Optimal Maturity of Government Debt with Incomplete Markets

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Abstract

In this paper we show how risk free bonds of di¤erent maturities can be used to replace state contingent debt in a general equilibrium dynamic optimal taxation problem. In particular, we show that if the state of the economy can only take a …nite number N of values each period, then the government can support the complete markets Ramsey allocation issuing bonds of $J \geq N$ di¤erent maturities. We also show that the optimal maturity structure does depend on teh relationship between the term strucutre of interest rates and goverenment expenditures. In the case that intreset rates are positively correlated with government expenditures in the Ramsey solution, then the government must hold short run assets and long term liabilites.

[*]PRELIMINARY AND INCOMPLETE (we do not have enough "maturity "!)

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1 Introduction

In this paper we show that in a dynamic economy where state contingent bonds cannot be issued, a Ramsey government may still implement the complete markets second best allocation by issuing risk-free bonds with a rich enough menu of di¤erent maturities.

The role for debt that pays in only some states of nature was emphasized by Lucas and Stokey (1983) as an instrument to smooth taxes across states. The active use of this …nancial instruments increases welfare and has been a key ingredient in most studies of optimal …scal policy and optimal debt management since then[1]. Lucas and Stokey also show that a proper management of the manturity structure of the debt makes the sequence of tax rates time consistent. In their model, however, if the government can fully commit to a tax policy, as we will assume in this paper, the maturity structure of the debt is irrelevant. In fact, this is the case for most general equilibrium models in the literature in which the government can fully commit to its future actions.

In a recent paper, Marcet, Sargent and Seppala (1999) solve a problem similar to the one in Lucas and Stokey but assume away the posisibility of state contingent debt. Thus, in their model, the government issues only one period risk free bonds. They show that the stochastic properties of the optimal taxes and the debt are very di¤erent to the case in which the governmnt can issue a full array of state contingent bonds[2]. In particular, they show that in the economy with complete markets, optimal taxes inherit the autocorrelation properties from the stochastic process of government expenditures only, while in the economy with one period risk free bonds only, there is an additional state variabl that introduces a unit root component on the solution of optimal taxes.

While Lucas and Stokey perform a purely normative excercise, MSS do attempt a positive excercise, since, they argue, their model without state contingent debt …ts better the experience of XVIII century England, while XVIII century France seems to behave more like the model with at least some state contingent bonds achieved through default.

The purpose of this paper is to show that as two bonds of di¤erent maturity issued on the same date are diferent assets, they allow to span a greater set of allocations than a single maturity bond can. In fact, we show that if

[1] See Chari, Christiano and Kehoe (1995) and the references therein.

[2] They alo show, however, that the welfare e¤ect of introducing state contingent bonds is very small.

the state of the economy has finite support, a rich enough set of maturities can complete markets. In particular, we show that in an infinite period model with uncertainty in which the shock can take on a finite number of values M each period, if the government can issue bonds with J different maturities, then if $J \geq M$; the government completes markets.

The intuition of the result is very simple. To complete markets, we require the same number of linerarly independet assets as state of nature. Note that a j period bond today is different from a j period bond tomorrow, thus if the number of different assets at a particular date, conditional on a particular state, is the same as the number of shocks in that date for that particular state, and this is true for any date and state, the economy has the same number of assets as states.

The dependence of the term structure of interst rates on the state of the economy explains why markets can be completed with different maturities. A promise made at time t to deliver one unit of consumption at $t + 1$ is worth one unit of consumption in $t + 1$: However, a promise made at time t to deliver one unit of consumption at $t + 2$ is worth the same as a one period bond in $t + 1$; which does depend on the state of nature. Thus, if one is concerned about transfering wealth from t to $t + 1$; a one period bond is a risk free asset, while a two period bond is a risky asset.

In an economy with contingent assets, as in Lucas and Stokey, the government issues bonds that pay only when government expenditures are low. Thus, it can smooth taxes across states of nature. To understand how the maturity structure can replicate the result, recall that the price of a bond will in general depend on the governmnt shock. If the Ramsey solution is such that private and public consumption are negatively correlated, then the interest rate will be relatively high when governemnt expenditure is high, and the price of a bond will be lower. Thus, the way to insure against a positive shock in governemnt expenditures is by holding a large amount of long term debt, whose value go down when govenrmnt expenditures go up. In fact, by explcitly solving a few examples, we show that the optimal maturity structure is characterized by holding short term assets and long term liabilities. In this way, the value of the outstanding debt will be negatively corellated with expenditures, as the Ramsey solution with complete markets indicate.

