

# Vintage capital and the dynamics of the AK model<sup>∞</sup>

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## Abstract

This paper analyzes the equilibrium dynamics of an AK-type endogenous growth model with vintage capital. The inclusion of vintage capital leads to oscillatory dynamics governed by replacement echoes, which additionally influence the intercept of the balanced growth path. These features, which are in sharp contrast to those from the standard AK model, can contribute to explaining the short-run deviations observed between investment and growth rates time series. To characterize the convergence properties and the dynamics of the model we develop analytical and numerical methods that should be of interest for the general resolution of endogenous growth models with vintage capital.

**Key words:** Endogenous growth, Vintage capital, AK model, Difference-differential equations

**JEL classification numbers:** E22, E32, O40

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# 1 Introduction

This paper focuses on the equilibrium dynamics of an AK-type endogenous growth model with vintage capital and non-linear utility. Several important considerations warrant the analysis of vintage capital growth models. First, vintage capital has become a key feature to be incorporated in growth models toward a satisfactory account of the postwar growth experience of the United States.<sup>1</sup> Second, most of the theoretical literature on this ground [e.g. Aghion and Howitt (1994), Parente (1994)] only focuses on the analysis of balanced growth paths. One of the main reasons underlying this circumstance is that dynamic general equilibrium models with vintage technology often collapse into a mixed delay differential equation system, which cannot in general be solved either mathematically or numerically.<sup>2</sup> Finally, it has been of some concern to us how vintages determine the long-term growth of an economy and the transitional dynamics to a given balanced growth path. A precise characterization of the role of vintages in the determination of the growth rate is still an open question in modern growth theory.

This paper proposes a first attempt towards the complete resolution of endogenous growth models with vintage capital. In doing so we incorporate a simple depreciation rule into the simplest approach to endogenous growth, namely the AK model. More precisely, by assuming that machines have a finite lifetime, the one-hoss-shay depreciation assumption, we add to the AK model the minimum structure needed to make the vintage capital technology economically relevant. This small departure from the standard model of exponential depreciation modifies dramatically the dynamics of the standard AK class of models. Indeed, convergence to the balanced growth path is no longer monotonic and the initial reaction to a shock affects the position of the balanced growth path.

The finding of persistent oscillations in investment is somewhat an expected result once non-exponential depreciation structures are incorporated into growth models. However, a complete model specification is needed to precisely characterize how the endogenous growth rate is affected by the determinants of the vintage structure of capital as well as to analyze the role of replacement echoes for the short-run dynamics. To achieve these results it turns out to be useful to proceed in two stages. We start by specifying a Solow-Swan version of the model where explicit results can be brought about. Then, we incorporate our technology assumptions into an oth-

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<sup>1</sup>For a recent review see Greenwood and Jovanovic (1998). Of course a similar growth experience should be found in most OECD countries, but it appears that still there are no systematic studies of the relevant evidence.

<sup>2</sup>There exist some well-known exceptions. First of all, Arrow (1962) proposes a vintage capital model in which learning-by-doing allows for a capital aggregator. Thus, integration with respect to time can be substituted by integration with respect to knowledge and explicit results can be brought out. A second example is provided by Solow (1960), where each vintage technology has a Cobb-Douglas specification. Under this assumption, it is also possible to derive an aggregator for capital.

erwise standard optimal growth framework. There are important insights we get from the Solow-Swan version of the model that we apply and extend in characterizing analytically the dynamics in the optimal growth version.<sup>3</sup> In solving for the Solow-Swan version of the model we are close to the strategy proposed by Benhabib and Rustichini (1991) since the vintage capital structure can be reduced to delayed differential equations with constant delays. However, the optimal growth version of the model requires an alternative strategy since the dynamic system augments to a mixed delayed-differential equation system. We have the advantage that some stability results can be proved in our setting though. In light of these results we are able to overcome the simultaneous occurrence of state dependent leads and lags by operating directly on the optimization problem without using the optimality conditions. We develop a numerical procedure that allows us in addition to deal with the important issue of the indetermination in levels that arises in an endogenous growth framework. Consequently, the analytical and numerical methods we present should be of interest in related applications.

Besides the methodological contribution there are some features we can learn from the AK vintage capital growth model, notwithstanding its simplicity as a theory of endogenous growth. First, with respect to the relevance of the AK model for the endogenous growth literature it is worth to say that the more precisely empirical evidence is revised the more the theory does not appear to be inconsistent with available data.<sup>4</sup> Second, and related, in particular for vintage capital we can build a case in favor of AK theory as far as deviations in trends of investment rates and growth rates are consistent with the patterns in postwar data, a testable prediction of our model specification. Finally, more elaborated theories of endogenous growth might be discussed as having constant social returns to capital as a limiting case. A lot of our procedures should be at work when reducing the level of aggregation by thinking more carefully about the economics of technology and knowledge.

The paper is organized as follows. We first specify in Section 2 the AK one-hoss-shay depreciation technology. In Section 3, we solve for the constant saving rate growth model, we characterize the balanced growth path and we prove non-monotonic convergence. An example is provided to explain the main economic properties of this type of model. In Section 4, the same type of analyses is carried out in the context of an optimal growth model. In Section 5, we show that a model with vintages of physical and human capital has the same reduced form that the

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<sup>3</sup>As emphasized in Boucekkine, Germain and Licandro (1997), there are important differences between a Solow and a Ramsey formulation of the vintage capital exogenous growth model, at least in the short-medium run.

<sup>4</sup>The AK class of models has been criticized as having little empirical support its main assumption: the absence of diminishing returns. This critique vanishes once technological knowledge is assumed to be part of an aggregate of different sorts of capital goods. More serious critiques analyze the testable predictions of this type of models [e.g. Jones (1995)]. However, such criticisms are themselves difficult to support when versions of the model and the data are compared appropriately [cf. McGrattan (1998)].

simple AK model, but it provides an explanation of growth in terms of embodied technological progress. Section 6 concludes.

## 2 The technology

We propose a very simple AK technology with vintage capital:

$$y(t) = A \int_{t-T}^t i(z) dz; \quad (1)$$

where  $y(t)$  represents production at time  $t$  and  $i(z)$  represents investment at time  $z$ , which corresponds to the vintage  $z$ . As in the AK model, the productivity of capital  $A$  is constant and strictly positive, and only capital goods are required to produce. Machines depreciate suddenly after  $T > 0$  units of time, the one-hoss shay depreciation assumption. As we show below, the introduction of an exogenous life time for machines changes dramatically the behavior of the AK model.

