Uncertainty Aversion and Backward Induction*

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Abstract

In the context of the centipede game this paper discusses a solution concept for extensive games that is based on subgame perfection and uncertainty aversion. Players who deviate from the equilibrium path are considered non-rational. Rational players who face non-rational opponents face genuine uncertainty and may have non-additive beliefs about their future play. Rational players are boundedly uncertainty averse and maximise Choquet expected utility. It is shown that if the centipede game is sufficiently long, then the equilibrium strategy is to play 'Across' early in the game and to play 'Down' late in the game.

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1 Introduction

The centipede game has become a benchmark both for the empirical adequacy and the theoretical consistency of game theoretic concepts. In any Nash equilibrium — and thus in every equilibrium refinement — the first player chooses 'Down' immediately; in the unique subgame-perfect equilibrium the players choose 'Down' everywhere.

Empirically, experimental evidence suggests that players do not act in this way (see, e.g. , McKelvey & Palfrey (1992)). Theoretically, subgame perfection applies equilibrium arguments, that hold for rational players, off the equilibrium path. This is consistent only under the assumption that deviations from rational play are not evidence of non-rationality, e.g. because rational players might tremble (Selten 1975). This aspect has led to a controversial debate about backward induction (see, e.g. , Basu (1988), Reny (1993), Aumann (1995), Binmore (1996), Aumann (1996)).

McKelvey & Palfrey (1992) are able to interpret experimental evidence in the sense of Kreps, Milgrom, Roberts & Wilson (1982, henceforth KMRW). In their model, the structure of the game is not mutual knowledge. Instead there is a small probability of being matched with an 'altruistic' opponent who always plays 'Across'. McKelvey & Palfrey (1992) show that, as a consequence, it is indeed rational to play 'Across' early in the game.

There are two arguments against this way of interpreting the experimental evidence. First, if taken as an explanation of evidence rather than an equilibrium effect, it relies on the actual existence of such altruists in the subject pool. The second, formulated by Selten (1991) in the context of the KMRW approach to the finitely repeated prisoner's dilemma, is that the analysis proceeds by changing the game, and not by analysing the same game in which the paradox arises. However, both criticisms do not apply if the players are assumed to know the game, but lack mutual knowledge of rationality, as suggested by Milgrom & Roberts (1982, p.303). If the rational players believe that non-rational opponents always play 'Across', the analysis of McKelvey & Palfrey (1992) is an explanation of the actual evidence in the original game.

Still, this approach to modelling lack of mutual knowledge of rationality leads to conceptual difficulties:

First, there is no reason why rational players should hold this specific belief about opponents that they do not consider to be rational. Therefore, not only is the specification of the belief that non-rational players always play 'Across' ad hoc, in the absence of a theory of non-rationality there is no basis for specifying any particular belief.

Secondly, this also holds in particular for the uniform distribution as a model of complete ignorance. There is no reason why a non-rational player should be assumed to choose all his strategies with equal probability. In addition, there is the well-known problem that a uniform probability depends on the description of the space of uncertainty: For instance, if a state is split into two sub-states, the combined probability of the two sub-states under the uniform distribution is higher than the probability of the original state.

Thirdly, and more fundamentally, if the Bayesian-Nash equilibrium is identified with rational play, then any deviation must be considered non-rational. This problem is related to, but different from the first: Not only need the players not have a particular belief about non-rational opponents, according to the rationality concept they must not have any particular belief. This consistency requirement follows from an identification of Bayesian-Nash equilibrium with rational play, because this implicitly defines all other strategies as non-rational.

Finally, the analysis of games under incomplete information on the basis of the Bayesian-Nash equilibrium assumes that the types of a player correspond to a consistent hierarchy of beliefs about the underlying uncertainty (Harsanyi 1967–68). This leads to the usual infinite regress. Thus in this analysis the rational player not only believes that a 'non-rational' opponent always plays 'Across', but also believes that the non-rational opponent believes a rational player to believe this, ... ad infinitum. But this means that a rational player must believe that his non-rational opponent has an infinite and consistent hierarchy of beliefs. This, of course, is at odds with the interpretation of this opponent as non-rational. It is for this reason that McKelvey & Palfrey (1992) refer to structural uncertainty and 'altruistic' types.