A qualification to the results must be made. In order to complete markets, the existing assets need to be linearly independent, i.e, the vector of returns of one and two period bonds must be different. We show that for this to be the case, the term structure of intererest rates in the Ramsey solution must be different across states, since it is the term structure that chages the prices of existing debt over time. However, we also show that as the term structure depends on the optimal tax rates, if there is enough assets, the government can always change slightly the taxes and approximate arbitrarely well the optimal state contingent allocation.

On the normative side, our paper shows that even though governments do not issue state contingent assets, a proper mangement of the debt maturity structure can do the job. On the positive side, our paper contributes to the literature on the properties of optimal debt and market structure. Since a wide range of maturities, as there is in real economies, approximates better than a sinlge bond a full array of state contingent bonds, it may undo the quantitative importance of the martingale properties of the taxes obtained by MSS.

The paper is organized in the following way. Section 1 presents the model and shows the main result of the paper. Section 2 provides a series of examples to illustrate how the maturity of the debt should be managed. In section 3 we simulate a model with a maturity structure like the one existing in the US to evaluate if the stochastic properties of the optimal taxes resemble more the complete markets or the sinlge one period risk free bond economies.

2 The Economy

The enviroment we analyze is very similar to the barter economy of Lucas and Stokey. This is a one perishable good economy with production. The representative agent is endowed with one unit of labor in every period. Labor is the single factor of production, an the technology is given by

$$c_t + x_t + g_t \leq 1; \qquad t = 0; 1; 2; \ldots; \text{ all } g^t. \tag{1}$$

where $x; c$ and g represent leisure and private and public consumption respectively.

As it is standard in the Ramsey literature, government consumption is exogenous. We assume that it follows a stochastic process such that a single realization $g \equiv (g_0; g_1; g_2 \ldots)$ has the joint distribution F. We assume g_t to be the only source of uncertainty in this economy. We also assume that goverenmnt expenditures can take one of a finite number N of values. This assumption allows us to make the point of the paper in a very clean way.

How the result can be extended to a continuum of values will be discussed in the end.

Preferences are represented by the following von Neumann-Morgenstern utility function

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t(g^t)) + v(x_t(g^t))] \tag{2a}$$

$\beta \in (0,1)$; u and v are stricly increasing and strictly concave. The assumption of separability is in no way essential, as it will be obvious, but some of the results are cleaner.

The only tax available to the government is a flat rate tax τ_t levied against labor income $1 - x_t$. The government can also issue debt (that can be negative). We consider two environments, with different asset structures.

A Ramsey problem is to choose the allocation $\{c_t, x_t\}_{t=0}^{\infty}$ that maximizes (2a) subject to (1) and subject to the constraint that the allocation can be decentralized as a complete markets competitive equilibrium with income taxes. As it is standard in the literature, this last restriction can be represented by a sinlge equation, the implementability constraint. This implementability constraint is obtained by replacing the complete markets competitive equilibrium prices in the life-time budget constraint of consumers[3]. In this case, the implementability constraint becomes

$$E_0 \sum_{t=0}^{\infty} \beta^t [c_t(g^t) \frac{U'(c_t(g^t))}{U'(c_0(g^0))} - (1 - x_t(g^t)) \frac{V'(x_t(g^t))}{U'(c_0(g^0))}] = b_{-1} \tag{3}$$

where b_{-1} represents the outstading liabilities inherited by the government[4].

We will consider two full commitment environments with different asset markets. In the first one, we let the government issue one period state contingent debt. Let $b_t(g_{t+1}; g^t)$ stand for the contingent debt issued by the government in period t, given history g^t, to pay one unit of the consumption good in period $t+1$ contingent on the shoch g_t being realized[5]. Alternatively, we let the government issue each period "risk free" bonds maturing at J dif-

[3] For details, see Lucas and Stokey (1983) or an appendix to this paper available from the authors upon request.

[4] Note that we are assuming that initial liabilities are all due on the first period. This is done just fro simplicity, it is in no way essential to the results.

[5] Note that we do not consider the case of contingent debt with different maturities. For the case with commitment there is no loss of generality by doing this.

ferent dates. Let $b_t^j(g^t)$ denote the promise, made at t to deliver one unit of the good in period $t+j$ in any state of nature, after history g^t.

2.1 The solution with state contingent bonds

In this subsection we discuss the structure of the optimal debt when one-period state contingent bonds can be issued. This is the solution implemented by Lucas and Stokey.