Technology (1) has some interesting properties. First, let us denote by  $k(t)$  the integral in the right hand side of (1). It can be interpreted as the stock of capital. Differentiating with respect to time, we have

$$\dot{k}(t) = i(t) - \delta(t)k(t);$$

where  $\delta(t) = \frac{i(t-T)}{k(t)}$ : In the standard AK model, the depreciation rate is assumed to be constant. However, in the one-hoss shay version, the depreciation rate depends on delayed investment, which shows the vintage capital nature of the model. Indeed, non-exponential depreciation schemes should be seen as a generalization of the classical view of capital. This view is related to the standard model of exponential depreciation and dramatically reduces the possible dynamics that an optimal growth model can describe.

Secondly, this specification of the production function does not introduce any type of technological progress. However, as in the standard AK model, the fact that returns to capital are constant results in sustained growth. Consequently, we have an endogenous growth model of vintage capital without (embodied) technical change. Notice that, even if vintage capital is a natural technological environment for the analyses of embodied technical progress these are two distinct concepts. Section 5 provides an interpretation of equation (1) in terms of human capital accumulation, that gives place to some type of embodied technological progress.

### 3 A constant saving rate

Let us start by analyzing an economy of the Solow-Swan type, where the saving rate,  $0 < s < 1$ , is supposed to be constant. The equilibrium for this economy can be written as a delayed integral equation on  $i(t)$ , i.e.,  $\forall t \geq 0$ :

$$i(t) = sA \int_{t-T}^t i(z) dz \quad (2)$$

with initial conditions  $i(t) = i_0(t) \geq 0$  for all  $t \in [-T; 0[$ . By differentiating (2), we can rewrite the equilibrium of this economy as a delayed differential equation (DDE) on  $i(t)$ ,  $\forall t \geq 0$ :

$$i'(t) = sA (i(t) - i(t-T)) \quad (3)$$

with  $i(t) = i_0(t) \geq 0$  for all  $t \in [-T; 0[$  and

$$i(0) = sA \int_{-T}^0 i_0(z) dz \quad (4)$$

From the definition of technology in (1), we know that changes in output depend linearly on the difference between creation (current investment) and destruction (delayed investment). Since investment is a constant fraction of total output, changes in investment are also a linear function of creation minus destruction, as specified in equation (3). This type of dynamics are expected to be non monotonic and to be governed by echo effects.

#### 3.1 Balanced growth path

A balanced growth path solution for equation (2) is a constant growth rate  $g \in \mathbb{R}$ , such that

$$g = sA \int_{-T}^0 e^{i g T} : \quad (5)$$

In what follows,  $g = g(T)$  refers to the implicit BGP relation, in (5), between  $g$  and  $T$ , for given values of  $s$  and  $A$ .

**Proposition 1**  $g > 0$  exists and is unique iff  $T > \frac{1}{sA}$

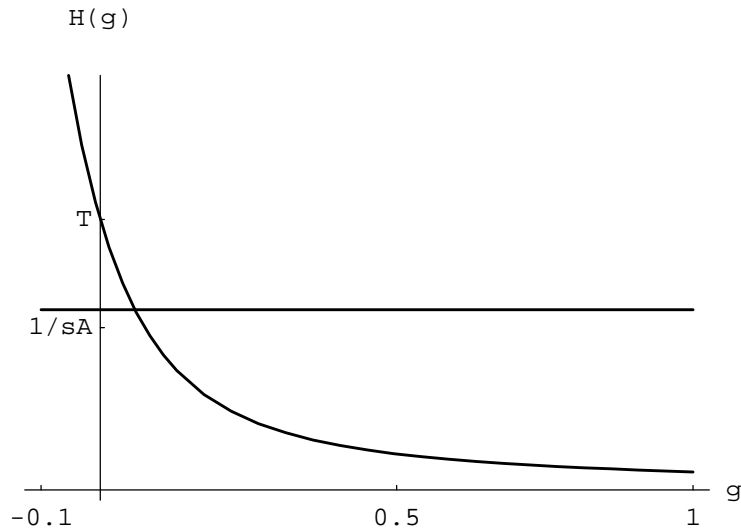


Figure 1: Determination of the growth rate on the BGP

Proof. From (5), we can write for  $g > 0$

$$H(g) = \frac{1}{sA};$$

where  $H(g) = \frac{1 - e^{-gT}}{g}$ : By l'Hôpital rule, we can prove that  $\lim_{g \rightarrow 0^+} H(g) = T$ . Moreover,  $\lim_{g \rightarrow 1^-} H(g) = 0$ : Additionally,  $H'(g) = \frac{(1+gT)e^{-gT} - 1}{g^2} < 0$ , because the numerator  $h(g) = (1+gT)e^{-gT} - 1$  is such that  $h(0) = 0$  and  $h'(g) = -gT^2 e^{-gT} < 0$  iff  $g > 0$ : Consequently, as it can be seen in Figure 1, if  $T > \frac{1}{sA}$  there exists a unique  $g > 0$  satisfying (5). ■

In what follows, we impose the restriction on parameters  $T > \frac{1}{sA}$ . Notice that a machine produces  $AT$  units of output during all its productive life and, given individuals' saving behavior, produces  $sAT$  units of capital. To have positive growth each machine must produce more than the one unit of good needed to produce it, i.e.,  $sAT$  should be greater than one.

**Proposition 2** Under  $T > \frac{1}{sA}$ ,  $\frac{\partial g}{\partial s}$ ,  $\frac{\partial g}{\partial A}$  and  $\frac{\partial g}{\partial T}$  are all positive

Proof. As we can see in Figure 1, the two first results are immediate. Notice that for any  $g > 0$ ,  $\frac{1 - e^{-gT}}{g} > \frac{1 - e^{-gT^0}}{g}$  if  $T > T^0$ . Then, we can still use Figure 1 to see that a proof for  $\frac{\partial g}{\partial T} > 0$  is immediate. ■

Therefore, as it is shown in Figure 2, there is a positive relation between the lifetime of machines and the growth rate. Since machines from all generations are

