Locally Complete Path Independent Choice Functions and Their Lattices

Mark R. Johnson and Richard A. Dean

Addresses:

Mark R. Johnson
A. B. Freeman School of Business
Tulane University
New Orleans, LA 70118

email: mrjohnso@mailhost.tcs.tulane.edu

Richard A. Dean
Department of Mathematics
California Institute of Technology
Pasadena, CA 91109

Mailing address:
Richard Dean
6349 Stone Bridge Rd.
Santa Rosa, CA 95409

email: rxdean@earthlink.net

Abstract

The concept of path independence (PI) was first introduced by Arrow (1963) as a defense of his requirement that collective choices be rationalized by a weak ordering. Plott (1973) highlighted the dynamic aspects of PI implicit in Arrow's initial discussion. Throughout these investigations two questions, both initially raised by Plott, remained unanswered. What are the precise mathematical foundations for path independence? How can PI choice functions be constructed? We give complete answers to both these questions for finite domains and provide necessary conditions for infinite domains. We introduce a lattice associated with each PI function. For finite domains these lattices coincide with locally lower distributive or meet-distributive lattices and uniquely characterize PI functions. We also present an algorithm, effective and exhaustive for finite domains, for the construction of PI choice functions and hence for all finite locally lower distributive lattices. For finite domains, a PI function is rationalizable if and only if the lattice is distributive. The lattices associated with PI functions that satisfy the stronger condition of the weak axiom of revealed preference are chains of Boolean algebras and conversely. Those that satisfy the strong axiom of revealed preference are chains and conversely.

KEY WORDS: Choice functions, Algebraic structure, Lattice, Locally Complete, Locally Distributive, Meet-Distributive, Path Independence, Algorithms, Rationalization.

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§1. INTRODUCTION.

The foundation for economic analysis is the act of choice. In modeling choice, it is standard to impose "consistency" or "path independence" requirements on how the choice made in one situation is related to the choice made in a similar situation. The results presented here identify the mathematical foundations for path independent choice. Specifically, given a few assumptions for arbitrary sized domains, we identify the necessary structure of path independent choice. This structure is that of the class of lattices having the property that each element is the join of a unique minimal set of join irreducible elements. For a set assigned a choice this unique minimal set of join irreducibles is the choice made from the set. For finite domains this coincides with the class lattices called lower locally distributive by Monjardet (1990) or meet-distributive by Edelman (1986).[1] We also provide refinements identifying the necessary structures for each major sub-category of path independence over infinite domains, Path Independent (PI), Rationalizable PI, Weak Axiom of Revealed Preference (WARP) and Strong Axiom of Preference (SAP). On finite domains these results are both necessary and sufficient characterizations for each category of path independent choice functions. Further, we offer an algorithm to construct all path independent choice functions on a finite domain and hence all finite locally lower distributive lattices.[2]

We mention three aspects of the fact that the fundamental structure of PI choice functions is revealed as a lattice. First, lattices exhibit the ordering properties commonly found in economic choice models. Specifically, the join (Plott's operation) and the meet (induced by the requirements of path independence) operations identify respectively the least upper bound and the greatest lower bound for any pair of elements. Second, the quotient property identified in Theorem 1 entails a simplification of the choice process by identifying an interval (defined by a biggest set and a smallest set and every set in between) for which the same choice is always made. Third, a similar simplicity property is present in the link between rationalizable path independent choice functions and distributive lattices. The question spotlighted is, when is the choice set itself a "sufficient statistic" for the choice process from some feasible set. For rational PI choice functions, the choice set is a "sufficient statistic" but for non-rationalizable PI choice functions, the choice set is not "sufficient."

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[1] We thank the referees who pointed out, during the review process for an early version of this paper, the coincidence of choice lattices over finite domains and lower locally distributive lattices.

[2] These results for finite domains were reported in Johnson and Dean (1996). Koshevoy (1998) has obtained independently the characterizations for PI and Rationalizable PI functions over finite domains.

The application of path independence to economics begins with Arrow (1963) who used the concept to defend his requirement that collective choices be rationalized by a weak ordering. Since then a number of advances were made, notably Sen's (1970) decompositions of path independence into alternative variants of "path independence" (e.g., his properties α, β and γ) and Suzumura's (1983) extention of the path independence concept into non-finite domains. Theorem 1 presents still another such variant true for the infinite domains we consider.

During these early investigations two questions addressed here were raised. First, what are the precise mathematical foundations for path independence. Plott (1973) provided an initial step toward answering this question by demonstrating a semigroup property (especially associativity) in path independent choice functions but Johnson (1990) demonstrated that the semigroup property was not sufficient.[3] Second, identification of non-rationalizable path independent choice functions raised the question of how can PI choice functions can be constructed. Although rationalizable choice functions can be constructed from their relations, there has been no easy means of constructing non-rationalizable path independent choice functions.

Algebraic properties have been used by others, including Kelly (1984), Aizerman (1985), Sertel (1988), and Sertel and Van Der Bellen (1979). Aizerman deletes the empty set and adds an identity element to Plott's system. For us, a choice function is a mapping from a portion of the power set of the universal set (including the empty set) into itself. The empty set is the identity element in a subsystem of Plott's original semigroup and so permits construction of the choice lattice. Because of differing treatment of the empty set, our choice lattices cannot be observed in the Aizerman framework.

The main results provided here differ from previous contributions in that path independent choice functions are characterized by purely mathematical properties (e.g., Theorem 2) and by identifying a specific mathematical structure with each class of path independent choice functions. Our structural results are of two types; (1) necessary strictures on sets of arbitrary size and (2) complete characterization of the mathematics on finite sets. For each type, we provide results for the four main classes of choice functions. To obtain these results, we make four assumptions about the domain of the choice function; loosely, these assumptions are a refinement of those Suzumura (1983) adopted in his investigations of path independence on large sets.

[3] Notably, Plott also suggested that the semigroup property he identified might be useful for extending the concept of path independence to non-finite domains. Our results lend support to his conjecture.

Malishevski (1994) studied the algebra considered by Plott and Johnson (1990, 1995) focusing on alternative operations rather than the implied structure. Johnson (1994) identified a link between PI choice functions and lattices and provided characterization results for the lattices associated with WARP and SAP choice functions on finite domains. Johnson and Dean (1996) provided complete characterization of the choice lattices for these four classes on finite domains. Additionally, they provided an algorithm for constructing all finite choice lattices.

Concurrent with these economic developments, the mathematics of lower locally distributive lattices was being developed. See the excellent survey articles by Monjardet (1990) and Edelman (1986). Koshevoy (1998) noted that these lattices may be considered from within the theory of convex geometries. He exploits this connection to establish the representation theorems for finite path independent choice functions and for finite rational choice functions from the perspective of the theory of convex geometries. For finite domains Koshevoy establishes the content of our Theorems 3, 4, 9 and 10.

The present exposition retains the intent of Johnson and Dean (1996) to make these algebraic (lattice) interpretations accessible to those interested in choice theory. For this reason we include proofs of results which could be derived from the convex geometry approach of Koshevoy just so they might be more easily assimilated. But more than that, our attack yields the results on WARP and SAP functions valid for infinite sets V. These results are not included in Koshevoy (1998). Our approach also leads to an algorithm for the construction of all PI choice functions on a finite set.

The algorithm starts with the identity choice function on a finite set V whose choice lattice is the Boolean algebra of all subsets of V. We show that through a sequence of contractions every PI choice function on V is constructed. We identify a link between our contraction operation and the deletable elements identified by Bordalo and Monjardet (1996). These contractions are reversible and therefore an alternative approach to constructing a PI choice function is to start from a total order and complete a sequence of expansions. Since finite choice lattices coincide with locally lower distributive lattices this algorithm gives a method of constructing all finite locally lower distributive lattices.

This paper is structured as follows. Section 2 contains preliminary definitions, notations and a characterization of PI choice functions using the quotient property. Choice lattices are introduced in Section 3 and examples are provided. Notably, while an exhaustive listing of choice lattices on 3 elements is offered, many of the examples provided do not assume finiteness. In Section 4 we give representation theorems for general choice functions. Section 5 details the construction of all PI choice functions on a finite set. In Section 6 we study rational path PI functions. Section 7 applies these results to WARP and SAP choice functions. A summary and discussion is presented in Section 8. Long proofs and technical lemmas are given in the Appendix. Routine proofs have been omitted. All proofs are available from the authors.

§ 2. PRELIMINARY RESULTS .

Following Suzumura (1983) we start from a universal set V of alternatives, and a collection $\mathcal{V}$ of subsets of V on which choices are to be made. We denote by 2^V the Boolean algebra of all subsets of V under set inclusion. Set inclusion is denoted by ($\subseteq$) and set containment by ($\supseteq$). Set union is denoted by ($\cup$) and set intersection by ($\cap$). If $\mathcal{K}$ is a collection of sets, $\bigcup\mathcal{K}$ denotes the set union of these sets; $\bigcap\mathcal{K}$ denotes their set intersection. The empty set is denoted by $\varnothing$. If V $\supseteq$ A $\supseteq$ B we denote by the quotient A/B the set {X: A$\supseteq$X$\supseteq$B}, sometimes called an interval in the Boolean algebra 2^V.

We make a selection of the subsets of V for which a choice is assigned. Let $\mathcal{V}$ denote the subsets of V comprising the domain of a choice function. Adopting Suzumura's (1983) property (a) and an extension of his property (b) we assume :

(a) $\mathcal{V}$ contains the empty set and all finite subsets of V.

(b) If A and B are sets in $\mathcal{V}$ then A$\cup$B and A$\cap$B are in $\mathcal{V}$.

From (a), if V is finite the domain $\mathcal{V}$ of the choice function is 2^V.

Definition 1. A ***choice function*** C on V is a function with domain $\mathcal{V}$ into 2^V satisfying the following properties.

(i) For all subsets A $\in\mathcal{V}$, C(A) $\in\mathcal{V}$ and C(A) $\subseteq$ A.

(ii) C(A) = $\varnothing$ if and only if A = $\varnothing$.

In this paper C always denotes a choice function, $\mathcal{R}$ will denote its range of. We often refer to the inverse sets under a choice function C. Suppose that A $\in\mathcal{V}$. Let

$$\mathrm{arc}(C(A)) = \{X \subseteq V\colon X \in \mathcal{V} \text{ and } C(X) = C(A)\} .$$

When V is infinite we assume two additional closure properties about the choice function C and the domain $\mathcal{V}$ of subsets of V on which C is defined.

(c) If A $\in\mathcal{V}$ then $\bigcup$arc(C(A)) $\in\mathcal{V}$.

(d) If A $\in\mathcal{V}$ and C(A) $\supseteq$ B then B $\in\mathcal{V}$.

In view of properties (i) and (b), if A and B both belong to $\mathcal{V}$ then so do C(A), C(B), A$\cup$B, and C(A$\cup$B), and A$\cap$B and C(A$\cap$B). Note that if V is finite, properties (b) – (d) follow directly from property (a).

Definition 2. A choice function is called ***path independent*** (PI) if for all A, B $\in \mathcal{V}$,

$$C(A\cup B) = C(C(A)\cup C(B))$$

Definition 3. A choice function on a set V with domain $\mathcal{V}$ is called ***locally complete*** if properties (a) - (d) hold with respect to C and $\mathcal{V}$.

Properties (c) and (d) are natural from a choice function perspective. Property (c) states that if a choice C(A) is made from A, i.e., $A\in\mathcal{V}$, then a choice can be made from the union of all the sets in arc(C(A)), i.e., $\bigcup \mathrm{arc}(C(A)) \in \mathcal{V}$. Lemma 4 shows that for PI choice functions this choice must be C(A).

Property (d) implies that if A $\in \mathcal{V}$ and the choice function is an identity on a set A, i.e., C(A) = A then any subset of A is also in $\mathcal{V}$. Lemma 6 says that if C is PI then the choice function is the identity function on any subset of A ,i.e., If C(A) $\supseteq$ B, then C(B) = B.

Lemmas 2, 4 and 6 suggest that while, for PI choice functions, properties (c) and (d) are not restrictive they do have consequences. For example, Plott (1973) showed by mathematical induction that path independence could be extended to finite collections of sets. Our Lemma 5 extends this result to any collection of sets in $\mathcal{V}$ of arbitrary size.[4]

Lemma 1: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$.