The first sequence of constraints of the Ramsey problem (1) ensures aggregate material balance for every period and every contingency. The second constraint (3) ensures that the resulting allocation can be implemented as a competitive equilibrium with complete markets. To be able to represent all these equilibrium conditions on a single equation, the assumtpion of complete markets is essentiall. Note that we could write a similar condition at time t

$$z_{t-1}(g_t, g^{t-1}) = E_t \sum_{s=t}^{\infty} \beta^{s-t}[c_s(g^s)\frac{U'(c_s(g^s))}{U'(c_t(g^t))} - (1-x_s(g^s))\frac{V'(x_s(g^s))}{U'(c_t(g^t))}] \quad (4)$$

The right hand side of this equation is the expected discounted value of the government surplus from period t (history g^t) on. That should be equal to the contingent liabilities the government has to honor on date t, given history g^t. In other words, for the government to be able to finance its deficits (surpluses) from date t on, conditioned on history g^t, a certain amount of wealth (debt) must be delived at t , conditioned on that history, which we denote as $z(g_t, g^{t-1})$.

In a world with complete markets, we let $b_{t-1}(g_t, g^{t-1})$ represent outstading debt obligations at time t; given that state g^t has realized, conditional on history g^{t-1}, so $b_{t-1}(g_t, g^{t-1}) = z_{t-1}(g_t, g^{t-1})$.

Note that given a Ramsey allocation, the right hand side of this equation is determined. The requirement of complete markets is equivalent to imposing no constraint liking the values of $b_{t-1}(g_t, g^{t-1})$ for alternative realizations of g_t, except for (3), that imposes a present value condition for all t, g^t. On the other hand, if we assume, as in Marcet et. al. that there is only uncontingent one-period bonds, the restrictions that $b_{t-1}(g_t, g^{t-1}) = b_{t-1}(g^{t-1})$ for all g_t, g^{t-1} and $t \geq 0$, must be added to the Ramsey problem. Thus, one way of understanding the Ramsey problem with complete markets is letting the $b_{t-1}(g_t; g^{t-1})$ be unconstrained[6]. In fact, once the Ramsey problem with complete markets has been solved, one can disentangle the optimal one-period state contingent debt structure for every period and every state using equation (4).

2.2 A rich maturity structure of risk free bonds

In this section, we want to show how a structure of debt maturities maps into a structure of state contingent bonds if J is large enough. Let $b_{t-1}^{t-1+j}(g^{t-1})$ be debt issued at time t that is due at $t+j$; and let p_t^{t+i} be the price, in units of time t goods, of a promise to deliver one unit of the good at time $t+i$. Then, the time t value of outstanding obligations at time t, given the realization g_t are given by

$$\sum_{i=0}^{J-1} p_t^{t+i}(g^t) b_{t-1}^{t+i}(g^{t-1}) + \dots + \sum_{i=0}^{1} p_t^{t+i}(g^t) b_{t-J+1}^{t+i}(g^{t-J+1}) + b_{t-J}^{t}(g^{t-J}) = \sum_{j=1}^{J} \sum_{i=0}^{J-j} p_t^{t+i} b_{t-j}^{t+i}$$

where the dependence of prices and quantities on teh histories has been left implicit on the right hand side. The first sum of the left hand side represents the value of the debt issued at time $t-1$; the second one is the value of the debt issued at $t-2$; and so on. Note that of all debt obligations, only b_{t-1}^{t+i} can depend on the history g^{t-1}. Thus, it is usefull to write

$$\sum_{j=1}^{J} \sum_{i=0}^{J-j} p_t^{t+i} b_{t-j}^{t+i} = \sum_{i=0}^{J-1} p_t^{t+i}(g^{t-1}; g_t) b_{t-1}^{t+i}(g^{t-1}) + D_{t-1}(g^t)$$

Thus, while the ampunt of debt issued does not depend on the state of nature at t, the value of the debt in terms of current consumption does, and therefore, the value of the debt obligations can be made contingent on the state of nature[7]. Note also, that while $D_{t-1}(g^t)$ does depend on the shock at t, it cannot be afected by governemnt decisions at $t-1$, since it is the value of the debt issued from period $t-J$ to $t-2$. Using equilibrium bond prices, the expression can also be written as

[6] Of course, (3) implies that the sequence of $b_{t-1}(g^t; g_{t-1})$ satisfies the relevant present value condition.