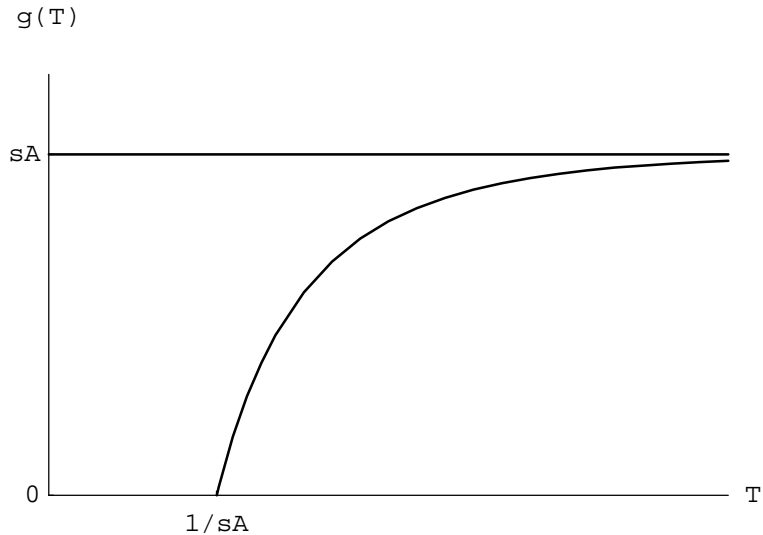


Figure 2: The BGP growth rate

equally productive, an increase on  $T$  is equivalent to a decrease in the depreciation rate in the AK model, which is positive for growth. Indeed, as  $T$  goes to infinity,  $g(T)$  is bounded above by  $sA$  which is the limit case for the AK model with zero depreciation rate: (5) reduces to  $g = sA$ . It turns out to be the case that property  $\frac{\partial g}{\partial T} > 0$  is crucial for the statement of the stability results below. Finally, the positive effect on growth of both the saving rate and the productivity of capital are obvious and they are present in the AK model as well.

With respect to the average age of capital, let us define it as:

$$m(t) = \frac{\int_{t-T}^t (t-z) R_t \frac{i(z)}{\int_{t-T}^t i(z) dz} dz;$$

that is, a weighted average of the ages of active vintages, the weights being equal to the relative participations of the successive active vintages in the total operating capital.

Under the BGP assumption that  $i(t)$  grows at the rate  $g$ , we can easily compute the BGP value for the average age:

$$m = \frac{1}{g} i \frac{\int_0^T e^{-gz} dz}{\int_0^T e^{-gz} dz}; \quad (6)$$

and show that, for a given  $T$ , the average age of capital is negatively related to the growth rate. Notice that when  $T = 1$ , (6) reduces to the AK model with zero

depreciation rate, where  $m = \frac{1}{g}$ . In this case, the average age of capital is negatively related to the growth rate. The reason is straightforward: given  $T$  and for a greater growth rate, the weight of new machines is larger and then the average age of capital is smaller. More in general, in the standard optimal growth model, if investment is growing at a constant rate on the BGP, there should be a negative relation between the average age of capital and the growth rate.<sup>5</sup>

## 3.2 Investment and output dynamics

### 3.2.1 Theoretical results on stability

In analyzing the stability properties of the DDE equation (3) we make use of a result in Hayes (1950).<sup>6</sup> Let us define detrended investment as  $\hat{i}(t) = i(t) e^{-gt}$ . From equations (3) and (5), we can show that

$$\hat{i}^0(t) = (sA - g) [\hat{i}(t) - \hat{i}(t - T)] \quad (7)$$

**Proposition 3** For  $g > 0$  all the nonzero roots of (7) are stable

**Proof.** The characteristic equation associated to (7) is

$$z - (sA - g) + (sA - g)e^{-zT} = 0:$$

By defining  $z = zT$  we obtain Hayes form:  $p e^z - p - z e^z = 0$ , with  $p = (sA - g)T$ . Consequently, in our case as in Benhabib and Rustichini (1991, example 4),  $z = 0$  is a root. For the remaining roots to have strictly negative real parts, we must prove  $p < 1$ . From (5), it can be easily shown that  $(sA - g)T = sAT e^{-gT}$ . Moreover, the first derivative of the implicit function  $g(T)$  in (5) is

$$g^0(T) = \frac{sgT e^{-gT}}{1 - sAT e^{-gT}};$$

which is strictly positive by Proposition 2.  $g^0(T) > 0$  implies  $p < 1$ , which completes the proof. ■

Given that the characteristic equation has only  $z = 0$  as a real root, the economy converges to the long-run growth trend by oscillations.<sup>7</sup>

<sup>5</sup>Consequently, the Denison (1964) claim on the unimportance of the embodied question is per-se irrelevant.

<sup>6</sup>The basic Hayes theorem (see Theorem 13.8 in Bellman and Cooke, 1963) is a set of two necessary and sufficient conditions for the real parts of all the roots of the characteristic equation to be strictly negative. See also Hale (1977, p. 109) for a complete bifurcation diagram for scalar one delay DDEs.

<sup>7</sup>Note that  $z = -g$  is also a root of the characteristic function of the DDE describing detrended investment dynamics. It corresponds to constant solution paths for  $i(t)$ . Since under Proposition



### 3.2.2 Numerical resolution of the dynamics

The DDE (7) can be solved using the method of steps described in Bellman and Cooke (1963, p. 45). To this end, we now single out a numerical exercise by choosing parameter values as reported in Table 1. In the BGP, the growth rate is equal to 0.0296. Concerning initial conditions, we have assumed  $i_0(t) = e^{g_0 t}$  for all  $t < 0$ ,  $g_0 = 0.0282$ . Exponential initial conditions are consistent with the economy being in a different BGP before  $t = 0$ . In this sense, this exercise is equivalent to a permanent shock in  $s$ ,  $A$  or  $T$ , which increases the BGP growth rate in a 5%. The nature of the shock has no effect on the solution, but it associates to  $i_0(t)$  different output histories. Figures 3 and 4 show the solution for detrended output and the growth rate. It is worth to remark that alternative specifications of initial conditions should have consequences for the transitional dynamics.

Table 1: Parameter values

$s$	$A$	$T$	$i_0$	$g_0$	$g$
0.2751	0.30	15	1	0.0282	0.0296

A first important observation, from Figure 4, is that the growth rate is non constant from  $t = 0$ , as it is in the standard AK model. It jumps at  $t = 0$ , is initially smaller than the BGP solution, increases monotonically over the first interval of length  $T$  and has a discontinuity in  $t = T$ . After this point the growth rate converges to its BGP value by oscillations. The behavior of the growth rate in the interval  $[0; T[$ , observed in Figure 4, is mathematically established in the following proposition:

**Proposition 4** If  $g_0 < g$ , then

- (a)  $g_0 < g(0) < g$
- (b)  $g'(t) > 0$  for all  $t \in [0; T[$
- (c)  $g(t)$  is discontinuous at  $t = T$
- (d)  $g \downarrow g(0)$  is increasing in  $g$

The Proposition is proved in the Appendix.

A permanent shock in  $A$  or in  $T$  makes output to jump at  $t = 0$ , thus investment also jumps. A permanent shock in  $s$  does affect investment directly. We have an equivalent jump in the AK model: under the same initial conditions but  $T = 1$ ,

$1, g > 0$ , the latter solution paths are incompatible with the structural integral equation (2), so that we have to disregard the this root.

