Linear approximation methods and international real business cycles with incomplete asset markets

January 2000

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Abstract

Most quantitative studies of international real business cycle (IRBC) models require the use of approximate solution methods. We solve an IRBC model with incomplete asset markets using King, Plosser and Rebelo's (1988) linear approximation method. We quantify the additional approximation error brought about by the existence of a unit root in the linear dynamic system and demonstrate that the symmetry of the model helps reduce this approximation error. A central finding is that the parametrizations which address the cross-country consumption correlation puzzle are precisely those where solutions may be least accurate.

JEL classification: E32, C63, F41

Keywords: real business cycles, incomplete markets

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The first attempt to model international real business cycles (IRBC) by Backus, Kehoe and Kydland (1992) pointed out a number of discrepancies between the data and the predictions of their model (see Backus *et al* (1995) and Ravn (1997)). First, the cross-country correlation of output is higher than the cross-country correlation of consumption in the data, while the opposite is observed in IRBC models (the cross-country consumption correlation puzzle). Second, IRBC models can account for only a fraction of the variability of relative prices found in the data. Third, productivity levels are more closely related across countries in the data than in IRBC models.

Many authors have turned their attention toward IRBC models with (exogenous) incomplete asset markets to try solving the cross-country consumption correlation puzzle. Some examples are Arvanitis and Mikkola (1996), Baxter and Crucini (1995), Crucini (1997) and Kollmann (1996). In their models, representative agents can trade only one-period bonds. van Wincoop (1996) also includes (exogenous) market incompleteness in a multi-country IRBC model with heterogenous agents. Asset market incompleteness can also be made endogenous as in Kehoe and Perri (1996) where international loans are not perfectly enforceable. Kehoe and Levine (1996) compare an (exogenous) incomplete markets model with a model where loans are not perfectly enforceable and conclude that the latter is simpler and should receive more attention.

This paper studies the accuracy of the linear approximation method commonly used to solve these models and documents the effect of their parametrization on their economic implications. A central finding is that the parametrizations which address the cross-country consumption correlation puzzle are precisely those where solutions may be least accurate. In section I, we contrast the complete markets IRBC model of Backus, Kehoe and Kydland (1995) with the incomplete markets IRBC models of Baxter and Crucini (1995) and Kollmann (1996). We also present the IRBC model we use in our analysis.

IRBC models are often approximated using the solution method of King, Plosser and Rebelo (1988) where the first-order conditions are linearized around the steady state. Dotsey and Mao (1992) investigated the accuracy loss due to that solution method. In section II, we add to this literature by showing the great degree of inaccuracy associated with a non-stationary productivity shock process.

IRBC models with incomplete financial markets introduce an additional source of inaccuracy. In such models there are an infinite number of steady-state equilibria so the (linearized) dynamic system has an endogenous unit root and therefore is not guaranteed to converge. In section III, we quantify this additional source of inaccuracy and show that it can be reduced by using a symmetric model. We also demonstrate that the degree of accuracy increases with the level of international spillovers in technology shocks and decreases with the level of persistence in technology shocks.

Baxter and Crucini (1995) and Kollmann (1996) were the first to consider incomplete asset markets in an IRBC model. Unfortunately, these two papers come to different conclusions regarding the effect of restricting asset markets (when using a stationary process for technology shocks) even though their models have very similar structure. In section IV we reconcile those apparently paradoxical results. We also show that the economic effect of the restrictions on the asset markets is difficult to measure due to the approximation error in the incomplete markets model. Section V concludes.

I. IRBC Models

Backus, Kehoe and Kydland (1995) present one of the simplest IRBC models. In this model the world is composed of two *ex ante* identical countries, denoted by $i = 1, 2$, in which identical agents produce and consume a single homogeneous good. Each country is represented by a consumer who seeks to maximize

$$Eu_i = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[c_{it}^{\mu}(1 - n_{it})^{1-\mu}\right]^{1-\sigma}}{1 - \sigma} \qquad (1)$$

where c_{it} and n_{it} denote consumption and hours worked in country i. Output in country i is given by the (constant returns to scale) production function

$$y_{it} = z_{it} k_{it}^{\theta} n_{it}^{1-\theta}, \qquad (2)$$

where z_{it} represents a shock to country i's technology and k_{it} the capital stock installed in country i. The law of motion for capital, incorporating the time-to-build structure, is given by

$$k_{it+1} = (1-\delta)k_{it} + s_{it}^1,$$

$$s_{it+1}^j = s_{it}^{j+1}, \quad \forall j = 1, ..., J-1,$$

where s_{it}^j denotes the value of investment projects that are j periods from completion at time t. Investment at time t is given by

$$x_{it} = \sum_{j=1}^{J} \phi_j s_{it}^j,$$

where ϕ_j denotes the fraction of value added to an investment project in the j^{th} period before completion. The technology shocks follow

$$\begin{bmatrix} z_{1t+1} \\ z_{2t+1} \end{bmatrix} = \begin{bmatrix} \rho_p & \rho_s \\ \rho_s & \rho_p \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix},$$

where ρ_p measures the persistence in technology shocks, ρ_s measures the level of international spillovers and the innovations $\epsilon = (\epsilon_1, \epsilon_2)$ have covariance matrix Ω. The transition matrix in the bivariate AR(1) process above is denoted P.