[7] Note that for this to be the case, it is essential to have debt of more than one period, since $p_t^t = 1$.

$$\sum_{j=1}^{J}\sum_{i=0}^{J-j} p_t^{t+i} b_{t-j}^{t+i} = \sum_{i=0}^{J-1}\frac{E_t[\beta^i u'(c_{t+i}) \mid g^{t-1}; g^t]}{u'(c_t)} b_{t-1}^{t+i}(g^{t-1}) + D_{t-1}(g^t)$$

where $E_t[\beta^i u'(c_{t+i}) \mid g^{t-1}; g^t]$ represents the time t expected utility of $t+i$, conditioned on history $[g^{t-1}; g^t]$. This equation represents the value of government assets at time t after history $[g^{t-1}; g^t]$. We are interested in finding conditions such that, by appropiately chosing the values of $b_{t-1}^{t+j}(g^{t-1})$ for $j = 0, 1, 2, ..., J-1$, the governemnt can reproduce the required Ramsey values $z_{t-1}(g^{t-1}; g^t)$ for all $t, (g^{t-1}; g^t)$. Thus, it has to be the case that

$$\sum_{j=1}^{J}\sum_{i=0}^{J-j} p_t^{t+i}(g^t) b_{t-j}^{t+i}(g^{t-j}) = z(g_t; g^{t-1})$$

or

$$z(g_t; g^{t-1}) = \sum_{i=0}^{J-1}\frac{E_t[u'(c_{t+j}) \mid g^{t-1}; g^t]}{u'(c_t)} b_{t-1}^{t+i}(g^{t-1}) + D_{t-1}(g^t)$$

for all g_t, g^{t-1}, and $t \geq 0$.

Let the vector of government debt of different maturities issued at time $t-1$ be

$$b_{t-1}(g^{t-1}) = \begin{pmatrix} b_{t-1}^{1}(g^{t-1}) \\ b_{t-1}^{2}(g^{t-1}) \\ \vdots \\ b_{t-1}^{J}(g^{t-1}) \end{pmatrix},$$

the matrix of returns of the debt in period t,

$$A_t(g^t) = \begin{pmatrix} 1 & \beta\frac{E_t[u'(c_{t+1}) \mid g^{t-1}; g_1]}{u'(c_t)} & \cdots & \beta^{J-1}\frac{E_t[u'(c_{t+J-1}) \mid g^{t-1}; g_1]}{u'(c_t)} \\ 1 & \beta\frac{E_t[u'(c_{t+1}) \mid g^{t-1}; g_2]}{u'(c_t)} & \cdots & \beta^{J-1}\frac{E_t[u'(c_{t+J-1}) \mid g^{t-1}; g_2]}{u'(c_t)} \\ & & \vdots & \\ 1 & \beta\frac{E_t[u'(c_{t+1}) \mid g^{t-1}; g_N]}{u'(c_t)} & \cdots & \beta^{J-1}\frac{E_t[u'(c_{t+J-1}) \mid g^{t-1}; g_N]}{u'(c_t)} \end{pmatrix},$$

and the vector of wealth transfers

$$Z_t(g^t) = \begin{pmatrix} Z_t(g^{t-1}; g_1) \\ Z_t(g^{t-1}; g_2) \\ \vdots \\ Z_t(g^{t-1}; g_N) \end{pmatrix};$$

Then, the system of equations for period t, given history g^{t-1} can be written as,

$$A_t(g^t)b_{t-1}(g^{t-1}) + D_{t-1}(g^t) = Z_t(g^t)$$

If the matrix of returns $A_t(g^t)$ is nonsingular we obtain

$$b_{t-1}(g^{t-1}) = A_t(g^t)^{-1}[Z_t(g^t) - D_{t-1}(g^t)]$$

which means that we can always find an intertemporal strategy of debt issues to support the Ramsey allocation. A necessary condition is that we have at least as many debt maturities as possible realizations of the government expenditure, i.e., $J \geq N$. A sufficient condition is that the yield curve differs across states of nature.

3 Some Examples

In this section we illustrate the structure of maturities associated with the Ramsey allocation for several examples, to give a flavor of what the optimal debt maturity structure ought to be. First, we show it using 3 examples from Lucas and Stokey plus an additional one with three states of nature.

Before doing so, it is worth emphazising a few properties of the Ramsey solution[8]. If the Ramsey solution is interior, it has to satisfy the following first order conditions

$$(1+\lambda)U'(c_t(g^t)) + \lambda c_t(g^t)U''(c_t(g^t)) = \mu_t(g^t), \qquad t = 0, 1, 2, ..., \text{all } g^t.$$

$$(1+\lambda)V'(x_t(g^t)) + \lambda(x_t(g^t) - 1)V''(x_t(g^t)) = \mu_t(g^t), \qquad t = 0, 1, 2, ..., \text{all } g^t.$$

Using these first order conditions together with equation (1) it is possible to solve for μ_t, c_t, x_t as a function of the multiplier λ and the value of the shock g_t only, so we can write them as $\mu_t(g_t, \lambda), c_t(g_t, \lambda), x_t(g_t, \lambda)$. This means that, as the multiplier is the same for every period and history, every time that governement expenditures takes the same value, the allocation c_t, x_t is the same.