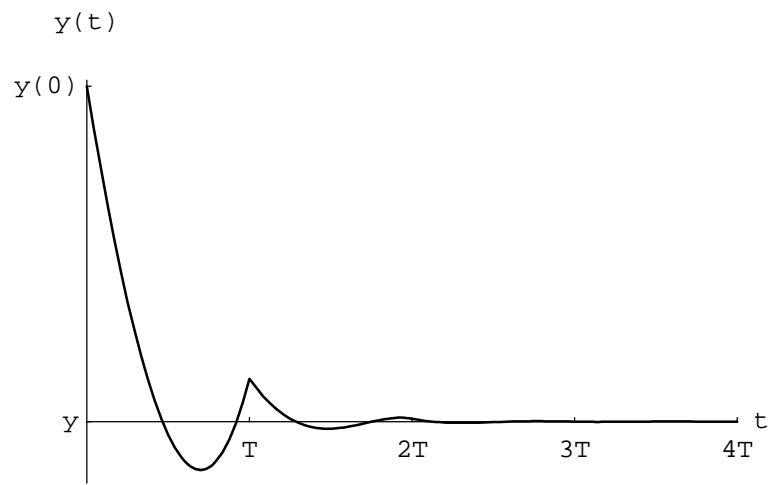


Figure 3: Constant saving rate: Detrended output

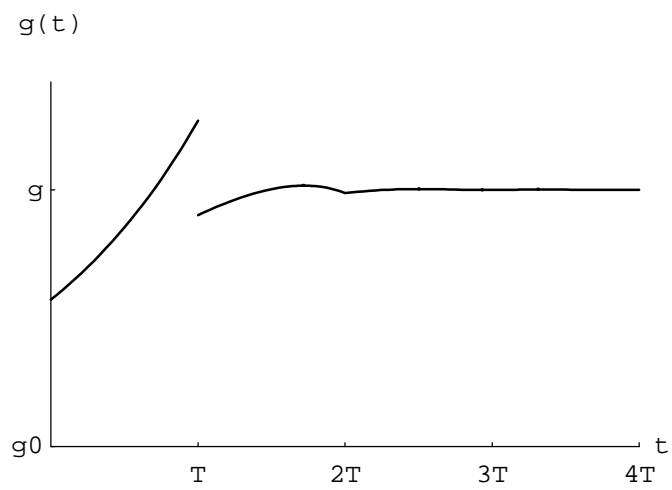


Figure 4: Constant saving rate: The growth rate

$g_0 < g$  iff  $s_0 A_0 < sA$ , then  $i(0) = \frac{sA}{g_0} > \frac{s_0 A_0}{g_0} = 1 = i_0$ . Investment jumps in order to allow the growth rate of the capital stock to jump at  $t = 0$ .

Output at  $t = 0$  is totally determined by initial conditions for investment. Moreover, the level of the new BGP solution depends crucially on the initial level of output. Since the adjustment is not instantaneous, the evolution of output on the adjustment period also influences the output level on the BGP as we can observe in Figure 3.

Finally, we perform numerical exercises for different values of the parameters. They indicate that the profile of both detrended output and the growth rate do not depend on  $g_0$  (of course, if  $g_0 > g$  the solution profile is inverted but symmetric) or on  $s$ ,  $A$  or  $T$ , provided that condition  $T > \frac{1}{sA}$  holds. The speed of convergence is always the same. Only the initial jump on the growth rate, the BGP level of detrended output and the amplitude of fluctuations depend on these parameters. As stated in part (d) of Proposition 4, the greater is  $g$  with respect to  $g_0$  the larger the distance between  $g(0)$  and  $g$ . When the permanent shock is important, the economy starts relatively far from the BGP growth rate and, even if the speed of convergence is always the same, this initial distance reduces the level of the BGP. Consequently, the greater is a positive shock, the larger is the slope of the BGP but the smaller is the intercept.

## 4 The optimal growth model

In the previous section, we have fully characterized the dynamics of the one-hoss shay AK model under the assumption of a constant saving rate. Under the same technological assumptions, in this section we generalize these results for an optimal growth model. Let a planner solve the following problem:

$$\text{Max} \int_0^{\infty} \frac{c(t)^{1-\frac{3}{4}}}{1-\frac{3}{4}} e^{-\frac{1}{2}t} dt \quad (8)$$

s.t.

$$y(t) = A \int_{t_i}^T i(z) dz \quad (1)$$

$$c(t) + i(t) = y(t) \quad (9)$$

$$0 \leq i(t) \leq y(t)$$

and given  $i(t) = i_0(t) \geq 0$  for all  $t \in [t_i, T; 0]$  with parameters  $\frac{1}{2} > 0$  and  $\frac{3}{4} > 0$ ,  $\frac{3}{4} \in [0, 1]$ .  $c(t)$  represents consumption. The optimal conditions for this problem are:

$$(y(t) - i(t))^{1-\alpha} = \dot{A}(t) \quad (10)$$

$$\dot{A}(t) e^{i \frac{1}{2} t} = A \int_t^{t+T} \dot{A}(z) e^{i \frac{1}{2} z} dz; \quad (11)$$

where  $\dot{A}(t)$  is the Lagrangian multiplier associated to the feasibility constraint.

Equation (11) says that at the optimum the cost of investment should be equal to its discounted flow of benefits, both evaluated at the marginal value of consumption.

## 4.1 Balanced growth path

From the previous equations, and assuming that  $y(t) = y e^{gt}$  and  $i(t) = i e^{gt}$ ,  $y > 0$  and  $i > 0$ , we obtain:

$$\frac{3}{4}g + \frac{1}{2} = A \frac{i}{y} \int_0^T e^{i(\frac{3}{4}g + \frac{1}{2})t} dt \quad (12)$$

$$g = \frac{i}{y} A \frac{i}{y} \int_0^T e^{i g t} dt; \quad (13)$$

Notice that equation (13) is equivalent to (5) if  $\frac{i}{y} = s$ . However,  $g$  is determined in equation (12), given the parameters  $\frac{3}{4}$ ,  $\frac{1}{2}$ ,  $A$  and  $T$ , and (13) determines the ratio  $\frac{i}{y}$ . In what follows, we still use the notation  $g = g(T)$  to refer to the equilibrium relation between  $g$  and  $T$  implicit now in equation (12).

**Proposition 5** If  $H(\frac{1}{2}) > \frac{1}{A}$ , then  $g > 0$ :

**Proof.** Using the function  $H(x) = \frac{(1 - e^{-xT})}{x}$ , whose properties were analyzed in the proof of Proposition 1, we can easily show that this proposition is true. ■

From equation (13), we know that if  $\frac{3}{4}$  and  $\frac{1}{2}$  are such that  $\frac{i}{y} = s$  in the BGP, for  $s$  defined in the previous section, the BGP of the optimal growth model is identical to the BGP of the constant saving rate model. Moreover, as a direct consequence of Proposition 2, it can be easily checked that  $g^0(T) > 0$ , as in the Solow-Swan version of the model.