For all A $\in\mathcal{V}$, C(A) $\in \mathcal{V}$ and C(C(A)) = C(A).

We say that C is an ***idempotent*** function.

Lemma 2: ***(Aizerman's Axiom)***: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$.

If A, B $\in\mathcal{V}$ and if A $\supseteq$ B $\supseteq$ C(A) then B$\in$ arc(C(A)).

Lemma 3: (***Chernoff's Axiom)*****:** If C is a locally complete PI choice function on V with respect to $\mathcal{V}$ then C satisfies Chernoff's Axiom:

If A , B $\in \mathcal{V}$ and A$\subseteq$B then C(A) $\supseteq$C(B)$\cap$A.

The next lemma describes inverse sets under a PI choice function.

Lemma 4: (***Quotient Property)*****:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Let A $\in\mathcal{R}$. Let A^ = $\bigcup$ arc(A). Then

$$\mathrm{arc}(A) = \{X \in\mathcal{V}: A\hat{} \supseteq X \supseteq A\}.$$

[4] One referee remarked that Lemma 5 is a kind of "continuity" for choice lattices. As Example 12 shows, it is not sufficiently strong to require infinite choice lattices to be compactly generated in the sense of Dilworth (1960).

Note that if V is finite then arc(A) is the entire quotient A^/A. This feature of PI choice functions is illustrated in the Appendix (Figures 15a and 15b) using Example 7 (Figure 7) of the next section. Further, if V is finite, Lemma 4 shows that the map A→A^ is a closure operator on V. This observation is fundamental in Koshevoy (1998) as it presents the choice lattice as the lattice of sets closed under this operator. However, as our example 8 shows, if A is infinite A^ need not be defined.

Lemma 4 along with Chernoff's Axiom leads us to the following equivalence result which is important because we find many instances when the proofs are made transparent by this alternate characterization of PI functions.

THEOREM 1: Let C be a locally complete choice function on V with respect to $\mathcal{V}$. The function C is path independent if and only if it satisfies Chernoff's Axiom and the Quotient property.

LEMMA 5: *(Extension of PI)*: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Let $\mathcal{K}$ be a collection of subsets K, $K\in\mathcal{V}$, such that $\bigcup\mathcal{K}\in\mathcal{V}$ and $\bigcup\{C(K): K \in \mathcal{K}\} \in\mathcal{V}$ then

$$C(\bigcup\mathcal{K}) = C(\bigcup\{C(K) : K\in\mathcal{K}\}).$$

Application of Chernov's Axiom leads to Lemma 6.

LEMMA 6: *(Hereditary Identity)*: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. If $A \in\mathcal{V}$ and $C(A)\supseteq B$ then $C(B) = B$.

Plott (1973) identified the binary product operation ($\bullet$) defined by

$$A\bullet B = C(C(A)\cup C(B)).$$

The connection between path independence and this operation was one of the significant contributions of Plott who in his 1973 paper proved that his operation ($\bullet$) was associative. We restate this result as Lemma 7 without further proof. Johnson (1995) showed for finite V that if C is PI then $<\mathcal{R}, \bullet>$ is an idempotent commutative semigroup with identity ($\emptyset$) and null element (C(V)) if $V \in \mathcal{V}$ and, hence, a lattice. This remains true for V infinite. Because of property (b), $\mathcal{R}$ is closed under the Plott product and if $A \in \mathcal{R}$ then $A\bullet A = A$. Algebraically speaking, A is an idempotent under the operation ($\bullet$). Note that if A and B are sets in $\mathcal{R}$ then

$$A\bullet B = C(A\cup B).$$

Lemma 7: (***Associativity)*:** Let C be a PI choice function on V with domain $\mathcal{V}$ satisfying properties (a) – (d). For all sets X,Y and Z $\in \mathcal{V}$, $(X\bullet Y)\bullet Z = X\bullet(Y\bullet Z)$.

§3. CHOICE LATTICES.

We have noted that the range of the choice function forms a commutative idempotent semigroup under the Plott product ($\bullet$). The crucial part of this structure is the associativity of the Plott product. In this section we show how the Plott product and its associativity lead to the construction of a lattice, which we call the choice lattice of the path independent choice function.

It is known (Clifford and Preston (1961)) that any idempotent commutative semigroup with an operation ($\circ$) becomes an upper semilattice in which the join of two elements A, B is $A\circ B$. In Lemmas 8 and 9 (see Appendix) we specialize this result to the semigroup $<\mathcal{R}, \bullet>$. These lemmas determine the least upper bound, $A\vee B$, of two elements in the choice lattice and the greatest lower bound, $A\wedge B$, of two elements in the choice lattice. One may say that the natural partial order of the semilattice determines the join of two elements, while the implications of PI determine the meets.

Our notation differentiates between set operations and lattice operations. The symbol ($\subseteq$) denotes set inclusion in 2^V. The partial ordering of the choice lattice is denoted by ($\leq$). Set union and intersection in 2^V are denoted by ($\cup$) and ($\cap$) respectively, lattice join and meet are denoted by ($\vee$) and ($\wedge$) respectively. If C is a locally complete choice function with respect to $\mathcal{V}$ we shall always denote $\bigcup \mathrm{arc}(C(A))$ as A^. The main theorem of this section is the following:

Theorem 2: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$ The range $\mathcal{R}$ forms a lattice in which:

(i) The lattice join of A and B is $A\vee B = A\bullet B = C(A\cup B)$.

(ii) The lattice meet of A and B is $A\wedge B = C(A\text{^} \cap B\text{^})$.

We refer to this lattice as the ***choice lattice*** for C.

In general a choice lattice is not a complete lattice (see example 8). Local completeness does however give the partial results of Lemma 10.

Lemma 10: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Let $\mathcal{K}$ be any set of elements in $\mathcal{R}$. If $\bigcup\mathcal{K}$ belongs to $\mathcal{V}$ then $C(\bigcup\mathcal{K})$ is the least upper bound for $\mathcal{K}$ in the choice lattice.

If A is a set of elements in a lattice it is convenient to write $\bigvee A$ for the join of all the elements in A when this join exists. If A is finite then this join always exists. Dually we write $\bigwedge A$ for the meet of all the elements of A when it exists, as it does if A is finite.

Lemma 10 has two immediate corollaries.

Corollary 1: If $A \in \mathcal{V}$ then C(A), as an element of the choice lattice, is the least upper bound of $\{\{x\} : x \in A\}$.

$$C(A) = \bigvee A.$$

Corollary 2: If $\mathcal{V} = 2^V$ then the choice lattice is a complete lattice.

For elements of the lattice, A and B, it is not the case that $A \geq B$ in the choice lattice entails $A \supseteq B$ as sets in the domain (e.g., Example 1 where $\{1,3\} \geq \{1,2\}$). However the following lemma holds.

Lemma 11: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$.

$A \geq B$ holds in the choice lattice if and only if $A\hat{} \supseteq B\hat{}$ holds in 2^V.

Corollary 3: If C is a locally complete rational PI choice function on V with respect to $\mathcal{V}$ and A and B are elements $\mathcal{R}$ then $(A \wedge B)\hat{} = A\hat{} \cap B\hat{}$.

In view of Corollary 1, the choice lattice contains the representation information (much like the binary relation for rational choice functions), it is often convenient to make a diagram of a finite lattice. The following convention is in common use. First, it is helpful to know when an element is covered by another. If x and y are distinct elements we say x is ***covered*** by y (or y covers x) if $x \leq y$ and there is no element "between" x and y, that is, if $x \leq z \leq y$ then either x = z or z = y. Now to make the diagram, use a circle for each element, placing the circle for an element x below the circle for an element y on the page if x is covered by y and draw a line from x to y.

Example 1: Let V = {1,2,3} Define a choice function as follows. Let

C{V) = C({1,3}) = {1,3} ; otherwise let C(A) = A.

Thus $\mathcal{R}$ consists of the subsets with fewer than three elements.

Since Arc({1,3}) = {1,2,3}/{1,3} and otherwise arc(A) = A/A, the Quotient Property holds. Chernoff's Axiom also can be easily verified and so C is PI.

From Lemma 8 we know that C(V)= {1,3} is the top element and that $\emptyset$ is the bottom element of the lattice. For any two element subset, {x,y}, we have

$\{x,y\} = \{x\} \vee \{y\}$. With this knowledge we may make a lattice diagram.

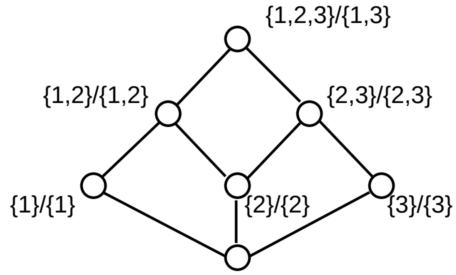

FIGURE 1: Choice lattice for Example 1.

EXAMPLES 2-7: Figures 2 – 7 show choice lattices of PI functions defined on the same domain as in Example 1. Each lattice is the choice lattice of a specific choice function whose definition can be read from the labels of the elements. Each element A of the lattice is labeled with the element of $\mathcal{R}$ that it represents. The quotient A^/A that it represents is given in parentheses unless A^ = A. The almost redundant braces and commas have been deleted to simplify the diagram. Thus, in Figure 7, the element labeled "1 (123/1)" conveys the information that the choice function giving rise to this lattice maps the quotient {1,2,3}/{1} to {1}. All of the Figure 7 quotients are depicted graphically in Figures 15a and 15b presented at the beginning of the Appendix. It will follow from Theorem 7 that every choice lattice of a PI choice function on the universal set {1,2,3} is isomorphic to one of these six lattices.

Extension of these principles to other finite sets is straightforward. For larger sets, see examples 8 through 13. Examples 8, 9, 10 ,12 and 13 have infinite ascending chains. Examples 11 and 13 have infinite descending chains. In examples 8 through 12 the underlying set V is the infinite set of positive integers.

EXAMPLE 8: Let $\mathcal{V}$ be the set of finite subsets of V. For a finite subset A, C(A) = the greatest integer in A, or $\varnothing$ if A = $\varnothing$. Properties (a) – (d) are easily verified. For an integer k, arc({k}) = {1, . . . , k}/{k}. The choice lattice is an infinite ascending chain without a maximal element. Note that the universal set V is not a member of $\mathcal{V}$.

$$\varnothing < \{1\} < \{2\} < \ldots$$

EXAMPLE 9: As in Example 8, let $\mathcal{V}$ be the collection of finite subsets of the positive integers V. For A $\in \mathcal{V}$ let C(A) = {minA, maxA} if A $\neq \varnothing$. (One may think of this as a consumer selecting coffee with alternatives of cream and sugar. Either the consumer prefers coffee black (espresso) or with the maximum of cream and sugar