Baxter and Crucini's (1995) model is very close to Backus, Kehoe and Kydland's. We outline only the differences between the two. A first difference is in the production function. For country i, this function, incorporating labor-augmenting technical change at gross rate γ, is

$$Y_{it} = z_{it} K_{it}^{\theta} \gamma^{(1-\theta)t} n_{it}^{1-\theta}.$$

Baxter and Crucini redefine the variables in order to remove the deterministic trend arising from the labor-augmenting technical change. They divide all variables, except hours worked, by γ^t and let lowercase letters denote the transformed variables. The production function in their transformed economy is then $y_{it} = z_{it} k_{it}^{\theta} n_{it}^{1-\theta}$, which is identical to the one in Backus, Kehoe and Kydland's model. Also, the adjusted discount factor used in the transformed economy is $\tilde{\beta} = \beta\gamma^{\mu(1-\sigma)}$.

A second difference is in the law of motion for capital. Instead of using time-to-build to slow down investment, Baxter and Crucini use a capital adjustment cost function $\psi(x_{it}/k_{it})$, where $\psi > 0$, $\psi' > 0$ and $\psi'' < 0$. The law of motion for capital is then

$$\gamma k_{it+1} = (1-\delta)k_{it} + \psi(x_{it}/k_{it})k_{it}. \tag{3}$$

When asset markets are restricted to one-period discount bonds, the budget constraint of country i is

$$\gamma P_t^B b_{it+1} + c_{it} + x_{it} = y_{it} + b_{it} \tag{4}$$

where b_{it+1} denotes the quantity of bonds purchased in period t and maturing in $t+1$, and P_t^B denotes the bond price.

The general form of the technology shock process is

$$\begin{bmatrix} \log z_{1t+1} \\ \log z_{2t+1} \end{bmatrix} = \begin{bmatrix} \rho_p & \rho_s \\ \rho_s & \rho_p \end{bmatrix} \begin{bmatrix} \log z_{1t} \\ \log z_{2t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix}. \tag{5}$$

Kollmann's (1996) model is very close to Baxter and Crucini's. The only differences are that Kollmann specifies explicitly the form of the capital adjustment cost function and does not consider labor-augmenting technical change. The parameter values for all three of these models are presented in appendix $A.1$.

To contrast the economic findings of these models and to study approximation error, we adopt a model similar to Baxter and Crucini's. There are only two trivial differences. First we do not consider labor-augmenting technical change, which implies $\gamma = 1$. Second, we specify a particular form for the capital adjustment cost function, $\psi(x_{it}/k_{it}) = (x_{it}/k_{it})^\tau$, where $0 < \tau < 1$, which satisfies the conditions $\psi > 0$, $\psi' > 0$ and $\psi'' < 0$. The greater τ the smaller the adjustment cost. As for any capital adjustment cost function, the parameter τ is calibrated to match the relative volatility of investment. In our case we set it to 0.977 so that investment is approximately three times more volatile than output.

Therefore, the representative consumer in country i seeks to maximize expected lifetime utility given by (1). Output in country i is given by the production function (2). The stock of capital evolves according to equation (3) and the technology shocks follow the process given by (5) where the innovations $\epsilon = (\epsilon_1, \epsilon_2)$ have covariance matrix Ω.

When asset markets are restricted to one-period bonds, agents in country i must satisfy the budget constraint (4) for all periods and states. Also the bond market clearing condition, $b_{1t} + b_{2t} = 0$, must hold for all periods and states. Appendix $A.2$ presents the derivation of the equilibrium system for this economy and discusses the solution method employed.

When asset markets are complete both countries can perfectly pool idiosyncratic risk and the optimal and competitive equilibrium solutions coincide. Appendix $A.3$ presents the derivation of the equilibrium system for this economy and discusses the solution method employed. Unless otherwise specified, the parameter values considered in this paper are the ones used by Backus, Kehoe and Kydland presented in appendix $A.1$, except for the absence of time-to-build.

II. Solution Method and Technology Shock Specification

Whether asset markets are complete or incomplete, it is important to be aware of the effect of the technology shock parametrization on the approximation error. For instance, the linear dynamic system following from our IRBC model with complete markets satisfies the saddle-path stability condition. Therefore, it fluctuates around its steady-state equilibrium under most circumstances. However, there are restrictions imposed on the bivariate AR(1) process utilized to model technology shocks. The usual specification is given in equation (5). Once the equilibrium system of equations is linearized and the fundamental dynamic system derived, we use the linearized version of the AR(1) process, which is

$$\begin{bmatrix} \hat{z}_{1t+1} \\ \hat{z}_{2t+1} \end{bmatrix} = \begin{bmatrix} \rho_p & \rho_s \\ \rho_s & \rho_p \end{bmatrix} \begin{bmatrix} \hat{z}_{1t} \\ \hat{z}_{2t} \end{bmatrix} + \begin{bmatrix} \epsilon_{1t+1} \\ \epsilon_{2t+1} \end{bmatrix}$$

