Duration Dependence in Stock Prices: An Analysis of Bull and Bear Markets*

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ABSTRACT

This paper investigates the presence of bull and bear market states in stock price dynamics. A new definition of bull and bear market states based on sequences of stopping times tracing local peaks and troughs in stock prices is proposed. Duration dependence in stock prices is investigated through posterior mode estimates of the hazard function in bull and bear markets. We find that the longer a bull market has lasted, the lower is the probability that it will come to a termination. In contrast, the longer a bear market has lasted, the higher is its termination probability. Interest rates are also found to have an important effect on cumulated changes in stock prices: increasing interest rates are associated with an increase in bull market hazard rates and a decrease in bear market hazard rates.

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1. Introduction

The bull and bear market terminology is widely used by financial analysts and stock market commentators to characterize the evolution in stock prices. Some early academic studies investigated simple definitions of bull and bear states. Fabozzi & Francis (1977), Kim & Zumwalt (1979) and Chen (1982) all consider definitions of bull markets based simply on returns in a given month exceeding a certain threshold value.[1]

Since the emergence of these initial studies, little progress has been made on formally modelling and estimating the evolution in bull and bear states. In this paper we attempt to rigorously define these concepts in terms of sequences of stopping times and we systematically investigate properties of returns in bull and bear states. Earlier definitions do not reflect long-run dependencies in stock prices and ignore information about the trend in stock price levels. We propose a definition of bull and bear markets that emphasizes movements in stock prices between *local* peaks and troughs. This definition essentially implies that the stock market switches from a bull to a bear state if stock prices have declined by a certain percentage since their previous (local) peak within that bull state. Likewise, the stock market switches from a bear to a bull state if stock prices experience a similar percentage increase since their previous local minimum within that bear state. This definition does not rule out sequences of negative (positive) price movements in stock prices during a bull (bear) market as long as their cumulated value does not exceed a certain threshold.

Application of these definitions to US stock prices generates a set of durations of bull and bear markets. These form the basis of our analysis and allow us to characterize the duration profile of bull and bear markets as well as their cumulated return distribution. We also model bull and bear hazard rates, i.e. the conditional probability that a bull or bear market will terminate given that it has lasted for a certain period. Inspection of these yields important new insights into long-run dependencies and deviations from the simple random walk model with a constant drift which is often treated as the natural 'null' model in studies of stock prices. We find evidence of very different duration dependence in bull and bear states. The longer a bull market has lasted, the lower is the hazard rate and hence the lower the probability that the bull state will terminate. In contrast, the longer a bear market has lasted, the higher is the probability that it will come to an end. We also find that interest rates can have an important effect on the cumulated movements in stock prices. Higher interest rates are associated with an increase in the probability of termination of bull markets but also with a decline in the probability that a bear market terminates.

Our paper is closely related to earlier work on deviations from simple random walk models and long-run correlations in stock returns. Lo & MacKinlay (1988), Fama & french (1988), Poterba & Summers (1988) and Richardson & Stock (1989) all find some evidence of slow

[1] Fabozzi & Francis (1977) considers an alternative definition of bull markets based on 'substantial' up and down movements. In this definition, a substantial move in stock prices occurs whenever the absolute value of stock returns in a given month exceeds half of one standard deviation of the return distribution.

mean reversion in stock prices. Although these papers make important progress towards understanding long-run correlations in asset returns, none of them has attempted to characterize long-run dependencies in terms of the duration profile of bull and bear markets or to link the autocorrelation estimates to bull and bear market states.

Some studies in the literature on long memory in asset prices are also closely related to our paper. Granger & Joyeux (1980) found long memory in absolute returns, while Bollerslev & Mikkelsen (1996) studied hyperbolic decay rates in volatility. Our paper studies long-run dependencies revealed in cumulated returns. Compared with long-run dependencies in the volatility or absolute value of returns, the type of dependence that we study accounts for the sign of returns and hence reflects the long-run direction of the market.

The plan of the paper is as follows. Section 2 presents our definition of bull and bear market states. Section 3 characterizes the unconditional distribution of the duration and returns in bull and bear market states using more than a century of daily stock prices from the US. Section 4 discusses our models of bull and bear hazard rates, while Section 5 presents our estimation methods. Finally Section 6 reports empirical results from estimation of the hazard models using US stock prices and Section 7 concludes.

2. Definition of Bull and Bear Markets

There is no generally accepted formal definition of bull and bear markets in the finance literature. This is surprising given how often these terms are used to describe the state of the stock market. One of the few sources that attempts a definition of bull and bear markets is Sperandeo (1990) who defines bull and bear markets as follows:

”Bull market: A long-term ... upward price movement characterized by a series of higher intermediate ... highs interrupted by a series of higher intermediate lows.

Bear market: A long-term downtrend characterized by lower intermediate lows interrupted by lower intermediate highs”. (p. 102).

To formalize the idea of a series of increasing highs interrupted by a series of higher intermediate lows, let I_t be a bull market indicator variable taking the value 1 if the stock market is in a bull market at time t, and zero otherwise. We assume that time is measured on a discrete scale. Suppose that at t_0 the stock market is at a local maximum and define the stochastic process $P_{max} = P[t_0]$, where $P[t_0]$ is the stock price at time t_0. Let c be a scalar defining the threshold of the movements in stock prices that triggers a switch between bull and bear markets. Also let $\tau \geq 1$ be a stopping time variable defined by the following condition:

$$\tau = \min_{i=1,\ldots} \{P[t_0 + i] \geq P_{\max} \ \vee \ P[t_0 + i] < (1 - c)P_{\max}\}. \tag{1}$$

If the first condition is satisfied, we update the local maximum in the current bull market state:

$$P_{\max} = P[t_0 + \tau], \quad t_{\max} = t_0 + \tau \tag{2}$$

and the bull market is said to have continued between t_0 and $t_0 + \tau$: $I_{t_0} = ... = I_{t_0+\tau} = 1$.

Conversely, if the second condition is satisfied so that the stock price at $t_0 + \tau$ has declined by a fraction c since its local peak

$$P[t_0 + \tau] < (1 - c)P_{\max},$$

then the bull market is said to have switched to a bear market and the bear market prevailed from time t_0 to $t_0 + \tau$: $I_{t_{max}} = ... = I_{t_0+\tau} = 0$. Parallel to the bull market definition, in the latter case we set $P_{min} = P[t_0 + \tau]$, $t_{min} = t_0 + \tau$.

If the starting point is a bear market state, by symmetry the stopping time is defined as follows:

$$\tau = \min_{i=1,...} \{P[t_{\min} + i] \leq P_{\min} \ \vee \ P[t_{\min} + i] > (1 + c)P_{\min}\}. \tag{3}$$

Notice that our definition of bull and bear markets partitions the full data sample into exclusive and mutually exhaustive bull and bear market subsets and also accounts for the underlying upward trend in real stock prices.

The stochastic process comprising the sequence of indicator variables gives rise to a random variable, T, measuring the duration of a particular bull or bear market. This variable is simply given as the time between successive switches in the indicator variable, I_t.[2]

3. Durations of bull and bear markets

To investigate the properties of bull and bear market states, we construct a data set of daily US stock prices from 2/17/1885 to 12/31/1997. From 2/17/1885 to 2/7/1962 the nominal stock price index is based on the daily returns provided by Schwert (1990). These returns include dividends. From 3/7/1962 to 12/31/1997 the price index was constructed from daily returns on the Standard & Poors price index, again including dividends and obtained from the CRSP tapes. Combining these series generates a time series of 31,412 daily nominal stock prices.

Inflation has varied considerably over the sample period and the drift in nominal prices does not have the same interpretation during low and high inflation periods. To deal with this issue, we constructed a daily inflation index as follows. From 1885 to 1913 our source for prices was chapter 26 in Shiller (1989). From January 1913 to December 1997, we used the Consumer Price Index (all urban consumers, not seasonally adjusted) from the Bureau of Labor Statistics. These series were converted into daily inflation rates by solving for the daily inflation rate such that the daily price index grows smoothly - and at the same rate - between subsequent

[2]Notice that the indicator variable need not be measurable with respect to the filtration generated by the sequence of stock prices $\Omega_t = \{P_0, P_1, ..., P_t\}$. This is because the current state may also depend on cumulated future changes in stock prices. To see this, suppose that stock prices have risen by two percent since the most recent local trough. This does not necessarily mean that the stock market is now in a bull market state if prices drop by more than two percent in the near future. However, if prices continue to rise beyond the trigger point determined by $P_{\min}$ and c, then a switch to a bull market will indeed have occurred.

monthly price indexes.[3] Finally we divided the nominal stock price (cum dividend) index by the consumer price index to get a daily index for real stock prices. This is the time-series we analyze in the following.

Several aspects of our data format are worth dwelling on before proceeding further. First, since our data sample terminates in 1997, we are dealing with right-censored data, although only a single data point is censored. Each duration spell thus consists of the length spent in the state and an indicator of whether this is an event time or a censoring time. Second, much of standard survival analysis in economics and finance assumes continuously measured data. However, since we use daily data and do not follow price movements continuously, our data is interval censored and the termination or censoring of our durations are only known to lie between consecutive follow ups.[4] Suppose that the measurement of T is divided into A intervals

$$[a_0, a_1), [a_1, a_2), \ldots, [a_{q-1}, a_q), [a_q, \infty) \text{ where } q = A - 1.$$

Only the discrete time duration $T \in \{1, \ldots, A\}$ is observed, where $T = t$ denotes termination within the interval $[a_{t-1}, a_t)$. Although we shall be drawing on approaches from the literature on economic duration data (see, eg, Kiefer (1988), Kalbfleisch & Prentice (1980) and Lancaster (1990)) this also means that we have to be careful in modifying the standard tools from continuous time analysis.

Insights into how our definition partitions real stock prices into bull and bear spells are gained from Figures 1a and 1b which show the sequence of consecutive bull and bear market durations over the full sample period 1885-1997. The 10 percent filter split the sample into 114 bull markets and 114 bear markets. As a means of better illustrating the individual episodes, we plot in eight separate windows the natural logarithm of the real stock price cum dividend index. Many of the bull markets are very long and it is clear that the bull market during the 1990s (lasting over seven years from 1990 to 1997) is in fact an outlier by historical standards.

Table 1 presents some basic descriptive statistics for the distribution of the time spent in bull and bear market states and Figures 2 and 3 show histograms of the unconditional durations of bull and bear markets, respectively. All results are based on a threshold value, c, of 10 percent. To facilitate interpretation of the results, we report properties of bull and bear market states in weeks although it should be recalled that our analysis was carried out using daily data. The mean bull market duration is 37 weeks against 19 weeks for bear market durations. The corresponding median values are 18 and 12 weeks for bull and bear markets, respectively. While the shortest bull and bear markets each lasted for only a week, the longest bull market, at 354 weeks or seven years, lasted four times longer than the longest bear market (88 weeks

[3] Since the volatility of daily inflation rates is likely to be only a fraction of that of daily stock returns, normalizing by the inflation rate has the effect of a time-varying drift adjustment and lack of access to daily inflation data is unlikely to affect our results in any important way.

[4] The possibility of ties for such data can cause problems in applications of partial likelihood methods for continuous time models

or a year and a half). Partly as a result of this, the dispersion of bull market durations is about two and a half times greater than that of bear markets.

Mean returns in bull markets are around 2.8 percent per week and -3.3 percent per week in bear markets. An even larger difference shows up in the median return per week which is 1.0 and -1.5 percent per week for bull and bear markets, respectively. This suggests that although bear markets last much shorter than bull markets, the rate of decline in bear markets proceeds more rapidly than the rate of increase during bull markets.

4. Discrete time models of bull and bear hazard rates

The previous analysis characterized the unconditional distribution of bull and bear market spells. However, during the long bull market of the mid-1990s, the concern was often expressed that the bull market would come to an end simply because it had lasted 'too long' by historical standards. Translated into statistical terms, this indicates a belief that the bull market termination probability depends positively on the duration of the bull market: Conditional on having lasted for a certain length of time, the probability that a bull market will terminate is believed to be an increasing function of time. The opposite view is that bull markets gain momentum: the longer a bull market has lasted for, the more robust it is, and hence the lower the hazard rate.