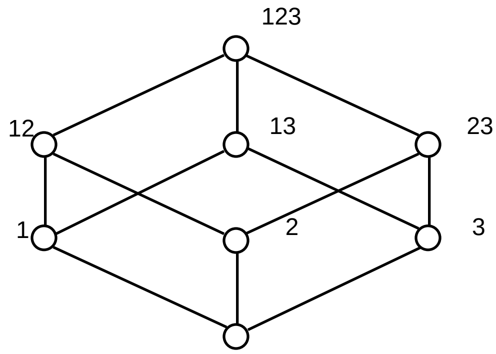

Figure 2

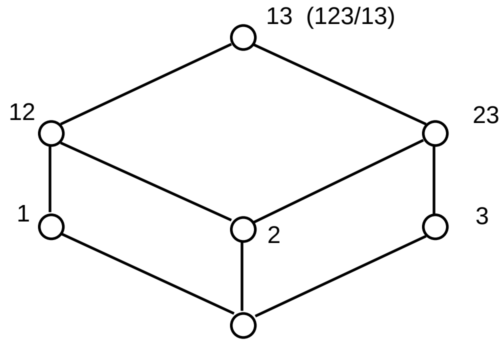

Figure 3

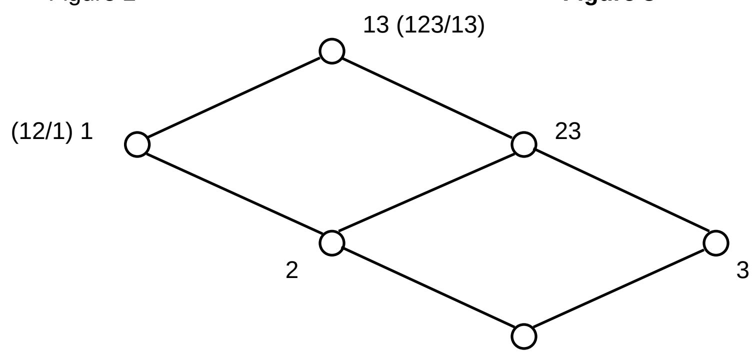

Figure 4

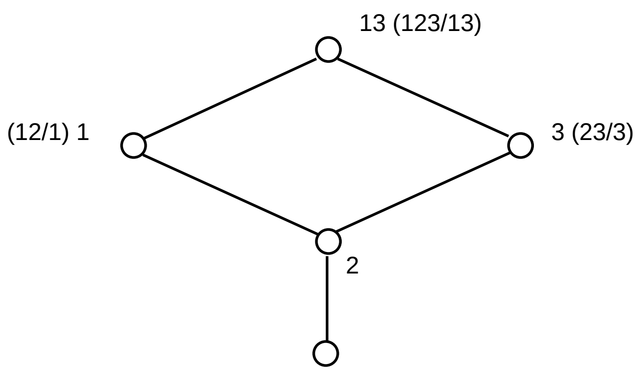

Figure 5

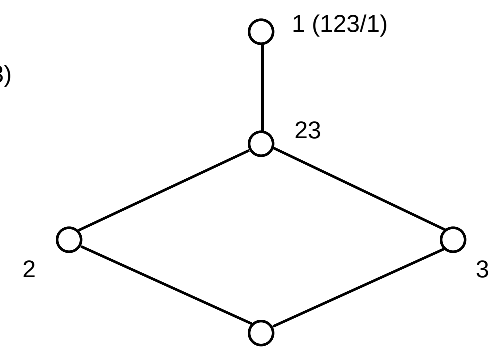

Figure 6

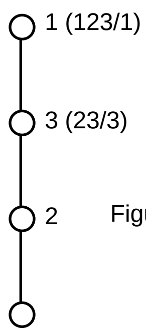

Figure 7

available(latte)! A straight-forward verification shows that properties (a) -(d) hold and that arc({r,s}) = [r,s]/{r,s} where [r,s] = {t: r≤t≤s}.

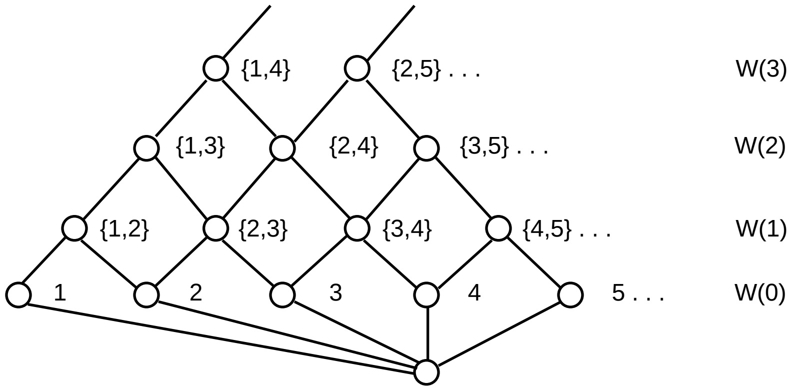

FIGURE 8

As shown in Figure 8, the lattice consists of rows W(k), for k = 0,1,..., whose elements are {a, a+k: a = 1, 2, 3, ...}. Thus W(0) consists of the join irreducibles {a}. There are many sublattices isomorphic to the non-distributive lattice of Figure 1. If r < s <t then {r, s,t} generate such a lattice.

EXAMPLE 10: The set $\mathcal{V}$ is the set of all finite subsets of V together with all subsets of V containing the integer 1. Thus A∈$\mathcal{V}$ if and only if 1∈ A or A is finite.

Verification of property (a) is immediate.

If A, B ∈$\mathcal{V}$ then A∪B ∈$\mathcal{V}$ if both A and B are finite or if 1 is contained in either A or B. A∩B ∈$\mathcal{V}$ if one is finite or if both contain 1. So Property (b) holds.

For A ∈ $\mathcal{V}$ define

C(A)= ∅ if A = ∅,

C(A) = 1 if 1 ∈ A, and

C(A) equals the greatest integer in A if 1 ∉ A.

Compute that arc({1}) = V/{1} while arc({k}) = {2, . . . ,k}/{k} if k ≥ 2. Property (c) holds since we have determined arc(C(A)) for all A ∈$\mathcal{V}$ and we see that $\bigcup$arc(C(A)) is in $\mathcal{V}$. Property (d) follows from similar observations. The choice lattice is an infinite ascending chain,

$$\varnothing < \{2\} < \{3\} < . \;. \;. < \{1\}$$

with top element {1}.

EXAMPLE 11: The set $\mathcal{V}$ is now the set of all subsets of V. For a non-empty subset

$A \subseteq V$, C(A) = the least integer in A. Again it is easy to see that all properties (a) – (d) are satisfied for $\mathcal{V}$. For an integer k, arc({k}) = {k, k+1,...}/{k}.
The choice lattice is an infinite descending chain:

$$\{1\} > \{2\} > \ldots > \{k\} > \{k+1\} > \ldots > \emptyset.$$

The next example is a generalization of Example 1. In this example the roles played by the elements 2 and 3 in Example 1 have been interchanged to make the definitions of the choice function in Example 12 seem more natural.

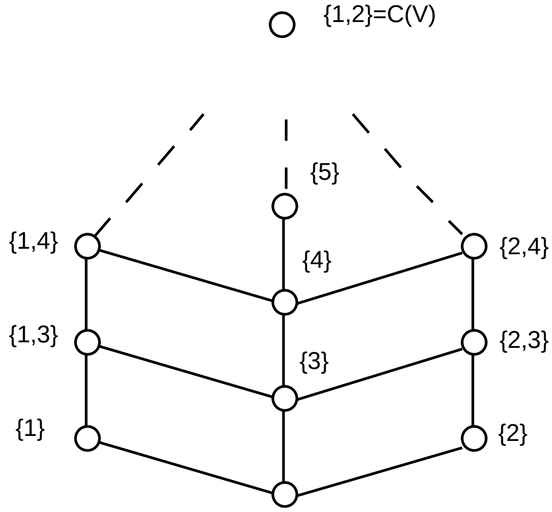

FIGURE 9: A schematic diagram of the choice lattice of Example 12

EXAMPLE 12: The class of sets $\mathcal{V}$ consists of all finite sets and any (infinite) set containing {1,2}. The verification that properties (a) and (b) hold is similar to the preceding example. The choice function is defined as follows:

If $A \supseteq \{1,2\}$ define C(A) = {1,2}

If $A \not\supseteq \{1,2\}$, define C(A) = $(\{1,2\} \cap A) \cup \{$largest integer in A$\}$

Note that the range of C is either the empty set, a singleton, or a doubleton. We compute the inverse sets:

arc({k}) = {k} if k = 1 or 2 and = {3, …k}/{k} if $k \geq 3$.

arc({a,k}) = {a, 3,…, k}/{a,k} for a = 1 or 2 and $k \geq 3$

arc({1,2}) = V/{1,2].

Now the arguments that properties (c) and (d) hold are similar to those for Example 10. The lattice is depicted in Figure 9.

We observe that the elements {1} (and {2}) are not compact in the sense of Dilworth (1960) because $\{1\} < \{1,2\} = \bigvee\{\{k : k \geq 3\}$ but {1} is not contained in the join of any finite subset of $\{\{k : k \geq 3\}$. Thus this lattice is not compactly generated.

EXAMPLE 13: This is the continuous analogue of Example 9. Let V be the set of real numbers [0,1] = {r: $0 \le r \le 1$}. Let $\mathcal{V}$ be the set of closed subsets of V. If A is a non-empty closed set let C(A) = {minA, maxA}. Another straight-forward verification shows that properties (a) - (d) hold. Again arc(A) = [minA,maxA]/{minA, maxA}. The lattice consists of rows indexed W(k) where now k is any real number in [0,1]. Hence there is a continuum of rows. Also, each row is a continuum consisting of {{a, a+k}: a,k $\in$ [0,1])}. As in Example 9 there is a sublattice isomorphic to the lattice of Example 1.

§4. REPRESENTATIONS.

In this section we show that finite choice lattices coincide with lower locally distributive lattices. But more generally even when V is infinite (and the choice function is a locally complete PI choice function on V with respect to $\mathcal{V}$) we show that each element of a choice lattice can be represented in a unique way as the join of join irreducible elements.[5] For finite lattices this is one of the equivalent characterizations of locally lower distributive lattices (see Monjardet (1990) or Edelman (1986)).

An element x of a lattice is called ***join irreducible*** if x = a$\vee$b implies x = a or x = b. It is traditional not to call the bottom element (if it exists) of a lattice a join irreducible element. Dually, an element y of a lattice is called ***meet*** irreducible if y is not the top element of the lattice and y = a$\wedge$b implies y = a or y=b.

In a finite lattice join irreducible elements always exist and every element is the join of the set of join irreducibles below it in the lattice. Infinite lattices may not have that property. The join irreducible elements of a choice lattice are easily identified.

LEMMA 12: *(Identification of join irreducibles)*: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. The join irreducibles of its choice lattice L are the singleton sets {x} for all x$\in$V.

In connection with Corollary 1, this lemma shows that in a choice lattice every element is the join of the set of join irreducibles below it in the choice lattice. This holds true whether or not V is finite. As an example of this lemma, consider again the choice lattice of Figure 1. There the top element {1,3} is the join of the join irreducibles {1}, {2}, and {3}. However in that join, {2} is redundant; {1,3} = {1} $\vee${2}$\vee${3} = {1}$\vee${3}. In the next lemma we show that redundant elements can always be deleted and that when all redundancy is gone, the representation is unique.

[5] The situation is similar to the way an integer can be represented as the product of primes. For lattices we may think of the "primes" as being the join irreducible elements.

The representation of an element of the lattice as the join of irreducibles is given in the next lemma. The terms "minimal" and "unique" are descriptive, standard terms in lattice theory and are made clear in the proof of the theorem.

Theorem 3: ***(Representation of elements in the choice lattice)*****:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Every element in the choice lattice can be uniquely expressed as the join of a minimal set of join-irreducibles.

Corollary 4. Every choice lattice arising from a PI choice function on a finite domain is locally lower distributive.

We conclude this section with a representation theorem (Theorem 4) which, in conjunction with Theorem 3, characterizes the class of finite choice lattices. For this result we need another lemma which strengthens Corollary 1. This lemma is valid for infinite sets V.

Lemma 13: (***Identification of A^ from the choice lattice***)**:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. For any element $A \in \mathcal{R}$,

$$A\hat{} = \{x \in V\text{: } A \geq \{x\}\}$$

In lattice terms A^ is the set of join irreducibles below A in the choice lattice. One payoff of this result is that the quotients of the choice function and therefore the choice function itself may be reconstructed from the choice lattice. A corollary is that two choice functions with isomorphic choice lattices are isomorphic.

The converse of Theorem 3 is the following strong representation theorem. We can prove it only when V is finite. It characterizes those finite lattices which can arise as choice lattices. For a constructive proof of this result see Johnson and Dean (1996). For a derivation from the perspective of convex geometry, see Koshavoy (1998).

Theorem 4: Every finite lattice in which every element has a unique representation as the join of an irredundant set of join irreducibles is the choice lattice for a PI choice function.

§5. CONTRACTIONS AND EXPANSIONS.

In this section we consider only finite sets. We establish two important ways to modify a PI function. Theorem 5 shows how certain elements may be deleted from the lattice and Theorem 6 shows how certain elements may be inserted in the lattice and so modify the choice function but still keeping its path independence and the same set of join irreducible elements. Indeed, each process can be used to undo the other.

We show that any PI function on a domain V consisting of n elements can be constructed by a sequence of contractions (Theorem 5) beginning with the Boolean algebra, 2^V, or alternatively from a sequence of expansions (Theorem 6) beginning with the total ordering (chain) of n elements.

