof** if there exist no $S \subseteq N$, $R \in \mathcal{R}^{\Pi^*}$, and $\tilde{R}_S \in \mathcal{R}^{\Pi^*}_S$ such that for all $i \in S$, $f_i(\tilde{R}_S, R_{-S}) R_i f_i(R)$, and for some $j \in N$, $f_j(\tilde{R}_S, R_{-S}) P_j f_j(R)$. It is a stronger property than strategyproofness since it rules out joint manipulations by coalitions. The group-strategyproofness of f as specified in Theorem 3 can be seen as follows. Suppose, by contradiction, that there exists a coalition S that can manipulate as described in the definition of group-strategyproofness. For all $t = 1, \ldots, m$, let $S_t = S \bigcap T_t^*$, where the unique stable coalition structure with respect to Π^* at R is $\sigma^* = M_1^* \bigcup \ldots \bigcup M_m^*$ and $T_t^* = \bigcup_{T \in M_t^*} T$ for all $t = 1, \ldots, m$. Then for all $i \in S_1$, $f_i(\tilde{R}_S, R_{-S}) = f_i(R)$, and thus $M_1 \subseteq \sigma^*$, given the algorithm in the proof of Part 1 of Theorem 1. Then for all $i \in S_2$, $f_i(\tilde{R}_S, R_{-S}) = f_i(R)$, and $M_2 \subseteq \sigma^*$. Repeating iteratively we obtain that for all $i \in \bigcup_{t=1}^m S_t$, $f_i(\tilde{R}_S, R_{-S}) = f_i(R)$. Since $\bigcup_{t=1}^m S_t = S$, this is a contradiction.

All previous results on strategyproof coalition formation and matching rules are negative,[10] with one exception. Sönmez (1999) shows that there is no strategyproof, Pareto efficient, and individually rational rule when all conceivable coalitions are admissible and preferences over coalitions are unrestricted. Bogomolnaia and Jackson (1998) prove that strategyproofness and individual stability (a weaker notion than stability) are incompatible on a rather restricted domain of preferences, namely, where only the size of a coalition matters, but not the identity of its members, and preferences over the size are single-peaked. The only possibility result reported previously is the finding by Alcalde and Barberà (1994) that if preferences satisfy a so-called *top dominance* condition for one side of a marriage market, then there exists a stable and strategyproof matching rule.[11] Our possibility result based on the single-lapping property is unrelated to this finding, since Alcalde and Barberà (1994) require a certain richness condition of the preference domain, which includes, among other requirements, a condition that every mate is admissible according to some preferences.

7 Concluding Remarks

In this study we provide a complete characterization of collections of admissible coalitions that guarantee the existence of a unique stable coalition structure for every preference profile. Our characterization result yields a simple and intuitive property of collections of coalitions, the single-lapping property, which is very easy to check. An alternative characterization based on a graph-theoretical equivalent of the single-lapping property is provided for additional insights, and the implications for simpler matching problems as well as for the existence of strategyproof coalition formation rules are also explored. A further question to investigate is whether a similar intuitive characterization of existence, but not necessarily uniqueness, of stable coalition structures, can be given.

[10] There is a parallel result with Theorem 3 for Shapley-Scarf indivisible goods markets (Shapley and Scarf (1974)), which is proved by Roth (1982) and Ma (1994).

[11] See also Roth and Sotomayor (1990) for a discussion of the incentives in matching problems and further references.

References

Abeledo, H., G. Isaak (1991), "A Characterization of Graphs that Ensure the Existence of Stable Matchings," *Mathematical Social Sciences* 22: 93-96.

Alcalde, J. (1995), "Exchange-Proofness or Divorce-Proofness? Stability in One-Sided Matching Markets," *Economic Design* 1: 275-287.

Alcalde, J., S. Barberà (1994), "Top Dominance and the Possibility of Strategy-Proof Stable Solutions to Matching Problems," *Economic Theory* 4: 417-435.

