Equilibrium Price Dispersion with Sequential Search.

J. Rupert J. Gatti

Trinity College

Cambridge, UK

31 January 2000

Initial Draft Version

(submitted for consideration for presentation at

Econometric Society World Congress, Seattle, Aug. 2000,

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Contact Address: Rupert Gatti

Trinity College

Cambrdige, CB2 1TQ

United Kingdom

e-mail: jrjg1@cam.ac.uk

tel: +44 (0)1223 339929

Abstract

Diamond's 'paradox' (1971) showed that in a market where consumers search sequentially and have strictly positive search costs the unique price equilibrium is where all firms charge the monopoly price. This paper demonstrates that Diamond's result depends crucially on the assumption of single commodity search and does not persist when the model is generalised to allow multi-commodity search. A model is presented where identical consumers search optimally (sequentially) and with positive search costs for two commodities. Firms supply only one of the commodity types so consumers are required to sample at least two firms to satisfy their consumption requirements. Within industries firms are identical, producing a homogenous product at the same, constant, marginal cost. The equilibrium is shown to display price dispersion, in fact no two firms charge the same price with positive probability. Comparative statics are conducted and it is demonstrated that the price dispersion depends solely on the search behaviour of consumers, converging to the competitive price as search costs converge to zero. Changes in industry demand effect equilibrium prices only through the indirect impact the change in demand has on the consumers' search behaviour.

JEL Classification No: D83

1. INTRODUCTION

Stigler (1961) observed that when consumers are not perfectly informed about prices they will search to discover favourable prices, and proposed that this search process could provide some explanation for the magnitude of price dispersion observed in real markets[1]. More recently, numerous commentators have suggested that one impact of the introduction of e-commerce will be a reduction in mark-ups that can be sustained by firms due to the reduction in the cost for consumers of comparing prices. Unfortunately existing search theory does not provide support for either proposition.

The dominant, and most challenging, result in the search literature was developed by Diamond (1971) who showed that when consumers search sequentially for one commodity, and search costs are strictly positive, the unique equilibrium will be at the monopoly price. When search costs are zero, however, the model reduces to a Bertrand pricing game for which the unique solution is at the competitive price.

Diamond’s result generates several uncomfortable implications. Firstly, when search costs are positive, all firms should charge the monopoly price irrespective of the size of the industry or the actual cost of search, a reduction in search cost would not change the equilibrium price charged. Secondly, as neither equilibrium displays price dispersion there is no role for search in equilibrium, and Stigler's conjecture that the search process will sustain price dispersion appears unfounded. Thirdly, there is a fundamental discontinuity in the equilibria, when search costs are strictly positive the monopoly price results but at zero search costs the competitive price results.

If price dispersion is to exist in equilibrium then Diamond's result suggested that some additional mechanism is required. Several alternatives have been suggested, usually

[1]Stigler provides evidence for the existence of price dispersion in markets for nearly homogeneous goods, see also Pratt *et al.* (1979) and Dahlby & West (1986).

relying on some form of exogenously specified heterogeneity amongst agents in the economy.

The most common technique is to assume heterogeneity in the consumers' cost of search. Diamond's discontinuity problem can be overcome if some consumers have zero search costs and others have positive search costs. It has been shown[2] that equilibrium price dispersions can be achieved if a distribution of search costs amongst consumers is allowed, and zero is an element of the support of the distribution. However, the resulting equilibria are not robust to changes in the distribution of search costs, particularly to changes close to zero.

Reinganum (1979) generates price dispersion through heterogeneity in producer costs, and many other authors[3] combine a distribution of producer costs with a distribution of consumer search costs to generate price dispersion. Heterogeneity in consumer tastes (Paulsen & von Ungern-Sternberg (1992)) or the consumers' willingness to pay for the commodity (Diamond (1987)) have also been used to generate price dispersion.

A second technique for generating price dispersion is to consider models where consumers do not search sequentially.[4] Burdett & Judd (1983) demonstrate that price dispersion will be generated if, in equilibrium, consumers have a strictly positive probability of receiving exactly one price quotation <u>and</u> a strictly positive probability of receiving more than one price quotation. They introduce the concept of 'Noisy search', where consumers cannot control the number of price quotations received each time they search, and show that price dispersion in equilibrium is assured when the probability distribution over price quotations satisfies stated condition.

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[2]Axell (1977), Rob (1985), Stahl (1989, 1996)
[3]e.g. MacMinn (1980), Carlson & McAfee (1982), Benabou (1993)
[4]Burdett (1990) reviews this literature.

All the works cited consider cases where consumers search for only one type of commodity. McAfee (1995) is the only paper that extends the analysis to multi-commodity search models[5]. McAfee develops a multi-commodity extension of the 'Noisy search' model originally presented by Burdett & Judd. He presents a model where consumers search amongst firms to minimise expenditure on a predetermined bundle of goods, all firms sell all the goods desired by consumers, and the search process is costly but noisy - that is, the marginal cost of making a sample is constant and strictly positive but there is a positive probability of receiving information from only one firm, and from more than one firm, each sample. McAfee characterises the equilibria in this model and shows that every equilibrium displays price dispersion.

There are several weaknesses with McAfee's model which provide a motivation for the model presented in this chapter. Firstly, the results of the model depend critically on the assumption of Noisy search, a sub-optimal search process which not only suggests that consumers are unable to control the amount of information they receive *and analyse* when making their purchase decisions, but also that the distribution of the quantity of such information is exogenously determined. Secondly, McAfee assumes that all firms sell every commodity desired by every consumer. Clearly this is not so, most stores specialise in the types of commodities supplied and so consumers must purchase from several different stores to satisfy their demand.

In this paper a multi-commodity search model is developed where consumers search to minimise total expenditure on a specific bundle of commodities, but firms do not sell all the commodities desired, so consumers must search amongst different firms to satisfy their demand. The model is a natural multi-commodity extension of the single commodity model analysed by Diamond, however the results are very different. It is

[5]In a different framework Albrecht, Axell & Lang (1986) provide an interesting model where consumers search simultaneously for wages and prices, and firms select wages and prices to maximise revenue. A price dispersion equilibrium is generated.

shown that all equibria in the model display price dispersion, with no two firms charging the same price with positive probability. Furthermore, as the cost of search falls to zero the equilibrium distribution of prices converges to the competitive price.