We call attention to two technical lemmas given in the Appendix. Lemma 14 describes coverings in a choice lattice and is needed for the proof of Theorem 5. Lemma 15 shows that the contraction described in Theorem 5 is equivalent to one of the deletions described by Bordalo and Monjardet (1996) if V is finite.

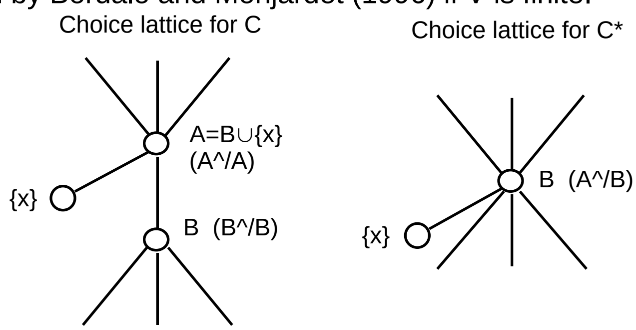

FIGURE 10: A comparison of the choice lattices for the choice functions C and C*. In C the covering from A to B has been contracted to form C*.

THEOREM 5: (***Contraction of a quotient***)**:** Let C be a PI choice function on a finite set V. Let B be a meet irreducible element in the choice lattice that is not equal to C(V) or to $\varnothing$. Let A be the unique element covering B. Suppose that A = B$\cup${x}. Let the function C* defined on $\mathcal{V} = 2^V$ as:

$$C^*(S) = C(S) \text{ if } C(S) \neq A,$$

$$C^*(S) = B \quad \text{ if } C(S) = A.$$

The function C* is a path independent choice function with respect to $\mathcal{V}$.

We say that C* is obtained from C by ***contracting*** the quotient A/B. Bordalo and Monjardet (1996) say that the element B is deletable, that the choice lattice for C* is obtained from the choice lattice for C by deleting the element B. It is important to note that in the choice lattice for C* the representation of the element A or B, whichever it is called in the modified lattice, as the join of a unique irredundant set of join irreducibles is still the set B. In symbols, B = $\bigvee$A in the choice lattice for C*. These relationships are shown schematically in Figure 10. This also shows that the set of join irreducibles in the lattice for C* is the same as that for C. This fact may also be observed from the construction of C*.

Examples. (Construction of Figures 2 – 7.)

Figure 2 shows the choice function for the identity (complete indifference) choice function on {1,2,3}. It is the Boolean algebra of all eight subsets of {1,2,3} under set inclusion. There are three meet irreducible elements in this lattice, each satisfying the hypothesis of Theorem 5. The results of contracting any one of these are isomorphic. If we choose to contract (123/13) we get the choice lattice of Figure 3.

In Figure 3 there are four meet irreducible elements. The quotients 13/12 and 13/23 do not satisfy the hypothesis of Theorem 5. However both the quotient 12/1 and 23/3 do. Again these alternatives are isomorphic. We have chosen to collapse 12/1. We get the choice lattice in Figure 4.

In Figure 4 there are three meet irreducible elements. The quotient 13/23 does not satisfy the hypothesis of Theorem 5. However both 13/1 and 23/3 do satisfy the hypothesis. The results of collapsing these quotients are not isomorphic. Collapsing quotient 23/3 leads to the choice lattice in Figure 5. Collapsing quotient 13/1 leads to the choice lattice in Figure 6. Note that the collapsing creates the amalgamation of the previous quotients 123/13 and 12/1 into the quotient 123/1 of Figure 6.

In Figure 5 there are two meet irreducible quotients both of which satisfy the hypothesis of Theorem 5. Collapsing either leads to a choice lattice isomorphic to the one in Figure 7. Collapsing 13/1 leads to Figure 7.

In Figure 6 there are three meet irreducible elements however 1/23 does not fulfill the requirements of Theorem 5. Either 23/2 or 23/3 do and the result of either collapsing is the four element chain of Figure 7. Collapsing 23/2 leads to the choice lattice of Figure 7.

Since each elements of the choice lattice of Figure 7 is represented by a singleton from V and hence is a join irreducible, no quotient will satisfy the hypothesis of Theorem 5 and the processing of generating choice lattices on {1,2,3} ends.

In the next lemma we show that we can expand the quotients corresponding to certain elements. You may find the schematic diagrams in Figure 11 depicting the choice lattices for C, *C and 2^V helpful. We suppose that C is choice function on a finite set V with range $\mathcal{R}$. Let B $\in \mathcal{R}$.

Under special conditions on B, we can split off an element x from arc(B) = B^/B to form a quotient B^/A where A = B$\cup${x} . Under the expanded choice function *C, arc(A) = B^/A and arc(B) = B^c/B where B^c is the relative complement of A in B^/B. The lattice for *C can be obtained by inserting a new element, B$\cup${x} in the lattice and adding the appropriate lines to show the new containment.

Finally we show that performing the contraction of Theorem 5 on *C returns the function C.

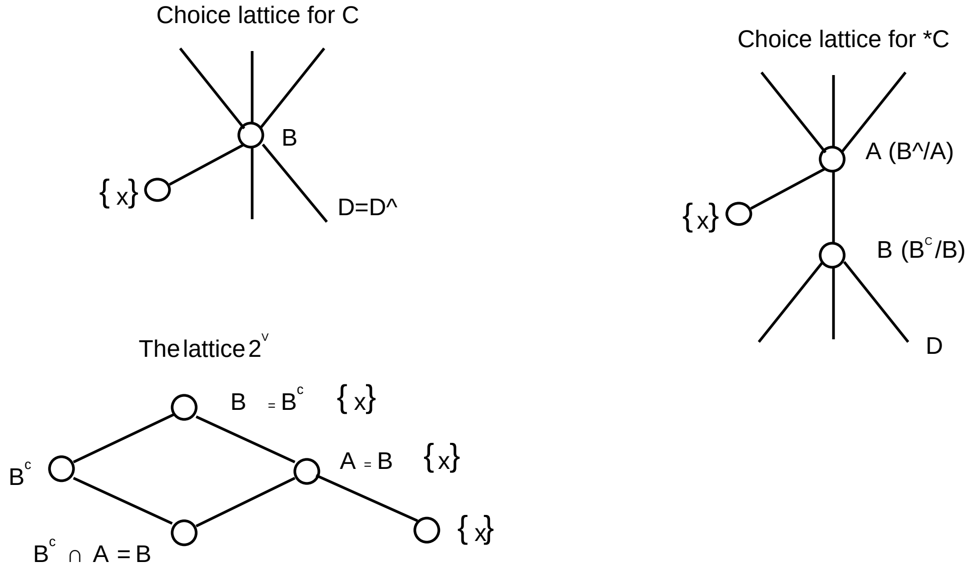

FIGURE 11: Expansion of the choice lattice for C to the choice lattice for *C.

THEOREM 6: (***Expansion of a quotient***)**:** Let C be a PI choice function on a finite set V. Let B belong to $\mathcal{R}$ be such that arc(B) =B^/B properly contains B. Choose $x \in$B^ satisfying the following two conditions: (1) $x \notin B$ and

(2) For all sets E such that B^$\supseteq$ E $\not\supseteq$ B$\cup$\{x\}, if $x \in E$ then $x \in C(E)$.

Let A = B$\cup$\{x\} and let B^c be the relative complement of A in B^/B.

Define a function *C as follows:

*C(D) = A if D $\in$ B^/A

*C(D) = C(D) otherwise. (Note that *C(B) = B.)

(i) The function *C is a path independent choice function.

(ii) B is meet irreducible under *C and is covered by A in the choice lattice for *C.

(iii) If a contraction is performed on *C using B and A as in Theorem 5, then the contracted function (*C)* = C.

(iv) If C is a PI choice function and C* is the result of a contraction of the covering A = B$\cup$\{x\}/B then C* may be expanded by \{x\} and *(C*) = C.

<u>Remark</u>. Under the notation of Theorem 6, if B is chosen to be a minimal element in the choice lattice for C, with respect to the condition B^$\neq$ B, then any element $x \in$ B^, $x \notin B$ satisfies condition (2) of Theorem 6.

THEOREM 7: The choice lattice and hence every PI choice function on a finite set V can be constructed by a sequence of contractions beginning with the identity choice function. Alternatively, every PI choice function can be constructed by a sequence of expansions beginning with a choice function whose choice lattice is a chain.

Here is a proof schema. Note that neither a contraction nor an expansion changes the set of join irreducibles. Now consider a choice function C and its choice lattice L. By Theorem 6 and the remark following it, C can be expanded to a choice function *C whose lattice contains one more element. By further sequence of expansions we can continue until reaching the identity function whose choice lattice is the Boolean algebra. This sequence will be finite if V is finite. Then by Theorem 6 (iii) that sequence can be reversed by a sequence of contractions described in Theorem 5 to reach C. Thus the process of contraction can produce any PI function. Alternatively from C we may carry out a sequence of contractions[6] until the choice lattice becomes a chain. Now by Theorem 6 (iv) that sequence may be reversed by a sequence of expansions.

An algorithm can also be provided for the construction of all PI choice functions on a finite set V. The algorithm proceeds by describing a definitive process, beginning with the identity choice function, whereby all contractions of a given finite choice lattice are constructed; a list of these lattices is built and then in an exhaustive routine, the process repeats. Since the number of elements in a choice lattice is decreased by 1 in the contraction process, the algorithm ends with all possible choice lattices on V. There will be many isomorphic lattices generated in this way but we know of no way to eliminate all of these in advance. When carried out on a set of four elements this algorithm constructs 35 non-isomorphic choice lattices.

§6. RATIONALIZED CHOICE FUNCTIONS AND DISTRIBUTIVITY.

In this section we consider choice functions that are locally complete with respect to a domain $\mathcal{V}$ and can be rationalized. We show that these choice lattices satisfy a strong lattice identity, the distributive law.

A lattice is said to be ***distributive*** if, for all elements a,b and c in the lattice

$$a\wedge(b\vee c) = (a\wedge b)\vee(a\wedge c).$$

[6] Unless the lattice is a chain it follows from Lemma 15 (see the Appendix) that any meet irreducible element that is minimal in the partially ordered set of meet irreducible elements together with its unique cover, satisfies the hypotheses of Theorem 5.

The lattices of Figures 2, 4,5,6, 7, 8,10 and 11 are distributive. By contrast the choice lattice of Figure 1 (equivalently Figure 3) is not distributive because

$$\{2\} = \{2\}\wedge\{1,3\} = \{2\}\wedge\{\{1\}\vee\{3\}\}$$

but $$(\{2\}\wedge\{1\})\vee(\{2\}\wedge\{3\}) = \varnothing\vee\varnothing=\varnothing.$$

The value $\varnothing$ is an artifact of performing the calculation on {1,2,3}. Examples of non-distributivity not involving the empty set exist if V has at least 4 elements. More important is why the result obtains. It occurs because $\{2\}\wedge\{\{1\}\vee\{3\}\}=C(\{2\}\hat{}\cap\{\{1\}\vee\{3\}\}\hat{})$ and the set {1,3} could have been chosen from ***either*** {1,3} or {1,2,3} while the sets {1} and {3] could have been chosen ***only*** from themselves. Because this choice lattice is not distributive and because the feasible set undergoes both expansion and contraction, the choice set is not a sufficient statistic for the prior choice acts. This example shows that not all choice lattices are distributive. It is known (e.g. Edelman 1986) that every locally lower distributive lattice that is not distributive has a sublattice isomorphic to the lattice of Figure 1.

Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. C is said to be ***rational*** if there is a relation R defined on V such that whenever $A\in\mathcal{V}$ then C(A) = {x : xRa for all $a \in A$}. In this case we also say the choice function C is or has been ***rationalized*** by R.

Note that nothing is required for sets not in the collection $\mathcal{V}$. In particular, $B\subseteq V$ then {x: xRb for all $b \in B$} may be empty. And even if it is not, B need not belong to the collection $\mathcal{V}$.