where hatted variables denote percent deviations from the steady state. That is, if we let $\bar{z}_i$ be the steady-state value for z_i, then $\hat{z}_{it} \equiv (z_{it} - \bar{z}_i)/\bar{z}_i$. For the complete markets model to fluctuate around its steady state, it must be the case that the shocks ultimately die out so that $\hat{z}_1$ and $\hat{z}_2$ come back to their steady-state value of 0. This occurs only if the eigenvalues of P are less than 1. The eigenvalues of P are $\rho_p + \rho_s$ and $\rho_p - \rho_s$. Therefore, we have the restrictions $\rho_p + \rho_s < 1$ and $\rho_p - \rho_s < 1$. Since ρ_p and ρ_s are (usually) positive constants we have the implicit restrictions that $\rho_p < 1$ and $\rho_s < 1$. Finally, if we want to prevent oscillatory behavior, we must have $\rho_p - \rho_s > 0$ so that no eigenvalue is negative. Therefore using a unit-root process without spillovers (as Baxter and Crucini do) violates the necessary conditions for the (complete asset markets) model to fluctuate around its steady state. With such a specification, each innovation in the technology shocks pushes the model toward a new steady-state equilibrium. In such a case, we expect the approximation error coming from the linear approximation to increase because we are linearizing around a particular steady state and the model moves away from that point.

One way to estimate this additional inaccuracy is to measure how far the new steady state is from the initial one. The experiments we perform are very simple. We choose a set of P matrices many of which violate the conditions $\rho_p + \rho_s < 1$ and $\rho_p - \rho_s < 1$. Then, we perform an impulse response exercise and verify the new equilibrium to which the model converges. More specifically, we simulate the complete markets model for 20,000 periods after one technology shock (set to one standard deviation of ϵ_1, denoted σ_ϵ) in country 1 in period 1. Table 1a presents the effects of this single shock. The numbers in the column GAP are calculated as follows: (1) calculate the difference (in percent) between the new steady state and the initial one, for all variables except net exports; (2) take the largest number (in absolute value) calculated in (1). From table 1a we see that the model does not come back to its initial steady state when the condition $\rho_p + \rho_s < 1$ is violated. Moreover, GAP is relatively large (2.339 percent) when $\rho_p = 1$.

Recall that the results in table 1a are generated by a single shock. When we perform a regular simulation there are many more shocks and therefore we expect the model to drift away from the initial steady state even more. To see this, we simulate the model for 20,000 periods but generate technology shocks (for both countries) in the first 100 periods only. From table 1b, we see that the new steady state achieved after 20,000 periods is always different from the initial one when the condition $\rho_p + \rho_s < 1$ is violated. Again we see that the new steady state is further from the initial one when $\rho_p = 1$ (GAP=22.183%).

In the case of the incomplete asset markets model, if the necessary conditions given above are violated, then there exist three sources of inaccuracy. The first is the loss of accuracy coming from the fact that we linearize the model (investigated by Dotsey and Mao (1992)). The second is the loss of accuracy coming from a non-stationary technology shock process (exogenous unit root). This is investigated in this section. The third is the loss of accuracy coming from the endogenous unit root in the linear dynamic system (investigated in the next section).

III. Solution Method and Incomplete Asset Markets

Our IRBC model with incomplete markets has an infinite number of steady-state equilibria. This is easily seen by looking at the equilibrium system derived in appendix $A.2$. In steady state, this system has 13 endogenous variables in 12 equations. In contrast, when markets are complete (see appendix $A.3$) the number of endogenous variables equals the number of equations and the model fluctuates around its unique steady state (as long as the stochastic process for the technology shocks is stationary). The infinite number of steady-state equilibria in the model with incomplete markets translates in a linearized system with an infinite number of steady-state equilibria. This implies the system has a unit root which is the discrete time analogue of a zero root studied by Giavazzi and Wyplosz (1985). The system violates the stability condition and exhibits zero-root dynamics as explained in Amable *et al* (1994).

Since the model is unstable, we do not expect the model to come back to its initial steady state. Usually, the equilibrium system of equations is linearized around a symmetric (initial) steady state where both countries do not trade. Since the equilibrium system is approximated around this initial steady state, the decision-rules calculated depend on it. When the economy moves toward another steady state, these decision-rules are inaccurate. The further away the economy moves from the initial steady state, the greater the inaccuracy. Therefore we examine whether the model actually fluctuates in the neighborhood of the initial steady state. If that is the case, then the system is not too inaccurate.

To investigate this question we proceed as follows. We simulate the economy for 20,000 periods but generate innovations in technology shocks in both countries in the first 100 periods only. Therefore, the economy has 19,900 periods to come back to a steady-state equilibrium. We repeat this exercise 5,000 times and present the results of two of these replications in the first two rows of table 2. Row 1 (2) presents what we call a "good draw" ("bad draw"). It is the realization where GAP was the smallest (largest). Based on these two rows we can conclude that the economy does not converge exactly to the initial steady state. We observe that the largest difference between the initial steady state and the new equilibrium (GAP) is 1.8557 percent for these two rows.

To get some insight into bad draws we perform 6 experiments where we control the sequences of innovations. The first one is similar to an impulse response. Again we simulate the model for 20,000 periods but this time there is a single innovation in technology shocks. It occurs in period 1 in country 1 and is equal to one innovation standard deviation (σ_ϵ). Row 3 of table 2 shows the results. The new steady state is very close to the initial one, the largest discrepancy being 0.03939 percent. We then increase the number of innovations set to one standard deviation. When the first five innovations to country 1's technology shocks are set to σ_ϵ the largest difference between the new and the initial steady state is 0.19696 percent as shown in row 4. When the first twenty-five innovations equal σ_ϵ, the largest difference increases to 0.98480 percent and to 4.92398 percent in the extreme case where 125 innovations are set to σ_ϵ (rows 5 and 6 respectively). Therefore we see that for a scenario where country 1 experiences 5 innovations which are not somehow compensated by innovations in country 2, the resulting steady state is close to the initial one, with GAP less than 0.20 percent.