Nevertheless, the KMRW approach has been extremely useful in helping to understand strategic interaction, particularly in industrial organization (Kreps & Wilson 1982, Milgrom & Roberts 1982) and, as in McKelvey & Palfrey (1992), in experimental game theory.

Our model is in the same spirit as KMRW (1982) and McKelvey & Palfrey (1992). We postulate that rationality is not mutual knowledge, i.e. an opponent may or may not be rational. We replace the assumption that players have a specific belief about non-rational play with the assumption that players are genuinely uncertain about the way non-rational opponents play. When facing uncertainty, players maximise Choquet expected utility (Schmeidler 1989, henceforth CEU). According to CEU, players act in face of uncertainty as if they maximise subjective expected utility. However, in contrast to a situation in which players face risk, players' beliefs do not have to be additive, i.e. the 'probabilities' that the players use to weigh consequences do not have to add to 1.

Our contribution in this paper is to define an equilibrium concept that extends subgame perfection to a game with genuine uncertainty due to lack of mutual knowledge of rationality. Thus we do not need to make any assumption about the behavior of non-rational players, and we can avoid modelling them as types. Instead, we can make an assumption about the rational players' attitude towards uncertainty. We assume that they are uncertainty averse, but only boundedly so. We show that this results in an equilibrium in the centipede game in which rational players play 'Across' early in the game and 'Down' late in the game. Moreover, it is subgame-perfect in the sense that decisions are optimal at every node in the game.

Our result is due to an interaction between the game-theoretic definition of strategy as a contingent plan and the players' attitude towards uncertainty. In calculating expected utilities, a player who is uncertainty averse will use 'probability weights' that do not add up to 1, and a 'probability residual' (the difference between the sum of the weights and 1) that he will allocate to the worst outcome. As long as the degree of uncertainty aversion is bounded, however, every strategy of the non-rational opponent will enter the calculation with some positive weight, however small. Since a strategy is a contingent plan, it specifies an action — 'Across' or 'Down' — after every history of the game, even those that are excluded by the strategy itself (because it specifies 'Down' very early). Consequently, the number of strategies increases exponentially in the length of the centipede game. This means that early in the game the 'probability residual' that is allocated to the worst outcome is small. Thus even uncertainty-averse players will find it profitable to go 'Across'. Late in the game, however, the number of remaining strategies is small, and uncertainty averse players will prefer ‘Down’. We show that this phenomenon is an equilibrium, i.e. it is stable even if other rational players act in a similar way.

CEU has been introduced into game theory by Dow & Werlang (1994) and Klibanoff (1993). Dow & Werlang (1994) show that in the presence of uncertainty the backward induction outcome may break down if the finitely repeated prisoner’s dilemma is analysed as a normal form game. Our model extends this result in two directions: First, we give an explicit reason for non-additive uncertainty, the lack of mutual knowledge of rationality. Secondly, we formulate a solution concept in the spirit of subgame perfection and show that the backward induction outcome breaks down in the subgame-perfect equilibrium of the centipede game, analysed in its extensive form. This allows the conclusion that these two concepts — backward induction and subgame perfection — differ fundamentally in the presence of uncertainty.

Other papers that combine the analysis of extensive form games with CEU are Eichberger & Kelsey (1995) and Lo (1995). In these papers there is no explicit distinction between rational and non-rational players. Eichberger & Kelsey (1995) use the Dempster-Shafer rule to update non-additive beliefs. Closest to the spirit of our analysis is Mukerji (1994), however he considers normal form games only.

The paper is organized as follows: Section 2 contains the model, section 3 an example, section 4 the result, and section 5 concludes. There is one appendix.