Plott (1973) has shown that rationalizability and path independence are independent of each other. The first result of this section is that rational locally complete PI choice functions always yield distributive choice lattices (Theorem 9). For finite lattices the converse is true: every finite distributive lattice arises as the choice lattice of rational PI choice function (Theorem 10). In other words, rational PI choice functions over finite domains can be completely characterized as the class of finite distributive lattices. This result has also been obtained by Koshevoy (1998).

First two remarks. Suppose that C is a choice function rationalized by the relation R. The relation R is ***reflexive*** (vRv for all $v \in V$) because C({v}) = {v}. It is also worth remarking at this point that if C is rational then the relation is determined by the choice function on the two element subsets. If C(x,y) = {x,y} then xRy and yRx; if C(x,y) = {x} then xRy and $y\neg Rx$. (The symbol $y\neg Rx$ means that y is not related to x under R.)

Our first result is of some interest by itself.

Lemma 16: Let C be a locally complete choice function on V with respect to $\mathcal{V}$. If C is a choice function rationalized by R then C satisfies Chernoff's Axiom.

The following theorem is a consequence of this lemma and Theorem 1.

Theorem 8: (***Criterion for a Rational Choice Function to be PI)*****:** A locally complete rational choice function on V with respect to $\mathcal{V}$ is path independent if and only if it satisfies the quotient property.

Lemma 18 is the clue to our proof of distributivity. It shows that in the choice lattice of a PI rational function lattice meets and joins are set intersections and unions of the sets {A^: A $\in \mathcal{V}$}, the tops of the quotients that map to the range of C. Its proof requires the technical result of Lemma 17 given in the Appendix. Theorems 9 and 10 now follow directly.

Lemma 18: If C is a locally complete rational PI choice function on V with respect to $\mathcal{V}$ and A and B are elements $\mathcal{R}$ then (i) $(A \vee B)$^ = A^$\cup$B^ and (ii) $(A \wedge B)$^=A^$\cap$B^.

Theorem 9: The choice lattice of a locally complete rational PI choice function on V with respect to $\mathcal{V}$ is distributive.

Theorem 10: (***Representation of Finite Distributive Lattices as Rational Choice Functions*****)**: Let D be a finite distributive lattice. There is a relation R and a PI choice function C rationalized by R such that its choice lattice is D.

Proof: See Johnson and Dean (1996) or Koshevoy (1998)

These theorems show that the lattices of Figures 2,4,5,6 and 7 are choice lattices of rational PI choice functions. The preponderance of these functions among the PI functions on {1,2,3} is a consequence of the smallness of three, the number of elements in V. If V has four elements, there are, up to isomorphism, 35 choice lattices 16 of which represent rational choice functions.[7] In general there are far more PI choice functions that cannot be rationalized than there are rational choice functions.

7 From Theorem 10 the number of non isomorphic rational path independent choice functions is the number of distributive lattices with n join irreducibles. In turn this is the number of distinct partially ordered sets of n elements. (See for example Birkhoff (1973)). Asymptotic bounds for the number of these sets are given in Kleitman and Rothschild (1970, 1975).

§7. CHOICE LATTICES FOR WARP AND SAP CHOICE FUNCTIONS.

In this section we investigate the choice lattices of choice functions satisfying WARP and SAP.

Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. C is said to satisfy the ***Weak Axiom of Revealed Preference*** (WARP) (Arrow, 1959) whenever the following condition holds.

$$\text{For all } A, B \in \mathcal{V} \text{ if } A \subseteq B \text{ then } C(A) = C(B) \cap A \text{ or } C(B) \cap A = \emptyset.$$

C is said to satisfy the ***Strong Axiom of Preference*** (SAP) if C satisfies WARP and in addition if, for all pairs $x,y \in V$, $C(\{x,y\}) = \{x\}$ or $\{y\}$.

Plott (1973) has shown that WARP implies PI but not conversely.

Theorems 11 and 12 show that the choice lattice of WARP function is a chain of sublattices which are either Boolean algebras or a single element while the choice lattice of a SAP function is a chain. For infinite sets these theorems offer strong necessary conditions on the lattice. For finite sets V, these conditions are also sufficient (Theorem 11(v) and Theorem 12(ii)).

Together with Theorems 9 and 10, these theorems show that the lattices of Figures 2, 5, 6 and 7 are WARP choice functions. The lattice of Figure 7 is SAP. Examples 8, 10 and 11 are easily verified to be SAP functions.

We begin with two lemmas which show the inherent structure of the choice lattice of a WARP choice function. Lemma 19 is not new but we indicate the short proof here for its applicability to sets V which are infinite and for completeness. From this lemma and Theorem 6 it follows that the choice lattice for a WARP choice function is distributive. Lemma 20 contains many of the technical details necessary for the proof of Theorems 9 and 10.

LEMMA 19: (***A WARP Induced Equivalence Relation***)**:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$ satisfying WARP.

(i) The relation (~) on V defined by

$$a \sim b \text{ if and only if } C(\{a,b\}) = \{a,b\}$$

is an equivalence relation on V. We let [a] denote the equivalence class to which an element $a \in V$ belongs.

(ii) If a relation R is defined on V by xRy if and only if x~y or x > y in the choice lattice then R is complete, reflexive and transitive. The choice function is rationalized by R.

LEMMA 20: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$ satisfying WARP.

(i) The set of equivalence classes {[a]: a$\in$ V} form a chain in the choice lattice with [a]>[b] if a > b. (We prove the ordering is independent of the representatives chosen from the equivalence classes.) Let [[a]] denote the sublattice of the choice lattice generated by [a]. The ordering of the equivalence classes extends to an ordering of the sublattices [[a]] with a $\in$ V.

(ii) If [a] contains two or more elements then either [a] is the minimal equivalence class in the chain or there is an equivalence class [b] such that [a] covers [b] in the chain of equivalence classes. In the first case, a$\wedge$a' = $\emptyset$, in the second a$\wedge$a' =[b], for any two elements a, a' in [a]. Thus $\emptyset$ or [b] is the bottom element of the sublattice [[a]] and, in the latter case, [b] is the top element of [[b]].

(iii) If [a] consists of the single element a then there may not be an equivalence class covered by [a] in the chain of equivalence classes.

THEOREM 11: (***Lattice of WARP Choice Functions***)**:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$ satisfying WARP.

(i) The choice lattice for C is composed of a single chain of distributive sublattices [[a]] generated by the equivalence classes [a], a $\in$ V.

(ii) If [a] $\in\mathcal{V}$ and contains at least two elements then [[a]] is isomorphic to the Boolean algebra $2^{[a]}$.

(iii) If [a] = {a} then there may or may not be an equivalence class covered by [a].

(iv) If V is finite then the choice lattice is a chain of Boolean algebras.

(v) Conversely, any finite lattice which is composed of a chain of Boolean algebras or single elements is the choice lattice of a choice function defined on a finite set and satisfying WARP.

If the choice function satisfies SAP then Theorem 12 follows immediately. As an example see Figures 15a and 15b in the Appendix.

THEOREM 12: ***(Lattice of a SAP choice function)*****:**

(i) The choice lattice of a locally complete PI choice function on V with respect to $\mathcal{V}$ satisfying SAP is a chain.

(ii) Any finite chain is the choice lattice of a choice function that satisfies SAP.

EXAMPLE 15: Let V be the set of positive integers. Let $\mathcal{V}$ consist of all the finite subsets of V. Define C(A) = A if A $\in \mathcal{V}$. There is only one equivalence class, V. The range of C consists of the sets in $\mathcal{V}$ and so for A$\in \mathcal{V}$, arc(A) = A. It is easy to verify that properties (a) – (d) are satisfied by $\mathcal{V}$ and that C satisfies WARP. The choice lattice has no top element and is ordered by set inclusion. Of course this is not a Boolean algebra.

EXAMPLE 16: Let V be the set of positive integers. Let E denote the set of even numbers. Let A$\in \mathcal{V}$ if A contains at most a finite set of odd numbers. Now define

C(A) = A if A $\subseteq$ E, otherwise

C(A) = the set of odd integers in A.

Note that no infinite set of odd numbers is in $\mathcal{V}$. Note that A is an element of the range of C if A $\subseteq$ E or if A is a finite set of odd integers.

Now it must be verified that properties (a) – (d) hold with respect to C. Properties (a) and (b) are straightforward to verify. Properties (c) and (d) follow readily once arc(C(A)) for A $\in \mathcal{V}$ has been determined. Suppose first that A$\subseteq$E. In this case C(A) = A and so arc(A) = A. In the second case, when C(A) is the (finite) set of odd integers in A, arc(C(A)) = {C(A)$\cup$B: B $\subseteq$ E }. Thus $\bigcup$arc(A) = A in the first case and C(A)$\cup$E in the second. Thus Property (c) holds. Property (d) now follows readily.

There are just two equivalence classes; the set of odd numbers which are above the set of even numbers. The top sublattice is isomorphic to that of Example 13, the bottom sublattice, generated by the even numbers is the Boolean algebra of all subsets of even numbers.

§8. SUMMARY AND CONCLUSIONS.

The main results are summarized in Figure 14. In a sense, this diagram confirms Plott's original conjecture that the associative property of the semigroup he identified would be useful in extending the path independence concept to situations without the finiteness he assumed. For finite domains, we have complete characterization as shown by the double arrows in Figure 14. The results of Koshavoy (1998) are summarized by the two double-headed arrows on the lower left.

The structures identified in Figure 14 form two nested sets. For finite domains, beginning with the class of chains, each system class is contained in the class immediately below it. More over, as the class of mathematical system is expanded, the system's powers increase. One difference in power can be seen by comparing rational PI choice functions and non-rational PI choice. In the first case, the fact that the associated lattice is distributive means that final choices ***must*** be invariant with respect to a sequence of expansions and contractions of the feasible sets. The examples provided on three elements demonstrate that for non-rational PI choice functions, this restriction is relaxed. A similar nesting of mathematical systems exists for the systems identified for infinite domains.

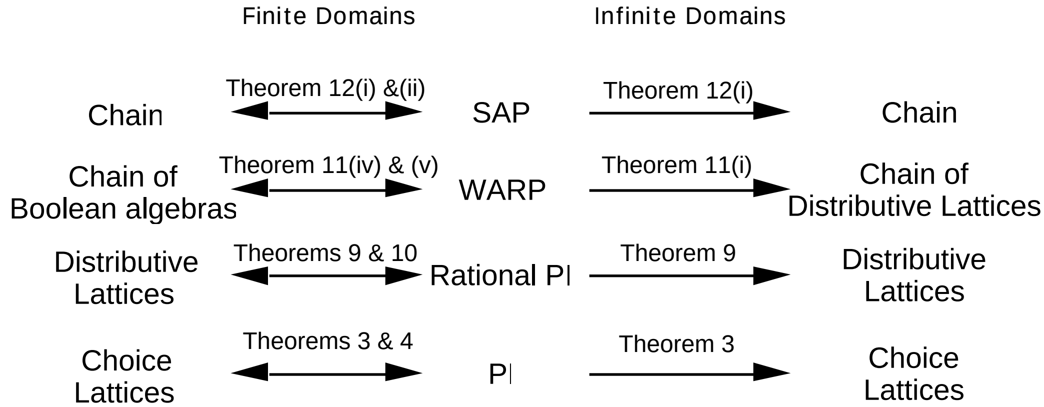

FIGURE 14: Summary of results.

Our algorithm for constructing choice lattices (and the associated choice function) also demonstrates a complexity difference. Rational choice functions can be easily generated from a binary relation. Non-rational PI choice functions can not; for them, a different, arguably more "complicated" process—our algorithm—is required.

We believe the extensions begun in this paper to infinite sets should be continued. For example, there are modifications of the contraction and expansion process that are valid for locally complete path independent choice functions with respect to its domain $\mathcal{V}$ even if V is infinite. More generally it has been suggested by one referee that our restrictions of the domain of the choice function over infinite sets might be a way to extend the theory of finite locally lower distributive lattices to infinite ones.