Testing these opposite hypotheses requires that we go beyond inspecting the unconditional probability that a bull market terminates in a particular interval. Instead we need to characterize the duration data in terms of the conditional probability that the bull or bear market ends in a short time interval following some period t, given that the bull market lasted up to period t. This is measured by the discrete hazard function

$$\lambda(t|x) = \Pr(T = t|T \geq t, x), \quad t = 1, \ldots, A, \tag{4}$$

which is the conditional probability of termination in interval $[a_{t-1}, a_t)$ given that the interval was reached in the first place.[5] x is a vector of additional conditioning information. Hypotheses concerning the probability that a bull or bear market is terminated as a function of its age are naturally expressed in terms of the shape of this hazard function.

Natural interest also lies in estimating the probability that a bull market lasts beyond a certain time horizon, that is, in estimating the discrete survivor function, defined as

$$S(t|x) = \Pr(T > t|x) = \prod_{j=1}^{t}(1 - \lambda(j|x)), \quad t = 1, \ldots, A. \tag{5}$$

[5]This is distinct from the *unconditional probability of termination* which is given by

$$\Pr(T = t|x) = \lambda(t|x)\prod_{j=1}^{t-1}(1 - \lambda(j|x)) = \lambda(t|x)\tilde{S}(t|x), \quad t = 1, \ldots, A.$$

This gives the probability of surviving on the interval $[a_{t-1}, a_t)$. Likewise, the probability of reaching this interval can be regarded as a survivor function, that is

$$\tilde{S}(t|x) = \Pr(T \geq t|x) = \prod_{j=1}^{t-1}(1 - \lambda(j|x)), \quad t = 1, \ldots, A,$$

and hence $\tilde{S}(t|x) = S(t-1|x)$.

The hazard models we are interested in estimating are all assumed to take the form

$$\lambda(t|x_i) = F(\mathbf{z}_{it}'\boldsymbol{\beta}), \tag{6}$$

where $\boldsymbol{\beta}$ comprises the parameters of interest and z_{it} is a (possibly time-varying) covariate that affects the hazard rate. The function $F(\cdot)$ is called the *Link-function*. This function must have the properties of a distribution function. Common choices are the Probit, the Logit and the Double Exponential link. Thoughout the paper we use a Logit-link.

4.1. Static Models

In this section we characterize hazard models when the underlying parameters linking the covariates to the hazard rate do not vary over time.

4.1.1. *Constant covariates*

Initially we characterize the baseline hazard rate of bull and bear market durations by considering the simple case with constant covariates. For this case the explanatory variables are fixed from the point of entry in the state. Some time-variation in the hazard rate is still possible, however, since the regression parameters are allowed to vary freely through the duration. The advantage of initially not considering any exogenous covariates is that our results are directly comparable to the large literature on univariate dynamics in stock prices.[6]

The data takes the form of $\{t_i, \boldsymbol{x}_i, \delta_i; i = 1, \ldots, n\}$, where $t_i = \min\{T_i, C_i\}$ is the minimum of the survival time and the censoring time C_i, $\boldsymbol{x}_i$ is a covariate observed at the beginning of the interval $[a_{t_{i-1}}, a_{t_i})$ and δ_i is a censoring indicator: $\delta_i = 1$ means termination in $[a_{t_{i-1}}, a_{t_i})$, while $\delta_i = 0$ means censoring in $[a_{t_{i-1}}, a_{t_i})$. Since the only censoring point occurs during the terminal interval T_i and C_i are independent and our data is randomly censored. This means that the probability of observing the termination of a duration is given by

$$\Pr(T_i = t_i, \delta_i = 1) = \Pr(T_i = t_i)\Pr(C_i > t_i), \tag{7}$$

while the probability of censoring at time t_i is given by[7]

$$\Pr(T_i = t_i, \delta_i = 0) = \Pr(T_i \geq t_i)\Pr(C_i = t_i), \tag{8}$$

[6] For a survey of this literature, see Campbell, Lo & MacKinlay (1997), chapter 2.

[7] In (7) and (8) it is assumed that censoring occurs at the beginning of the interval. If it is assumed to occur at the end then (7) and (8) must be changed to $\Pr(T_i = t_i, \delta_i = 1) = \Pr(\mathrm{T_i = t_i})\Pr(C_i \geq t_i)$ and $\Pr(T_i = t_i, \delta_i = 0) = \Pr(\mathrm{T_i > t_i})\Pr(C_i = t_i)$.

Combining (7) and (8) with the assumption of non-informative censoring, we get the likelihood contribution of observation i:[8]

$$\begin{aligned}
\mathcal{L}_i &= \Pr(T_i = t_i)^{\delta_i} \Pr(T_i \geq t_i)^{1-\delta_i} \underbrace{\Pr(C_i > t_i)^{\delta_i} \Pr(C_i = t_i)^{1-\delta_i}}_{= c_i} \\
&= c_i \Pr(T_i = t_i)^{\delta_i} \Pr(T_i \geq t_i)^{1-\delta_i} \\
&= c_i \lambda(t|x_i)^{\delta_i} \prod_{j=1}^{t_i-1} (1 - \lambda(j|x_i)). \qquad (9)
\end{aligned}$$

Before proceeding further, it is convenient to set up equation (9) using notation similar to that used from the literature on discrete choice models. For this purpose we construct the following discrete indicator variable:

$$y_{ij} = \begin{cases} 1, & \text{bull or bear market terminates in } [a_{j-1}, a_j) \\ 0, & \text{bull or bear market survives through } [a_{j-1}, a_j) \end{cases} \qquad j = 1, \ldots, t_i.$$

An observation censored at t_i will thus be represented by $y_i = (y_{i1}, \ldots, y_{it_i-1}) = (0, \ldots, 0)$, whereas failure at t_i means that the observation is represented by $y_i = (y_{i1}, \ldots, y_{it_i}) = (0, \ldots, 0, 1)$. Using this notation, the contribution to the likelihood function from the i'th observation is

$$\mathcal{L}_i \propto \prod_{j=1}^{t_i-(1-\delta_i)} \lambda(j|x_i)^{y_{ij}} (1 - \lambda(j|x_i))^{1-y_{ij}}. \qquad (10)$$

Summing across duration spells, the total log-likelihood for the model $\lambda(t|x_i) = F(\mathbf{z}'_{it}\boldsymbol{\beta})$ is given by[9]

$$\ln \mathcal{L} \propto \sum_{i=1}^{n} \sum_{j=1}^{t_i-(1-\delta_i)} y_{ij} \ln(\lambda(j|x_i)) + (1 - y_{ij}) \ln(1 - \lambda(j|x_i)). \qquad (11)$$

Recalling that censoring occurs for the final bull or bear state, this equation needs to be modified slightly to

$$\ln \mathcal{L} \propto \sum_{i=1}^{n} \sum_{j=1}^{t_i} y_{ij} \ln(\lambda(j|x_i)) + (1 - y_{ij}) \ln(1 - \lambda(j|x_i)). \qquad (12)$$

where

$$y_{ij} = (y_{i1}, \ldots, y_{it}) = \begin{cases} (0, \ldots, 0), & \delta_i = 0 \\ (0, \ldots, 1), & \delta_i = 1 \end{cases} \qquad j = 1, \ldots, t_i.$$

[8] It follows from the definition of the discrete hazard function that the last two terms in (9) do not depend on the parameters determining the survival time.

[9] This is identical to the log-likelihood of $\sum_i (t_i - 1 + \delta_i)$ observations from the binary response model which is given by $Pr(y_{ij} = 1 \mid x_i) = F(\mathbf{z}'_{ij}\boldsymbol{\beta})$.

4.1.2. *Time-varying covariates*

One could reasonably expect that switches between bull and bear markets occur due to changes in the underlying economic environment. For example, the drift in stock prices may turn from positive to negative as a result of increased interest rates or worsening economic prospects. To account for such effects, we need to extend the setup from the previous section and allow $\boldsymbol{z}_{it}$ to be a vector that incorporates time-varying covariates. Now the data for the i'th duration spell take the form

$$\{\underbrace{t_i}_{\text{duration}}, \overbrace{x_i(a_0), x_i(a_1), \ldots x_i(a_{q-1}), x_i(a_q)}^{\equiv \boldsymbol{x}_i\text{: time-varying covariates}}, \underbrace{\delta_i}_{\text{censoring}}\}.$$

Given the discreteness of our data, the covariates follow a step function with jumps at the follow-up times. We will be using $\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{it}$ as short-hand notation for the sequence of observations of covariates for the ith duration spell preceding time t. Hence $\boldsymbol{x}_{it}$ is assumed to be a vector observed at the beginning of interval $[a_{t-1}, a_t)$. Within this interval the history of covariates

$$\boldsymbol{X}_i(t) = (\boldsymbol{x}_{i1}, \boldsymbol{x}_{i2}, \ldots, \boldsymbol{x}_{it}),$$

is allowed to influence the hazard rate:

$$\lambda(t|\boldsymbol{X}_i(t)) = \Pr(T = t|T \geq t, \boldsymbol{X}_i(t)) = F(\boldsymbol{z}'_{it}\boldsymbol{\beta}).$$

Several specifications are possible for the functional form of the covariate effect $(\boldsymbol{z}'_{it}\beta)$. If the parameters are allowed to vary over (duration) time, an attractively simple specification is

$$\boldsymbol{z}'_{it}\boldsymbol{\beta} = \gamma_{0t} + \boldsymbol{x}'_{it}\boldsymbol{\gamma}_t,$$

$$\text{where} \quad \begin{aligned} \boldsymbol{z}'_{it} &= (0, \ldots, 1, \ldots, 0, 0, \ldots, \boldsymbol{x}'_{it}, \ldots, 0) \\ \boldsymbol{\beta}' &= (\gamma_{01}, \ldots, \gamma_{0q}, \boldsymbol{\gamma}'_1, \ldots, \boldsymbol{\gamma}'_q). \end{aligned}$$

This can be extended to include several time lags,

$$\boldsymbol{z}'_{it}\boldsymbol{\beta} = \gamma_{0t} + \boldsymbol{x}'_{it}\boldsymbol{\gamma}_0 + \ldots + \boldsymbol{x}'_{it-r}\boldsymbol{\gamma}_{-r},$$

$$\text{where} \quad \begin{aligned} \boldsymbol{z}'_{it} &= (0, \ldots, 1, \ldots, 0, \boldsymbol{x}'_{it}, \boldsymbol{x}'_{it-1}, \ldots, \boldsymbol{x}'_{it-r}) \\ \boldsymbol{\beta}' &= (\gamma_{01}, \ldots, \gamma_{0q}, \boldsymbol{\gamma}'_0, \boldsymbol{\gamma}'_{-1}, \ldots, \boldsymbol{\gamma}'_{-r}),. \end{aligned}$$

For the case with time-varying covariates, the log-likelihood function is constructed from the extended data set $(t_i, \delta_i, \boldsymbol{X}_i(t_i))$, $i = 1, \ldots, n$:

$$\ln \mathcal{L} \propto \sum_{i=1}^{n} \sum_{j=1}^{t_i-(1-\delta_i)} y_{ij} \ln(\lambda(j|\boldsymbol{X}_i(j))) + (1 - y_{ij}) \ln(1 - \lambda(j|\boldsymbol{X}_i(j))). \tag{13}$$

For further details on the construction of this log-likelihood function, and for a discussion of external and internal covariates see Fahrmeir & Tutz (1994, p. 327-330).

4.2. Dynamic models of bull and bear market durations

The models in the previous section are *static* in the sense that they treat baseline hazard coefficients and covariate parameters as *fixed effects.* As such they are appropriate if the number of intervals is relatively small. However, in applications such as ours with many intervals, but not enough to apply continuous time techniques, such unrestricted models and estimation of hazard functions can lead to nonexistence and divergence of maximum likelihood estimates due to the large number of parameters.