2 The Model

2.1 The Centipede Game

Consider the following version of the centipede game:

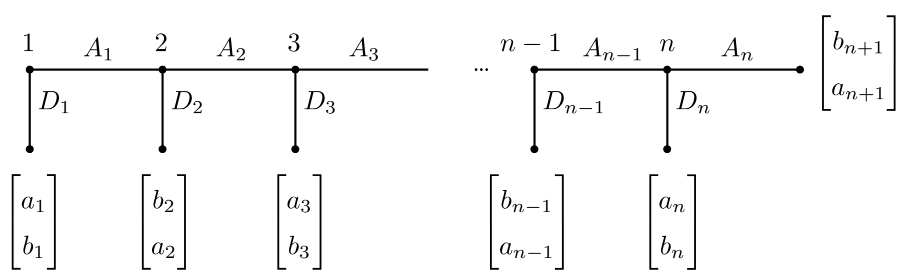

Figure 1

The decision nodes are numbered from 1 to n. For definiteness we assume that n is odd. Player P_1 moves at odd nodes, player P_2 at even nodes. At node i, a player chooses between 'Across' A_i and 'Down' D_i. The leader payoff is a_i, i.e. a_i is the payoff to the player who plays D_i. The follower payoff is b_i.

The payoffs are such that the game is a centipede game, i.e.

(1) a_i and b_i are strictly increasing in i,

(2) $a_i > b_{i+1}$,

(3) $\eta_i := \frac{a_i - b_{i+1}}{a_{i+2} - b_{i+1}}$ is weakly increasing in i,

(4) $\eta_i \leq \frac{1}{8}$ for all $i \in N$.

Thus the game corresponds to a situation in which two players can share a certain profit, but only in unequal terms. Overall profit $a_i + b_i$ is increasing, but every player prefers to be the leader now than to be the follower in the next stage. If the opponent could be relied upon to play 'Across', however, each player would play 'Across' earlier[1]. The centipede game is due to Rosenthal (1981), its name is due to Binmore (1987–88).

A pure strategy of player j is a mapping that associates with each of his decision nodes i an action A_i or D_i. Thus, if a player has m decision nodes he has 2^m many pure strategies, i.e. the number of strategies grows exponentially in the length of the game.

The players are assumed to have a prior probability that specifies the probability that the opponent is non-rational. For simplicity we assume that this prior is common to both players[2]. We denote this prior probability by ϵ, and assume $0 < \epsilon < 1$.

Our equilibrium concept aims to capture the optimal strategies of rational players. Thus a rational player must not have an incentive to deviate from his equilibrium strategy, as long as a rational opponent does not deviate either. However, a ra-

[1] Conditions (3) and (4) are conditions on the payoff increases. It means that the sure gains from playing 'Down' in relation to the possible gains from playing 'Across' increase, i.e. that playing 'Down' does not become less attractive in relative terms (3). Condition (4) says that these gains must not be too high; this is sufficient, but not necessary, to ensure that playing across does not result from uncertainty love alone. In their experiments, McKelvey & Palfrey (1992) assume that η_i is constant with $\eta_i = \frac{1}{7}$ and $n = 4$, resp. $n = 6$. (See also footnote 9.)

[2] Allowing different priors only introduces one more degree of freedom. This would not make the analysis conceptually deeper, and would make it easier to generate different equilibria.

tional player does not know what a non-rational opponent will do, and so faces genuine uncertainty. We assume that, when facing this uncertainty, rational players maximise Choquet expected utility in the sense of Schmeidler (1989).

2.2 Choquet Expected Utility Theory

According to CEU, players act in the face of uncertainty as if they possess a utility function over consequences and subjective beliefs over the domain of uncertainty, and maximise subjective expected utility. However, in contrast to a situation in which players face risk, players' beliefs do not have to be additive, i.e. representable by a probability measure. Instead, players' beliefs are represented by a capacity, i.e. a not necessarily additive 'probability' measure.