These results are achieved in a model without any of the heterogeneity amongst agents or the 'non-optimal' search processes which proved necessary in single commodity search models. The equilibrium price dispersion is a repercussion of the search process itself, lending support to Stigler's original conjecture and suggesting that Diamond's 'monopoly price' equilibrium is an artefact of the assumption of single commodity search. Interestingly, changes in demand effect the equilibrium only through their impact on consumers' search behaviour. So, for example, a doubling of the quantities demanded by each consumer has an identical effect on the consumers' search behaviour as a halving of the cost of search, and consequently the effect on the distribution of prices charged is also identical. An increase in the number of consumers per firm, however, has no impact on search behaviour and so no impact on the distribution of prices charged.

Intuitively, there are three important forces driving the results. Firstly, as stated earlier, firms are not supplying all commodities – so consumers must enter at least two firms to satisfy demand. Secondly, consumers do not know exactly which commodity will be supplied by a firm prior to sampling.[6] Consequently, when the making their purchase decision a consumer may have sampled one, or more than one, firm supplying each commodity type. As Burdett & Judd (1983) pointed out, this occurrence generates a price dispersion in equilibrium. Finally, the potentially counter-intuitive result that prices 'fall' as consumer demand increases is explained by the assumption of constant marginal costs of production, so changes in demand have

[6] Of course, if consumers were perfectly informed about this the model would resort to a single commdoity search model, as analysed by Diamond.

no impact upon the competitive price and prices above marginal cost are maintained only because consumers are not perfectly informed.

The paper is divided into three further sections. In the following section the basic model is presented, equilibria in the model are shown to exhibit price dispersion (Proposition 1), and one particular equilibrium is characterised (Proposition 2). Comparative statics are conducted for this equilibrium in Section 3, where the impact of changes in search costs (Proposition 3) and demand (Proposition 4) on the equilibrium are analysed. Conclusion are contained in the final section, and most of the proofs are contained in the Appendix.

2. THE MODEL

The model analysed throughout this paper is based on the following three assumptions:

Assumption 1. There are two types of commodities, denoted by $i \in \{1,2\}$.

Assumption 2. There exists a continuum of firms, with mass M. Each firm supplies one commodity type at a constant marginal cost which, without loss of generality, is set equal to zero. Each firm selects a sale price $p \in \Re_+$ for the commodity supplied to maximise expected profits (revenue), and satisfies all demand received. Let $F_i(p_i)$ be the distribution of prices charged by firms supplying commodity i, and M_i be the mass of firms supplying commodity i (referred to collectively as industry i).

Assumption 3. Consumers have inelastic demand for n_i units of commodity i and may, at any time, sample one firm randomly at a cost $0 \leq k < \infty$ per sample.[7] There is a positive probability of sampling a firm from either industry; let π, $0<\pi<1$, denote the probability of sampling a firm from industry 1, and $1-\pi$ the probability of sampling a firm from industry 2. Consumers know π, have perfect recall, and search optimally to minimise the cost of purchasing the desired consumption bundle (n_1,n_2) given their beliefs over the distribution of prices in each industry, F_1 and F_2.

[7] It has been standard in the search literature to assume unit demand for commodities. Elsewhere (Gatti (1999)) I have criticised this tendency, pointing out that the consumers' search behaviour depends on the nature of the consumers' indirect utility function. In this model the inclusion of an elastic demand function would require the explicit inclusion of an indirect utility function. Providing the indirect utility function is submodular, so the consumers' acceptance sets are comprehensive (ref. Gatti (1999)), and the consumers expenditure function is concave, the main results in this paper can be shown to hold – however the extra analytical effort involved is considerable and provides little additional insight. It is also worth noting that we do not need to assume that a maximum acceptable price exists, as required for example by Diamond (1971), and consequently the difficulty of ensuring a positive consumer surplus in equilibrium is also avoided.

The equilibrium concept is Perfect Baysian so, in equilibrium, consumers hold the correct beliefs over the distribution of prices charged, and firms maximise expected revenue given correct beliefs over the search behaviour of consumers.

Let $R_i(p_i : \Sigma, F_1, F_2)$ be the revenue generated by a firm in industry i charging price p_i when consumers adopt the search strategy Σ and the distribution of prices charged by other firms is given by F_1 and F_2.

Definition: An *equilibrium* is a triple (Σ, F_1, F_2) where Σ is an optimal search strategy for consumers given the distribution of prices charged, and $F_1$and F_2 are probability distributions over price such that $R_i(p_i : \Sigma, F_1, F_2) \leq R_i^*$ for all $p_i \in \Re_+$, where $R_i^* \in \Re_+$ and $R_i(p_i : \Sigma, F_1, F_2) = R_i^*$ for all p_i in the support of F_i .

Before analysing the model further it will be useful to introduce the two more definitions.

Definition: The *best price vector*, $\boldsymbol{q} = (q_1,q_2)$, records the lowest price a consumer has observed for each of the commodity types. For notational ease, set $q_i = \infty$ when no firm in industry i has been observed.

Definition: A set $A \subseteq \Re^2$ is a *comprehensive set* if and only if $\boldsymbol{q} \in A$, and $\boldsymbol{q'} \leq \boldsymbol{q}$ means $\boldsymbol{q'} \in A(k)$.[8]

We now consider the optimal search strategy adopted by consumers.

[8] The inequality denotes the standard vector inequality in Euclidean Space.

Lemma 1:

Given Assumptions 1-3, the consumers' optimal search strategy is to search sequentially until a best price vector $\boldsymbol{q} \in A(k)$ *is discovered, where*

$$A(k) = \{\boldsymbol{q} \in \Re_+^{\,2} \mid \pi\, n_1 \int_0^{q_1} F_1(p_1)dp_1 + (1-\pi)n_2 \int_0^{q_2} F_2(p_2)\,dp_2 \leq k\,\} \qquad (1.1)$$

Furthermore, for all k>0, A(k) is a convex, compact and comprehensive set.

Proof:

The result is a simple extension of Burdett & Malueg (1981), Theorem 1, pp.369-70 and the discussion following, the only difference being the explicit inclusion of n_i units of each commodity. Gatti (1999, Proposition 7, p.235) provides a direct proof; setting the consumer's indirect utility function $u(\boldsymbol{q}) = -(n_1q_1 + n_2q_2)$ and noting that, integrating by parts,

$$\int_0^{q_i} (q_i - p_i)dF(p_i) \quad = \quad \int_0^{q_i} F(p_i)dp_i \qquad \textit{QED}$$

Having identified the consumers' optimal search behaviour we now show that any equilibrium in the model must display price dispersion when search costs are positive.