To get around these problems, we follow Fahrmeir (1994) and adopt state space techniques. An advantage of this approach is that simultaneous estimation and smoothing of the baseline and covariate effects becomes possible. The general framework and notation follows the previous section. However, we also need to define risk indicators r_{it} $(i, t \geq 1)$ by

$$r_{it} = \begin{cases} 1, & \text{if the } i'th \text{ bull or bear market is at risk in } [a_{t-1}, a_t) \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, we define the risk vector $\boldsymbol{r}_t = (r_{it}, i \geq 1)$, and the risk set $\mathcal{R}_t = \{i : t \leq t_i - (1 - \delta_i)\}$ at time t, i.e., the set of duration spells that are at risk in the interval $[a_{t-1}, a_t)$.[10] Covariates and failure indicators for all $i \in \mathcal{R}_t$ in $[a_{t-1}, a_t)$ are collected in the vectors

$$\begin{aligned} \boldsymbol{x}_t &= (x_{it}, i \in \mathcal{R}_t) \\ \boldsymbol{y}_t &= (y_{it}, i \in \mathcal{R}_t). \end{aligned}$$

Finally we denote the histories of covariates, failure and risk indicators up to period t by

$$\begin{aligned} \boldsymbol{x}_t^* &= (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{t-1}) \\ \boldsymbol{y}_t^* &= (\boldsymbol{y}_1, \ldots, \boldsymbol{y}_{t-1}) \\ \boldsymbol{r}_t^* &= (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_{t-1}). \end{aligned}$$

In the presence of time-varying covariates, the general hazard rate model (6) can be stated as follows

$$\lambda(t|\boldsymbol{X}_i(t)) = F(\boldsymbol{z}_{it}'\boldsymbol{\alpha}_t). \tag{14}$$

In the usual state space terminology, this is the *measurement equation*. The components of the state vector $\boldsymbol{\alpha}_t$ comprise both the baseline parameter and the covariate effects and the design vector $\boldsymbol{z}_{it}$ is a function of the covariates. In the simplest case we set $\boldsymbol{z}_{it} = (1, \boldsymbol{x}_{it})$ and $\boldsymbol{\alpha}_t =$

[10] Using this notation the log-likelihood function can be written as

$$\ln \mathcal{L} \propto \sum_{t=1}^{q} \sum_{i \in \mathcal{R}_t} y_{it} \ln(\lambda(t|\boldsymbol{X}_i(t))) + (1 - y_{it}) \ln(1 - \lambda(t|\boldsymbol{X}_i(t))), .$$

(γ_{0t}, γ_t), and use the first-order random walk as our choice of *transition equation*

$$\boldsymbol{\alpha}_t = \boldsymbol{\Phi}\boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t, \quad \Leftrightarrow \begin{pmatrix} \gamma_{0t} \\ \boldsymbol{\gamma}_t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_{0t-1} \\ \boldsymbol{\gamma}_{t-1} \end{pmatrix} + \begin{pmatrix} \xi_{0t} \\ \boldsymbol{\xi}_{1t} \end{pmatrix},$$

$$\text{where} \begin{pmatrix} \xi_{0t} \\ \boldsymbol{\xi}_{1t} \end{pmatrix} \sim \mathcal{N}(\mathbf{0}, diag(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)), \quad p = \dim(\boldsymbol{\gamma}_t), \tag{15}$$

$$\text{and } \boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{a}_0, \boldsymbol{Q}_0).$$

The random walk model has the advantage of not imposing mean reversion on the parameters and allows the parameters to differ at various durations (although neighboring points cannot be too far from each other) if the data supports such variation.

5. Estimation

The following set of standard assumptions are sufficient to guarantee that the models are fully specified in terms of their likelihoods:

(A1) Conditional on $\boldsymbol{\alpha}_t$, $\boldsymbol{y}_{t-1}^*$, and $\boldsymbol{x}_t^*$, current $\boldsymbol{y}_t$ is independent of $\boldsymbol{\alpha}_{t-1}^* = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{t-1})$:[11]

$$p(\boldsymbol{y}_t|\boldsymbol{\alpha}_t^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*) = p(\boldsymbol{y}_t|\boldsymbol{\alpha}_t, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*), \quad t = 1, 2, \ldots \tag{16}$$

(A2) Given $\boldsymbol{y}_{t-1}^*$, $\boldsymbol{r}_{t-1}^*$ and $\boldsymbol{x}_{t-1}^*$, the covariate $\boldsymbol{x}_t$ and risk vector $\boldsymbol{r}_t$ are independent of $\boldsymbol{\alpha}_{t-1}^*$:[12]

$$p(\boldsymbol{x}_t, \boldsymbol{r}_t|\boldsymbol{\alpha}_{t-1}^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_{t-1}^*) = p(\boldsymbol{x}_t, \boldsymbol{r}_t|y_{t-1}^*, \boldsymbol{x}_{t-1}^*), \quad t = 1, 2, \ldots \tag{17}$$

(A3) The parameter process is Markovian:[13]

$$p(\boldsymbol{\alpha}_t|\boldsymbol{\alpha}_{t-1}^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*) = p(\boldsymbol{\alpha}_t|\boldsymbol{\alpha}_{t-1}), \quad t = 1, 2, \ldots \tag{18}$$

(A4) Given $\boldsymbol{\alpha}_t$, $\boldsymbol{y}_{t-1}^*$, and $\boldsymbol{x}_t^*$, individual responses y_{it} within $\boldsymbol{y}_t$ are conditionally independent:[14]

$$p(\boldsymbol{y}_t|\boldsymbol{\alpha}_t^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*) = \prod_{i \in \mathcal{R}_t}^{n} p(y_{it}|\boldsymbol{\alpha}_t, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*), \quad t = 1, 2, \ldots \tag{19}$$

[11] This assumption is standard in state space modelling. It simply states that the conditional information in $\boldsymbol{y}_t$ about $\boldsymbol{\alpha}_t^*$ is exclusively contained in the current parameter $\boldsymbol{\alpha}_t$.

[12] Thus we assume that the covariate and censoring processes contain no information on the parameter process. This assumption holds for non-informative random censoring and for fixed or external covariates.

[13] This assumption is implied by the transition model and the assumption on the error sequence.

[14] This is weaker than the usual unconditional independence assumption, since it allows for interaction via the common history. It is likely to hold if a common cause is incorporated in the covariate process.

In practice, the error variances, $\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2$, and the initial values, $\boldsymbol{a}_0$, $\boldsymbol{Q}_0$, in the transition equation are typically unknown hyper-parameters. They can be treated as nuisance parameters which appear only in the transition equation, and may be interpreted as smoothing constants, all of which can be either subjectively chosen or estimated from the data.

5.1. Posterior mode smoothing and penalized likelihood estimation

We first consider estimation of $\boldsymbol{\alpha}^* = (\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_q)$ when the hyper-parameters $\boldsymbol{a}_0$, $\boldsymbol{Q}_0$, and $\boldsymbol{Q}$ are assumed to be either known or given. An optimal (Bayesian) solution relies on determining the posterior density,

$$p(\boldsymbol{\alpha}^*|\boldsymbol{y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*) = p(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_q|\boldsymbol{y}_1, \ldots, \boldsymbol{y}_q, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_q, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_q), \tag{20}$$

Since our measurement equation is non-normal, solving for the posterior generally requires using numerical or Monte Carlo integration. A simpler strategy, advocated by Fahrmeir (1992), is to base estimation on posterior modes and to use filtering and smoothing algorithms. By repeated application of Bayes theorem we have

$$\begin{aligned} p(\boldsymbol{\alpha}^*|\boldsymbol{y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*) \;=\; & \prod_{t=1}^{q} p(\boldsymbol{y}_t|\boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*, \boldsymbol{r}_t^*; \boldsymbol{\alpha}_t^*) \prod_{t=1}^{q} p(\boldsymbol{\alpha}_t|\boldsymbol{\alpha}_{t-1}^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*, \boldsymbol{r}_t^*) \\ & \cdot \prod_{t=1}^{q} p(\boldsymbol{x}_t, \boldsymbol{r}_t|\boldsymbol{\alpha}_{t-1}^*, \boldsymbol{y}_{t-1}^*, \boldsymbol{x}_{t-1}^*, \boldsymbol{r}_{t-1}^*) \frac{p(\boldsymbol{\alpha}_0)}{p(\boldsymbol{y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*)}. \end{aligned}$$

Under assumptions (A1)-(A4), we get

$$p(\boldsymbol{\alpha}^*|\boldsymbol{y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*) \propto \prod_{t=1}^{q} \prod_{i \in \mathcal{R}_t} p(y_{it}|\boldsymbol{y}_{t-1}^*, \boldsymbol{x}_t^*, \boldsymbol{r}_t^*; \boldsymbol{\alpha}_t^*) \prod_{t=1}^{q} p(\boldsymbol{\alpha}_t|\boldsymbol{\alpha}_{t-1}^*) \cdot p(\boldsymbol{\alpha}_0). \tag{21}$$

Taking logarithms and using (14) and (15), estimation of $\boldsymbol{\alpha}^*$ by posterior modes, that is maximization of posterior densities, is equivalent to maximizing the following *penalized log-likelihood function:*[15]

$$\begin{aligned} \ln \mathcal{L}(\boldsymbol{\alpha}^*) \;=\; & \sum_{t=1}^{q} \sum_{i \in \mathcal{R}_t} l_{it}(\boldsymbol{\alpha}_t) - \frac{1}{2}(\boldsymbol{\alpha}_0 - \boldsymbol{a}_0)' \boldsymbol{Q}_0^{-1} (\boldsymbol{\alpha}_0 - \boldsymbol{a}_0) \\ & - \frac{1}{2} \sum_{t=1}^{q} (\boldsymbol{\alpha}_t - \boldsymbol{\alpha}_{t-1})' \boldsymbol{Q}^{-1} (\boldsymbol{\alpha}_t - \boldsymbol{\alpha}_{t-1}), \end{aligned} \tag{22}$$

where

$$l_{it}(\boldsymbol{\alpha}_t) = y_{it} \ln(F(\boldsymbol{z}_{it}' \boldsymbol{\alpha}_t)) + (1 - y_{it}) \ln(1 - F(\boldsymbol{z}_{it}' \boldsymbol{\alpha}_t)) \tag{23}$$

is the log-likelihood contribution of the i'th observation.

For the general case Appendix A provides additional details on the numerical optimization of the penalized likelihood function.

[15]The first term measures the goodness of fit of the model, while the second and third terms - both of which are introduced by the smoothness prior specified by the transition model - penalize large deviations between successive parameters and lead to smoothed estimates.

5.1.1. *A Simple Example*

Intuition for the method is perhaps furthered by considering the simplest case which only requires estimating the baseline hazard. The measurement equation simplifies to

$$\lambda(t|\boldsymbol{X}_i(t)) = F(\gamma_{0t}) = \frac{\exp(\gamma_{0t})}{1+\exp(\gamma_{0t})}, \tag{24}$$

and the transition equation for the baseline hazard is the first-order random walk

$$\gamma_{0t} = \gamma_{0t-1} + \xi_{0t}, \quad \xi_{0t} \sim \mathcal{N}(0, \sigma_1^2), \quad \text{and} \;\; \gamma_{00} \sim \mathcal{N}(g_0, \sigma_0^2).$$

For this case the penalized log-likelihood function becomes

$$\begin{aligned} \ln \mathcal{L}(\gamma_{0t}) \;\; = \;\; & \sum_{t=1}^{q} \sum_{i \in \mathcal{R}_t} \{y_{it} \ln(F(\gamma_{0t})) + (1-y_{it}) \ln(1-F(\gamma_{0t}))\} \\ & - \frac{1}{2\sigma_1^2} \sum_{t=1}^{q} (\gamma_{0t} - \gamma_{0t-1})^2 - \frac{1}{2\sigma_0^2} \sum_{t=1}^{q} (\gamma_{00} - g_0)^2. \end{aligned} \tag{25}$$

It is easily seen that the contribution of the failure indicator y_{it} to the score is given by

$$\boldsymbol{u}_{it}(\gamma_{0t}) = y_{it} - \frac{\exp(\gamma_{0t})}{1+\exp(\gamma_{0t})} \;, \;\text{and}\;\; \boldsymbol{u}_t(\gamma_{0t}) = \sum_{i \in \mathcal{R}_t} \boldsymbol{u}_{it}(\gamma_{0t}), \tag{26}$$

while the contribution of the expected information matrix is:

$$\boldsymbol{U}_{it}(\gamma_{0t}) = -\frac{\exp(\gamma_{0t})}{\{1+\exp(\gamma_{0t})\}^2} \;, \;\text{and}\;\; \boldsymbol{U}_{\mathrm{t}}(\gamma_{0\mathrm{t}}) = \sum_{\mathrm{i} \in \mathcal{R}_{\mathrm{t}}} \boldsymbol{U}_{\mathrm{it}}(\gamma_{0\mathrm{t}}). \tag{27}$$

6. THE HAZARD RATE OF US BULL AND BEAR MARKETS

Using the estimation techniques and hazard models described in Sections 4 and 5, we first estimated the hazard function for bull and bear markets in a model without time-varying covariates. The outcome of this exercise is, in the form of the baseline hazard, plotted in Figures 4 (bull market) and 5 (bear market). The baseline hazard in bull markets is initially slighly above four percent per week but it quickly drops to under two percent (for bull markets that have lasted a year and a half) only to increase to almost four percent again for bull markets whose age exceed three years. For bear markets, there is weaker evidence of duration dependence in the baseline hazard. Only for very old bear markets is there some evidence of an increasing hazard rate. However, at these long durations the standard error bands are also much wider than at shorter durations, so this evidence should be interpreted cautiously.