Intuitively, the reason why the new steady state is close to the initial one when we perform a regular simulation is that innovations in technology shocks across countries have offsetting effects over time. Consider the case where country 1 experiences an innovation equal to one standard deviation in period 1 and country 2 an innovation of one standard deviation in period 2. The simulation length is still 20,000 observations. The results are in row 7 of table 2. The largest discrepancy is very close to zero, which confirms our intuition. Moreover, the innovation in country 2 need not be very close (in time) to the one in country 1. When country 2's innovation is postponed until period 50, the largest difference is again close to zero (row 8). Therefore, it is clear that, when the innovation sequences in both countries have similar empirical distributions, the new steady state is close to the initial one. Therefore, the symmetry of the covariance matrix used to generate innovations is important if we want the economy to come back to an equilibrium close to the initial one.

To prove the latter statement, we compare rows 9 and 10 in table 2. Row 9 is the base case where the model is perfectly symmetric. We simulate the model for 20,000 periods but this time we generate innovations in both countries for the first 100 periods based on a particular seed. When the model is symmetric, GAP=0.34381%. However when we double country 1's innovations standard deviation, so that

$$\Omega = \begin{bmatrix} 0.01704^2 & 0.258 \times 0.01704 \times 0.00852 \\ 0.258 \times 0.01704 \times 0.00852 & 0.00852^2 \end{bmatrix},$$

the resulting GAP is 0.73989 percent (row 10). This increase in GAP is not solely from the increase in volatility since doubling the standard deviations in both countries exactly doubles the GAP to 0.68762 percent.

The symmetry of the matrix governing the levels of persistence and international spillovers in technology shocks (P) is also important. Rows 9 and 11 demonstrate this point. Again we simulate the economy for 20,000 periods but generate innovations in technology shocks for both countries only in the first 100 periods. When Ω and P are symmetric the new and initial steady states are close. The largest discrepancy is 0.34381 percent. When the same sequence of innovations is used but matrix P is

$$\mathrm{P} = \begin{bmatrix} 0.904 & 0.052 \\ 0.149 & 0.908 \end{bmatrix}$$

then the new steady state is relatively far from the initial one with the largest difference being 1.72993 percent as shown in row 11. Note that the asymmetric P above has the same eigenvalues as the symmetric P.

The symmetry of the model is therefore very important. Another example of this importance arises when we consider different country sizes. When country 2 represents one tenth of the world the gap almost doubles (0.61886, row 12) compared to the case where both countries are equal (0.34381). In contrast, the symmetry of an IRBC model with complete asset markets is irrelevant. All the rows in table 2 would have GAP equal to zero.

There are two other factors affecting the accuracy of the IRBC model with incomplete markets. Those are the levels of persistence (ρ_p) and of international spillovers (ρ_s) in the technology shock process. Except for row 11, all the results presented in table 2 depend on the parameters $\rho_p = 0.906$ and $\rho_s = 0.088$. However, changing these parameters modifies importantly the results in table 2. When we perform the experiments in table 2 but using $\rho_p = 0.906$ and $\rho_s = 0$, we have to multiply the GAPs by a factor of (approximately) 1.97. When we use $\rho_p = 0.95$ and $\rho_s = 0.044$ the factor is still 1.97 but when we set $\rho_p = 0.95$ and $\rho_s = 0$, then the factor is 3.59. Therefore we see that reducing the level of international spillovers reduces the accuracy of the solution method. Also, if we set $\rho_p = 0.99$ and $\rho_s = 0$ then we have to multiply the numbers in table 2 by 11.53 showing that the level of inaccuracy increases with the degree of persistence.

The conclusion we can draw from our analysis is that when we model an economy with incomplete financial markets, we have to pay attention to the symmetry of the model. Using symmetric P, country-size, and Ω, and respecting the conditions $\rho_p + \rho_s < 1$ and $\rho_p - \rho_s < 1$ increases the likelihood of having an economy fluctuating around steady-state equilibria in the neighbourhood of the initial one. However, these conditions are not sufficient to guarantee the accuracy of the solution method since the levels of persistence and international spillovers play a significant role.

IV. Cross-Country Consumption Correlation and Incomplete Asset Markets

In this section we reconcile the main results of Baxter and Crucini (1995) and Kollmann (1996). Baxter and Crucini showed that when Backus, Kehoe and Kydland's matrix P is used, restricting asset markets has little impact on predicted moments. According to them, this result comes from the fact that international spillovers are so large that wealth effects are almost the same in both countries, whether markets are restricted or not. They also showed that when technology shocks are permanent and there are no international spillovers, then restricting asset markets does affect the predictions of their model. However, we demonstrated in section III that using such a P matrix generates unreliable statistics (even in the complete markets model) as the approximation around a steady state is inaccurate. Instead, we look at a persistence level consistent with a stationary stochastic process for technology shocks.