APPENDIX: PROOFS OF LEMMAS AND THEOREMS

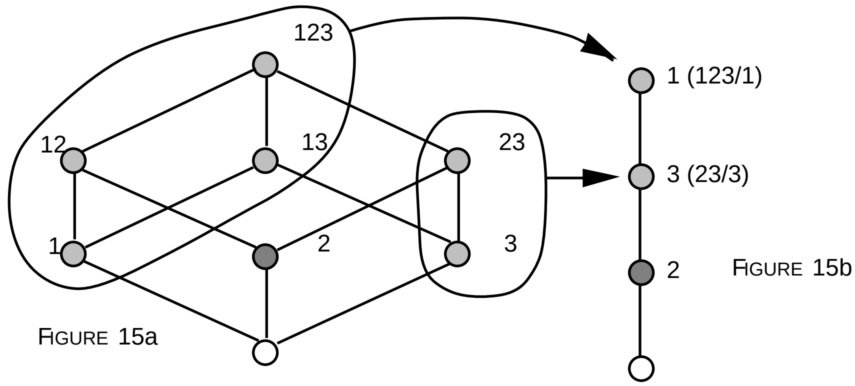

Figure 15a: Boolean algebra on V={1,2,3}. Figure 15b: Choice lattice for the choice function. Quotients in the domain are shaded consistent with the elements in the range.

<u>Proof of Lemma 4</u>.

$\hat{A} \in \mathcal{V}$ by property (c). It suffices to show that

(*) $$\hat{A} \supseteq A \supseteq C(\hat{A})$$

for then, by Lemma 2, $C(\hat{A}) = C(A) = A$ and so if X belongs to $\mathcal{V}$ and to $\hat{A}/A = \hat{A}/C(\hat{A})$ then, by Lemma 2 again, $C(X) = C(\hat{A}) = A$. Conversely, if $X \in \mathcal{V}$ and $C(X) = A$ then $X \supseteq C(X) \supseteq A \supseteq C(A)$ while $X \in \mathrm{arc}(A)$ and so $X \subseteq \bigcup \mathrm{arc}(A) = \hat{A}$. Thus $\mathrm{arc}(A) = \{X \in \mathcal{V}: \hat{A} \supseteq X \supseteq A\}$.

Now to prove (*). First, $\hat{A} \supseteq A$ because $A \in \mathrm{arc}(A) \subseteq \bigcup \mathrm{arc}(A) = \hat{A}$.

Second, to prove that $A \supseteq C(\hat{A})$ we argue as follows. If $X \in \mathrm{arc}(A)$ then $C(X) = A$ and so $\hat{A} \supseteq X$. Now, using Chernoff's Axiom, $A = C(X) \supseteq C(\hat{A}) \cap X$. So for all $X \in \mathrm{arc}(A)$, $A \supseteq C(\hat{A}) \cap X$.

Hence, taking set unions over arc(A) and using the distributivity of set intersection over set union, $A \supseteq \bigcup \{C(\hat{A}) \cap X: X \in \mathrm{arc}(A)\} = C(\hat{A}) \cap \bigcup \{X \in \mathrm{arc}(A)\} = C(\hat{A}) \cap \hat{A} = C(\hat{A})$.$\square$

<u>Proof of Theorem 1</u>.

We need only establish here the sufficiency of Chernoff's Axiom and the Quotient Property for PI. Let C be a choice function satisfying these two properties. We are to prove that for all $A,B \in \mathcal{V}$, $C(A \cup B) = C(C(A) \cup C(B))$. For ease of computation let $D = A \cup B$. $D \in \mathcal{V}$ by condition (b). Since $D \supset A$ it follows from Chernoff's Axiom that $C(A) \supseteq C(D) \cap A$ and similarly $C(B) \supseteq C(D) \cap B$. Thus $C(A) \cup C(B) \supseteq [C(D) \cap A] \cup [C(D) \cap B] = C(D) \cap [A \cup B] = C(D)$. Thus $D = A \cup B \supseteq C(A) \cup C(B) \supseteq C(D)$. It follows from property (b) and the Quotient Property that $C(D) = C(C(A) \cup C(B))$.$\square$

<u>Proof of Lemma 5</u>.

For brevity, let $A = \bigcup \mathcal{K}$. For all $K \in \mathcal{K}$, $A \supseteq K$ and so by Chernoff's Axiom, $C(K) \supseteq C(A) \cap K$ for all $K \in \mathcal{K}$. Now, taking set union over $\mathcal{K}$,

$$\bigcup\{C(K): K\in\mathcal{K}\} \supseteq \bigcup\{C(A)\cap K: K\in\mathcal{K}\} = C(A) \cap \bigcup\{K: K\in\mathcal{K}\} = C(A)\cap A = C(A);$$

the first equality holds because of the infinite distributivity of set intersection over unrestricted set union. On the other hand, $A = \bigcup\{K: K\in\mathcal{K}\} \supseteq \bigcup\{C(K): K\in\mathcal{K}\}$ and thus $A\hat{} \supseteq A \supseteq \bigcup\{C(K): K\in\mathcal{K}\} \supseteq C(A)$. From the Quotient Property, $C(\bigcup\{C(K): K\in\mathcal{K}\}) = C(A)$.□

LEMMA 8: Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$.

(i) The semigroup $<\mathcal{R}, \bullet>$ whose binary operation is the Plott product $A\bullet B = C(C(A) \cup C(B))$ is a partially ordered set under the definition:

$$A \leq B \text{ if and only if } B = A\bullet B.$$

(ii) In this partially ordered set $A\bullet B$ is the least upper bound for A and B and is denoted $A\vee B$. In this case, since $C(A) = A$ and $C(B) = B$

$$A\vee B = C(C(A)\cup C(B)) = C(A\cup B).$$

(iii) If $V \in\mathcal{V}$ then $C(V)\geq A$ for all $A \in\mathcal{R}$; that is, C(V) is the top element.

(iv) The bottom element is the empty set $\varnothing$.

Proof. Conclusions (i) and (ii) follow from the general result cited in Clifford and Preston (1961). Now suppose that $V\in\mathcal{V}$. If $A \in\mathcal{R}$ then, since $A \subseteq V$,

$$A\bullet C(V) = C(A \cup C(V)) = C(C(A)\cup C(V)) = C(A\cup V) = C(V).$$

Thus C(V) is the top element of the partially ordered set.

The empty set $\varnothing$ acts as an identity element for if $A \in\mathcal{R}$, $\varnothing\bullet A = C(\varnothing \cup A) = C(A) = A$ and hence $\varnothing$ is the bottom element under the partial ordering.□

LEMMA 9: ***(Identification of meets in the lattice)*:** Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Let A and B belong to $\mathcal{R}$. Under the partial ordering of $\mathcal{R}$ defined in Lemma 8, the greatest lower bound of A and B, is

$$\text{glb}(A, B) = C(A\hat{} \cap B\hat{}),$$

consequently we may define the meet operation:

$$A\wedge B = C(A\hat{}\cap B\hat{}).$$

Proof. Using property (c) and then (b) we know that $A\hat{}\cap B\hat{} \in\mathcal{V}$. We prove first that $C(A\hat{}\cap B\hat{})$ is a lower bound for A and B. Begin by noting that $A\hat{} \supseteq A\hat{}\cap B\hat{}$ and $A\hat{} \supseteq A$ so that $A\hat{} \supseteq (A\hat{}\cap B\hat{})\cup A \supseteq C(A\hat{}\cap B\hat{}) \cup A \supseteq A$. From the Quotient Property, $C[C(A\hat{}\cap B\hat{})\cup A] = A$. Now compute: $C(A\hat{}\cap B\hat{})\bullet A = C[C(C(A\hat{}\cap B\hat{})\cup C(A))] = C[C(A\hat{}\cap B\hat{})\cup A] = A$, and so $C(A\hat{}\cap B\hat{}) \leq A$. Similarly $C(A\hat{}\cap B\hat{}) \leq B$, and so $C(A\hat{}\cap B\hat{})$ is a lower bound for A and B.

Now suppose that an element $W \in\mathcal{R}$ is a lower bound for A and B in the semilattice. From $W \leq A$ we have. $A = W\bullet A = C(W\cup A)$ and hence that $A\hat{} \supseteq W\cup A \supseteq A$; in particular $A\hat{} \supset W$.

Similarly, from $W \leq B$, we have $B\hat{} \supseteq W$. Thus if W is a lower bound for A and B, it follows that $A\hat{}\cap B\hat{} \supseteq W$. Now since $A\hat{}\cap B\hat{} \supseteq C(A\hat{}\cap B\hat{})$ it follows that

$$A\hat{}\cap B\hat{} \supseteq W\cup C(A\hat{}\cap B\hat{}) \supseteq C(A\hat{}\cap B\hat{})$$

and so $W\cup C(A\hat{}\cap B\hat{})$ belongs to $arc(C(A\hat{}\cap B\hat{}))$. That means

$$C(W\cup C(A\hat{}\cap B\hat{})) = C(A\hat{}\cap B\hat{}) \text{ or } W\bullet C(A\hat{}\cap B\hat{}) = C(A\hat{}\cap B\hat{})$$

and hence for any lower bound W for A and B, $W \leq C(A\hat{}\cap B\hat{})$.

Thus $C(A\hat{}\cap B\hat{})$ is the greatest lower bound of the pair (A, B).□

<u>Proof of Theorem 2</u>.

From Lemma 9 it follows that $\mathcal{R}$, partially ordered by the definitions of Lemma 8, forms a lattice and so Theorem 2 is established.□

<u>Proof of Lemma 10</u>.

Suppose that $\mathcal{K}$ is any collection of elements in $\mathcal{R}$ such that $\bigcup\mathcal{K} \in \mathcal{V}$. Let $W = C(\bigcup\mathcal{K})$. We claim that W is the least upper bound in the choice lattice of the set $\mathcal{K}$. First to show that W is an upper bound for $\mathcal{K}$ we must show $W\bullet H = W$ for all $H\in\mathcal{K}$. Compute:

$$W\bullet H = C(W\cup H) = C(C(\bigcup\mathcal{K})\cup C(H)) = C((\bigcup\mathcal{K}) \cup H) = C(\bigcup\mathcal{K}) = W,$$

The penultimate equality holds in view of Lemma 5, and so $W \geq H$ for all $H \in\mathcal{K}$.

Second we must show that W is the least upper bound of $\mathcal{K}$. Suppose that X is an upper bound; i.e. $X \geq K$ for all $K\in\mathcal{K}$. This means that $X = C(X) = C(X\cup K)$ for all $K\in\mathcal{K}$ and hence $X\hat{} \supseteq X\cup K$ for all $K\in\mathcal{K}$ therefore, by taking set union over all $K\in\mathcal{K}$,

$X\hat{} \supseteq X\cup\bigcup\mathcal{K} \supseteq X = C(X)$ and so $C(X\cup\bigcup\mathcal{K}) = C(X)$; hence $X \geq \bigcup\mathcal{K} = W$.

Thus W is the least upper bound for $\mathcal{K}$ in the choice lattice.□

<u>Proof of Corollary 2</u>.

Lemma 10 shows that every subset of elements in the choice lattice has a least upper bound. Since in addition the lattice has a bottom element, ∅, it follows that the lattice is complete; in particular every set of elements in the choice lattice has a greatest lower bound. See Birkhoff (1973) for an exposition of this result.□

<u>Proof of Lemma 11</u>.

First, suppose $A\geq B$. Then $C(A\cup B) = A$. Now compute, using PI,

$$C(A\hat{}\cup B\hat{}) = C(C(A\hat{})\cup C(B\hat{})) = C(A\cup B)$$

Thus $A\hat{}\cup B\hat{} \in arc\ A$ and so $A\hat{}\supseteq A\hat{}\cup B\hat{}\supseteq B\hat{}$.

Conversely, suppose that A and B are such that $A\hat{}\supseteq B\hat{}$. Then $A\hat{}\supseteq B\hat{}\cup A \supseteq A$ so that $C(B\hat{}\cup A) = A$. On the other hand, using PI,

$C(B\hat{}\cup A) = C(C(B\hat{})\cup C(A)) = C(B\cup A) = A\bullet B$ and so $A = A\bullet B$, or $A \geq B$.□

Proof of Lemma 12.

First we show that $\{x\}$ is join irreducible. If $\{x\} = A \vee B$ in L, then $\{x\} = C(A \cup B)$ and so $x \geq A \cup B$. Without loss of generality, suppose that $x \geq A$. Since $\{x\} \geq A$ in L, it follows that $\{x\} = \{x\} \vee A = C(\{x\} \vee A\}$, but since $x \in A$, $C(\{x\} \cup A) = C(A) = A$. Hence $\{x\} = A$.