Figure 6 plots the difference between the baseline hazard rate in bear and bull markets estimated from a bivariate random walk model with a logit link function. This setup allows us to directly evaluate differences in bear and bull market hazards since we can compute standard errors for the difference in hazard rates. Such standard errors are plotted along the point estimates. The figure shows that bear markets are associated with a higher hazard rate across all durations. At any length of time, a bear market has a higher probability of termination than a bull market of the same duration. In fact, the excess hazard rate of bear over bull markets appears to be increasing as a function of duration and is about four times higher for long durations compared with shorter ones. This of course is a key determinant of the historically high mean returns on US stocks. Our findings suggest that it is the absence of very long bull markets that account for these high mean returns not differences between bull and bear markets at the short end of the duration distribution.

To shed light on how the hazard rates depend on the underlying state of the economy, we next included interest rates as a time-varying covariate. Interest rates have been widely documented to be one of the most precise indicators of the state of the business cycle and appears to be a key determinant of monthly stock returns.[16] For post-world war data, the level of interest rates tracks the business cycle very well. However, because we have such a long sample in which the inflation rate has varied considerably, we also include changes in interest rates. Interest rate *levels* may not contain the same information over the sample, while interest rate *changes* are more likely to track changes in the business cycle across the full sample. Our set of covariates is thus $z_{it}^{'} = (1, i_t, \Delta i_t)$, where i_t is the interest rate level and Δi_t is the interest rate change. The hazard specification is

$$\lambda(t|z_i(t)) = F(\gamma_{0t} + \gamma_{1t} i_t + \gamma_{2t} \Delta i_t). \tag{28}$$

There is no continuous data series on daily interest rates from 1885 to 1997, so we constructed our data from four separate sources. From 1885 to 1889 the source is Chapter 26 in Shiller (1989). From 1890 to 1925, we use the interest rate for 90-day stock exchange time loans as reported in Banking and Monetary Statistics, Board of Governors of the Federal Reserve System (1943). These rates are reported on a monthly basis and we convert them into a daily series by simply applying the interest rate reported for a given month to each day of that month. From 1926 to 1954, we use the one-month T-bill rates from the Fama/Bliss risk-free rates CRSP file, again reported on a monthly basis and converted into a daily series. Finally, from July 1954 to 1997, we use the daily Federal Funds rate. These three sets of interest rates are concatenated to form one time series.

Figure 7 presents the baseline hazard for bull markets after controlling for interest rate and interest rate change effects. Comparing Figure 7 to Figure 4, it is clear that controlling for interest rates has a significant effect on the shape of the baseline hazard. In contrast with the $U-$shaped pattern from the model without interest rate effects, now the baseline hazard rapidly drops from six to two percent per week as the bull market duration is extended to half a year and remains constant for longer durations. This suggests that young bull markets are substantially more at risk of termination than bull markets that have lasted for a minimum of six months.

Figure 8 shows that higher interest rates are associated with a lower bull market hazard rate for very short durations, but that the sign of this covariate parameter switches and later is associated with a higher hazard rate. We believe that the initial negative sign should be interpreted

[16]See, e.g., Fama & French (n.d.), Jagannathan & Runkle (1993) and Pesaran & Timmermann (1995).

with caution: interest rates tend to be high towards the beginning of a new business cycle expansion and this is often the beginning of a bull market in stock prices.[17] More importantly, perhaps, positive interest rate changes are associated with large increases in hazard rates: a one percentage point increase in the interest rate is associated with an increase in the hazard rate by three percentage points. This represents more than a doubling of the hazard rate for durations that exceed half a year.

Turning next to the bear markets and comparing Figures 5 and 10, it is clear that the shape of the baseline hazard does not change as a result of including interest rate effects. However, interest rate effects appear to have a large effect on the hazard rate of bear markets: the hazard rate is lower, the higher is the level of interest rates (Figure 11) and the larger the change in interest rates (Figure 12). Hence a bear market tends to last longer in an environment with high and increasing interest rates. Interestingly, the size of the covariate effects is smaller in bear markets than in bull markets.

7. Conclusion

This paper has investigated a new type of long-run dependence in stock prices based on the distribution of time spent in markets where cumulated returns exceed some positive threshold value (bull states) or fall below some negative threshold value (bear states). Our measure of dependence is based on cumulated prices and hence is different from the long memory properties of absolute returns or volatility of returns found in earlier studies. We find strong evidence contradicting standard models for the underlying stock price process. Bull market hazard rates decline for low durations while they increase for long bear market durations.

Several additional points need to be addressed in future analysis. Earlier studies have found some evidence of negative autocorrelation in long-horizon stock returns but have mostly failed to formally reject the random walk model. We conjecture that duration-based tests of violations of the random walk model for asset prices may have power in directions where standard tests based on autocorrelations fail to be powerful. It is true that converting stock returns into a sequence of bull and bear market states and considering duration spells instead of prices discards information that is present in the full sequence of price changes. However, to the extent that the long-run dependence in bull and bear markets takes the form of a duration dependent hazard function, our method is likely to be more powerful in detecting deviations from the random walk model than methods based on the autocorrelogram. We intend to investigate this point further in future work.

Another point of obvious interest from an asset pricing point of view is whether long run portfolio performance can be improved by accounting for the duration dependence reported in this paper. The finding that bull markets are particularly fragile when they are relatively young, while bear market hazard rates increase as a function of duration should be useful information

[17] However, the variation in the sign of the parameter for this covariate, and its steep slope as a function of the duration, indicate the advantage of using a flexible approach such as ours which can detect for such variation.

for stock market investors.

Third, we intend to use the distribution of the length and size of bull and bear markets as a diagnostic for standard processes for the underlying stock prices. For instance, one may reasonably expect that ARCH effects can account for some of the shorter bull and bear durations, but that such effects cannot account for the longer end of the duration distribution. Although the bull and bear state terminology is meant to identify long-run dependencies in the drift in stock prices, it is clear that periods with clustering of high volatility can trigger a switch in the state. For example, the one-day drop in stock prices on October 19, 1987 would in itself be sufficient to trigger a bear market even if it were subsequently followed by positive returns. In future work we intend to use simulation experiments of a GARCH model to shed light on the extent to which our bull and bear market findings are driven by ARCH effects.

Appendix A: Numerical maximization for the Generalized Kalman Filter and Smoother

This appendix briefly explains some of the details of the numerical optimizations. To perform numerical optimization of the penalized log-likelihood function, we use the generalized extended Kalman filter and smoother suggested by Fahrmeir (1992). Denote by $d_{it}(\boldsymbol{\alpha}_t)$ the first derivate $\partial F(\eta)/\partial\eta$ of the response function $F(\eta)$ evaluated at $\eta = \mathbf{z}_{it}'\boldsymbol{\alpha}_t$. The contribution to the score of the failure indicator y_{it} is given by

$$\boldsymbol{u}_{it}(\boldsymbol{\alpha}_t) = \frac{\partial l_{it}(\boldsymbol{\alpha}_t)}{\partial\boldsymbol{\alpha}_t} = \boldsymbol{z}_{it}\frac{d_{it}(\boldsymbol{\alpha}_t)}{F(\mathbf{z}_{it}'\boldsymbol{\alpha}_t)\{1-F(\mathbf{z}_{it}'\boldsymbol{\alpha}_t)\}}\{y_{it}-F(\mathbf{z}_{it}'\boldsymbol{\alpha}_t)\},$$

and the contribution of the expected information matrix is

$$\boldsymbol{U}_{it}(\boldsymbol{\alpha}_t) = \frac{\partial^2 l_{it}(\boldsymbol{\alpha}_t)}{\partial\boldsymbol{\alpha}_t'\partial\boldsymbol{\alpha}_t} = \boldsymbol{z}_{it}\boldsymbol{z}_{it}'\frac{(d_{it}(\boldsymbol{\alpha}_t))^2}{F(\mathbf{z}_{it}'\boldsymbol{\alpha}_t)\{1-F(\mathbf{z}_{it}'\boldsymbol{\alpha}_t)\}}.$$

The sums $\boldsymbol{u}_t(\boldsymbol{\alpha}_t) = \sum_{i\in\mathcal{R}_t}\boldsymbol{u}_{it}(\boldsymbol{\alpha}_t)$ and $\boldsymbol{U}_t(\boldsymbol{\alpha}_t) = \sum_{i\in\mathcal{R}_t}\boldsymbol{U}_{it}(\boldsymbol{\alpha}_t)$ are then contributions of the risk set to the score vector and the information matrix in the interval $[a_{t-1}, a_t)$.

Smoothing estimates $\boldsymbol{a}_{t|q}$ $(t = 0, \ldots, q)$ of $\boldsymbol{\alpha}_t$ can now be obtained as numerical approximations to posterior modes given all the data $(\boldsymbol{y}^*, \boldsymbol{x}^*, \boldsymbol{r}^*)$ up to q. Approximate error covariance matrices $\boldsymbol{V}_{t|q}$ are obtained as the corresponding numerical approximations to curvatures, i.e. inverses of expected negative second derivatives of $\ln\mathcal{L}(\boldsymbol{\alpha}^*)$, evaluated at the mode.

Finally $\boldsymbol{a}_{t|t-1}$ and $\boldsymbol{a}_{t|t}$ are the prediction and filter estimates for $\boldsymbol{\alpha}_t$ given the data up to $t-1$ and t, with corresponding error matrices $\boldsymbol{V}_{t|t-1}$ and $\boldsymbol{V}_{t|t}$.