Considering Baxter and Crucini's and Kollmann's results together would lead to the following conclusion. Baxter and Crucini showed that when there are international spillovers, restricting asset markets has no effects on the IRBC model's predictions. Kollmann showed that with larger persistence and no international spillovers, restricting asset markets does modify the model's predictions. Therefore, it must be that reducing the level of international spillovers or increasing persistence is the factor explaining whether incomplete markets can help resolve the cross-country consumption correlation puzzle. We demonstrate that the effects of restricting asset markets are highly dependent on the parametrization of the technology shock process. Since one goal of imposing restrictions on the asset markets is to reduce $\mathrm{corr}(c_1, c_2)$, we focus on that moment only. Note that other moments act similarly. That is, when $\mathrm{corr}(c_1, c_2)$ is not much affected by restrictions then so are the other moments (standard deviations, autocorrelations and so on).

Table 3 shows the effect of the restrictions on the cross-country correlation of consumption. First, when there are no spillovers (rows 1 to 3) restricting asset markets does lower $\mathrm{corr}(c_1, c_2)$. Moreover, the larger the level of persistence, the larger the effect. Row 3 is consistent with the large effect of the restrictions found by Baxter and Crucini when they set $\rho_p = 1$ and $\rho_s = 0$. Second, decreasing the level of international spillovers (compare row 1 with 4 and 2 with 5) greatly increases the effect on $\mathrm{corr}(c_1, c_2)$. When $\rho_s \neq 0$ the changes in $\mathrm{corr}(c_1, c_2)$ are not statistically significant. These results are consistent with Baxter and Crucini's wealth effect argument. The larger the spillovers, the more similar are the wealth effects across countries and therefore the closer to the complete markets model we get. Therefore, Kollmann's result is perfectly consistent with Baxter and Crucini's.

In section III we showed that both a reduction in spillovers and an increase in persistence reduce the accuracy level of the solution method. Therefore, it is hazardous to conclude that the restrictions on the asset markets have a large economic effect when there are no spillovers and when there is a lot of persistence because the portion of the change in cross-country consumption correlation due to the approximation error is unknown. This loss of accuracy can certainly explain the large standard deviations on $\mathrm{corr}^I(c_1, c_2)$ in the first three cases.

V. Conclusion

We measured the accuracy of King, Plosser and Rebelo's (1988) solution method when applied to an IRBC model with incomplete markets. We showed that it is necessary for the sum of the levels of persistence and international spillovers to be less than one to obtain accurate results in the complete markets setting and reduce the inaccuracy in a model with incomplete markets. We demonstrated that the use of a unit root process without spillovers greatly reduces the accuracy of the solution method.

We presented measures of the inaccuracy generated by the presence of an endogenous unit root in the (linearized) dynamic system when markets are incomplete and showed that this inaccuracy can be reduced by having a model as symmetric as possible. We also showed that the lower the degree of international spillovers in technology shocks and the higher the level of persistence, the greater the inaccuracy of the solution method when applied to the incomplete asset markets model.

Finally we demonstrated that the effect of restricting asset markets on predicted moments is highly dependent on the levels of persistence and spillovers. This dependence explains why Baxter and Crucini (1995) conclude that the restrictions on asset markets have little impact on the cross-country consumption correlation when using a stationnary process for technology shocks while Kollmann (1996) shows the opposite.

The finding that the economic effect of asset markets restrictions depends on the persistence in the shocks to income was also demonstrated in the asset-pricing literature. For instance, Telmer (1993) who specifies a labour income process with little persistence does not find much effect from asset markets restrictions whereas Constantinides and Duffie (1996) show that in a model where the shocks to income are random walks, an economy with incomplete markets is different from one with complete markets.

As shown by Kollmann (1996) and van Wincoop (1996), restrictions on asset markets actually improve the model's predictions regarding cross-country correlation of output compared to the cross-country correlation of consumption. However, the results depend heavily on the levels of persistence and spillovers assumed or estimated which in turn influence greatly the degree of accuracy of the solution method of King, Plosser and Rebelo (1988). Moreover, since estimates of the level of international spillovers are somewhat imprecise it might be desirable to map this uncertainty into uncertainty about the model's predictions.

APPENDIX A.1 Parametrizations of Previous Models

1.1 Backus, Kehoe and Kydland (1995)

preferences: $\beta = 0.99$, $\mu = 0.34$, and $\sigma = 2.0$;

technology: $\theta = 0.36$ and $\delta = 0.025$;

time-to-build: $J = 4$, $\phi_j = 0.25 \quad \forall j$;

technology shock process:

$$\mathrm{P} = \begin{bmatrix} 0.906 & 0.088 \\ 0.088 & 0.906 \end{bmatrix}, \qquad \Omega = 0.00852^2 \begin{bmatrix} 1 & 0.258 \\ 0.258 & 1 \end{bmatrix}.$$

1.2 Baxter and Crucini (1995)

preferences: $\beta = 0.98$, $\mu = 0.2$ and $\sigma = 2$

technology: $\theta = 0.42$, $\delta = 0.025$ and $\gamma = 1.004$

capital adjustment cost: $1/\psi' = 1$, $-(\psi'/\psi'') \div (x/k) = 15$

the two technology shock processes considered are:

First

$$\mathrm{P} = \begin{bmatrix} 0.906 & 0.088 \\ 0.088 & 0.906 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 1 & 0.258 \\ 0.258 & 1 \end{bmatrix}.$$

Second

$$\mathrm{P} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Omega = \begin{bmatrix} 1 & 0.258 \\ 0.258 & 1 \end{bmatrix}.$$

1.3 Kollmann (1996)

utility function: $U(c, 1-n) = (1/(1-\sigma)\left[\left(c\ (1-n)^{\tilde{\mu}}\right)^{1-\sigma} - 1\right]$

preferences: $\beta = 0.9828$, $\tilde{\mu} = 0.39$ and $\sigma = 2$

technology: $\theta = 0.36$ and $\delta = 0.021$.

capital adjustment cost: The law of motion for capital is

$$k_{it+1} + \phi(k_{it+1}, k_{it}) = (1-\delta)k_{it} + x_{it},$$

where

$$\phi(k_{it+1}, k_{it}) = 0.5 \cdot \phi \cdot \left[k_{it+1} - k_{it}\right]^2 / k_{it}, \quad \phi = 3.$$

technology shock process:

$$\mathrm{P} = \begin{bmatrix} 0.95 & 0 \\ 0 & 0.95 \end{bmatrix}, \qquad \Omega = 0.007^2 \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}.$$

APPENDIX A.2 Equilibrium System of Equations: Incomplete Markets

Agent in country i chooses sequences $\{c_{it}, n_{it}, k_{it+1}, b_{it}, x_{it}\}$ to solve the problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t U(c_{it}, 1 - n_{it}) = E_0 \sum_{t=0}^{\infty} \beta^t \frac{\left[c_{it}^{\mu}(1 - n_{it})^{1-\mu}\right]^{1-\sigma}}{1 - \sigma}$$

subject to

$$c_{it} + x_{it} + P_t^B b_{it+1} = z_{it} k_{it}^{\theta} n_{it}^{1-\theta} + b_{it}$$

$$k_{it+1} = (1 - \delta) k_{it} + \left(\frac{x_{it}}{k_{it}}\right)^{\tau} k_{it}$$

This maximization problem is solved by Lagrange's method. Define the Lagrangean

$$E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \frac{\left[c_{it}^{\mu}(1 - n_{it})^{1-\mu}\right]^{1-\sigma}}{1 - \sigma} + \lambda_{it} \left[z_{it} k_{it}^{\theta} n_{it}^{1-\theta} + b_{it} - c_{it} - P_t^B b_{it+1} - x_{it} \right] + \nu_{it} \left[k_{it+1} - (1 - \delta) k_{it} - \left(\frac{x_{it}}{k_{it}}\right)^{\tau} k_{it} \right] \right\}$$

where λ_{it} and ν_{it} are Lagrange multipliers. Defining $U(c_{it}, 1 - n_{it}) \equiv U_{it}$ and $y_{it} \equiv z_{it} k_{it}^{\theta} n_{it}^{1-\theta}$, the first-order conditions are

(A1) $\quad (c_{it}): \quad \mu(1 - \sigma)\frac{U_{it}}{c_{it}} = \lambda_{it}$

(A2) $\quad (n_{it}): \quad (1 - \mu)(1 - \sigma)\frac{U_{it}}{1 - n_{it}} = \lambda_{it}(1 - \theta)\frac{y_{it}}{n_{it}}$

(A3) $\quad (x_{it}): \quad \lambda_{it} + \tau \nu_{it} \left(\frac{x_{it}}{k_{it}}\right)^{\tau - 1} = 0$

(A4) $\quad (b_{it+1}): \quad \lambda_{it} P_t^B = \beta E_t \lambda_{it+1}$

(A5) $\quad (k_{it+1}): \quad \nu_{it} + \beta E_t \left\{ \lambda_{it+1} \theta \frac{y_{it+1}}{k_{it+1}} - \nu_{it+1} \left[1 - \delta + (1 - \tau) \left(\frac{x_{it+1}}{k_{it+1}}\right)^{\tau} \right] \right\} = 0$

$$(A6) \qquad (\lambda_{it}): \qquad c_{it} + x_{it} + P_t^B b_{it+1} = z_{it} k_{it}^{\theta} n_{it}^{1-\theta} + b_{it}$$

$$(A7) \qquad (\nu_{it}): \qquad k_{it+1} = (1-\delta)k_{it} + \left(\frac{x_{it}}{k_{it}}\right)^{\tau} k_{it}$$

Therefore, the equilibrium system is composed of 15 equations: (A1) to (A7) for $i = 1, 2$ and market clearing condition

$$(A8) \qquad b_{1t} + b_{2t} = 0.$$

Note that equation $(A8)$ in conjunction with the budget contraints imply the world market clearing condition

$$(A9) \qquad c_{1t} + c_{2t} + x_{1t} + x_{2t} = y_{1t} + y_{2t}.$$

Since the numerical solution method prevents us from imposing both budget constraints, we follow Baxter and Crucini (1995) and replace country 1's budget constraint by $(A9)$. The equilibrium system can be simplified by using $(A8)$ to substitute out b_{1t} and $(A4)$ to substitute out P_t^B. Therefore, we are left with an equilibrium system in the endogenous variables $(c_1, c_2, n_1, n_2, x_1, x_2, k_1, k_2, b_2, \lambda_1, \lambda_2, \nu_1, \nu_2)$ composed of equations $(A1)$, $(A2)$, $(A3)$, $(A5)$ and $(A7)$ for both countries, equation $(A9)$, equation $(A6)$ for country 2 and

$$(A10) \qquad \frac{E_t \lambda_{1t+1}}{\lambda_{1t}} = \frac{E_t \lambda_{2t+1}}{\lambda_{2t}}.$$