Proposition 1:

Given Assumptions 1-3,

a. when k = 0, the unique equilibrium requires all firms to charge the competitive (marginal cost) price.

b. when k > 0; every equilibrium displays price dispersion, no equilibrium price distribution will have a mass point, and the competitive price is not an element of the support of any equilibrium price distribution.

Proof:

a. When search costs equal zero consumers will continue searching until they have discovered a firm charging the lowest price. The model becomes a Bertrand pricing game, for which the unique equilibrium has all firms charging the marginal cost.

b. The proof is contained in the Appendix. It is shown that the marginal cost is not an element of the support of equilibrium price distribution, and that for any price above marginal cost the expected revenue function will increase discontinuously with a reduction in price if a positive mass of firms are charging that price. The proof formalises the intuition proposed by Burdett & Judd (1983), showing that in equilibrium some consumers sample only one firm in an industry and others sample more than one.

Proposition 1 demonstrates that the results for multi-commodity search models with positive search costs differ significantly from the single commodity model analysed by Diamond (1971). In Diamond's model all firms charged the same (monopoly) price in equilibrium while in this 2-commodity model no two firms will charge the same price with positive probability. Of course we have not, as yet, shown that an equilibrium price distribution actually exists when search costs are positive. Proposition 2 characterises one equilibrium price distribution.

Proposition 2:

Given Assumption 1-3 and $k > 0$, there exists an equilibrium (Σ, F_1, F_2) where consumers select the search strategy described in Lemma 1 and, for $i \in \{1,2\}$, the distribution of prices charged by firms is given by

$$F_i(p_i) \;=\; \min\left\{ 1 \;,\; \max\left\{ \frac{\sqrt{\dfrac{\pi_i p_i}{(1+\pi_i)h_i - p_i}} - (1-\pi_i)}{\pi_i} \;,\; 0 \right\}\right\} \qquad (2.1)$$

where π_i is the probability of sampling a firm from industry i, h_i is the largest element in the support of the distribution F_i ,

and h_1 and h_2 satisfy the equations:

$$\pi\, n_1 \int_0^{h_1} F_1(p_1)\mathrm{d}p_1 + (1-\pi)n_2 \int_0^{h_2} F_2(p_2)\mathrm{d}p_2 = k \qquad (2.2)$$

and

$$\frac{h_1}{h_2} = \frac{n_2(1-\pi)^3(1+\pi)}{n_1\pi^3(2-\pi)} \qquad (2.3)$$

Furthermore, the supports of the equilibrium price distributions are convex, with lowest price

$$l_i = \frac{(1-\pi_i)^2(1+\pi_i)h_i}{\pi_i + (1-\pi_i)^2} \qquad (2.4)$$

and the expected revenue earned by all firms in industry i is

$$R_i* = \frac{C(1-\pi_i)(1+\pi_i)n_i h_i}{M_i \pi_i} \qquad (2.5)$$

Proof:

Contained in the Appendix.

The proof to Proposition 2 considers only price distributions where (h_1, h_2) is an element of the consumer's acceptance set, effectively considering only equilibria with the property that consumers stop searching once they have obtained a price quotation from each industry. It shows that an equilibrium satisfying this condition exists, but does not rule out the possibility of additional equilibria not satisfying the condition. While we can be certain (from Proposition 1) that such equilibria display price dispersion we do not analyse these possible equilibria further.

It is worth noting that, given k, the values of h_1 and h_2 are uniquely defined in Proposition1. These are the highest prices that can be charged in equilibrium, and are determined by the search behaviour of consumers. Consumers in this model have inelastic demand without an upper limit on price, so are prepared to pay any price for the required consumption bundle. The degree to which firms are able to exploit this depends on the difficulty consumers have in comparing prices. As Stigler (1961) conjectured, the equilibrium distribution of prices is sustained by the imperfect information held by consumers, and their search behaviour. In the next section we consider directly the effects changes in the cost of search and demand have on the equilibrium distribution of prices.

3. COMPARATIVE STATICS

We now wish to consider how the equilibrium identified in Proposition 2 is effected by changes in the cost of search and the level of demand. Before doing so the following definitions are introduced.

Definition: Consider two distributions defined on the same domain *(X)*, *G(x)* and *H(x)*; the distribution *H First Order Stochastic Dominates* (FOSD) distribution *G* if and only if, for all $x \in X$, $H(x) \leq G(x)$.

Definition: Consider a distribution *G(x:z)* defined on a domain *X* and dependent on a coefficient $z \in \Re$. I will say that the distribution *G(x:z)* is *increasing (decreasing) in z* if, for any $z' > z$, $G(x:z')$ First Order Stochastic Dominates $G(x:z)$ ($G(x:z)$ FOSD $G(x:z')$).

Proposition 3: (Change in the cost of search)

For $i \in \{1,2\}$, let $F_i*(p_i:k)$ denote the equilibrium price distribution identified in Proposition 2 for a given value of k, and let $h_i*(k)$ be the highest element in the support of $F_i*(p_i:k)$:

(i) $F_i*(p_i:k)$ *is increasing in k so, for* $k' > k$, $F_i*(p_i:k')$ *FOSD* $F_i*(p_i:k)$

(ii) $h_i*(k)$ *is an increasing function of k*

(iii) $\lim_{k\to 0} h_i*(k) = 0$.

Proof:

Contained in the appendix.

Proposition 3 shows that the equilibrium distribution of prices 'decrease' as search costs fall, converging to the competitive price as search costs converge to zero. Clearly this is a very different result to the dramatically discontinuous behaviour of prices in Diamond's (1971) single commodity search model.

Proposition 4: (Changes in Demand)

For $i \in \{1,2\}$, let $F_i*(p_i:n_i)$ denote the equilibrium price distribution identified in Proposition 2 for a given value of n_i, and let $h_i*(n_i)$ be the largest element in the support of $F_i*(p_i:n_i)$:

(i) *Changes in C, M_1 or M_2 have no impact on the distribution of prices charged in equilibrium*

(ii) $F_i*(p_i:n_i)$ *is decreasing in n_i, so for* $n_i' > n_i$, $F_i*(p_i: n_i)$ *FOSD* $F_i*(p_i:n_i')$

Proof:

Contained in the Appendix

Proposition 4 highlights important differences in the effect on the equilibrium distribution of prices from changes in demand. An increase in demand generated by an increase in the mass of consumers per store will increase revenue but have no impact on the distribution of price charged. An increase in demand generated by an increase in the quantity desired by individual consumers will '*lower*' the equilibrium distribution of prices. The result highlights the fact that it is the search behaviour of consumers which is generating the equilibrium price distribution – as was initially suggested by Stigler (1961). A change in the mass of consumers or firms has no impact on the search behaviour of any individual consumer, and so has no impact on the equilibrium prices. An increase in the demand for one commodity does, however, alter search behaviour – encouraging further search. Consequently the price distribution lowers in response to the increase in consumer search. It is obvious from Eqn (2.2) in Proposition 2 that the effects on the equilibrium distribution of prices from doubling the quantity demanded of each commodity is precisely equivalent to halving the search costs; the consumers' search behaviour is identical in these two cases.