Filtering and smoothing of our sample data proceed in the following steps:

1. <u>Initialization:</u>

$$\boldsymbol{a}_{0|0} = \boldsymbol{a}_0,$$
$$\boldsymbol{V}_{0|0} = \boldsymbol{Q}_0.$$

2. FILTER PREDICTION STEPS:

   For $t = 1, \ldots, q$:

$$\begin{aligned} \boldsymbol{a}_{t|t-1} &= \boldsymbol{\Phi}\boldsymbol{a}_{t-1|t-1}, \\ \boldsymbol{V}_{t|t-1} &= \boldsymbol{\Phi}\boldsymbol{V}_{t-1|t-1}\boldsymbol{\Phi} + \boldsymbol{Q}. \end{aligned}$$

3. FILTER CORRECTION STEPS:

   For $t = 1, \ldots, q$:

$$\begin{aligned} \boldsymbol{a}_{t|t} &= \boldsymbol{a}_{t|t-1} + \boldsymbol{V}_{t|t}\boldsymbol{u}_t, \\ \boldsymbol{V}_{t|t} &= (\boldsymbol{V}_{t|t-1}^{-1} + \boldsymbol{U}_t)^{-1}. \end{aligned}$$

4. BACKWARD SMOOTHING STEPS:

   For $t = 1, \ldots, q$:

$$\begin{aligned} \boldsymbol{a}_{t-1|q} &= \boldsymbol{a}_{t-1|t-1} + \boldsymbol{B}_t(\boldsymbol{a}_{t|q} - \boldsymbol{a}_{t|t-1}), \\ \boldsymbol{V}_{t-1|q} &= \boldsymbol{V}_{t-1|t-1} + \boldsymbol{B}_t(\boldsymbol{V}_{t|q} - \boldsymbol{V}_{t|t-1})\boldsymbol{B}_t', \end{aligned}$$

   where

$$\boldsymbol{B}_t = \boldsymbol{V}_{t-1|t-1}\boldsymbol{\Phi}'\boldsymbol{V}_{t|t-1}^{-1}.$$

The algorithm requires that initial values $\boldsymbol{a}_0$, $\boldsymbol{Q}_0$ and error covariances $\boldsymbol{Q}$ of the transition equation are known or given. The hyper-parameters $\boldsymbol{\alpha}_0, \ldots \boldsymbol{\alpha}_d$, $\boldsymbol{a}_0$, $\boldsymbol{Q}_0$ and $\boldsymbol{Q}$ can be jointly estimated by an EM-type algorithm which can be summarized as follows:

1. Choose starting values $\boldsymbol{a}_0^{(0)}$, $\boldsymbol{Q}_0^{(0)}$ and $\boldsymbol{Q}^{(0)}$.

   Iterate the following step 2 and 3 for $p = 1, 2, \ldots$

2. SMOOTHING:
   Compute $\boldsymbol{a}_{t|q}^{(p)}$ and $\boldsymbol{V}_{t|q}^{(p)}$ $(t = 1, \ldots, q)$ by the generalized Kalman filter and smoothing, with unknown parameters replaced by their current estimates $\boldsymbol{a}_0^{(p)}$, $\boldsymbol{Q}_0^{(p)}$ and $\boldsymbol{Q}^{(p)}$.

3. EM STEP:
   Compute $\boldsymbol{a}_0^{(p+1)}$, $\boldsymbol{Q}_0^{(p+1)}$ and $\boldsymbol{Q}^{(p+1)}$ by

$$\begin{aligned} \boldsymbol{a}_0^{(p+1)} &= \boldsymbol{a}_{0|q}^{(p)} \\ \boldsymbol{Q}_0^{(p+1)} &= \boldsymbol{V}_{0|q}^{(p)} \\ \boldsymbol{Q}^{(p+1)} &= \frac{1}{q}\sum_{t=1}^{q}\{(\boldsymbol{a}_{t|q}^{(p)} - \boldsymbol{\Phi}\boldsymbol{a}_{t-1|q}^{(p)})(\boldsymbol{a}_{t|q}^{(p)} - \boldsymbol{\Phi}\boldsymbol{a}_{t-1|q}^{(p)})' + \boldsymbol{V}_{t|q}^{(p)} \\ &\quad -\boldsymbol{\Phi}\boldsymbol{B}_t^{(p)}\boldsymbol{V}_{t|q}^{(p)} - \boldsymbol{V}_{t|q}^{(p)\prime}\boldsymbol{B}_t^{(p)\prime}\boldsymbol{\Phi} + \boldsymbol{\Phi}\boldsymbol{V}_{t-1|q}^{(p)}\boldsymbol{\Phi}'\} \end{aligned}$$

   with $\boldsymbol{B}_t^{(p)}$ defined as in the smoothing steps.

4. Stop when some termination criterion is reached.

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APPENDIX: TABLES AND FIGURES

Table 1: Summary Statistics. The 10 percent filter split the sample into 114 bull markets and 114 bear markets.

Series	Mean	Median	Standard deviation	Minimun	Maximum
Bull market durations (weeks)	36.83	18	50.2	1	354
Bear market durations (weeks)	19	12	19.01	1	88
Log-return (%) bull markets	2.77	0.98	5.7	0.23	42.5
Log-return (%) bear markets	-3.3	-1.54	6	-46	-0.23

FIGURE SUMMARY

Figure 1 Bull and Bear market classifications.

Figure 2 Histogram of Bull market durations.

Figure 3 Histogram of Bull market durations.

Figure 4 Unconditional hazard rates of Bull market durations.

Figure 5 Unconditional hazard rates of Bear market durations.

Figure 6 Parameter reflecting the difference in unconditional hazard rates between Bear and Bull market durations.

Figure 7 Baseline hazard rates for Bull markets, controlling for interest rates and interest rate change effects.

Figure 8 Parameter reflecting the interest rate effect on Bull markets.

Figure 9 Parameter reflecting the effect of interest rate change over the duration of Bear markets.

Figure 10 Baseline hazard rates for Bear markets, controlling for interest rates and interest rate change effects.

Figure 11 Parameter reflecting the interest rate effect on Bear markets.

Figure 12 Parameter reflecting the effect of interest rate change over the duration of Bull markets.

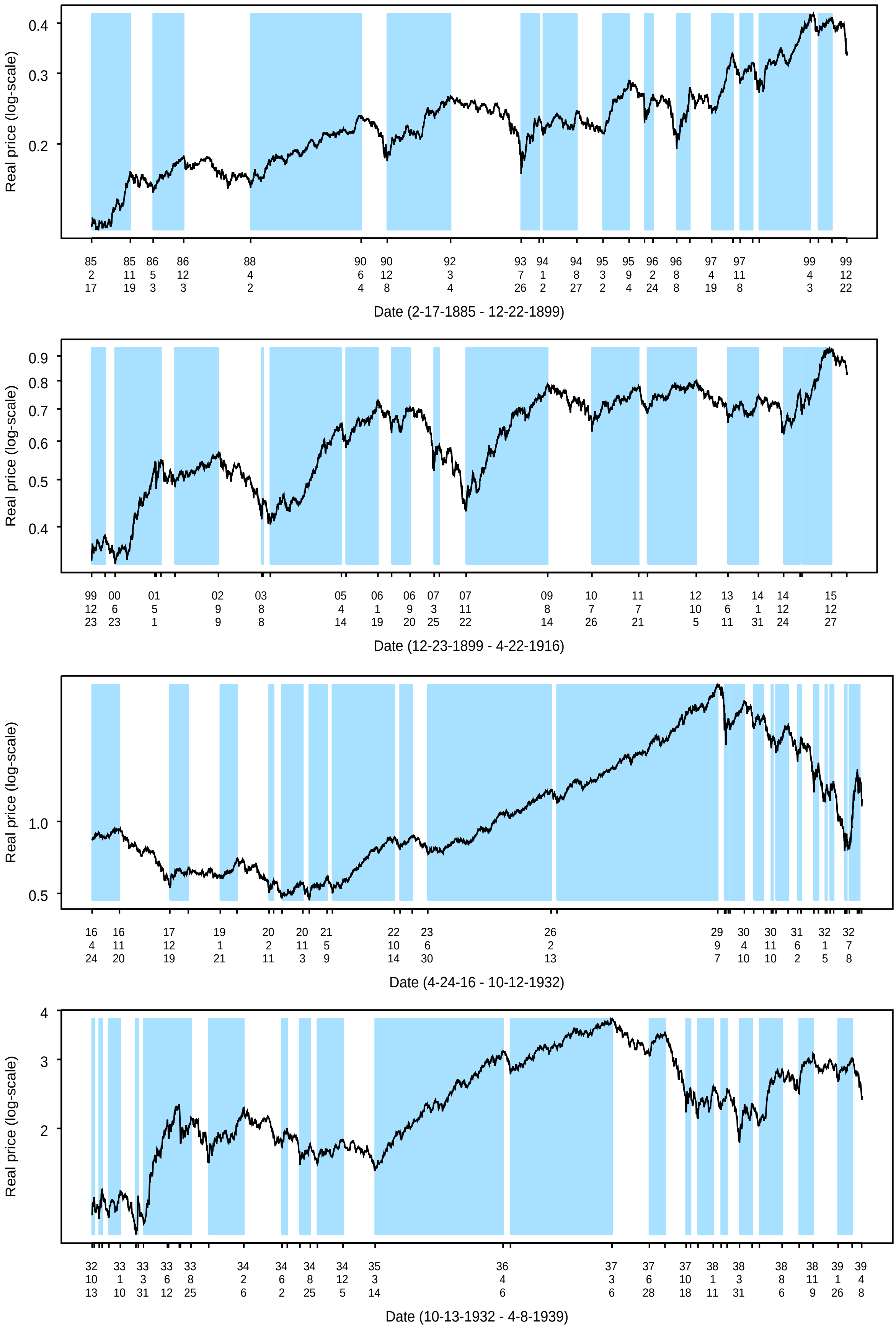

Figure 1a: Bull and Bear markets defined from real S&P-500 stock index with a stopping rule of 10%.

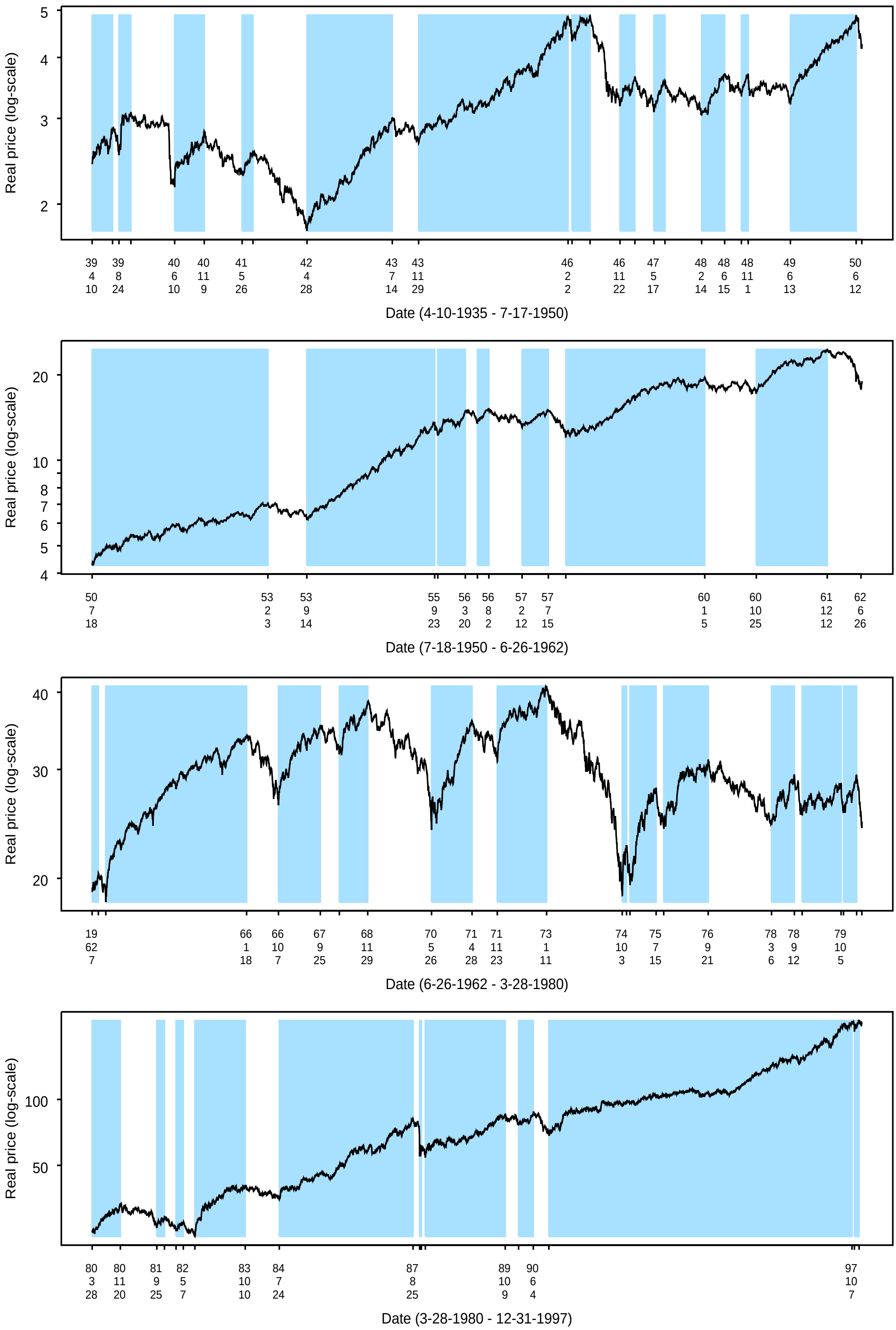

Figure 1b: Bull and Bear markets defined from real S&P-500 stock index with a stopping rule of 10%.