The condition  $(1 - \frac{3}{4})g < \frac{1}{2}$  is needed for utility to be bounded along the BGP. Under this condition, it can be shown that  $\frac{i}{y} < 1$ . Along the BGP the saving rate should be strictly smaller than one.

## 4.2 Investment and output dynamics

### 4.2.1 Theoretical results on stability

Notice that condition (11) only depends on the Lagrangian multiplier  $\hat{A}(t)$ , which grows at the rate  $\frac{1}{3}g$  on the BGP. Let us define  $x(t) \equiv \hat{A}(t) e^{-\frac{1}{3}gt}$  and rewrite (11) as

$$x(t) e^{i(\frac{1}{3}g + \frac{1}{2})t} = A \int_t^{t+T} x(z) e^{i(\frac{1}{3}g + \frac{1}{2})z} dz; \quad (14)$$

This advanced integral equation is forward looking and forms a top block of the system, implying that the detrended marginal value of consumption,  $x(t)$ , can be solved first. By differentiating (14), we get the following advanced differential equation (ADE):

$$x'(t) = -\left(x(t+T) + \frac{1}{2}x(t)\right); \quad (15)$$

where  $\frac{1}{2} \equiv A + \frac{1}{3}g + \frac{1}{2}$ , strictly positive for (12). In analyzing the stability of the ADE (15) we build upon similar arguments as in Section 3.2.1.

**Proposition 6**  $x(t) = x$  constant, for all  $t \geq 0$ , is the only stable solution of (15)

**Proof.** The characteristic equation is  $z + \frac{1}{2} - e^{zT} = 0$  and defining  $z = \frac{1}{T} \ln q$  we can easily obtain Hayes' form with  $p = -T$  and  $q = \frac{1}{2}$ . This implies a stability condition  $-T < 1$  which it can be easily checked it is equivalent to  $g^0(T) > 0$ . Note this result is obtained for  $\frac{1}{3}z$  so that all the roots but  $z = 0$  have strictly positive real parts. ■

Moreover, since  $x(t)$  has to converge to  $(y + i)^{\frac{1}{3}}$ , Proposition 6 implies  $x(t) = (y + i)^{\frac{1}{3}}$  for all  $t \geq 0$ . Detrended consumption is also constant and equal to  $c(t) = c = x^{\frac{1}{1-\frac{1}{3}}}$ . The value of  $c$  is determined by the initial conditions. The optimality of this result is straightforward. The block recursive structure of the problem allows the planner to choose detrended consumption without any restriction other than (14). It seems obvious that, from concavity of the utility function, he must prefer a constant detrended consumption path. Observe that, irrespective of the value of the intertemporal elasticity of substitution, the planner always chooses a constant detrended consumption, as it does in the standard AK model. However, in our model he needs to let the saving rate to fluctuate to compensate for fluctuations in output due to echo effects.

To analyze the transitional dynamics of detrended production and investment, we need to solve equations (1) and (10) jointly with the definition of  $x(t)$  and the solution  $x(t) = x$ . Before doing that, let us define  $\tilde{y}(t) = y(t) e^{i \frac{1}{3}gt}$  and  $\tilde{i}(t) = i(t) e^{i \frac{1}{3}gt}$ . By combining (1) and (10), the definition of  $x(t)$  and Proposition 6, we can show that the dynamics of detrended investment are given by:

$$\dot{y}(t) = \lambda y(t) + (A - \lambda I) y(t - T) + c \quad (16)$$

with initial conditions  $y(t) = y_0(t) e^{\lambda t}$  for all  $t \in [-T; 0[$  and  $y(0) = y_0$ , where

$$y_0 = A \int_{-T}^0 y_0(z) dz$$

Since the constant  $\lambda y(t) + c$  adds only constant partial solutions, the stability of detrended investment depends upon the homogeneous part of equation (16).

**Proposition 7** Any stable solution of the DDE (16) has the form:

$$y(t) = \bar{y} + \sum_{r \in E_s} c_r e^{s_r t}$$

where  $\bar{y} = \frac{c}{\lambda - A e^{-\lambda T}}$ ,  $E_s$  is the set of stable roots of the characteristic function of the homogenous part of the DDE (16), and  $c_r$  are constant terms determined by the initial conditions.

The general solution form stated above is merely an application of the superposition principle to the non-homogenous DDE (16).  $\bar{y}$  is a constant solution of the DDE and  $s_r$  are the roots of  $f(z) = z - (A - \lambda I) e^{-zT} - \lambda$ , which turns out to be the characteristic function of the homogenous part of the DDE (16). The expansion representation of the stable solutions of the homogenous part of (16) is an application of Theorem 3.4 in Bellman and Cooke (1963). Note that the expansion involves constant terms  $c_r$  because the roots of  $f(z)$  are all simple. Indeed, a multiple root arises if and only if  $f(z) = f'(z) = 0$ . It is trivial to show that this situation cannot occur in our case. On the other hand, one can put the characteristic function  $f(z)$  into the form of Hayes with  $p = (A - \lambda I)T$  and  $q = A e^{-\lambda T}$ . Since equation (13) can be rewritten as  $(A - \lambda I)T = A e^{-\lambda T}$ , it turns out that  $p > q$  as far as the long run saving rate is strictly lower than one. Hence, one of the two necessary and sufficient conditions of Hayes theorem does not hold and the characteristic function admits generally both stable and unstable roots. For stability requirements (of detrended investment), we rule out the unstable roots. But still the constant terms  $c_r$  and the consumption term  $c$  cannot be fully determined if no initial function  $y_0(t)$ ,  $t \in [-T; 0[$  is specified. But even if the latter function was specified, we would not be able to compute analytically the solution paths since this would require the computation of the entire set of the stable roots of function  $f(z)$ , which is typically infinite. So we resort to numerical resolution.

#### 4.2.2 Numerical resolution of the dynamics

The computational procedure that we use to find the equilibrium paths of the optimal growth model is of the cyclic coordinate descent type (see Luenberger (1973), p. 158) and operates directly on the optimization problem. It is an extension of the algorithm proposed by Boucekine, Germain, Licandro and Magnus (1999). The Appendix contains a description of the algorithm used to compute the optimal solution. Roughly, it consists of finding a fixed point vector  $i(t)$  by sequentially maximizing the objective with respect to coordinate variables at time  $t$ . We perform a comparable experiment to that of the Solow-Swan version of the model and parameter values are chosen correspondingly. This implies parameter values as those reported in Table 2.