This result, that prices fall as demand increases, may at first seem counter-intuitive. It is worth reminding readers that these results are generated in a model with constant marginal costs of production and no capacity constraints, so the perfectly competitive price is not effected by changes in demand. Prices above marginal cost can be considered as 'mark-ups', sustained by the imperfect information held by consumers. As consumer demand increases the consumers' incentive to collect information rises, and consequently the sustainable 'mark-up' falls.

4. CONCLUSIONS & EXTENSIONS

In this paper we have demonstrated the validity of Stigler's (1961) proposal that the observed distribution of prices for apparently homogenous commodities can be explained by the imperfect information held by consumers, and their search behaviour. Diamond's problematic 'paradox' appears to be an artefact of his assumption that consumers search for individual commodities independently. The paper also provides theoretical support for the proposal that 'mark-ups' will be cut due to the increased ability consumers now have to compare prices using the internet.

There are a number of extensions to this model that I am presently working on:

1. Multiple search techniques.

At present consumers are restricted to one search tool – giving probability π of sampling a firm from industry 1. Obviously consumers have a number of different search techniques available to them, and they may choose 'optimally' between them.

Strong Conjecture: With two search techniques available a continuum of equilibrium price distributions exist – all satisfying the same comparative static behaviour as observed in Section 3.

2. Joint Production Costs:

One possible criticism of the model presented is that the firms produce one commodity type *by assumption*, even though it would be more profitable for them to sell both types. Including production costs that increase with the number of commodities sold will ensure that specialisation is an equilibrium condition for sufficiently low search costs.

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APPENDIX

Proof of Proposition 1b:

First we show that zero cannot be an element of the support of any equilibrium price distribution, then we show that the expected revenue can be increased discontinuously with a price reduction if a positive mass of firms are charging a price greater than zero.

Let $R_i(p_i)$ be the expected revenue by a firm in industry i charging price p_i. If F_1* and F_2* are equilibrium distributions of prices then all prices in the supports of these distributions must have the same expected revenue, i.e.

for all $p_i \in \text{supp}\{F_i{*}\}$, $R_i(p_i) = R_i{*}$.

We show that R_i* must be strictly greater than zero and therefore that zero cannot be an element of the support of an equilibrium price distribution.

From Lemma 1 we know that there exists a pair of prices b_1, $b_2 > 0$ where

$$\pi\, n_1 \int_0^{b_1} F_1{*}(p_1)dp_1 + (1-\pi)n_2 \int_0^{b_2} F_2{*}(p_2)\,dp_2 = k\ \}$$

and, for some $\varepsilon > 0$, $F_1{*}(b_1) > \varepsilon$ and $F_2{*}(b_2) > \varepsilon$.

A firm in industry 1 will sell to any consumer who, when searching, samples only a firm in industry 2 charging a price $p_2 \le b_2$ prior to discovering the firm. The expected number of consumers following this sample path is $C(1-\pi)F_2{*}(b_2)\pi / M_1$, and so

$$R_1(b_1) \quad = \quad R_1{*} \quad \geq \quad \left(\frac{C(1-\pi)F_2{*}(b_2)\pi}{M_1} \right) n_1 b_1 \quad > \quad 0$$

as $R_1(0) = 0 < R_1{*}$, $0 \notin \text{supp}\{ F_1{*}\}$.

A similar analysis for industry 2 gives $0 \notin \text{supp}\{ F_2{*}\}$.

We now show that an equilibrium price distribution cannot give a strictly positive probability to any price greater than zero. The proof is by contradiction, and is constructed for industry 1. Obviously it extends directly to industry 2.

Assume that there exists $b_1 \in \text{supp}\{F_1*\}$, such that Prob$\{p_1 = b_1\} = P > 0$. The mass of consumers initially discovering a firm in industry 1 charging b_1 is $C\pi P$, and a proportion π of these then sample another firm in industry 1. Consider a firm in industry 1 charging a price b_1, $C\pi P\pi/M_1$ consumers will enter another firm in industry 1 charging price b_1 prior to entering the firm – and so any sales made to these consumers will be shared between the two firms. We may assume, without loss of generality, that the firms' share is less than one, so a marginal reduction in the price charged will ensure sales are made to all these consumers – and that revenue generated from these consumers increases discontinuously. From Lemma 1, the consumers' acceptance set is comprehensive so a firm reducing prices marginally will not loose any of their existing consumers. Consequently, we may conclude that a discontinuous increase in revenue has been generated by the marginal price reduction, and so $b_1 \notin \text{supp}\{F_1*\}$, a contradiction. *QED*

Proof of Proposition 2:

Recall, an equilibrium is a triple (Σ, F_1, F_2) where Σ is an optimal search strategy for consumers given the distribution of prices charged, and $F_1$and F_2 are probability distributions over price such that $R_i(p_i : \Sigma, F_1, F_2) \leq R_i*$ for all $p_i \in \Re_+$, where $R_i* \in \Re_+$ and $R_i(p_i : \Sigma, F_1, F_2) = R_i*$ for all p_i in the support of F_i .

The proof takes the following structure. Initially we make to two simplifying assumptions (Assumptions A1 and A2) which are shown to be satisfied in the equilibrium obtained. Given these assumptions, the firms' revenue functions are obtained (Lemma A1). In equilibrium the revenue generated must be equal for all prices charged, Lemma A2 uses this requirement to characterise the price distributions necessary for this to be true. Given these price distributions and the cost of search, Lemma 1 characterises the consumers' optimal acceptance set – and this is shown to satisfy Assumption A1. The remainder of the proof obtains conditions for a pair of price distributions, together with the resultant consumer acceptance set, to constitute an equilibrium, ensuring that prices outside the support of the distributions do not generate higher revenue.