This model thus corresponds to a situation in which uncertainty cannot be reduced to probability. This model allows a parsimonious explanation of the Ellsberg paradox that people do not act as if their beliefs can be represented by probability measures. CEU retains the useful notion of belief and explains lack of probabilistic sophistication as a result of the players' attitude towards uncertainty.

Formally, let S be a set of states of nature. Let $s \in S$ and let $\Sigma \subseteq 2^S$ be a σ-algebra of events $E \in \Sigma$. A capacity associates with each event a real number such that[3]

(1) $v(\emptyset) = 0$,

(2) $v(S) = 1$, and

(3) $E \subseteq E' \implies v(E) \leq v(E')$.

The expected utility with respect to a capacity is defined as the Choquet (1953) integral: Let X be a simple positive random variable, i.e. X takes the positive values $x_1, x_2, ..., x_k$ on the events $E_1, E_2, ..., E_n$. The sets are measurable, pairwise disjoint, and their union is S. Without loss of generality, assume $x_1 > x_2 > ... > x_k$ and set $x_{k+1} := 0$. As usual, let $v(X \geq t) := v(\{\omega \in \Omega | X(\omega) \geq t\})$.

Then the Choquet integral is defined as[4]

$$\int X dv \quad := \quad \int_0^\infty v(X \geq t) dt$$

[3]The monotonicity property (iii) weakens the finite-additivity axiom $E \cap E' = \emptyset \implies v(E \cup E') = v(E) + v(E')$ for finitely-additive measures.

[4]The integral on the right hand side is the extended Riemann integral.

$$= \sum_{i=1}^{k}(x_i - x_{i+1})v(\cup_{j=1}^{i} E_j).$$

If v is additive this is the usual expectation. Thus the Choquet integral generalizes the usual formula for the expectation in terms of the decumulative distribution function E $X = \int_0^\infty F(X \geq t)dt$. It is a natural definition for an integral because it assigns to a characteristic function 1_E of an event E the capacity $v(E)$ of this event, and preserves monotonicity, i.e. if $X(s) \leq X'(s)$ for all $s \in S$ then $\int_S X dv \leq \int_S X' dv$.

2.3 Uncertainty Aversion

The non-additivity of v allows the formalisation of the player's attitude towards uncertainty. According to the definition of the integral, if probability weights are not additive then the probability residual is allocated to the worse outcome: Consider two events E and E' Let $E \cap E' = \emptyset$ and $E \cup E' = S$. Assume that the random variable X takes value x_1 on E and x_2 on E', and that $x_1 > x_2$. Let $v(E)+v(E') < 1$. Then by the definition of the integral $\int_S X dv = x_1 \cdot v(E) + x_2 \cdot (1 - v(E))$. This means that the probability residual $1 - v(E) - v(E')$ is allocated to the worse outcome. Thus subadditivity of a players' beliefs corresponds to his uncertainty aversion when facing genuine uncertainty. A decision-theoretic axiomatisation of uncertainty aversion in terms of preferences over acts is due to Schmeidler (1989)[5].

When a player faces a non-rational opponent his relevant space of uncertainty is the opponent's pure strategy set.Therefore, we assume that a rational player assigns to each of his opponent's pure strategies $s_j \in S_j$ some "probability weight" $\theta_{s_j} \geq 0$. Since any deviation from rationality is as non-rational as another, the player has no reason to regard any of a non-rational opponent's strategies more likely than another. For this reason we assume $\theta_{s_j} = \theta$, for all $s_j \in S_j$. This also simplifies the analysis. For simplicity we also assume that the players are identical, i.e. that the θ is the same for both of them[6].

[5]For related axiomatisations see, e.g. Gilboa & Schmeidler (1989) and Sarin & Wakker (1994)

[6]As before, introducing a different degree of uncertainty aversion for the second player corresponds to an additional degree of freedom. We think it is desirable not to introduce any ad hoc asymmetry.