Table 2: Parameter values

$\frac{3}{4}$	$\frac{1}{2}$	A	T	$i_0$	$g_0$	g
8.0	0.06	0.30	15	1	0.0282	0.0296

We set  $\frac{3}{4}$  and  $\frac{1}{2}$  that correspond at the BGP value for  $s$  (0.2751) used in Section 3. Notice that the implied value of  $\frac{3}{4}$  is relatively high. It can be easily checked that this quantitative peculiarity comes from the AK model and it is not a result of the one-hoss shay depreciation assumption.

Figures 5 and 6 are plotted in the same scale as Figures 3 and 4 above, respectively. They depict the solution path for output and the growth rate, which behave very similar as in the constant saving rate model. From Proposition 6, we know that the planner optimally chooses to have a constant detrended consumption. For this reason, the saving rate rises at the beginning, increasing the growth rate (with respect to the Solow-Swan case) and therefore allowing output to converge to a higher long-run level. As a consequence, the planner generates longer lasting fluctuations than those that were obtained in the constant saving rate model. Indeed, in the optimal growth model it is the saving rate that bears most of the adjustment to the BGP.

As stated in Proposition 6, detrended consumption should be constant from  $t = 0$ , but its level should be determined by initial conditions. Figure 7 compares the numerical solution obtained for detrended consumption in both models, the dashed line corresponds to the optimal growth solution and the solid line to the constant saving rate model. In the optimal growth model our numerical procedure illustrates on the fact that the planner is optimally choosing the stable solution, and the algorithm succeeds in calculating the constant detrended consumption level. In order to have a constant detrended consumption, the saving rate must increase at the beginning and fluctuate around its BGP solution afterward, as it is shown in Figure

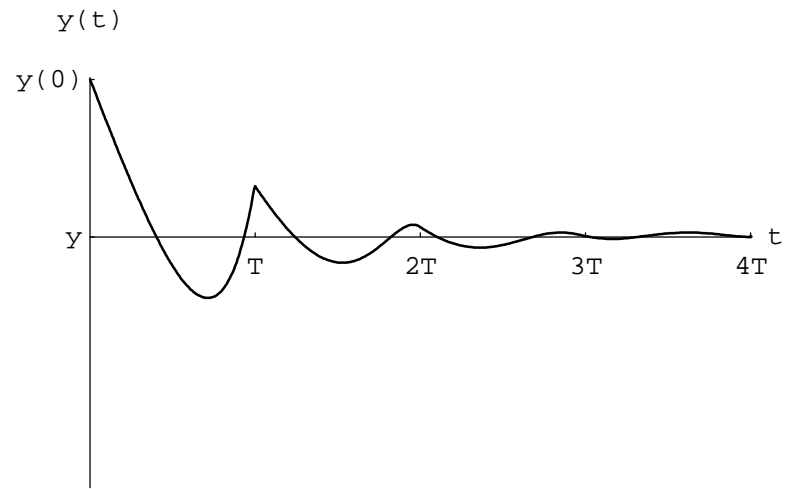


Figure 5: Optimal growth model: Detrended output

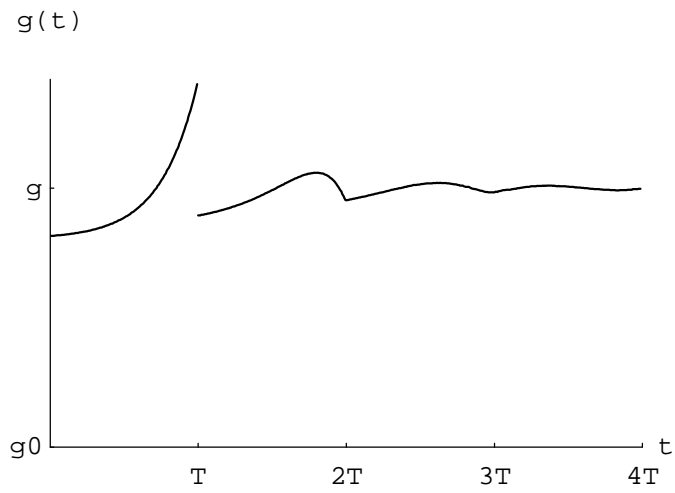


Figure 6: Optimal growth model: The growth rate



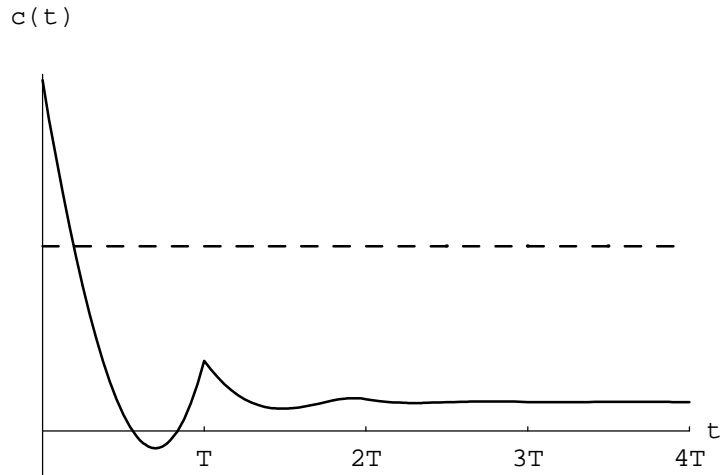


Figure 7: Consumption: optimal growth vs constant saving rate

8. Alternatively, in the Solow-Swan version of the model detrended consumption is just a constant fraction of output and fluctuates likewise.

Finally, in the context of our simple model, we can further derive implications in terms of the empirical relevance of the AK class of models. In particular, incorporating vintage capital into an otherwise standard optimal AK growth model contributes to break the close connection between investment and growth in the short-medium run. This is a feature of the data which has been stressed the AK model contradicts [cf. Jones (1995)].<sup>8</sup> Figure 8 summarizes the short-run dynamics of the investment share (dashed line) and the growth rate (solid line): investment rates do not move in lock step with growth rates. The intuition is straightforward. Compared with the standard version of the model we move from  $g(t) = A i(t) - y(t)$  to  $g(t) = A i(t) - y(t) + \lambda (i(t-T) - i(t))$  being  $\lambda > 0$ . The growth rate depends not only upon the current investment rate but also on delayed investment. Temporary changes in investment will imply temporary changes in growth rates from their long-run trend. Thus, the sort of fluctuations the model generates is not merely a mathematical property but derives testable implications for the AK theory.

A further analysis on stability can be achieved by computing numerically a subset of the infinite roots of the homogeneous part of (16), those with a negative real part near to zero [cf. Engelborghs and Roose (1999)]. We have found that this subset is non empty and therefore supports the convergence by oscillations result in Figures 5 and 6. For the optimal growth model and the parameter values in Table 2, Figure 9 shows the real parts in the x axis and the imaginary parts in the y axis.