We start the analysis by making two simplifying assumptions;

Assumption A1: The consumers adopt an acceptance set $A \subset \Re_+^2$ which is a convex, compact and comprehensive set of best price vectors, and search sequentially until a best price vector $\boldsymbol{q} \in A$ is obtained.

Assumption A2: The distributions F_1 and F_2 have no mass points, and the vector of highest prices charged is accepted by consumers, i.e. $(h_1, h_2) \in A$.

Lemma A1:

Given Assumption A1, and distributions (F_1, F_2) satisfying Assumption A2, for i=1,2
$R_i(p_i) =$

$$R_i(p_i) = \left(\frac{\pi_i C}{M_i}\right)(1-\pi_i)\left(\frac{1}{(1-\pi_i[1-F_i(p_i)])^2} + \frac{1}{\pi_i}\right)n_i p_i \qquad \text{when } (p_i, h_{-i}) \in \text{A}$$

$$= \left(\frac{\pi_i C}{M_i}\right)(1-\pi_i)\left(\frac{F_{-i}(a(p_i))}{(1-(1-\pi_i)[1-F_{-i}(a(p_i))])^2} + \frac{1}{\pi_i} - \frac{[1-F_{-i}(a(p_i))]}{1-(1-\pi_i)[1-F_{-i}(a(p_i))]}\right)n_i p_i$$
$$\text{when } (p_i, h_{-i}) \notin \text{A but } (p_i, l_{-i}) \in \text{A}$$

$= \quad 0 \qquad$ when $(p_i, l_{-i}) \notin$ A

Proof:

The proof is shown for i = 1, the proof for i = 2 follows directly.

Let

$$A^{-1}(p_1) = \{ p_2 \mid (p_1, p_2) \in A \}$$

be the set of prices for commodity 2 which would induce the consumer to stop searching and purchase both commodities when the best price discovered for commodity 1 is p_1. Given that A is compact, convex and comprehensive so to is $A^{-1}(p_1)$, although for p_1 sufficiently large $A^{-1}(p_1) = \{\emptyset\}$. Let $a(p_1)$ be the largest element of $A^{-1}(p_1)$, when it exists.

i) If $(p_1, l_2) \notin A$;

then $a(p_1) < l_2$ and $F_2(a(p_1)) = 0$, so the probability of any consumer discovering a price for commodity 2 sufficiently low for that consumer to purchase commodity 1 at price p_1 is zero. Consequently $R_1(p_1) = 0$.

ii) If $(p_1, h_2) \in A$;

we consider the histories a consumer may have when entering a firm i in industry 1, and then consider the expected number of consumers having that history, and the probability that the firm makes a sale to such a consumer.

The sample history, H, a consumer possesses when entering the firm records the industry types and prices the consumer has observed prior to discovering firm i.

So $H \in \{(I^t, p^t);\ t = 0,1,2,\ldots.\}$, where $(I^0, p^0) = \{\emptyset\}$.

There will be, on average, $\pi C/M_1$ consumers entering firm i who have just started their search process and have no previous history, $H = \{\varnothing\}$. Having entered firm i and observed p_i these consumers continue to search until they have sampled a firm in industry 2 whereupon, as $(p_1, h_2) \in A$, they complete their search and purchase both commodities. So, a sale will be made by firm i if

a) the consumer next samples a firm from industry 2, which occurs with probability (1-π)
b) the consumer samples one firm from industry 1, charging a price greater than p_i (which occurs with probability $\pi[1\text{-}F_1(p_i)]$), and then samples a firm from industry 2
c) the consumer samples two firms from industry 1, both charging prices greater than p_i , and then samples a firm from industry 2
d) etc.

The expected revenue obtained by a firm in industry 1 charging price p_1 from a consumer with history H=$\{\varnothing\}$ is

$$R_1(p_1: \{\varnothing\}) = n_1 p_1(\pi C/M_1)(1-\pi)\sum_{n=0}^{\infty}(\pi[1-F_1(p_1)])^n$$
$$= \frac{n_1 p_1(\pi C/M_1)(1-\pi)}{1-\pi[1-F_1(p_1)]}$$

Similarly, a consumer who has entered m firms from industry 1 prior to entering firm i will purchase from firm i if all firms previously sampled charged a price greater than p_i, and the consumer subsequently samples in the same way as described above. The expected revenue earned from such consumers is

$$R_1(p_1: \{\varnothing,(1,p^1),\ldots(1,p^m)\}) = \frac{n_1 p_1(\pi[1-F_1(p_1)])^m(\pi C/M_1)(1-\pi)}{1-\pi[1-F_1(p_1)]}$$

The firm will also sell to any consumer who has entered only firms from industry 2 prior to entering firm i. The expected revenue obtained from the consumers who have entered m firms in industry 2 is

$R_1(p_1$: $\{\varnothing,(2,p^1),\ldots(2,p^m)\})$ $\quad = n_1 p_1 (1-\pi)^m (\pi C/M_1)$

The firm will never have the opportunity to sell to a consumer who has sampled firms from both industries prior to entry as, from Assumption A1, $(h_1, h_2) \in A$; and we need not consider the possibility of shared sales as, from Assumption A2, F_1 has no mass points.

Adding all these term together we get

$$R_1(p_1) = \sum_{m=0}^{\infty} \frac{n_1 p_1 (\pi[1-F_1(p_1)])^m (\pi C/M_1)(1-\pi)}{1-\pi[1-F_1(p_1)]} \quad + \quad \sum_{m=1}^{\infty} n_1 p_1 (1-\pi)^m (\pi C/M_1)$$

$$= n_1 p_1 (1-\pi)(\pi C/M_1)\left(\frac{1}{(1-\pi[1-F_1(p_1)])^2} \quad + \quad \frac{1}{\pi} \right)$$

iii) If $(p_1, h_2) \notin A$ and $(p_1, l_2) \in A$;

then $p_1 > h_1$ and the firm will sell only to consumers who observe a price for commodity 2 which is less than or equal to $a(p_1)$ prior to sampling any other firm in industry 1.