If a rational player is completely uncertainty averse, we have $\theta = 0$, and in evaluating one of his pure strategies the player will assign probability 1 to the opponent's strategy that minimizes his utility. As long as $\theta > 0$, the player is only boundedly uncertainty averse, in that he gives some weight, however small, to other strategies of his opponent. Formally, this means that a rational player's beliefs about the strategy choice of a non-rational opponent is given by the capacity[7]

$$v(E) = \begin{cases} 1 & , \quad E = S_j \\ \theta |E| & , \quad E \subset S_j. \end{cases}$$

The assumption that the rational player is uncertainty averse thus translates into $\theta < \frac{1}{|S_j|}$. The main point of this paper is that there is an interaction between uncertainty aversion and the game-theoretic definition of strategy, as long as the uncertainty aversion is bounded.

2.4 Expected Payoffs

The specification of this capacity now allows us to define the payoff, that a rational opponent expects if he plays his pure strategy $s_i \in S_i$ and believes that his opponent is non-rational, as the CEU of his utility:

$$u(s_i, v) := \int_{S_j} u(s_i, s_j) dv.$$

Since a player does not know, however, if his opponent is rational or not, but has a prior belief ϵ that the opponent is non-rational, his expected payoff from his strategy s_i given that a rational opponent uses strategy s_j^* is given by

$$(1 - \epsilon) u(s_i, s_j^*) + \epsilon u(s_i, v).$$

A rational player will choose a strategy that maximises his payoff not only at the beginning of the game, but also in each subgame. It thus remains to specify how a rational player's beliefs change during the course of the game.

2.5 Updating and the Dempster-Shafer Rule

An updating rule has to generalize Bayes' Rule to non-additive probability measures. We assume that non-additive beliefs are updated through the Dempster-Shafer rule.

[7]Here $|E|$ denotes the cardinality of the set E.

Formally, let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

The Dempster-Shafer rule (Dempster 1968, Shafer 1976) corresponds to Bayes' Rule if the capacity is additive. When it is not, it reflects the uncertainty aversion, or pessimism, of the player (Gilboa & Schmeidler 1993).

The main use we make of the Dempster-Shafer rule is that it allows the formalization of the updating process after an action that is only taken by a non-rational player: Let ϵ be the prior probability that the opponent is not rational. Assume that the opponent has two actions A and D, and that a rational opponent chooses action A with probability p. Then the posterior belief ϵ' about the opponents' rationality is given by

$$\epsilon' := \frac{\epsilon \cdot (1 - |S_j|\theta)}{1 - \epsilon|S_j|\theta - (1-\epsilon)(1-p)},$$

where $|S_j|$ is the number of the opponents' strategies, in the subgame starting at the given node, that specify D. This is formally derived in the appendix.

Note that, first, if $p = 0$ and only a non-rational player chooses A the Dempster-Shafer rule gives the result that $\epsilon' = 1$. Secondly, if $p = 1$ then $\epsilon' < \epsilon$, i.e. a rational action is interpreted as evidence of rationality. Finally, as long as there is some doubt about the rationality of the opponent at the beginning of the game, there are no probability zero events.

We can now define the solution concept.

2.6 The Equilibrium Concept

An equilibrium is a strategy combination from which no rational player has an incentive to deviate unilaterally. We are considering an extensive game in which rationality is not mutual knowledge, so we have to extend this definition in two ways: First, we incorporate the assumption that rational players face genuine uncertainty, maximise Choquet expected utility, are boundedly uncertainty averse and update their beliefs according to the Dempster-Shafer rule. Secondly, in the spirit of subgame perfection we require optimality at each decision node.

A *θ-perfect Choquet-Nash equilibrium* is a pair of behavior strategies (σ_1^*, σ_2^*) such that

(1) at each node, each pure strategy of a rational player in their support maximises his expected utility given his beliefs about the opponent's rationality, the rational opponent's strategy, and the degree of uncertainty aversion,

(2) the beliefs about rational opponents are correct, and

(3) the beliefs about the opponent's rationality are updated according to the Dempster-Shafer rule.