[8] For details, see Lucas and Stokey (1983).

As it may be clear already from last section, a key element to determine the optimal debt structure is the behavior of the yield curve associated with the Ramsey allocation. This will be given by the holding return of the different debts, which in turn depends on the correlation between consumption and governement expenditures.

Lemma 1 : Given an interior solution, the value of consumption will be relatively low when g is relatively high if $V'''(x) \geq 0$.

Proof: Using the necesary conditions for an optimun and the resource constraint we get an equation relating c_t and g_t;

$$(1+\lambda)U'(c_t) + \lambda c_t U''(c_t) = (1+\lambda)V'(1 - c_t - g_t) - \lambda(c_t + g_t)V''(1 - c_t - g_t)$$

applying the implicit function theorem to get an expression for $\frac{\partial c_t}{\partial g_t}$;

$$\frac{\partial c_t}{\partial g_t} = -\frac{(1+2\lambda)V''(\cdot) - \lambda(c_t+g_t)V'''(\cdot)}{(1+2\lambda)U''(\cdot) + \lambda c_t U'''(c_t) + (1+2\lambda)V''(\cdot) - \lambda(c_t+g_t)V'''(\cdot)}$$

From the second order conditions we know that the denominator is negative. As $V'''(\cdot) \geq 0$ and $\lambda > 0$[9], the numerator is also negative.

Example 2 :(corresponds to example 5 in LS) Let $b_0^j = 0$; $j = 1, 2$; let $g_t = 0$ for all $t \neq T$ and let $g_T = G > 0$ with probability α and $g_T = 0$ with probability $1-\alpha$.

As we argued above, the first order conditions of the Ramsey problem imply that the allocation will be the same every period and state where g_t is the same. Thus, using the implementability constraint (3)

$$\sum_{t=0}^{\infty} \beta^t [cU' + (x-1)V'] + \alpha\beta^T \{[cU' + (x-1)V'] - [\tilde{c}\tilde{U}' + (\tilde{x}-1)\tilde{V}']\} = 0$$

where $c = c_t(g_t = 0)$; $\tilde{c} = c_t(g_t = G)$; and the same interpretation holds for all other variables. If we let $\omega = cU' + (x-1)V'$; we can writte the equation above as

[9] See Lucas and Stokey equation 3.3, or the appendix to this paper, available form the authors upon request.

$$\sum_{t=0}^{\infty} \beta^{t}\omega + \alpha\beta^{T}\{\omega - \phi]\} = 0 \tag{5}$$

Note also that if we multiply the first order conditions by the corresponding variable for the periods and states where the govenment expenditure is zero, we obtain

$$(1+\lambda)cU' + \lambda c^2 U'' = c\mu;$$

$$(1+\lambda)(x-1)V' + \lambda(x-1)^2 V'' = (x-1)\mu;$$

and adding the two

$$(1+\lambda)[cU' + (x-1)V'] + \lambda[c^2U'' + (x-1)^2V''] = [c+x-1]\mu = 0$$

or

$$(1+\lambda)\omega + \lambda[c^2U'' + (x-1)^2V''] = [c+x-1]\mu = 0$$

But $\lambda > 0$ and both U and V are concave, so $\omega > 0$. Thus, it follows from (5) that $\phi < 0$.

With one-period state contingent debt, the optimal debt structure has to satisfy , for $t = 0, 1, \ldots, T-1$

$$-(1-\beta^{t-1})\frac{\omega}{1-\beta} = \beta^{t}U' b_{t-1}$$

Thus, for the first periods, the government runs surpluses and accumulates assets. Note that as there is no uncertainty, only risk free assets are issued. For $t > T$; the debt satisfies

$$\frac{\omega}{1-\beta} = U' b_{t-1}$$

since there is no uncertainty form there on. Note that this must also hold for period T if $g_T = 0$; since all periods are alike from there on. Finally, at period T; the debt must satisfy

$$\omega + \alpha b^{\phi}_{T-1}\beta U^{\phi\prime} + (1-\alpha)b_{T-1}\beta U' = b_{T-2}U'$$

and recall that b_{T-2} is negative and ω positive. Also b_{T-1} is positive, so $\bar{b}_{T-1}$ is negative.

Thus, the goverment will be running budget surpluses, accumulating assets, up to date T. At date $T-1$ the goverment will be selling bonds that pay on the event that the expenditure is low, and buying bonds that pay contingent on the expenditure being high.