The system can now be linearized by taking a first-order Taylor series approximation around its steady state. After substituting out $(c_1, c_2, n_1, n_2, x_1, x_2, \lambda_1)$ using the linearized version of $(A1)$, $(A2)$, $(A3)$ and $(A9)$ we obtain the fundamental dynamic system

$$E_t \begin{bmatrix} \hat{k}_{1t+1} \\ \hat{k}_{2t+1} \\ \hat{b}_{2t+1} \\ \hat{\lambda}_{2t+1} \\ \hat{\nu}_{1t+1} \\ \hat{\nu}_{2t+1} \end{bmatrix} = W \begin{bmatrix} \hat{k}_{1t} \\ \hat{k}_{2t} \\ \hat{b}_{2t} \\ \hat{\lambda}_{2t} \\ \hat{\nu}_{1t} \\ \hat{\nu}_{2t} \end{bmatrix} + Q \begin{bmatrix} \hat{z}_{1t} \\ \hat{z}_{2t} \end{bmatrix} + R\ E_t \begin{bmatrix} \hat{z}_{1t+1} \\ \hat{z}_{2t+1} \end{bmatrix}$$

where hatted variables denote percent deviations from steady state. That is, if we let $\bar{z}_i$ be the steady-state value for z_i, then $\hat{z}_{it} = (z_{it} - \bar{z}_i)/\bar{z}_i$. Since asset holdings are assumed to be zero in steady state we define $\hat{b}_{2t} = b_{2t}/\bar{y}_2$.

Matrix W is 6×6 and matrices Q and R are 6×2. $\hat{k}_{1t}$, $\hat{k}_{2t}$ and $\hat{b}_{2t}$ are predetermined at time t (state variables) while $\hat{\lambda}_{2t}$, $\hat{\nu}_{1t}$ and $\hat{\nu}_{2t}$ are not (co-state variables). Matrix W governs the system dynamics. For the system to have a unique solution, W must have as many roots outside the unit circle as there are co-state variables. For the system to be stable, W must have as many roots on or outside the unit circle as there are co-state variables. Therefore we need W to have 3 eigenvalues greater than one (in absolute value) for uniqueness and 3 eigenvalues greater or equal to one (in absolute value) for stability. The roots are 0.8888, 0.9666, 1, 1.0101, 1.0450 and 1.1365. Therefore the system is unstable and has a unique solution given by Blanchard and Kahn (1980).

APPENDIX A.3 Equilibrium System of Equations: Complete Markets

When financial markets are complete we know the competitive equilibrium is Pareto optimal. Therefore, we can conveniently derive the equilibrium system using an equal weight planner problem since both countries are *ex ante* identical. The planner maximizes the sum of expected lifetime utilities subject to the world resource constraint

$$c_{1t} + x_{1t} + c_{2t} + x_{2t} = z_{1t} k_{1t}^{\theta} n_{1t}^{1-\theta} + z_{2t} k_{2t}^{\theta} n_{2t}^{1-\theta}$$

This maximization problem is solved by Lagrange's method. Define the Lagrangean

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \frac{\left[c_{1t}^{\mu}(1-n_{1t})^{1-\mu}\right]^{1-\sigma}}{1-\sigma} + \frac{\left[c_{2t}^{\mu}(1-n_{2t})^{1-\mu}\right]^{1-\sigma}}{1-\sigma} \\ & + \lambda_t \left[z_{1t} k_{1t}^{\theta} n_{1t}^{1-\theta} + z_{2t} k_{2t}^{\theta} n_{2t}^{1-\theta} - c_{1t} - x_{1t} - c_{2t} - x_{2t} \right] \\ & + \nu_{1t} \left[k_{1t+1} - (1-\delta) k_{1t} - \left(\frac{x_{1t}}{k_{1t}} \right)^{\tau} k_{1t} \right] \\ & + \nu_{2t} \left[k_{2t+1} - (1-\delta) k_{2t} - \left(\frac{x_{2t}}{k_{2t}} \right)^{\tau} k_{2t} \right] \Bigg\} \end{aligned}$$

Defining $U(c_{it}, 1-n_{it}) \equiv U_{it}$ and $y_{it} \equiv z_{it} k_{it}^{\theta} n_{it}^{1-\theta}$, the first-order conditions are

(*A*11) $$(c_{it}): \qquad \mu(1-\sigma)\frac{U_{it}}{c_{it}} = \lambda_t, \qquad i = 1, 2$$

(*A*12) $$(n_{it}): \qquad (1-\mu)(1-\sigma)\frac{U_{it}}{1-n_{it}} = \lambda_t (1-\theta)\frac{y_{it}}{n_{it}}, \qquad i = 1, 2$$

(*A*13) $$(x_{it}): \qquad \lambda_t + \tau \nu_{it} \left(\frac{x_{it}}{k_{it}} \right)^{\tau - 1} = 0, \qquad i = 1, 2$$