Alkan, A. (1988), "Nonexistence of Stable Threesome Matchings," *Mathematical Social Sciences* 16: 207-209.

Alkan, A. (1999), "On the Properties of Stable Many-to-Many Matchings under Responsive Preferences," mimeo.

Banerjee, S., H. Konishi, T. Sönmez (1999), "Core in a Simple Coalition Formation Game," mimeo.

Bogomolnaia, A., M.O. Jackson (1998), "The Stability of Hedonic Coalition Structures," mimeo.

Boros, E., V. Gurvich, A. Vasin (1997), "Stable Families of Coalitions and Normal Hypergraphs," *Mathematical Social Sciences* 34: 107-123.

Cechlárová, K., A. Romero-Medina (1998), "Stability in One-Sided Matching Markets," mimeo.

Chung, K.-S. (1999), "On the Existence of Stable Roommate Matchings," *Games and Economic Behavior* forthcoming.

Demange, G. (1994), "Intermediate Preferences and Stable Coalition Structures," *Journal of Mathematical Economics* 23: 45-58.

Drèze, J.H., J. Greenberg (1980), "Hedonic Coalitions: Optimality and Stability," *Econometrica* 48: 987-1003.

Gale, D., L.S. Shapley (1962), "College Admissions and the Stability of Marriage," *American Mathematical Monthly* 69: 9-15.

Greenberg, J., S. Weber (1986), "Strong Tiebout Equilibrium under Restricted Preferences Domain," *Journal of Economic Theory* 38: 101-117.

Kalai, E., A. Postlewaite, J. Roberts (1979), "A Group Incentive Compatible Mechanism Yielding Core Allocations," *Journal of Economic Theory* 20: 13-22.

Kaneko, M., M.H. Wooders (1982), "Cores of Partitioning Games," *Mathematical Social Sciences* 3: 313-327.

Le Breton, M., G. Owen, S. Weber (1992), "Strongly Balanced Cooperative Games," *International Journal of Game Theory* 20: 419-427.

Ledyard, J.O. (1977), "Incentive Compatible Behavior in Core-Selecting Organizations," *Econometrica* 45: 1607-1621.

Ma, J. (1994), "Strategy-Proofness and the Strict Core in a Market with Indivisibilities," *International Journal of Game Theory* 23: 75-83.

Myerson, R.B. (1977), "Graphs and Cooperation in Games," *Mathematics of Operations Research* 2: 225-229.

Owen, G. (1986), "Values of Graph-Restricted Games," *SIAM Journal of Algebraic and Discrete Methods* 7: 210-220.

Quint, T. (1991), "Necessary and Sufficient Conditions for Balancedness in Partitioning Games," *Mathematical Social Sciences* 22: 87-91.

Roth, A.E. (1982), "Incentive Compatibility in a Market with Indivisible Goods," *Economics Letters* 9: 127-132.

Roth, A.E., A. Postlewaite (1977), "Weak versus Strong Domination in a Market with Indivisible Goods," *Journal of Mathematical Economics* 4: 131-137.

Roth, A.E., M. Sotomayor (1990), *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis,* Econometric Society Monograph 18, Cambridge: Cambridge University Press.

Shapley, L., H. Scarf (1974), "On Cores and Indivisibility," *Journal of Mathematical Economics* 1: 23-37.

Sönmez, T. (1999), "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica* 67: 677-689.

Sotomayor, M. (1996), "A Non-constructive Elementary Proof of the Existence of Stable Marriages," *Games and Economic Behavior* 13: 135-137.

Sotomayor, M. (1999), "Three Remarks on the Many-to-Many Stable Matching Problem, Dedicated to David Gale on His 75th Birthday," *Mathematical Social Sciences* 38: 55-70.

Tan, J.J.M. (1991), "A Necessary and Sufficient Condition for the Existence of a Complete Stable Matching," *Journal of Algorithms* 12: 154-178.