We now have the following results:

Result 1:

Every centipede game has at least on θ-perfect Choquet-Nash equilibrium, for every common degree of uncertainty aversion θ and every degree ϵ of mutual knowledge of rationality.

Result 2:

However small the degree ϵ of lack of mutual knowledge of rationality, and however small the degree of uncertainty aversion, as long as they are positive, in the θ-perfect Choquet-Nash equilibrium the first player will not play 'Down' with probability 1.

The results are formally stated and in section 4. In the next section we illustrate them by an example.

3 An Example

Consider the following centipede game:

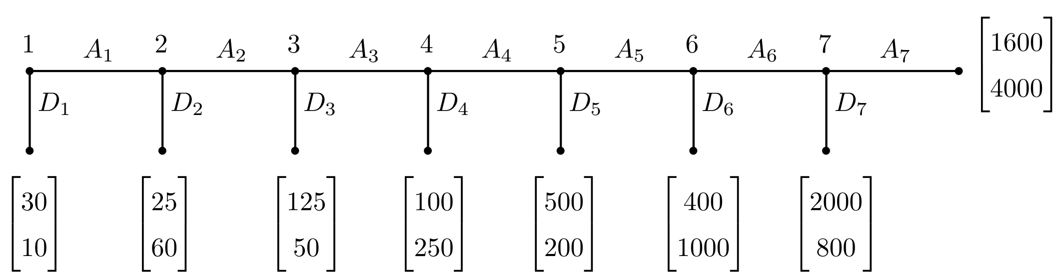

Figure 2

We assume that players have a common prior $\epsilon = \frac{1}{3}$ that the opponent is non-rational. We assume that players are boundedly uncertainty averse with degree of uncertainty aversion $\theta = \frac{1}{20}$ for both players.

By backward induction, we analyse this game starting from node 7.

At **node 7**, player P_1 will achieve 2000 if he plays D_7 as opposed to 1600 if he plays A_7. He is no longer in a situation of strategic interaction but in a pure decision situation. Therefore D_7 is his optimal choice.

At **node 6**, player P_2 faces both risk and uncertainty. He faces the risk that the opponent is non-rational, which is given by player P_2's belief ϵ_6 at node 6. Moreover, he faces the uncertainty what a non-rational opponent might play. The opponent has two strategies at node 7. Since P_2 is uncertainty averse, each of these strategies receives probability weight θ. The residual $1 - 2\theta$ is allocated to the strategy that is worst for P_2. Thus his Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_2(v_6, A_6) &= (1 - 2\theta)800 + \theta 800 + \theta 4000 \\ &= 800 + \theta(4000 - 800) \\ &= 960. \end{aligned}$$

In calculating his overall payoff from A_6, P_2 knows, by backward induction, that a rational player P_1 will play D_7, which results for P_2 in a payoff of 800. Thus his overall payoff is given by

$$(1 - \epsilon_6)800 + \epsilon_6 960.$$

P_2 can ensure 1000 by playing D_6, so D_6 is optimal.

At **node 5**, it follows by the same reasoning that D_5 is optimal.

At **node 4**, player P_2 knows that a non-rational opponent has four strategies in the continuation game, and that it is optimal to play D_6 at node 6. Thus P_2's Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_2(v_4, A_4) &= (1 - 4\theta)200 + 2\theta 200 + 2\theta 1000 \\ &= 200 + 2\theta(1000 - 200) \\ &= 280. \end{aligned}$$

Thus his overall payoff is given by

$$(1) \qquad (1-\epsilon_4)200+\epsilon_4 280.$$

P_2 can only ensure 250 by playing D_4, so the optimal strategy depends on his beliefs ϵ_4.