Similar reasoning to that conducted in (ii) shows that

$$R_2(p_2) = \sum_{m=0}^{\infty} \frac{n_1 p_1 ((1-\pi)[1-F_2(a(p_1))])^m (\pi C/M_1)(1-\pi) F_2(a(p_1))}{1-(1-\pi)[1-F_2(a(p_1))]}$$

$$+ \quad \sum_{m=1}^{\infty} n_1 p_1 (1-\pi)^m (1-[1-F_2(a(p_1))]^m)(\pi C/M_1)$$

$$= n_1 p_1 (\pi C/M_1)(1-\pi)\left(\frac{F_2(a(p_1))}{(1-(1-\pi)[1-F_2(a(p_1))])^2} \quad + \quad \frac{1}{\pi} \quad - \quad \frac{[1-F_2(a(p_1))]}{1-(1-\pi)[1-F_2(a(p_1))]} \right)$$

Having obtained the firm's revenue function, we now use the equilibrium requirement that the revenue for all prices charged must be equal to characterise the equilibrium distribution of prices.

Lemma A2:

Given Assumption A1, and distributions (F_1, F_2) satisfying Assumption A2, the expected revenue is equalised for all elements in the supports of F_1 and F_2 if and only if for prices in the support of these distributions

$$F_i(p_i) = \left(\frac{\sqrt{\dfrac{\pi_i p_i}{(1+\pi_i)h_i - p_i}} - (1-\pi_i)}{\pi_i} \right) \quad , \quad i = 1,2 \qquad \text{(A2)}$$

Proof:

Again we present the proof when i =1, the case when i =2 follows directly.

From Lemma A1, for all $p_1 \leq h_1$

$$R_1(p_1) = \left(\frac{\pi C}{M_1} \right)(1-\pi)\left(\frac{1}{(1-\pi[1-F_1(p_1)])^2} + \frac{1}{\pi} \right) n_1 p_1$$

We require that for all p_1 in the support of F_1, $R_1(p_1)=R_1(h_1)$, so

$$\left(\frac{1}{(1-\pi[1-F_1(p_1)])^2} + \frac{1}{\pi} \right) n_1 p_1 \quad = \quad \left(1 + \frac{1}{\pi} \right) n_1 h_1$$

and so

$$F_1(p_1) = \left(\frac{\sqrt{\dfrac{\pi p_1}{(1+\pi)h_1 - p_1}} - (1-\pi)}{\pi} \right)$$

$$\left(\frac{\pi_i C}{M_i}\right)(1-\pi_i)\left(\frac{1}{(1-\pi_i[1-F_i(p_i)])^2}+\frac{1}{\pi_i}\right)p_i$$

Lemma A3:

Given Assumption A1, and distributions (F_1, F_2) given by Eqn.A2 and satisfying Assumption A2, a necessary condition for F_1 and F_2 to be equilibrium distributions is that (h_1, h_2) is a boundary element of A.

Proof:

We show that if (h_1, h_2) is an interior element of A , then there exists a price b_1 such that (b_1, h_2) $\in$ A , and $R_1(b_1) > R_1(h_1)$. Consequently, a requirement for the distributions F_1 and F_2 given in Lemma A2 to be equilibrium distributions and to satisfy Assumption A2 is that (h_1, h_2) is on the boundary of A.

If (h_1, h_2) is an interior element of A , then there exists a price $b_1 > h_1$ such that (b_1, h_2) $\in$ A, and $F_1(b_1) = 1$.

From Lemma A1

$$R_1(b_1)=\left(\frac{\pi C}{M_1}\right)(1-\pi)\left(1+\frac{1}{\pi}\right)n_1 b_1>\left(\frac{\pi C}{M_1}\right)(1-\pi)\left(1+\frac{1}{\pi}\right)n_1 h_1=R_1(h_1)$$

We can now specify the optimal search behaviour for consumers and ensure that Assumptions A1 and A2 are satisfied.

Lemma A4:

Given (h_1, h_2) and the distributions F_1 and F_2 given in Lemma A2, there exists a unique k such that (h_1, h_2) is a boundary element of $A(k)$, where $A(k)$ is the optimal consumers' acceptance set, and Assumptions A1 and A2 are satisfied.

Proof:

Select k such that

$$\pi n_1 \int_0^{h_1} F_1(p_1)dp_1 + (1-\pi)n_2 \int_0^{h_2} F_2(p_2)\,dp_2 = k$$

where $F_1(p_1)$ and $F_2(p_2)$ are the distributions given in Lemma A2.

From Lemma 1

$$A(k) = \{\boldsymbol{q} \in \Re_+^{\,2} \mid \pi n_1 \int_0^{q_1} F_1(p_1)dp_1 + (1-\pi)n_2 \int_0^{q_2} F_2(p_2)\,dp_2 \le k\ \}$$

is the optimal consumer's acceptance set, and is compact, convex and comprehensive – thus Assumption A1 is satisfied.

As, from Lemma A2, F_1 and F_2 have no mass points, and (h_1, h_2) is a boundary element of $A(k)$, Assumption A2 is satisfied.

Lemma A5:

Given (h_1, h_2), the distributions F_1 and F_2 given in Lemma A2 and the acceptance set $A(k)$ defined in Lemma A4 constitute an Perfect Baysian equilibrium if and only if

$$\frac{h_1}{h_2} = \frac{(1-\pi)^3(1+\pi)}{\pi^3(2-\pi)}$$

Proof:

From Lemma 1, $A(k)$ specifies the optimal acceptance set for consumers with search cost k determined by Lemma A4. So consumers have no incentive to change their search strategy. Given (h_1, h_2) and $A(k)$, the distributions F_1 and F_2 determined in Lemma A2 ensure that $R_i(p_i) = R_i^*$ for all $l_i \le p_i \le h_i$. To ensure that ($A(k)$, F_1, F_2) is an equilibrium, we need to check that $R_i(p_i) \le R_i^*$ for all p_i outside that range.

i) Consider $p_1 < l_1$: so $(p_1,h_2) \in A(k)$ and $F_1(p_1) = 0$.

From Lemma A1

$$R_1(p_1) = \left(\frac{\pi C}{M_1}\right)(1-\pi)\left(\frac{1}{(1-\pi)^2} + \frac{1}{\pi}\right)n_1 p_1 < \left(\frac{\pi C}{M_1}\right)(1-\pi)\left(\frac{1}{(1-\pi)^2} + \frac{1}{\pi}\right)n_1 l_1 = R_1(l_1) = R_1 *$$

as required.

ii) Consider $h_1 < p_1 \le a(l_2)$: so $(p_1, l_2) \in A(k)$ and $F_1(p_1) = 1$.