On the other hand suppose $A \in \mathcal{R}$, A is not the empty set and is not a singleton, say, $A = \{a\} \cup B$ where $a \notin B \neq \emptyset$. Since $B \subset A$ and $C(A) = A$ it follows that $C(B) = B$. Therefore $A = \{a\} \vee B$ in L where neither $\{a\}$ nor B is equal to A. So A is not join irreducible in L.□

Proof of Theorem 3.

Let C be a locally complete PI choice function with respect to $\mathcal{V}$ and let $\mathcal{R}$ be its range. Let $A \in \mathcal{R}$ so that $A = C(A)$. From Corollary 1 we know that $A = \bigvee \{\{x\}: x \in A\}$; i.e. that A is the least upper bound of the singleton sets $\{x\}$ for $x \in A$. This shows that every element of the choice lattice is the join of join irreducibles.

Next we show that this representation of A as the join of irreducibles is minimal in the sense that A is not the join of a proper subset of A. To prove this, suppose that $A = \bigvee T$ where $T \subseteq A$. By property (d), $T \in \mathcal{V}$. From Lemma 10, $\bigvee T = C\{\{t\}: t \in T\}$ and so

$$C(A) = A = C\{\{t\} : t \in T\} \subseteq T.$$

Thus $A = T$.

Finally we must show this representation is unique in the following sense: Suppose there are sets S and T of join irreducibles such that $A = \bigvee S = \bigvee T$. If these representations of A as the join of join irreducibles are minimal, then $S = T$. It suffices to prove that $A = C(A) = S = T$. In any event, since the members of S are singleton sets, we know from Lemma 10 that the least upper bound of S is $C(\bigcup S) = C(S)$. On the other hand, by assumption A is the least upper bound of S. Hence $A = C(S)$. Because the join $\bigvee S$ is minimal no subset of S can be deleted from this join and so $S = C(S) = A$. Similarly $T = A$ and so $S = T$.□

Proof of Lemma 13 .

If $A \geq \{x\}$ then $A = A \bullet \{x\}$ and so $A = C(A \cup \{x\})$ and so $A \cup \{x\} \in A^\wedge/A$. Therefore $A^\wedge \supseteq \{x\}$ if $A \geq \{x\}$. Hence $A^\wedge \supseteq \{x \in V: A \geq \{x\}\}$. Conversely suppose that $y \in A^\wedge$. Therefore $A^\wedge \supseteq A \cup \{y\} \supseteq A$ and so $A \bullet \{y\} = C(A \cup \{y\}) = A$; i.e. $A \geq \{y\}$ and so $A^\wedge \subseteq \{x \in V: A \geq \{x\}\}$.□

LEMMA 14: (*Coverings, a necessary condition*): Let C be a locally complete PI choice function on V with respect to $\mathcal{V}$. Let A and B belong $\mathcal{R}$.

(i) If $x \notin B^\wedge$ then $C(B \cup \{x\}) \geq B$ and $C(B \cup \{x\}) \neq B$ in the choice lattice.

(ii) If $A \geq B$, $A \neq B$ then there exists $x \in A$ and $x \notin B^\wedge$, and so

$$A \geq C(B \cup \{x\}) \geq B \text{ and } C(B \cup \{x\}) \neq B.$$

(iii) If A covers B then there exists one and only one $x \in A$, $x \notin B^\wedge$ such that

$A = C(B \cup \{x\})$.

Proof of Lemma 14.

Proof of (i). The calculation

$$B \bullet C(B \cup \{x\}) = C(C(B) \cup C(B \cup \{x\})) = C(B \cup B \cup \{x\}) = C(B \cup \{x\})$$

shows that $B \leq C(B \cup \{x\})$. If $B = C(B \cup \{x\})$ then $B \cup \{x\} \in \mathrm{arc}B$ and so $B \cup \{x\} \subset B^\wedge$; in particular $x \in B^\wedge$ contrary to assumption.

Proof of (ii). To guarantee the existence of $x \in A$ and $x \notin B^\wedge$, suppose to the contrary that $A \subset B^\wedge$. From Lemma 11, since $B \leq A$, we know that $B^\wedge \subseteq A^\wedge$ and hence $C(B^\wedge) = A$; but $C(B^\wedge) = B$ and $B \neq A$ by assumption. Thus an x exists. The rest of (ii) follows from (i)

Proof of (iii). We know from (ii) that there is an $x \in A$ such that $C(B \cup \{x\}) \geq B$ and unequal to B. Calculate

$$A \bullet C(B \cup \{x\}) = C(A \cup C(B \cup \{x\})) = C(A \cup B \cup \{x\}) = C(A \cup B) = A \bullet B = A$$

to show that $A \geq C(B \cup \{x\})$. Since A covers B, $A = C(B \cup \{x\})$.

To prove the uniqueness of this element, suppose that $y \in A^\wedge$, $y \notin B^\wedge$. From (i) we know $C(B \cup \{y\}) > B$ and is unequal to B. The calculation

$$A \bullet C(B \cup \{y\}) = C(A \cup C(B \cup \{y\})) = C(C(A^\wedge) \cup C(B \cup \{y\})) = C(A^\wedge \cup B \cup \{y\}) = C(A^\wedge \cup B)$$
$$= C(C(A^\wedge) \cup B) = C(A \cup B) = A \vee B = A$$

shows that $A \geq C(B \cup \{y\})$. Since A covers B, $A = C(B \cup \{y\})$. Hence both $B \cup \{x\}$ and $B \cup \{y\}$ are in arcA and therefore so is their set intersection:

$$(B \cup \{x\}) \cap (B \cup \{y\}) = B \cup (\{x\} \cap \{y\}) = B$$

if $x \neq y$. Thus if $x \neq y$, $B \in$ arc A, or $C(B) = A$; a contradiction since $C(B) = B$. Thus $x = y$.□

Lemma 15. Let L be the choice lattice of a PI choice function C defined on a finite set V. Suppose that for elements A and B in $\mathcal{R}$, A covers B and that B is meet irreducible in L. Then $A = \{B_1, x\}$ where B_1 is a proper subset of B if and only if B covers a meet irreducible element.

Proof. From Lemma 14 we know that $A = C(B \cup \{x\})$ where $x \notin B$.

Suppose first that $A = B \cup \{x\}$. We are to show that B covers no meet irreducible element. Suppose then that B covers D. From Lemma 14, $B = C(D \cup \{y\})$ for some $y \in B - D$.

We are to show that D is meet reducible. We claim that $D = B \wedge (D \vee \{x\})$. Note that $D \neq B$ and $D \neq D \cup \{x\}$ else $B^\wedge \supseteq D^\wedge \supseteq \{x\}$ contrary to the choice of x. In any event, $B \geq B \wedge (D \vee \{x\}) \geq D$ and because B covers D one of these containments is an equality. If $B = B \wedge (D \vee \{x\})$ then

$D \vee \{x\} \geq B$ and $D \vee \{x\} = B \vee D \vee \{x\} = B \vee \{x\} = B \cup \{x\}$. Since $y \in B$, $y \in B \cup \{x\}$ while $y \notin D \cup \{x\}$; a contradiction. Thus $D = B \wedge (D \vee \{x\})$ is meet reducible.

Suppose second that $A = \{B_1, x\}$ where B_1 is a proper subset of B. Let $y \in B - B_1$ and let $D = B - \{y\}$. Because of Lemma 6 (Hereditary Identity), $C(D) = D$ and by its construction B covers D. We will now show that D is meet irreducible.

If to the contrary then D must be covered by an element E not contained in A and $D = B \wedge E$. From Lemma 14 we know that $E = C(D \cup \{z\})$ for some element z. Note that $A \geq A \wedge (B \vee E) \geq B$ and since A covers B it follows that one of these containments is an equality. We examine both possibilities.

Suppose that $B = A \wedge (B \vee E)$. Since B is meet irreducible this equality implies that $B = B \vee E$ or that $B \geq E$ contrary to the choice of E. But then $B \geq E \geq D$ and since B covers D one of these containments must be an equality which is impossible.

Suppose that $A = A \wedge (B \vee E)$. This means that $B \vee E \geq A$ and hence $B \vee E \geq A \vee E \geq B \vee E$, so that $A \vee E = B \vee E$ or that

$$C(B_1,\{x\},D,\{z\}) = C(B,\{z\}) .$$

However, y is not a member of $(B_1,\{x\},D,\{z\})$ and hence $y \notin C(B_1,\{x\},D,\{z\})$ and so $y \notin C(B,\{z\})$. Therefore $C(B,\{z\}) = C((B-\{y\}),\{z\}) = C(D,\{z\}) = E$. But then $A \vee E = E$ and therefore $E \geq A \geq B$, contrary to the assumption that E not contain B. □

Lemma 15 permits us to make a connection with the work of Bordalo and Monjardet (1996) and so prove Theorem 5.

Proof of Theorem 5.[8] In their Theorem 10[d], Bordalo and Monjardet (1996) give three conditions, any one of which is sufficient while one must be necessary for B to be deletable. The sufficiency of two of these are applicable here: (i) B is both meet and join irreducible, (ii) B is meet irreducible but not join irreducible, A, the unique element covering B is not join irreducible and every element covered by B is meet reducible.

If B is both meet and join irreducible then condition (i) applies and B can be deleted. Lemma 14 shows that $A = C(B \cup \{x\})$ which is equal to $B \cup \{x\}$ in this case since B is a singleton.

If B is not join irreducible then Lemma 15 shows that under the hypotheses of Theorem 5 condition (ii) applies. Thus B can be deleted. The definition of the new choice function C* follows easily and we omit the details.□

Proof of Theorem 6. Part (i). An inspection of the conditions shows that *C is clearly a choice function on V. Second, the inverse sets of *C are those of C except for the inverse

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[8] For our original proof of this result see Johnson and Dean (1996).

sets of A and B. Because B^C is the relative complement of $B\cup\{x\} = A$ in B^/B, every set in B^/B belongs either to B^/A or to B^C/B, but not to both. Thus the *C inverse set of A is B^/A since *C(X) = A if and only if $X \in$ B^/A and the *C inverse set of B is B^C/B. From these arguments we have that *C satisfies the quotient property .

To complete the proof that *C is path independent we verify Chernoff's Axiom.

Suppose $D \supseteq E$. We must prove $*C(E) \supseteq *C(D)\cap E$. We know $C(E) \supseteq C(D)\cap E$. We consider four cases based on the relationship of D and E to the quotient B^/A.

Case 1. Neither D nor E belong to B^/A. Then *C = C for D and E. The implication is inherited from the condition on C.

Case 2. Both D and E belong to B^/A. Then *C(E) = *C(D) = A and so the implication holds.

Case 3. $D\in$ B^/A and $E\in$B^/A.

Thus *C(D) = C(D) and $*C(E) = A \supseteq B = C(E)$ so that the condition to be verified becomes $A\supseteq C(D)\cap E$. Because C is path independent, $C(E) \supseteq C(D)\cap E$ and so $B \supseteq C(D)\cap E$. Since $A\supseteq B$ the condition to be verified holds.

Case 4. $D\supseteq E$, $D \in$ B^/A and $E \notin$B^/A.

In this case the condition to be verified becomes $C(E) \supseteq A\cap E$.

We know from the choice function C that $C(E) \supseteq C(D)\cap E = B\cap E$. Now compute

$$A\cap E = (B\cup\{x\})\cap E = (B\cap E)\cup(\{x\}\cap E) = (B\cap E)\cup\{x\}$$

since $x \in E$ by hypothesis. Since $C(E) \supseteq B\cap E$ it suffices to show that $A\cap E\supseteq \{x\}\cap E$ and for this it suffices to show $C(E) \supseteq\{x\}\cap E = \{x\}$. But by the condition on E, $x \in C(E)$. The proof of Part (i) is complete.

The proofs of Parts (ii) and (iii) are routine and the details are omitted.