Histograms of Bull and Bear market durations

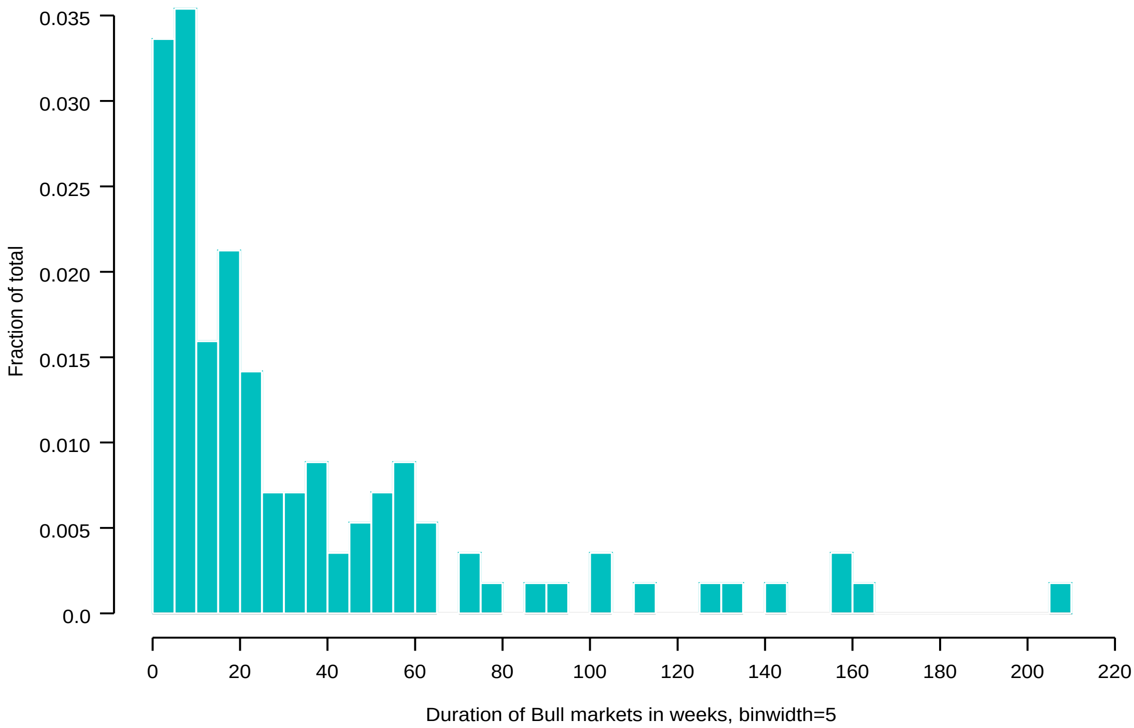

Figure 2: Histogram of Bull market durations. The confidence bands are ± 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%.

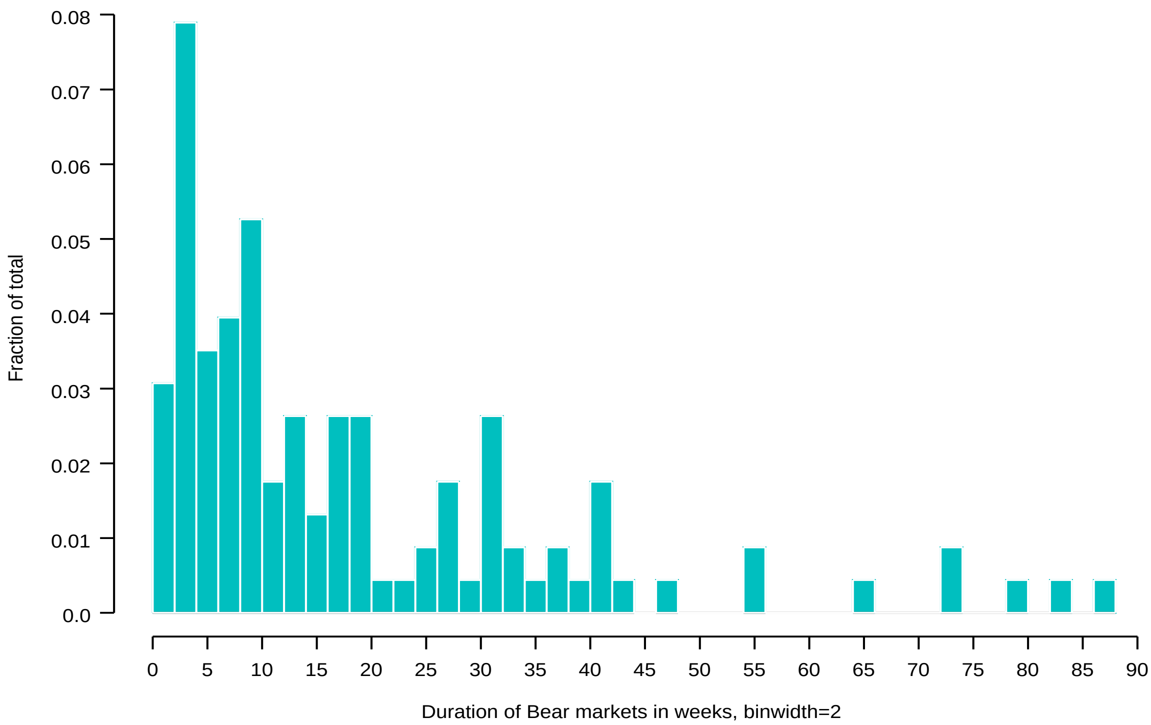

Figure 3: Histogram of Bear market durations. The confidence bands are ± 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%.

Unconditional hazard rates

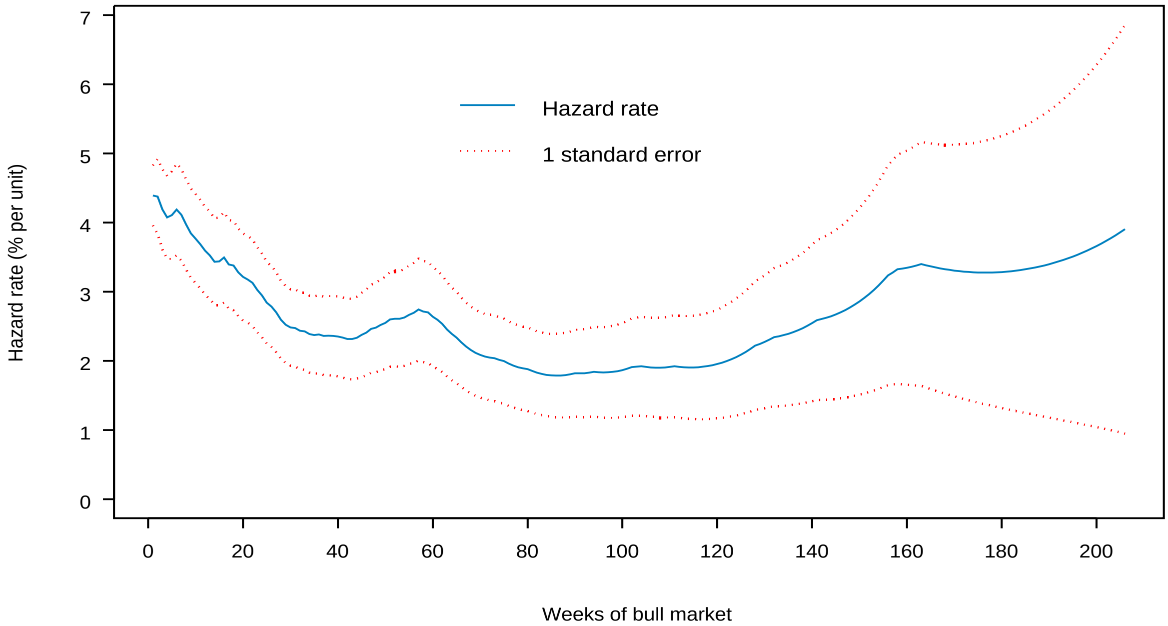

Figure 4: Unconditional hazard rates of Bull market durations. The confidence bands are ± 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%. The model is the simple random walk hazard rate with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(z'_{it}\alpha_t)$, where $z'_{it} = 1$ and $\alpha_t = \gamma_{0t}$, with $\gamma_{0t} = \gamma_{0t-1} + \xi_{0t}$, $\xi_{0t} \sim N(0,\sigma_1^2)$, and $\gamma_{00} \sim N(g_0, \sigma_0^2)$.

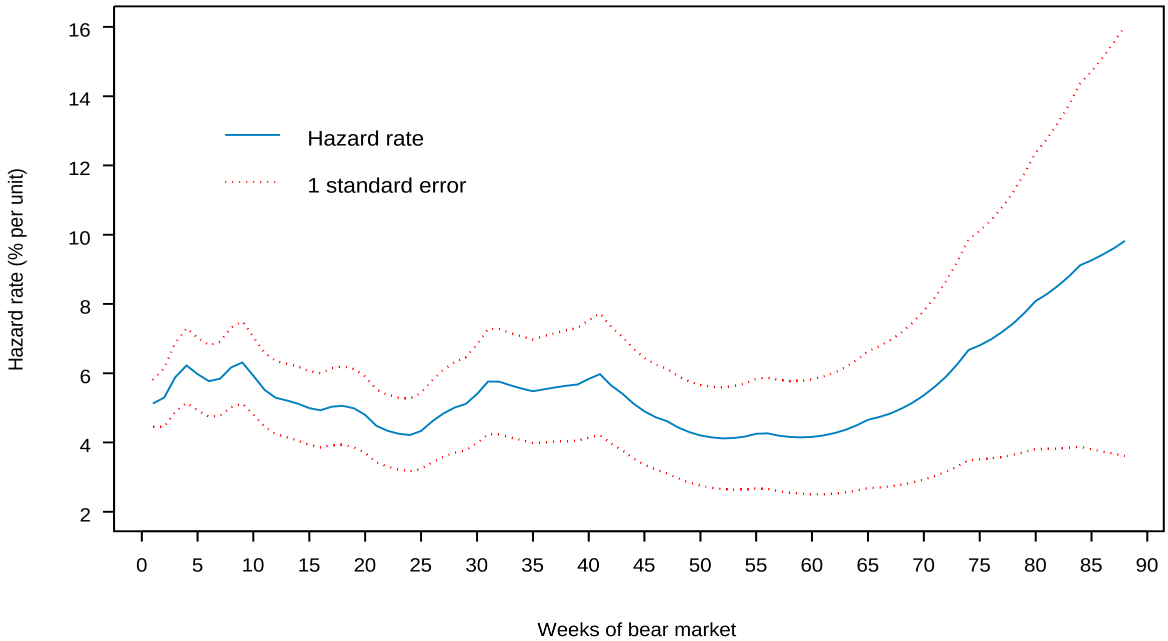

Figure 5: Unconditional hazard rates of Bear market durations. The confidence bands are ± 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%. The model is the simple random walk hazard rate with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(z'_{it}\alpha_t)$, where $z'_{it} = 1$ and $\alpha'_t = \gamma_{0t}$, with $\gamma_{0t} = \gamma_{0t-1} + \xi_{0t}$, $\xi_{0t} \sim N(0,\sigma_1^2)$, and $\gamma_{00} \sim N(g_0, \sigma_0^2)$.