From Lemma A1

$$\left(\frac{\pi_i C}{M_i}\right)(1-\pi_i)\left(\frac{F_{-i}(a(p_i))}{(1-(1-\pi_i)[1-F_{-i}(a(p_i))])^2} + \frac{1}{\pi_i} - \frac{[1-F_{-i}(a(p_i))]}{1-(1-\pi_i)[1-F_{-i}(a(p_i))]}\right)p_i$$

$$R_1(p_1) = \left(\frac{\pi C}{M_1}\right)(1-\pi)\left(\frac{F_2(a(p_1))}{(1-(1-\pi)[1-F_2(a(p_1))])^2} + \frac{1}{\pi} - \frac{[1-F_2(a(p_1))]}{1-(1-\pi)[1-F_2(a(p_1))]}\right)n_1 p_1$$

(A5.1)

we show that this function is concave, and so $R_1(p_1) \le R_1(h_1)$ for all $p_1 > h_1$ if and only if $(\mathrm{d}R_1/\mathrm{d}p_1)(h_1) \le 0$.

Let
$$\begin{aligned} x(p_1) &= \pi + (1-\pi)F_2(a(p_1)) \\ &= 1-(1-\pi)[1-F_2(a(p_1))] \\ &= \sqrt{\frac{(1-\pi)a(p_1)}{(2-\pi)h_2 - a(p_1)}} \quad \text{, from Lemma A2.} \qquad \text{(A5.2)} \end{aligned}$$

substituting into Eqn. (A5.1) gives

$$\begin{aligned} R_1(p_1) &= \left(\frac{\pi C}{M_1}\right)\left(\frac{x(p_1)-\pi}{x(p_1)^2} + \frac{1-\pi}{\pi} - \frac{1-x(p_1)}{x(p_1)}\right)n_1 p_1 \\ &= \left(\frac{\pi C}{M_1}\right)\left(\frac{1}{\pi} - \frac{\pi}{x(p_1)^2}\right)n_1 p_1 \end{aligned}$$

Differentiating gives

$$\frac{\mathrm{d}R_1(p_1)}{\mathrm{d}p_1} = \left(\frac{n_1 \pi C}{M_1}\right)\left(\frac{1}{\pi} - \frac{\pi}{x(p_1)^2} + \frac{2p_1\pi}{x(p_1)^3}\left(\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1}\right)\right) \qquad \text{(A5.3)}$$

$$\frac{\mathrm{d}^2 R_1(p_1)}{\mathrm{d}{p_1}^2} = \left(\frac{n_1 \pi C}{M_1}\right)\left(\frac{2\pi}{x(p_1)^3}\right)\left(2\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1} - \frac{3p_1\pi}{x(p_1)}\left(\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1}\right)^2 + p_1\frac{\mathrm{d}^2 x(p_1)}{\mathrm{d}{p_1}^2}\right) \qquad \text{(A5.4)}$$

Differentiating Eqn. A5.2 gives

$$\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1} = \left(\frac{(2-\pi)(1-\pi)h_2}{2x(p_1)[(2-\pi)h_2 - a(p_1)]^2}\right)\left(\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1}\right) \quad \text{(A5.5)}$$

$$\frac{\mathrm{d}^2 x(p_1)}{\mathrm{d}{p_1}^2} = \left(\frac{(2-\pi)(1-\pi)h_2}{2x(p_1)[(2-\pi)h_2 - a(p_1)]^2}\right)\left[\left(\frac{\mathrm{d}^2 a(p_1)}{\mathrm{d}{p_1}^2}\right) - \frac{1}{x(p_1)}\left(\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1}\right)\left(\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1}\right) + \left(\frac{2}{(2-\pi)h_2 - a(p_1)}\right)\left(\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1}\right)^2\right] \quad \text{(A5.6)}$$

Substituting A5.6 and A5.5 into A5.4 and rearranging terms gives

$$\frac{\mathrm{d}^2 R_1(p_1)}{\mathrm{d}{p_1}^2} = \frac{2Cn_1\pi^2(2-\pi)(1-\pi)}{M_1 x(p_1)^4[(2-\pi)h_2 - a(p_1)]^2}\left\{\frac{p}{2}\frac{\mathrm{d}^2 a(p_1)}{\mathrm{d}{p_1}^2} + \frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1} - \frac{p_1}{a(p_1)}\left(\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1}\right)^2\right\}$$

As the acceptance set defined by Lemma 1 is both comprehensive and convex we have that $\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1} \leq 0$ and $\frac{\mathrm{d}^2 a(p_1)}{\mathrm{d}{p_1}^2} \leq 0$

and so

$$\frac{\mathrm{d}^2 R_1(p_1)}{\mathrm{d}{p_1}^2} \leq 0$$

Consequently, to ensure that $R_1(p_1) \leq R_1(h_1)$ for all $h_1 < p_1 \leq a(l_2)$ we need only show that

$$\lim_{p1 \to h1} \frac{\mathrm{d}R_1(p_1)}{\mathrm{d}p_1} = \left(\frac{n_1\pi C}{M_1}\right)\left(\frac{1}{\pi} - \pi + 2h_1\pi\left(\frac{\mathrm{d}x(h_1)}{\mathrm{d}p_1}\right)\right) \leq 0$$

$$\frac{\mathrm{d}x(p_1)}{\mathrm{d}p_1} = \left(\frac{(2-\pi)(1-\pi)h_2}{2x(p_1)[(2-\pi)h_2 - a(p_1)]^2}\right)\left(\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1}\right)$$

From Eqns. A5.2 and A5.5 we have that $x(h_1) = 1$ and

$$\frac{\mathrm{d}x(h_1)}{\mathrm{d}p_1} = \left(\frac{(2-\pi)}{2(1-\pi)h_2}\right)\left(\frac{\mathrm{d}a(h_1)}{\mathrm{d}p_1}\right)$$

From Burdett &Malueg, we know that

$$\frac{\mathrm{d}a(p_1)}{\mathrm{d}p_1} = -\frac{\pi n_1 F_1(p_1)}{(1-\pi)n_2 F_2(a(p_1))}$$

and the equilibrium condition is satisfied if

$$\frac{1}{\pi} - \pi - \frac{h_1 n_1 \pi^2 (2-\pi)}{h_2 n_2 (1-\pi)^2} \leq 0$$

$$(1-\pi)\left(\frac{1+\pi}{\pi}\right) \leq \frac{h_1 n_1 \pi^2 (2-\pi)}{h_2 n_2 (1-\pi)^2}$$

$$\frac{h_1}{h_2} \geq \frac{n_2 (1-\pi)^3 (1+\pi)}{n_1 \pi^3 (2-\pi)}$$

Identical analysis in industry 2 gives the equilibrium condition

$$\frac{h_1}{h_2} \leq \frac{n_2 (1-\pi)^3 (1+\pi)}{n_1 \pi^3 (2-\pi)}$$

and these conditions hold simultaneously only when

$$\frac{h_1}{h_2} = \frac{n_2 (1-\pi)^3 (1+\pi)}{n_1 \pi^3 (2-\pi)}$$

as stated in Lemma A5

iii) Consider $p_1 > a(l_2)$

From Lemma A1 the expected revenue equals zero for all prices in this range, so will never be charged in equilibrium as $R_i^* = R_i(h_i) > 0$.