(*A*14)
$$(k_{it+1}): \qquad \nu_{it} + \beta E_t \left\{ \lambda_{t+1} \theta \frac{y_{it+1}}{k_{it+1}} - \nu_{it+1} \left[1 - \delta + (1-\tau) \left(\frac{x_{it+1}}{k_{it+1}} \right)^{\tau} \right] \right\} = 0, \qquad i = 1, 2$$

$$(A15) \qquad (\lambda_t): \qquad c_{1t} + x_{1t} + c_{2t} + x_{2t} = y_{1t} + y_{2t}$$

$$(A16) \qquad (\nu_{it}): \qquad k_{it+1} = (1-\delta)k_{it} + \left(\frac{x_{it}}{k_{it}}\right)^{\tau} k_{it}, \qquad i = 1,2$$

Therefore, we have an equilibrium system composed of equations $(A11)$ to $(A16)$ in the endogenous variables $(c_1, c_2, n_1, n_2, x_1, x_2, k_1, k_2, \lambda, \nu_1, \nu_2)$. The system can now be linearized by taking a first-order Taylor series approximation around its steady state. After substituting out $(c_1, c_2, n_1, n_2, x_1, x_2, \lambda)$ using the linearized version of $(A11)$, $(A12)$, $(A13)$ and $(A15)$ we obtain the fundamental dynamic system

$$E_t \begin{bmatrix} \hat{k}_{1t+1} \\ \hat{k}_{2t+1} \\ \hat{\nu}_{1t+1} \\ \hat{\nu}_{2t+1} \end{bmatrix} = W \begin{bmatrix} \hat{k}_{1t} \\ \hat{k}_{2t} \\ \hat{\nu}_{1t} \\ \hat{\nu}_{2t} \end{bmatrix} + Q \begin{bmatrix} \hat{z}_{1t} \\ \hat{z}_{2t} \end{bmatrix} + R \; E_t \begin{bmatrix} \hat{z}_{1t+1} \\ \hat{z}_{2t+1} \end{bmatrix}$$

where hatted variables denote percent deviations from steady state.

Matrix W is 4×4 and matrices Q and R are 4×2. $\hat{k}_{1t}$, $\hat{k}_{2t}$ are predetermined at time t (state variables) while $\hat{\nu}_{1t}$ and $\hat{\nu}_{2t}$ are not (co-state variables). Matrix W governs the system dynamics. For the system to have a unique solution, W must have as many roots outside the unit circle as there are co-state variables. For the system to be stable, W must have as many roots on or outside the unit circle as there are co-state variables. Therefore we need W to have 2 eigenvalues greater than one (in absolute value) for uniqueness and 2 eigenvalues greater or equal to one (in absolute value) for stability. The roots are 0.8888, 0.9666, 1.0450, and 1.1365. Therefore the system is stable and has a unique solution given by Blanchard and Kahn (1980).

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Table 1a. Complete Markets Model — One Shock

Case	Persistence	Spillovers	Sum	GAP (%)
1	0.906	0.088	0.994	0.00000
2	0.906	0.094	1.000	0.89194
3	0.950	0.050	1.000	0.89194
4	1.000	0.000	1.000	2.33942

Table 1b. Complete Markets Model — 200 Shocks

Case	Persistence	Spillovers	Sum	GAP (%)
1	0.906	0.088	0.994	0.00000
2	0.906	0.094	1.000	10.14903
3	0.950	0.050	1.000	10.14903
4	1.000	0.000	1.000	22.18271

Notes: GAP is the largest difference, for any variable, between the new steady-state value and the initial one. The seed is fix across all cases in table 1b.

Table 2. Incomplete Asset Markets Model

Case	Description of the Experiment	GAP (%)
1	Good Draw	2.895E-05
2	Bad Draw	1.85568
3	1 Shock	0.03939
4	5 Shocks	0.19696
5	25 Shocks	0.98480
6	125 Shocks	4.92398
7	1 period	4.37e-10
8	50 periods	2.19e-08
9	Symmetric P, Ω and Country Size	0.34381
10	Asymmetric Ω	0.73989
11	Asymmetric P	1.72993
12	Asymmetric Country Size	0.61886

Notes: GAP is the largest difference, for any variable, between the new steady-state value and the initial one. The seed is fix across cases 9 to 12.

Table 3. Effect of Asset Markets Restrictions on $\mathbf{corr(c_1, c_2)}$

Case	ρ_p	ρ_s	$\text{corr}^C(c_1, c_2)$	$\text{corr}^I(c_1, c_2)$
1	0.906	0.000	0.770	0.592
			(0.082)	(0.127)
2	0.950	0.000	0.805	0.527
			(0.074)	(0.143)
3	0.990	0.000	0.863	0.183
			(0.056)	(0.183)
4	0.906	0.088	0.912	0.872
			(0.032)	(0.045)
5	0.950	0.044	0.896	0.804
			(0.040)	(0.070)

Notes: The cross-country consumption correlation is denoted $\text{corr}^C(c_1, c_2)$ in the complete markets model and $\text{corr}^I(c_1, c_2)$ in the incomplete markets model. Moments are calculated using HP filtered percent deviations from steady state. They are averages over 1000 replications, each 100 periods in length. Standard deviations are in parentheses.