Part (iv). We start from a choice function C and its lattice L. In L we have a covering, $A=B\cup\{x\}\geq B$ and B is meet irreducible. The quotient for A is A^/A and for B it is B^/B. This covering is collapsed under C* to form the lattice L*. The quotient under C* is A^/B. It must be shown that the conditions (1) and (2) of Theorem 6 are met for C*. We know $x \notin B$. So suppose that E is a set such that $x\in E$, $A\hat{} \supseteq E$ but $E\not\supseteq B\cup\{x\}$. We must show that $x \in C^*(E)$. Now $C^*(E) = C(E)$. In L, $C(E)\geq\{x\}$ since $C(E)\vee\{x\} = C(E\cup\{x\}) = C(E)$. Consider $B\vee C(E)$ in L. Since $A\hat{}\supseteq B\cup E$ it follows that $A \geq B\vee C(E) \geq B$. But A covers B so one of these must be an equality. Suppose that $A = B\vee C(E)$. We know $x\in A$, $x\notin B$ so it must follow that $x\in C(E)$.

Suppose that $B\vee C(E) = B$ so that $B\geq C(E)$ but $C(E)\geq\{x\}$ and hence $B\vee\{x\} = B$ contrary to the assumption that $B\vee\{x\} = A$. Hence we can perform the expansion on C*.

Now the rest of the details are routine and are omitted.□

Now we state a technical lemma needed for the proof of Lemma 18.

LEMMA 17: Let C be a PI choice function rationalized by R on the universal set V and domain $\mathcal{V}$.

If $a \in A \in \mathcal{V}$ then either aRA or there exists $x \in C(A)$ such that $a \neg Rx$.

We often use this in the form: If A = C(A) and $a \in A\hat{}$ then either $a \in A$ or $a \neg Rx$ for some $x \in A$.

Proof of Lemma 18.

In view of Corollary 3 we need only prove (i). Since $A \vee B = C(A \cup B) = C(C(A\hat{}) \cup C(B\hat{}))$ $= C(A\hat{} \cup B\hat{})$, the last equality holding by PI, it follows that $C(A\hat{} \cup B\hat{}) \in \text{arc}(A \vee B)$. and so $(A \vee B)\hat{} \supseteq A\hat{} \cup B\hat{}$. To prove the reverse containment, suppose that $x \in (A \vee B)\hat{} = (C(A \cup B))\hat{}$. By Lemma 17 either $x \in C(A \cup B)$ or there exists $d \in C(A \cup B)$ such that $x \neg Rd$. In the former case, $x \in (A\hat{} \cup B\hat{})$. In the latter, since $C(A \cup B) \subseteq A \cup B$ we may suppose without loss of generality, that $d \in A$. Then $C(A \cup \{x\}) = A$ since $x \neg Rd$. But then the quotient property implies $A\hat{} \supseteq (A \cup \{x\}) \supseteq A$, so that $x \in A\hat{} \subseteq A\hat{} \cup B\hat{}$.

Proof of Theorem 9.

We continue the notation as before. The universal set is V, if C(A) = A then arc A = A^/A.

We prove that the mapping Φ from the choice lattice L into 2^V defined by $\Phi(A) = A\hat{}$ is an injection. This means that L is isomorphic to a sublattice of the Boolean algebra, 2^V, which is, among other things, distributive. Now the mapping Φ is one to one for if A^ = B^ then A =C(A^) = C(B^) = B. Lemma 18 shows that Φ preserves meets and joins.□

Proof of Lemma 19.

It is easy to see that (~) is reflexive and symmetric. To prove transitivity suppose that for elements a,b and c in V, a~b and b~c. We are to prove that a~c. We suppose that $b \neq a$ and $b \neq c$; otherwise transitivity is trivial. By hypothesis we know that C({a,b}) = {a,b} and C({b,c}) = {b,c}. Next we gain information on C({a,b,c}). Using the hypothesis and WARP, since $\{a,b\} \subseteq \{a,b,c\}$ it follows that {a,b} = C({a,b} = C{a,b,c}∩{a,b} unless the later intersection is empty. But if C({a,b,c}∩{a,b} = ∅ then C({a,b,c} = {c}. But then, using WARP again together with the hypothesis,

$$\{b,c\} = C(\{b,.c\}) = C(\{a,b,c\}) \cap \{b,c\} = \{c\};$$

a contradiction. Thus it follows that C({a,b,c}) ⊇{a,b}. Interchanging a and c in the above argument shows that C({a,b,c} ⊇{b,c} and hence C({a,b,c}) = {a,b,c}. But now using WARP again C({a,c}) = C({a,b,c}∩{a,c} = {a,c} and so a~c; completing the proof of transitivity.

Finally it is routine to prove that R has its requisite properties. We omit the details.□

Proof of Lemma 20.

First we show given two distinct classes [a] and [b] then either a > b or b > a and moreover whichever holds, say a>b, then a'>b' for all a' ∈ [a] and all b' ∈ [b]. To see this, since a is not equivalent to b, C({a,b}) = {a} or C({a,b}) = {b} . Suppose the first alternative holds, then $\{a\}\vee\{b\} = C(\{a\}\cup\{b\}) = \{a\}$ and so a > b. (We continue our "abuse of notation" to identify {a} without the braces when we speak of lattice elements.) We continue with the proof assuming that a> b without loss of generality.

Now suppose a~a'. We compute

$$C(\{a', a, b\}) = C(\{a'\}\cup\{a,b\}) = C(\{a'\}\cup C(\{a,b\}) = C(\{a'\}\cup\{a\}) = \{a',a\}.$$

Now using WARP,

$$C(\{a',b\}) = C(\{a',a,b\})\cap\{a',b\} = \{a'\}$$

and thus a' > b. By interchanging the roles of a and b we find C({a,b'}) = {a} and so a > b' if $b' \in [b]$. Next we compute using path independence that

$$C(\{a',b',a\}) = C(C\{a'\} \cup C\{b',a\}) = C(\{a', a\} = \{a',a\}.$$

Finally then, using WARP,

$$C(a',b') = C(\{a', b', a\})\cap\{a',b'\} = \{a'\} \text{ and so } a'>b' \text{ in the lattice.}$$

The important fact about the choice lattice for C is that the set of its join irreducibles (which coincide with the elements of V) break into disjoint equivalence classes and that in the choice lattice these equivalence classes form a chain under the ordering of the lattice.

Let [[a]] denote the sublattice generated by the join irreducibles in the equivalence class [a]. Then the choice lattice is the set union of these sublattices [[a]] for $a \in V$. Moreover if a > b and $[a]\neq[b]$, then H > K if $H\in [[a]]$ and $K \in [[b]]$. This is so because each element of the choice lattice is the join of join irreducibles and the comparability of elements H and K is inherited from the comparability of the join irreducibles in their representations (Theorem 4). Thus these sublattices [[a]] form a chain and the choice lattice is the set union of these sublattices.

Now we consider the case that [a] contains two or more elements. It is useful to determine for distinct elements a and a', both in [a], the lattice meet $a\wedge a'$. We prove that

$$a\wedge a' = C(\{\{t\}: a>t, a\neq t\}).$$

(In this proof we will write a>t if $a\geq t$ and $a\neq t$ in the choice lattice to simplify notation.)

From the characterization of meets in the choice lattice given by Lemma 9 we must first determine a^ and a'^. In any event C({a,r}) ={a} if a > r so that arc(a) contains

$$\bigcup\{\{a, r\} : a > r\} = \{a\}\cup\bigcup\{r : a > r\} = \{a\}\cup\bigcup\{r : a' > r\} .$$

the last equality holding because the set of elements below a in the choice lattice is the same as the set below a' if and only if a~a'. Conversely, suppose $K \in$ arc(a) and that $k \in K$, $k\neq\{a\}$. Then, using WARP, $C(\{a,k\}) = C(K\}\cap\{a,k\}) = \{a\}$ and so a > k. Thus $K \subseteq \{a\}\cup \bigcup\{r: a > r\}$. This means that $\text{arc}(a) = \{a\}\cup\bigcup\{r : a > r\}$. By property (c), $\text{arc}(a) \in \mathcal{V}$ since in this case $\bigcup\text{arc}(a) = \text{arc}(a)$. It follows immediately that a^ = arc(a). Similarly a'^= arc(a') and so

$$a \wedge a' = C(a\hat{}\cap a'\hat{}) = C(\bigcup\{r : a > r\})$$

If the set {r: a>r} is empty, then [a] is the minimal equivalence class in the chain of equivalence classes. In this case $a \wedge a' = \emptyset$. In either case $a \wedge a'$ is in the distributive sublattice [[a]] and is the bottom element of this lattice because all elements of [a] lie above it.

Next we argue that if $a \wedge a' \neq \emptyset$, this bottom element is precisely one equivalence class [b]. Let b be any element in C(a^∩a'^). By Theorem 4, $a \wedge a'$ has a unique representation as the irredundant join of join irreducibles. These irreducibles are just the elements in C(a^∩a'^). In this case we know that these join irreducibles come from equivalence classes which form a chain. Thus if b>r, the element r cannot occur in the irredundant representation for a∩a' and hence is not in C(a^∩a'^). So the elements in C(a^∩a'^) must belong to [b].

Conversely we show that any element in [b] must belong to C(a^∩a'^). Suppose b' ∈ [b]. Then {b,b'} ⊆ a^∩a'^ and so we may calculate, using WARP

$$\{b,b'\} = C(\{b,b'\}) = C(a\hat{}\cap a'\hat{})\cap\{b,b'\}$$

since the latter set intersection contains b and so cannot be empty. Thus b' ∈ C(a^∩a') and so C(a^∩a'^) = [b].

Again from the minimality of the representation it follows that [b] must be the maximal equivalence class of those classes below [a] in the chain of classes. Thus [a] covers [b] in that chain. We have immediately that [b] is the top element of the sublattice [[b]] generated by [b].

For the third part of Lemma 20 we refer to Example 10 to show what may occur when [a] is a single element. It is easy to verify that the choice function of this example satisfies WARP and SAP yet the top element {1} = [1] covers no equivalence class.□

<u>Proof of Theorem 11</u>.

Lemma 19 and Theorem 9 show that the choice lattice for C is distributive. The singletons {a} for a ∈ V constitute the join irreducibles (Lemma 12). The equivalence relation of Lemma 19 organizes them into equivalence classes [a]. Lemma 20 shows that the equivalence classes [a], a ∈V form a chain and this ordering extends to the sublattices [[a]], a∈V generated by the equivalence classes. Because each element in the choice lattice has a representation as the join of a minimal set of join irreducibles, each element of the lattice belongs to one of these sublattices. Hence the lattice consists of this chain of sublattices. These sublattices are of course distributive.

The top element, if it exists, of the sublattice [[a]] is C([a]). It may not exist since there is no condition forcing the union of an infinite number of elements to belong to $\mathcal{V}$.

However the bottom element of [[a]] always exists. Lemma 20 shows that if [a] consists of at least two elements this bottom element is an equivalence class [b]. It follows that in the chain of equivalence classes, [a] covers [b]. If [a] is not a member of $\mathcal{V}$ then little more can be said about [[a]]. If [a] = {a}, then there is no guarantee that [a] covers another equivalence class.

Suppose then that [a] $\in \mathcal{V}$ and contains at least two elements. From the argument in the proof of Lemma 20 it follows that C([a]) = [a] and is the top element in the sublattice [[a]]. Applying this inference to [[b]] we see that the top element of [[b]] is [b] so that the bottom element of [[a]] is the top element of [[b]].

From Property (d) and Lemma 6 it follows that C(E) = E for all subsets E $\subseteq$ [a] and so in the sublattice [[a]] lattice meets and joins are set intersections and set unions. From this it follows that the sublattice is isomorphic to $2^{[a]}$.

Conversely, suppose D is a finite lattice which is a chain of Boolean algebras as defined in the theorem. It is routine to prove that D is distributive. By Theorem 10 there is a PI choice function defined on the set of join irreducibles whose choice lattice is D. We omit the details of the proof that in this case the choice function satisfies WARP.□

<u>Proof of Theorem 12</u>.

Choice functions satisfying SAP of course also satisfy WARP and can be rationalized by the linear order R of Lemma 20 which, in addition to the conditions that R must satisfy by rationalizing a WARP function, also satisfies the anti-symmetry required by SAP:

$$\text{If } xRy \text{ and } yRx \text{ then } x = y$$

This means that the equivalence classes of Lemma 19 are singletons. If V has a finite number n of elements this lattice is a chain of length n+1.□

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