This completes the Proof of Proposition 2. Eqn.(2.1) is determined by Lemma A2, Eqn.(2.2) specifies the consumers' optimal acceptance set in Lemma A4, and Eqn.(2.3) is obtained from Lemma A5. Equations (2.4) and (2.5) follow directly from Eqns.(2.1), setting $F_i(l_i) = 0$, and Lemma A1.

QED

Proof of Proposition 3:

Let $G_i(h_i)$ = $\int_0^{h_i} F_i(p_i)\mathrm{d}p_i$, where $F_i(p_i)$ is given by Eqn. (2.1).

First we show that $G_i(h_i)$ is an increasing function of h_i and that $\lim_{h_i \to 0} G_i(h_i) = 0$, and then use these results to prove Proposition 3.

Lemma 3.1:

a. $G_i(h_i)$ is a strictly increasing function of h_i

b. $\lim_{h_i \to 0} G_i(h_i) = 0$

c. $\lim_{h_i \to \infty} G_i(h_i) = \infty$

Proof of Lemma 3.1.

a. Differentiating $G_i(h_i)$,

$$\frac{\mathrm{d}G_i(h_i)}{\mathrm{d}h_i} = \int_0^{h_i} \left(\frac{\mathrm{d}F_i(p_i)}{\mathrm{d}p_i} - \frac{\mathrm{d}F_i(p_i)}{\mathrm{d}h_i} \right) dp_i$$

From Eqn (2.1)

$$\frac{\mathrm{d}F_i(p_i)}{\mathrm{d}p_i} = \frac{1}{2\pi} \left(\frac{\pi_i p_i}{(1+\pi_i)h_i - p_i} \right)^{-\frac{1}{2}} \left(\frac{(1+\pi_i)h_i}{[(1+\pi_i)h_i - p_i]^2} \right)$$

and

$$\frac{\mathrm{d}F_i(p_i)}{\mathrm{d}h_i} = \frac{1}{2\pi} \left(\frac{\pi_i p_i}{(1+\pi_i)h_i - p_i} \right)^{-\frac{1}{2}} \left(\frac{\pi_i (1+\pi_i) p_i}{[(1+\pi_i)h_i - p_i]^2} \right) \qquad \text{(A3.1)}$$

So,

$$\frac{\mathrm{d}G_i(h_i)}{\mathrm{d}h_i} = \int_0^{h_i} \frac{1}{2\pi} \left(\frac{\pi_i p_i}{(1+\pi_i)h_i - p_i} \right)^{-\frac{1}{2}} \left(\frac{(1+\pi_i)}{[(1+\pi_i)h_i - p_i]^2} \right) (h_i - \pi_i p_i)\, dp_i$$
$$> 0$$

and $G_i(h_i)$ is a strictly increasing function of h_i

b. As $h_i = \int_0^{h_i} dp_i \geq \int_0^{h_i} F_i(p_i) dp_i \geq 0$, $\lim_{h_i \to 0} G_i(h_i) = 0$.

c. Using Eqn. 2.1

$$G_i(h_i) \quad \geq \quad \int_0^{h_i}\left(\frac{1}{\pi_i}\right)\left[\left(\frac{\pi_i p_i}{(1+\pi_i)h_i}\right)^{\frac{1}{2}} - \quad (1-\pi_i)\right]dp_i$$

$$=$$

From Eqn(2.3), in equilibrium

$$h_1 = f(h_2) = \frac{n_2(1-\pi)^3(1+\pi)h_2}{n_1\pi^3(2-\pi)} \tag{A3.2}$$

and let

$$A(h_2) = \pi\, n_1 G_1(f(h_2)) + (1-\pi)n_2\; G_2(h_2) \tag{A3.3}$$

From Lemma 3.1 $A(h_2)$ is a strictly increasing function of h_2, $A^{-1}(k)$ exists, and $\lim_{k\to 0} A^{-1}(k) = 0$. Setting $h_2{*}(k) = A^{-1}(k)$ and $h_1{*}(k) = f(h_2{*}(k))$ completes the proof of parts (ii) and (iii).

Part (i) follows as $h_i{*}(k)$ is increasing in k and,

from Eqn. (A3.1),

$$\frac{\mathrm{d}F_i(p_i)}{\mathrm{d}h_i} \geq 0$$

so,

$$\frac{\mathrm{d}F_i(p_i)}{\mathrm{d}k} \quad = \quad \frac{\mathrm{d}F_i(p_i)}{\mathrm{d}h_i}\frac{\mathrm{d}h_i{*}}{\mathrm{d}k} \quad \geq \quad 0$$

QED

Proof of Proposition 4:

(i) Eqns. (2.1),(2.2) and (2.3) are unaffected by changes in C, M_1 or M_2 , and so the equilibrium defined by Proposition 2is unaffected also.

(ii) We show the proof for i =2, the proof when i = 1 follows similarly.

Using Eqns (A3.2) and (A3.3) from the Proof of Proposition 3,

$$\frac{\mathrm{d}A(h_2)}{\mathrm{d}n_2} \quad = \quad \pi\, n_1 \frac{\mathrm{d}G_1(h_1)}{\mathrm{d}h_1} \frac{\mathrm{d}f\,(h_2)}{\mathrm{d}n_2} + (1-\pi)\, G_2(h_2) \quad \geq \quad 0$$

and

$$\frac{\mathrm{d}h_2 * (k)}{\mathrm{d}n_2} \quad = \quad \frac{\mathrm{d}A^{-1}(k)}{\mathrm{d}n_2} \quad \leq \quad 0$$

so

$$\frac{\mathrm{d}F_2(p_2)}{\mathrm{d}n_2} \quad = \quad \frac{\mathrm{d}F_2(p_2)}{\mathrm{d}h_2} \frac{\mathrm{d}h_2 * (k)}{\mathrm{d}n_2} \quad \leq \quad 0$$

QED