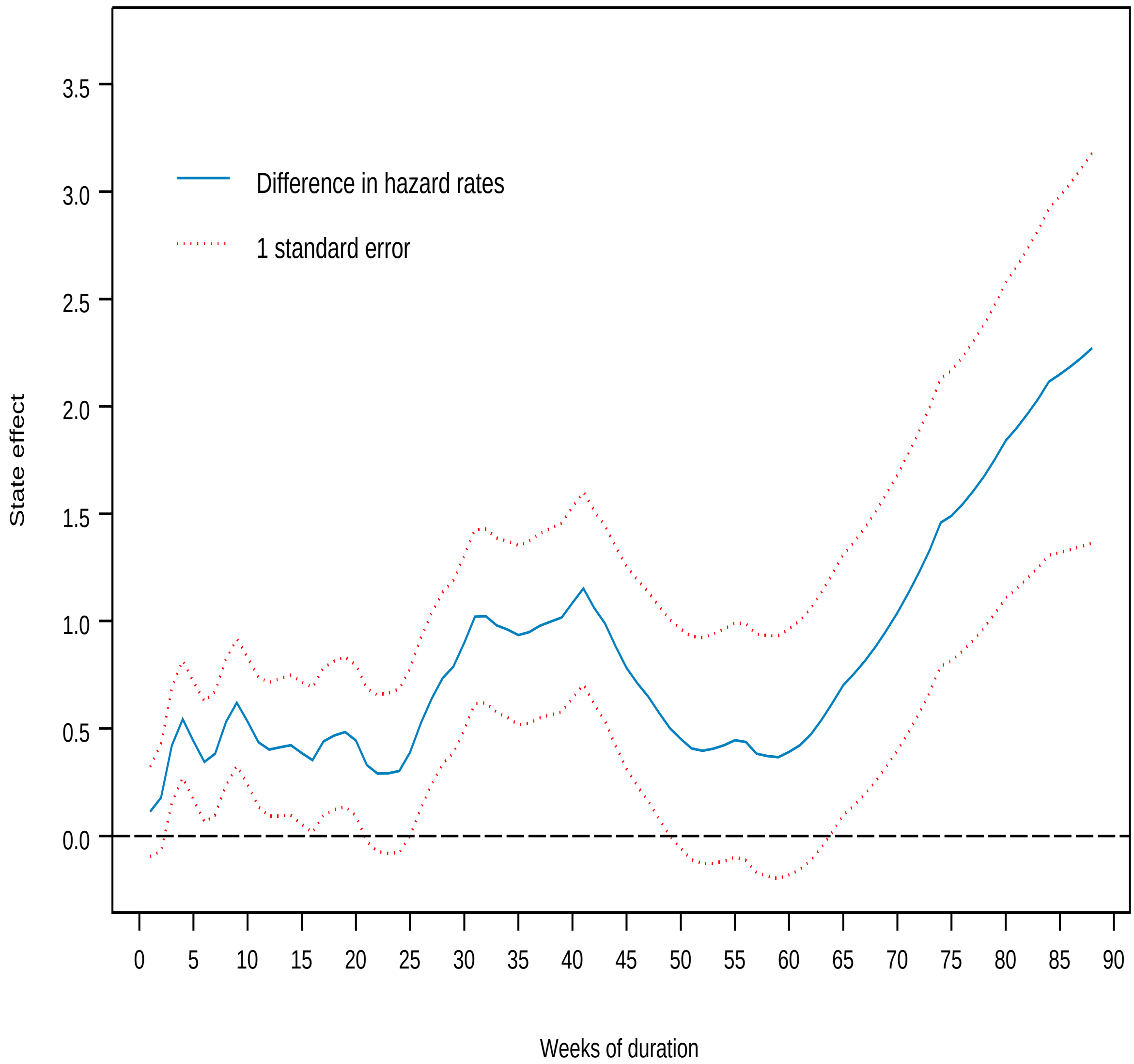

Figure 6: Parameter reflecting the difference in hazard rates between Bear and Bull market durations. The confidence bands are ± 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%. The model is a bivariate simple random walk hazard rate with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\boldsymbol{z}'_{it}\boldsymbol{\alpha}_t)$, where $\boldsymbol{z}'_{it} = (1, w_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \beta_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_t \sim N(\boldsymbol{0}, \boldsymbol{Q})$, and $\boldsymbol{\alpha}_0 \sim N(\boldsymbol{g}_0, \boldsymbol{Q}_0)$. β_t gives the difference between Bull markets ($w_{it} = 1$), and Bear markets ($w_{it} = 0$).

Interest rate effects for Bull markets

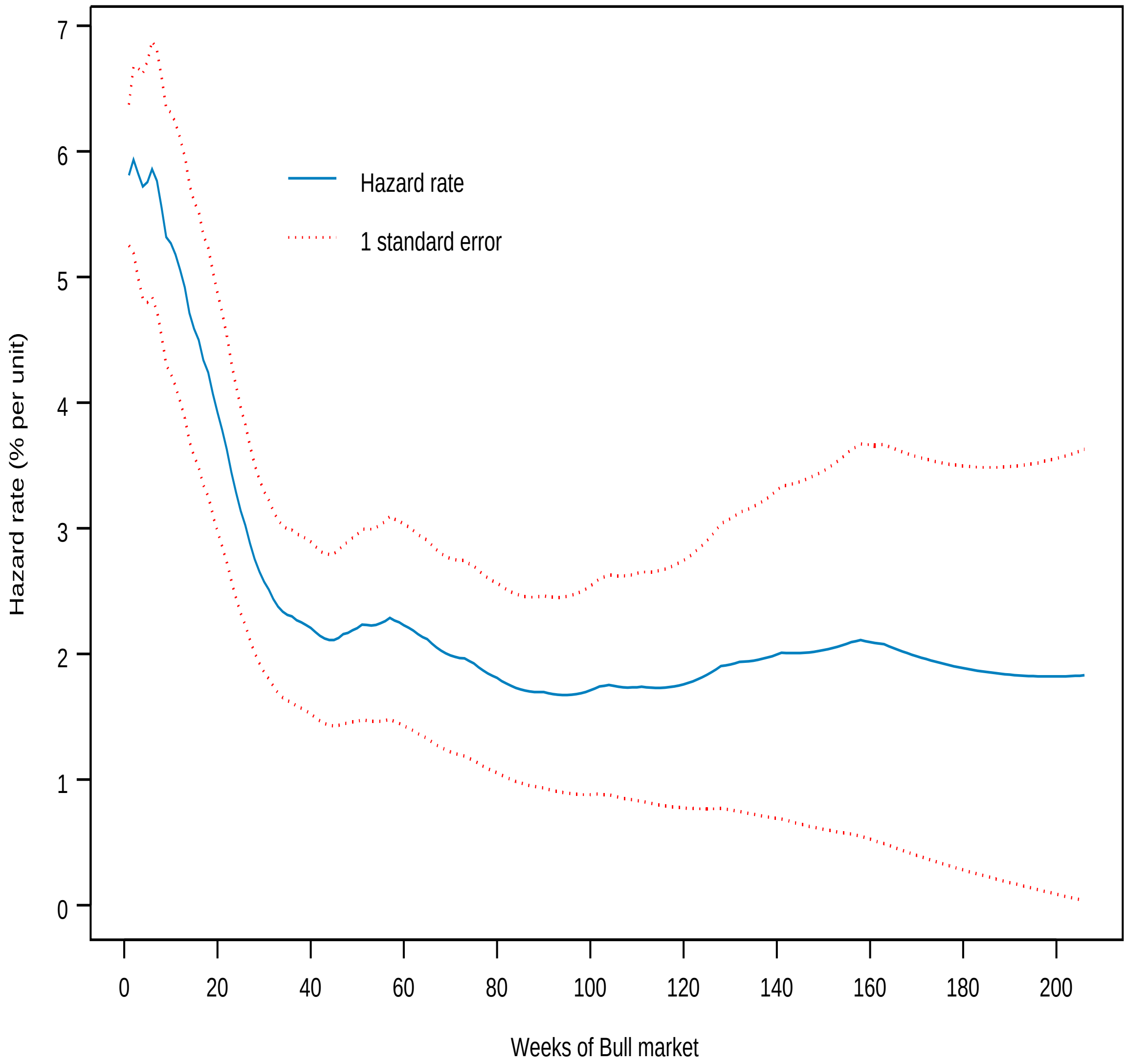

Figure 7: Baseline hazard rates for Bull markets, controlling for interest rate and interest rate change effect. The confidence bands are $\pm$ 1 standard error. Defined from the real S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\boldsymbol{z}'_{it}\boldsymbol{\alpha}_t)$, where $\boldsymbol{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t$, $\boldsymbol{\xi}_{0t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$, and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \boldsymbol{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bull markets.

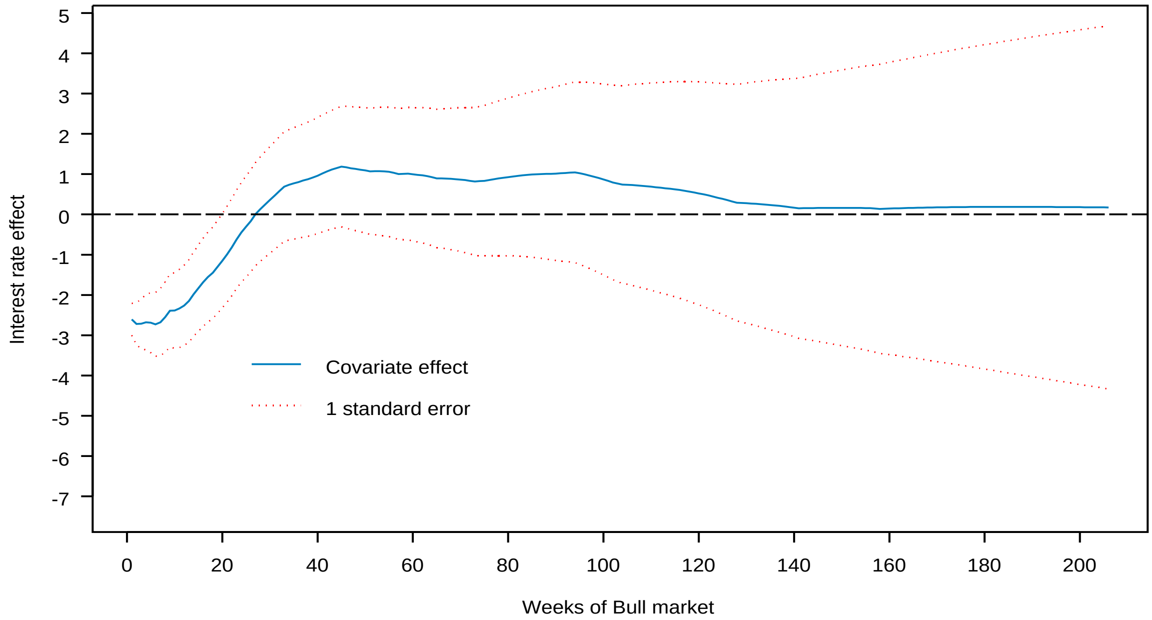

Figure 8: Parameter reflecting the interest rate effect on Bull markets. The confidence bands are $\pm$ 1 standard error. Defined from the real S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\boldsymbol{z}'_{it}\boldsymbol{\alpha}_t)$, where $\boldsymbol{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_{0t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}), \quad$ and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \boldsymbol{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bull markets.