We now show that it is possible to replicate the same allocation with one and two period risk free bonds. The implementability constraints, evaluated at T in both states, must satisfy

$$\begin{aligned}
(b^T_{T-2} + b^T_{T-1})\bar{U}' + \beta U' b^{T+1}_{T-1} &= \bar{\omega} + \frac{\beta}{1-\beta}\omega \\
(b^T_{T-2} + b^T_{T-1})U' + \beta U' b^{T+1}_{T-1} &= \omega + \frac{\beta}{1-\beta}\omega
\end{aligned}$$

As there is no uncertainty until period T; we assume that the government issues only one period debt (actually assets!) for $t < T-1$; so $b^T_{T-2} = 0$: We can writte the system as

$$\begin{pmatrix} \bar{U}' & \beta U' \\ U' & \beta U' \end{pmatrix} \begin{pmatrix} b^1_{t-1} \\ b^2_{t-1} \end{pmatrix} = \begin{pmatrix} \bar{\omega} + \frac{\beta}{1-\beta}\omega \\ \omega + \frac{\beta}{1-\beta}\omega \end{pmatrix}$$

so the solution for the values of debt issued at $T-1$ is

$$\begin{aligned}
b^1_{T-1} &= \frac{\beta U'(\bar{\omega} - \omega)}{\beta U'(\bar{U}' - U')} \\
b^2_{T-1} &= \frac{\frac{1}{1-\beta}(\bar{U}'\omega - [(1-\beta)\bar{\omega} + \beta\omega]U')}{\beta U'(\bar{U}' - U')}
\end{aligned}$$

Recall that $\omega > 0$ and $\bar{\omega} < 0$: By Lemma 1, consumption is lower when governmnt expenditures are higher, so $\bar{U}' > U'$: Thus, the government accumlates assets in the short run and issues debt in the long run.

With one and two period bonds, we can replicate the payoff of the contingent debt, provide that the conditions of the Theorem 1 are satisfied, being long on the debt worth more on the state with high expenditure, and short on the other debt. The relevant price to determine the optimal maturity structure of government debt is the interest rate of period T, since this will determine the price of the one period debt of period T, that will be the price at period T of the two period debt issued on period $T - 1$.

Lemma 1 established that consumption is relatively low when the expenditure is relatively high, implying that the interest rate is higher on this state (i.e., the price of the two period bonds that are being carried from the previous period is low). Then, the government at $t = T - 1$ will be issuing long term debt and accumulating short term assets.

Given that the interest rate at T does depend on the value of g_T, the matrix of returns is nonsingular and the system has a (unique) solution. Note however, that the matrix of returns can be singular. If V is linear in leisure, then the value of consumption[10] is constant across states and time periods, so $U' = \tilde{U}'$. If this is the case, the government can approximate arbitrarely well the Ramsey solution by letting the income tax at time T if $g_T = G$ be equal to the Ramsey tax plus epsilon, and reduce accordinlgy the tax in all other sates and periods such that the budget constraint holds. Note that for all positive values of epsilon the matrix of returns will be nonsingular. As epsilon goes to zero, welfare goes to Ramsey.

Example 3 :(corresponds to example 7 in LS) Let $b_0^j = 0$ j=1,2; let $g_0 = G > 0$. If $g_t = G$, then $g_{t+1} = G$ with probability α and $g_{t+1} = 0$ with probability 1-α. If $g_t = 0$, then $g_{t+1} = 0$.

Again, there will be only two possible taxes in equilibrium and, correspondingly, two possible allocations. Let ω and Δ have the same interpretations as before. The implementability constraint at time zero can be writen as

$$E_0\left[\sum_{t=0}^{\infty}\beta^t\omega_t\right] = 0$$

or

$$\frac{\Delta}{1-\beta} + \frac{\omega\beta(1-\alpha)}{(1-\beta)(1-\beta\alpha)} = 0$$

As before, $\omega > 0$ and $\Delta < 0$. Now, at $t > 0$, it has to be the case that when expenditures are high,

$$b_{t-1}(G)\tilde{U}' = \frac{\Delta}{1-\beta} + \frac{\omega\beta(1-\alpha)}{(1-\beta)(1-\beta\alpha)} = 0$$

[10] This trivially follows from the first order conditions of the Ramsey problem.

while

$$b_{t-1}(0)\bar{U}' = \frac{\omega}{1-\beta}$$

if expenditures are low. Thus, if expenditures are high, the government runs a deficit and finances it with debt that only pays if next period expenditures are low. While expenditures are high, the government repeats the policy. If at some t expenditures are low,the government will be runing a surplus for ever, to pay the interest on the debt and will roll over the debt.