<sup>8</sup>For a review see McGrattan (1998). Considering evidence over longer time periods and more countries that Jones does she finds the long-run trends that AK theory predicts and that our model economy preserves. McGrattan also provides examples suggesting that the relationship which forms the basis of Jones' (1995) time series tests does not generally hold for the AK model.

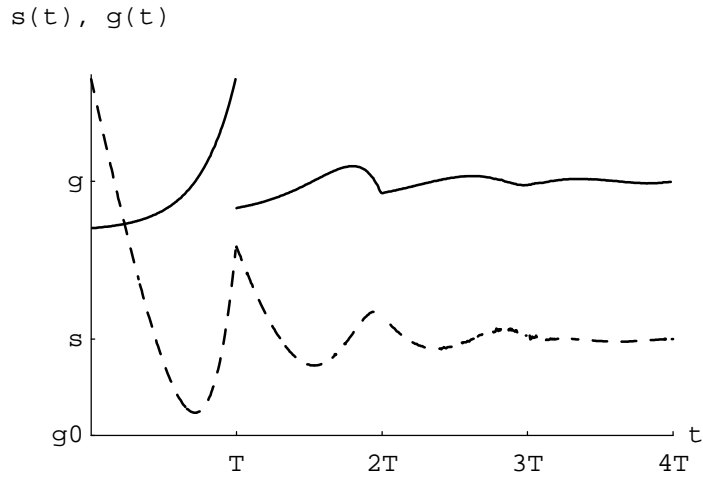


Figure 8: The growth and the saving rates

Figure 10 does the same for the constant saving rate model and parameters in Table 1. We can evaluate the convergence speed of the economy using the computed roots: the closer to zero is the smallest real part of the nonzero computed eigenvalues, the slower is convergence. These figures confirm that the Solow-Swan version of the model converges more rapidly.

## 5 A Solow (1960) interpretation

The AK model can also be seen as a reduced form of a more general economy with both physical and human capital. This result is obtained in a one sector model using a constant returns to scale technology in both types of capital. In such a model output can be used on a one-for-one basis for consumption, for investment in physical capital and for human capital accumulation. In this section we investigate what are the implications of considering this stylized representation in a vintage capital framework. For this purpose we aggregate over vintage technologies following Solow (1960).

Let us assume that the technology of a vintage  $z$  is given by

$$y(z) = B i(z)^{1-\alpha} h(z)^\alpha; \quad (17)$$

where  $B > 0$  and  $0 < \alpha < 1$ .  $h(z)$  represents human capital associated to vintage  $z$ . Let us assume that both physical and human capital are vintage specific and have the same lifetime  $T > 0$ . Machines use specific human capital, which is destroyed when machines are scrapped. Thus, given the one-for-one allocation structure of our setting the price of each type of capital would be fixed at unity. Under these assumptions, the representative plant of vintage  $z$  solves the following problem:

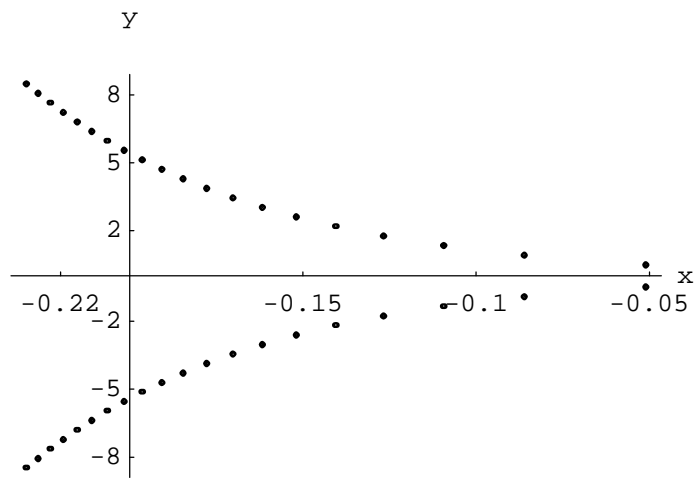


Figure 9: Eigenvalues of the optimal growth model

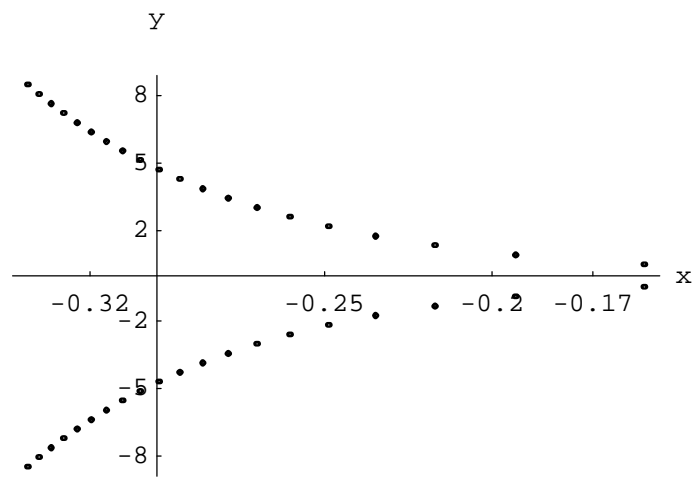


Figure 10: Eigenvalues of the constant saving rate model

$$\max_{f(z):h(z)g} B i(z)^{1-\theta} h(z)^{\theta} i(z) i(z) h(z)$$

where

$$i(z) = \int_z^{z+T} e^{-\int_z^v r(v) dv} dz;$$

Given our irreversibility assumption, a plant of vintage  $z$  produces the same output from  $z$  to  $z + T$ . The interest rate is denoted by  $r(t)$  and  $i(z)$  is the discounted value of a flow of one unit of output produced during the plant life. Given that both forms of capital face the same user cost, it is very easy to show that the optimal ratio of physical to human capital is

$$\frac{i(z)}{h(z)} = \frac{1-\theta}{\theta};$$

the same for all vintages. Substituting it in (17), and aggregating over all operative plants at time  $t$ , we get that aggregate production is equal to

$$y(t) = A \int_{t-T}^t i(z) dz;$$

where  $A = B \frac{1-\theta}{\theta} \theta^{\theta}$ .