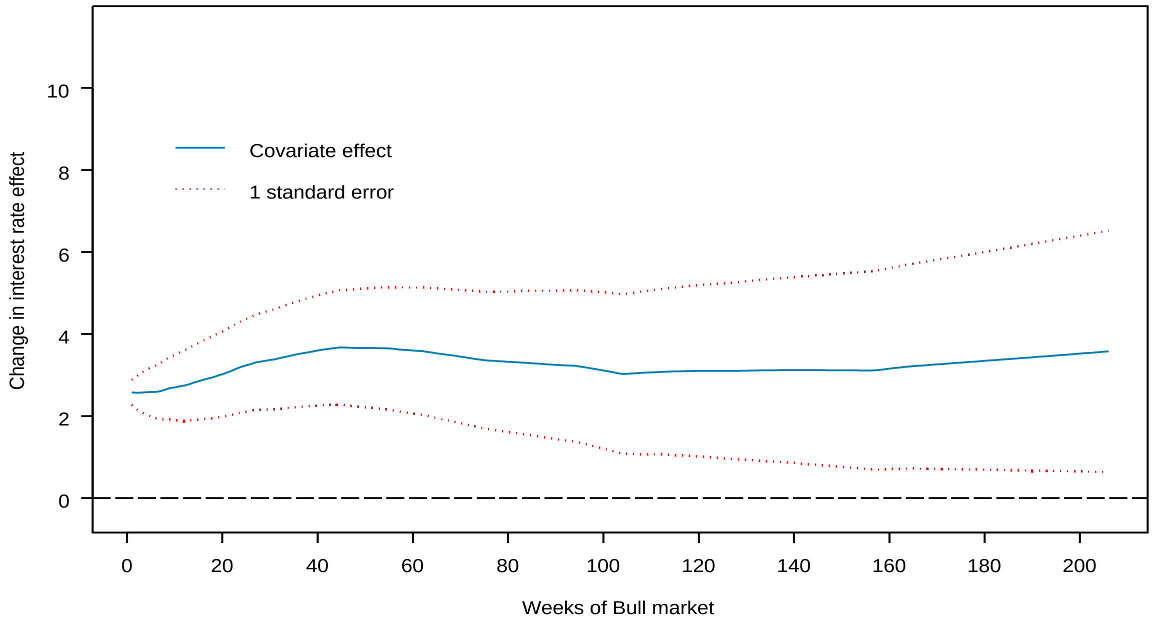

Figure 9: Parameter reflecting the effect of interest rate change on Bull markets. The confidence bands are $\pm$ 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\boldsymbol{z}'_{it}\boldsymbol{\alpha}_t)$, where $\boldsymbol{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t, \quad \boldsymbol{\xi}_{0t} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}), \quad$ and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \boldsymbol{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bull markets.

Interest rate effects for Bear markets

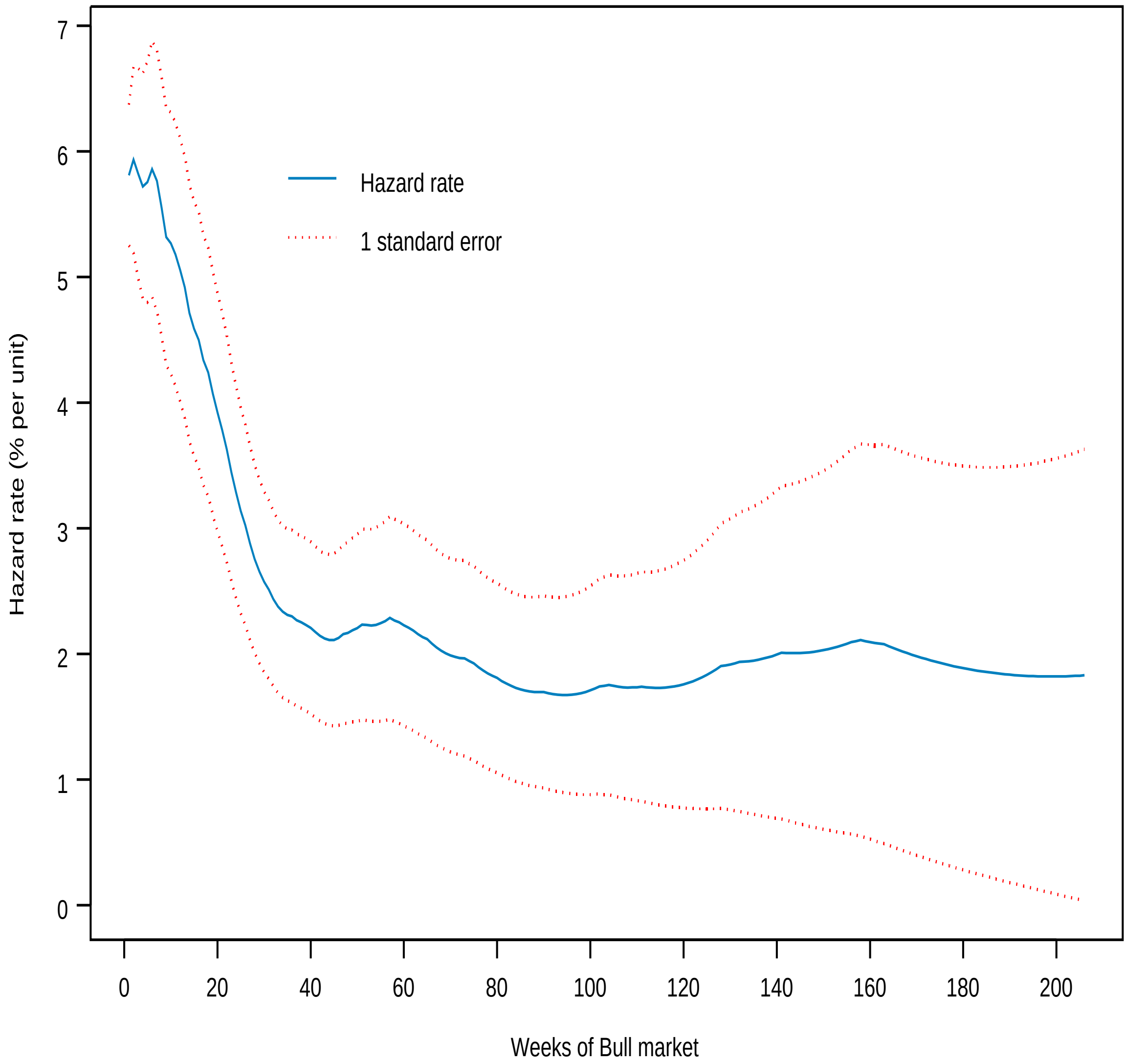

Figure 10: Baseline hazard rates for Bear markets, controlling for interest rate and interest rate change effect. The confidence bands are $\pm$ 1 standard error. Defined from the real S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\mathbf{z}'_{it}\boldsymbol{\alpha}_t)$, where $\mathbf{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t$, $\boldsymbol{\xi}_{0t} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q})$, and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \boldsymbol{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bear markets.

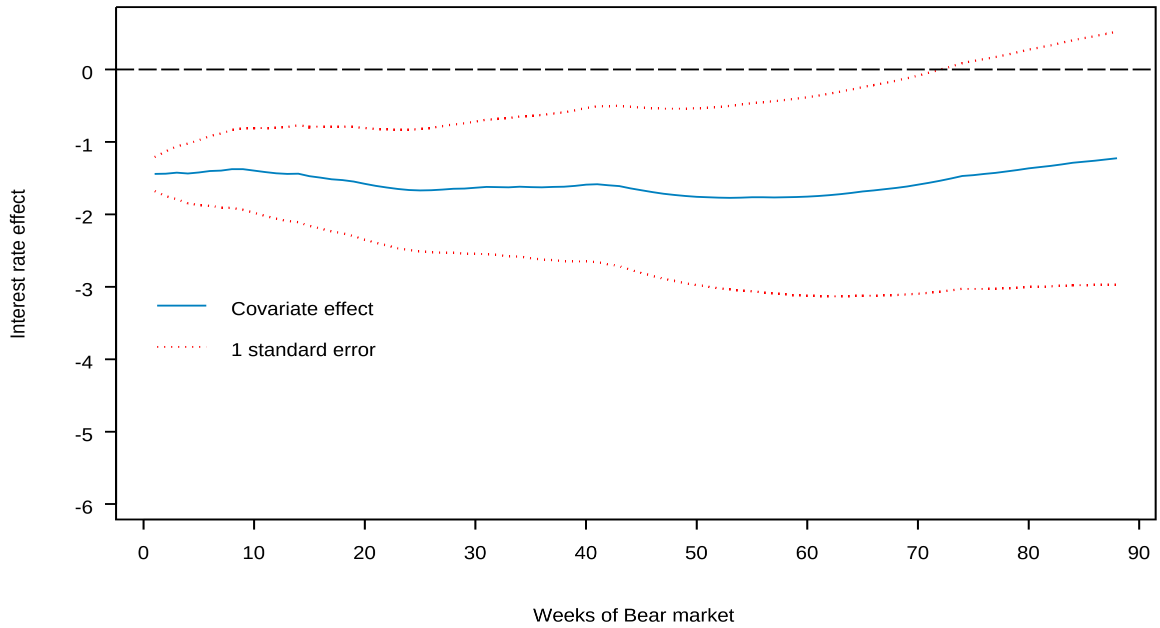

Figure 11: Parameter reflecting the interest rate effect on Bear markets. The confidence bands are $\pm$ 1 standard error. Defined from the real S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\mathbf{z}'_{it}\boldsymbol{\alpha}_t)$, where $\mathbf{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t$, $\boldsymbol{\xi}_{0t} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \mathbf{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bear markets.

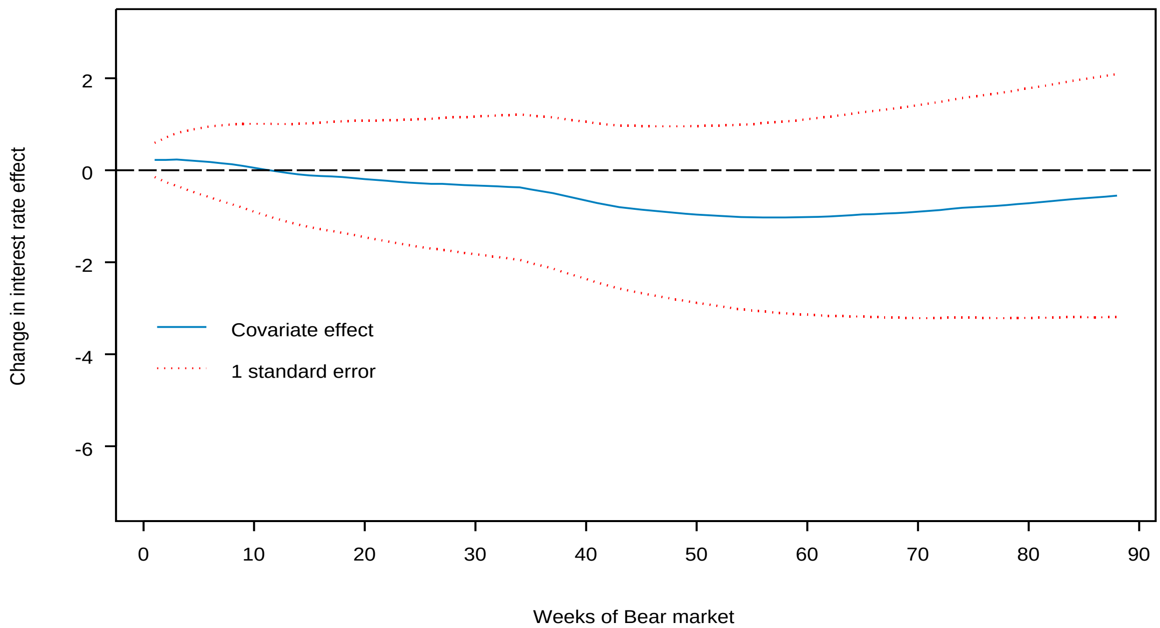

Figure 12: Parameter reflecting the effect of interest rate change on Bear markets. The confidence bands are $\pm$ 1 standard error. Defined from S&P-500 stock index with a stopping rule of 10%. The model is the random walk model with a logit link function. That is $\lambda(t|\boldsymbol{X}_i(t)) = F(\mathbf{z}'_{it}\boldsymbol{\alpha}_t)$, where $\mathbf{z}'_{it} = (1, i_{it}, \Delta i_{it})$ and $\boldsymbol{\alpha}'_t = (\gamma_{0t}, \boldsymbol{\beta}_t)$, with $\boldsymbol{\alpha}_t = \boldsymbol{\alpha}_{t-1} + \boldsymbol{\xi}_t$, $\boldsymbol{\xi}_{0t} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$, and $\boldsymbol{\alpha}_0 \sim \mathcal{N}(\boldsymbol{g}_0, \mathbf{Q}_0)$. i_{it} is the interest rate at the begining of the week in question, Δi_{it} is the change in the interest rate from the duration origin to the begining of the week in question. $\boldsymbol{\beta}_t$ gives the covariate effect on the hazard rate of Bear markets.