Similarly to the previous example, the gobvernment will be running budget surpleses when expenditures are zero and deficits when they are positive. Initial net liabilities will be zero if expenditures are positive and positive in case they become zero. Thus, the government issues debt that pays only if expenditures become zero.

Also as in the previous example, the value of the two period debt is low if g is high, so the same intuition applies, and the optimal debt structure is similar: long term liabilities and short term assets holdings.

We show now how the same allocation can be implemented by appropiately issuing one and two period bonds. In this case, the implementability constraints at time one, in each state, must satisfy

$$\begin{aligned}
b_0^1\bar{U}' + \beta E U_2' b_0^2 &= \frac{\Delta}{1-\beta\alpha} + \frac{\omega\beta(1-\alpha)}{(1-\beta)(1-\beta\alpha)} \\
b_0^1 U' + \beta E U_2' b_0^2 &= \frac{\omega}{1-\beta}
\end{aligned}$$

or

$$\begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix} = \begin{pmatrix} \bar{U}' & \beta E U_2' \\ U' & \beta E U_2' \end{pmatrix}^{-1} \begin{pmatrix} \Delta + \frac{\beta}{1-\beta}\omega \\ \omega + \frac{\beta}{1-\beta}\omega \end{pmatrix}$$

and the solution will be

$$\begin{aligned}
b_0^1 &= \frac{1}{1-\beta\alpha}\frac{\Delta-\omega}{\bar{U}'-U'} \\
b_0^2 &= \frac{1}{\beta(\bar{U}'-U')(\alpha\bar{U}'+(1-\alpha)U')}\left[\frac{\bar{U}'\omega}{1-\beta} - \frac{U'}{1-\beta}\left(\frac{\Delta(1-\beta)}{1-\beta\alpha} + \frac{\omega\beta(1-\alpha)}{1-\beta\alpha}\right)\right]
\end{aligned}$$

As before, $\Delta < 0$; $\omega > 0$ and $\tilde{U}' - U' > 0$. Also, $\left(\frac{\Delta(1-\beta)}{1-\beta\alpha} + \frac{\omega\beta(1-\alpha)}{1-\beta\alpha}\right) < \omega$; since it is a weigthed sum of Δ and ω. Thus, the optimal policy implies to hold short term assets and long term liabilities.

How about the optimal debt policy for general t? Note that the solution is not stationary, since at the begining of time the government does not inherit a two period bond, while this will be the case from period one on. Thus, the optimal debt structure at time $t \geq 2$ while government expenditure is high, must satisfy

$$
\begin{aligned}
(b^t_{t-2} + b^t_{t-1})\tilde{U}' + \beta E_t U'_{t+1} b^{t+1}_{t-1} &= \frac{\Delta}{1-\beta\alpha} + \frac{\omega\beta(1-\alpha)}{(1-\beta)(1-\beta\alpha)} \\
(b^t_{t-2} + b^t_{t-1})U' + \beta E_t U'_{t+1} b^{t+1}_{t-1} &= \frac{\omega}{1-\beta}
\end{aligned}
$$

The only difference with period 1 is that $b^t_{t-2} \neq 0$. The solution now will be

$$
\begin{pmatrix} b^t_{t-1} \\ b^{t+1}_{t-1} \end{pmatrix} = \begin{pmatrix} \tilde{U}' & \beta E U' \\ U' & \beta E U' \end{pmatrix}^{-1} \left\{ \begin{pmatrix} \frac{\Delta}{1-\beta\alpha} + \frac{\omega\beta(1-\alpha)}{(1-\beta)(1-\beta\alpha)} \\ \frac{\omega}{1-\beta} \end{pmatrix} - \begin{pmatrix} b_0^2 \tilde{U}' \\ b_0^2 U' \end{pmatrix} \right\}
$$

Thus, the amount of long term assets issued is the same as before, while the amount of short term debt issued is larger, given by

$$
b^t_{t-1} = b^1_0 - b^t_{t-2}
$$

The intuition is simple. Since what matters is the net value of assets due at t, regardless of when they were issued, the accumulation of assets after period one has to compensate for the two period debt issued two periods before.

Example 4 :Let $b_0^j = 0$ j=1,2; $g_0 = g^h$ and assume the government expenditures follows a markov process with two states, g^h and 0, where $g^h > 0$;with transition matrix $\Pi = \begin{pmatrix} \pi_{hh} & \pi_{hl} \\ \pi_{lh} & \pi_{ll} \end{pmatrix}$.