Aggregate production in this model clearly reduces to the AK technology presented in the previous sections. The interest of this Solow (1960) version of our one-hoss shay AK model is that we can interpret it in terms of embodied technological progress. On the BGP, human and physical capital are both growing at the positive rate  $g$ . Consequently, labor associated to the representative plant of vintage  $z$  has  $h(z)$  as human capital, which is greater than the human capital of all previous vintages. Under this interpretation, technical progress is embodied in new plants.<sup>9</sup>

The key difference with Solow's paper comes from the specificity of human capital. In the Solow paper, labor is an homogeneous good and technological progress

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<sup>9</sup>From constant returns to scale in production, the number of plants is undetermined. Moreover, our assumption on human capital accumulation makes the number of workers undetermined also, since we can associate any amount of human capital to any small unit of labor. Without any loss of generality, we can assume that the measure of firms and the measure of labor are both one. In this sense, a plant is always associated with one worker. Since the human capital investment of a plant is increasing, we can interpret it as technological progress embodied in the labor resource. Of course, since human capital is vintage specific and associated to a particular vintage of capital, we could in a large sense say that technical progress is embodied in physical capital too, but it is still labor saving. Arrow (1962) is an example of labor saving technical progress embodied in new machines. However, this model makes an important difference with respect to the recent literature on embodied technical progress, as in Greenwood, Hercowitz and Krusell (1998), which follows Solow (1960) by assuming that technical change is capital saving.

is embodied in the physical capital. The first assumption implies that the equilibrium wage is the same for all vintages. From the second assumption, to restore the equality of labor productivities across vintages, we must associate less labor to older vintages. Under these conditions, Solow shows that the aggregate production from adding vintage specific Cobb-Douglas technologies is also Cobb-Douglas. In our model, human capital is vintage specific, implying that the capital-labor ratio of a particular vintage is not varying over time, and it is the same for all vintages. Under this alternative assumption, aggregate production is of the one-hoss shay AK type.

## 6 Conclusions

Recent discussions on growth theory emphasize the ability of vintage capital models to explain growth facts. However, there is a small number of contributions endogenizing growth in vintage models, and most of them focus on the analysis of balanced growth paths. The model analyzed here goes part way toward developing the methods for a complete resolution of endogenous growth models with vintage capital. For analytical convenience it is limited to a case in which the engine of growth is simple: returns to capital are bounded below. However, the basic properties of the model are common to most endogenous growth models. Our framework represents a minimal departure from the standard model with linear technology: we impose a constant lifetime for machines. Under this assumption we show that some key properties of the AK model change dramatically. In particular, convergence to the BGP is no more instantaneous. Instead, convergence is non monotonic due to the existence of replacement echoes. As a consequence, investment rates do not move in lock step with growth rates.

## Appendix

In this appendix we prove Proposition 4 and we present an outline of the algorithm used to compute equilibrium paths of the optimal growth model.

### Proof of Proposition 4

(a) From (2) we can show that

$$g(0) = sA_j \frac{g_0 e^{i g_0 T}}{1 - e^{i g_0 T}} \quad (A1)$$

From (5), we can show that

$$g = sA \left( 1 - \frac{g e^{i g T}}{1 - e^{i g T}} \right) \quad (A2)$$

Since  $G(g) = \frac{g e^{i g T}}{1 - e^{i g T}}$  is such that  $G'(g) < 0$ , then  $g(0) < g$ . Finally, from Proposition 2, we know that the relation between  $g$  and  $s$ , implicit in (5), is decreasing. Consequently, there exists a  $a < sA$ , such that

$$g_0 = a \left( 1 - \frac{e^{i g_0 T}}{1 - e^{i g_0 T}} \right) = a \left( \frac{g_0 e^{i g_0 T}}{1 - e^{i g_0 T}} \right) < g(0):$$

(b) From (3)

$$g(t) = \frac{i'(t)}{i(t)} = sA \left( \frac{i(t - T)}{i(t)} \right):$$

Differentiating with respect to time gives, for all  $t \in [0; T[$

$$g'(t) = g(t) - g_0:$$

Since  $g(0) > g_0$ ,  $g'(t) > 0 \forall t \in [0; T[$ .

(c) Given that  $H'(g) < 0$  and  $g_0 < g$ , from (4) and (5),  $i(0) > \lim_{t \rightarrow 0^+} i_0(t) = 1$ . From (3),  $i'(t)$  has a discontinuity at  $t = T$ .

(d) Combining (A1) and (A2), we get

$$g - g(0) = G(g_0) - G(g) > 0:$$

At given  $g_0$ , an increase in  $g$  rises  $g - g(0)$  since  $G'(g) < 0$ :  $\forall$

### Algorithm

The planner's problem can be redefined in terms of variables for which its long-run is known.

Let define  $i(t) = \frac{i(t)}{i_0(i - T)}$  and  $z(t) = \frac{y(t)}{i(t)}$ , then (8) reads:

$$\max \int_0^T \frac{[z(t) - 1]^{1-\frac{3}{4}}}{1 - \frac{3}{4}} i(t)^{1-\frac{3}{4}} e^{i \frac{1}{2} t} dt$$

subject to

$$z(t) = A \int_{t-T}^t \frac{i(z)}{i(t)} dz \quad (A3)$$

$$\frac{i^0(t)}{i(t)} = g(t) \quad (A4)$$

given initial conditions  $i(t) = i_0(t) = \frac{i_0(t)}{i_0(t)} \rightarrow 0$  for all  $t < 0$

The numerical procedure operates on this transformation of the problem and the optimization relies upon the objective. In line with the cyclic coordinate descent algorithm proposed by Boucekine, Germain, Licandro and Magnus (1999), the unknowns are replaced by piecewise constants on intervals  $(0; \Phi)$ ,  $(\Phi; 2\Phi)$ , ..., and iterations are performed to find a fixed-point  $g(t)$  (and/or state variable  $i(t); y(t)$ ) vector up to tolerance parameter 'Tol'. An outline of the algorithm used to compute an approximate solution of problem above is the following:

Step 1: Initialize  $g^0(t)$ , the base of the relaxation, with dimension  $K$  sufficiently large. For  $t \in [K; N]$ ,  $N > K$  and large enough, set  $g(t) = g$  (the BGP solution). Notice that knowing  $g(t)$  we can compute  $i(t)$  and  $z(t)$  using (A3) and (A4).

Step 2: Maximization step by step:

- 2 Step 2.0: maximize with respect to coordinate  $g_0$  keeping unchanged coordinates  $g_i, i > 0$
- 2 Step 2.k: maximize with respect to coordinate  $g_k$  keeping unchanged coordinates  $g_i, i > k$ , with coordinates  $g_i, 0 \leq i < k$  updated
- 2 Step 2.K: last  $k < K$  step, get  $g^1(t)$

Note that at each  $k$  step states must be updated.

Step 3: If  $g^1(t) = g^0(t)$ , we are done. Else update  $g^0(t)$  and go to Step 2.

Table 3: Algorithm parameters

N	K	$\Phi$	Tol
10 T	4 T	0:1	$10^{-5}$

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