The implementability constraint in period zero implies that

$$
E_0\left[\sum_{t=0}^{\infty} \beta^t \omega_t\right] = 0 = E_t\left[\sum_{t=0}^{\infty} \beta^t \omega_t \mid g_t = g^h\right] = E_h
$$

and this will also be the case for every t such that $g_t = g^h$; given the recursivve structure of the economy. Similarly

$$k = E_t\left[\sum_{t=0}^{\infty}\beta^t \omega_t \mid g_t = 0\right] = E_l$$

where k is constant to be determined. Given the transtion matrix, and letting ω and $\bar{\omega}$ be the value of the surplus when government expenditures are low and high respectively, as before, $E_h; E_h$ must satisfy

$$\begin{aligned} 0 &= E_h = \bar{\omega} + \beta[\pi_{hh}E_h + \pi_{hl}E_l] \\ k &= E_l = \omega + \beta[\pi_{lh}E_h + \pi_{ll}E_l] \end{aligned}$$

or

$$\begin{aligned} 0 &= E_h = \bar{\omega} + \beta\pi_{hl}k \\ k &= E_l = \omega + \beta\pi_{ll}k \end{aligned}$$

which means that $k > 0$. Thus, the optimal maturity structure the first period will satisfy

$$\begin{aligned} b_0^1\bar{U}' + \beta E U_2' b_0^2 &= 0 \\ b_0^1 U' + \beta E U_2' b_0^2 &= k \end{aligned}$$

which can be written

$$\begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix} = \begin{pmatrix} \bar{U}' & \beta E U_2' \\ U' & \beta E U_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ k \end{pmatrix}$$

and the result will be

$$\begin{aligned} b_0^1 &= \frac{-k}{\bar{U}' - U'} \\ b_0^2 &= \frac{\bar{U}'}{\beta E(U_2')}\left[\frac{-k}{\bar{U}' - U'}\right] \end{aligned}$$

Again, the optimal structure involves holding short run assets and long run liabilities. The optimal structure for $t > 1$ satisfies

$$
\begin{aligned}
(b^t_{t-2} + b^t_{t-1})\tilde{U}' + \beta E U'_{t+1} b^{t+1}_{t-1} &= 0 \\
(b^t_{t-2} + b^t_{t-1})U' + \beta E U'_{t+1} b^{t+1}_{t-1} &= k
\end{aligned}
$$

and the solution will be as before,

$$
\begin{aligned}
b^t_{t-1} &= b^1_0 - b^t_{t-2} \\
b^{t+1}_{t-1} &= b^2_0
\end{aligned}
$$

Discussion of the results In all the examples discussed, optimality calls for accumulating short run assest and holding long run liabilities. Lemma 1 is behind this result. Note that in the three cases, in the solution of b^t_{t-1}; i.e., the optimal amount of one period debt to issue, the term $(\tilde{U}' - U')$ is in the denominator. Lemma 1 establishes teh conditions under which that term is larger than zero. Recall that $\tilde{U}'$ is the marginal utility of consumption when government expenditures are high, which, if the conditions of Lemma 1 are satisfied, is larger than U'. The reason for this is that otimality calls for a larger income tax when governement expenditures are higher. Being the solution of a Ramsey problem, is has to be the case that the demand elasticity of consumption is higher when government consumption is higher under the conditions of Lemma1.

Recall that tax smoothing across states with contingent debt is achieved by issuing debt that does not pay off when government expenditures are high. To reproduce the result with short and long term debt, and if the interest rate goes up when governemnt expenditures go up, it is optimal to have a portfolio whose valueis increasing with the current interest rate, i.e., a portfolio with short run assets and long run debt.

4 Simulations. IN PROGRESS

As we mentioned before, Marcet et al have shown that the stochastic properties of optimal policies depend crucially on the market structure. When markets are complete, as in Lucas and Stokey, the serial correlation of optimal taxes and governemnt debt are tied closely to that one of goverenment expenditures. On the other hand, when the only asset is a one-period risk free bond, a series of constraints must be imposing, that ensure that $z(g_t; g^{t-1}) = z(g^{t-1})$ for all $t; g^{t-1}$: They show that the multiplier of this constraint imparts a unit-root component to the solution of the optimal tax rate.

As we showed in this paper, the existance of several maturities allows the government to at least partially complete markets. In a discrete time model like the one we have it would be impossible to complete markets in the environment computed by Marcet et al, since the support of the shock is a continuum. Thus, allowing for the maturities that the US governemnt issues will complete only patially markets.

The numericall question we adress in this section is if a model like the one in Marcet et al implies that optimal taxes behave as random walks when teh government uses optimaly the maturity structure to better approximate the Ramsey allocation.

TO BE DONE.

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