By the Dempster-Shafer rule, ϵ_4 and ϵ_2 are related as follows:

$$(2) \qquad \epsilon_4 := \frac{\epsilon_2 \cdot (1-\frac{8}{20})}{1-\frac{8}{20}\epsilon_2-(1-\epsilon_2)(1-p_3^*)}.$$

At **node 3**, player P_1 knows that a non-rational opponent has four strategies in the continuation game, and that it is optimal to play D_5 at node 5. Thus P_1's Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_1(A_3, v_3) &= (1-4\theta)100+2\theta 100+2\theta 500 \\ &= 100+2\theta(500-100) \\ &= 140. \end{aligned}$$

Thus his overall payoff is given by

$$(3) \qquad (1-\epsilon_3)[p_4^* 500+(1-p_4^*)100]+\epsilon_3 140,$$

where p_4^* is the probability with which a rational player P_2 plays A_4. P_1 can only ensure 125 by playing D_3, so the optimal strategy depends on his beliefs ϵ_3 and on P_2's optimal strategy p_4^*.

By the Dempster-Shafer rule, ϵ_3 and ϵ are related as follows:

$$(4) \qquad \epsilon_3 := \frac{\epsilon \cdot (1-\frac{8}{20})}{1-\frac{8}{20}\epsilon-(1-\epsilon)(1-p_2^*)}.$$

At **node 2**, player P_2 knows that a non-rational opponent has eight strategies in the continuation game, and that it is optimal to play A_4 with probability p_4^* at node 4. However, he also knows that at node 4 he can ensure 250, so that p_4^*, due to its optimality, ensures at least as much. Thus P_2's Choquet expected utility from a non-rational opponent is bounded below:

$$\begin{aligned} u_2(v_2, A_2) &\geq (1-8\theta)50+4\theta 50+4\theta 250 \\ &= 50+4\theta(250-50) \\ &= 90. \end{aligned}$$

Thus his overall payoff is bounded below by

$$(1-\epsilon_2)[p_3^*250 + (1-p_3^*)50] + \epsilon_2 90, \tag{5}$$

where p_3^* is the probability with which a rational player P_1 plays A_3. This payoff is bounded below by $50 + 40\epsilon_2$ for $p_3^* = 0$.

By the Dempster-Shafer rule, ϵ_2 and ϵ are related as follows:

$$\epsilon_2 := \frac{\epsilon \cdot (1 - \frac{16}{20})}{1 - \frac{16}{20}\epsilon - (1-\epsilon)(1-p_1^*)}. \tag{6}$$

At **node 1**, player P_1 knows that a non-rational opponent has eight strategies in the continuation game, and that it is optimal to play A_3 with probability p_3^* at node 3, which gives at least 125. Thus P_1's Choquet expected utility from a non-rational opponent is bounded below:

$$\begin{aligned} u_1(A_1, v_1) &\geq (1-8\theta)25 + 4\theta 25 + 4\theta 125 && (7)\\ &= 25 + 4\theta(125 - 25) && (8)\\ &= 45. && (9) \end{aligned}$$

Thus his overall payoff is bounded below by

$$(1-\epsilon_1)[p_2^*125 + (1-p_2^*)25] + \epsilon_1 45,$$

where p_2^* is the probability with which a rational player P_2 plays A_2.

Since $\epsilon_1 := \epsilon = \frac{1}{3}$, it follows that

$$(1-\epsilon_1)[p_2^*125 + (1-p_2^*)25] + \epsilon_1 45 = \frac{95}{3} + \frac{200}{3}p_2^*.$$

Since D_1 gives 30, P_1 will prefer A_1.

From the Dempster-Shafer rule, this implies

$$\begin{aligned} \epsilon_2 &:= \frac{\epsilon \cdot (1 - \frac{16}{20})}{1 - \frac{16}{20}\epsilon - (1-\epsilon)(1-p_1^*)} && (10)\\ &= \frac{1}{11}. && (11) \end{aligned}$$

This, in turn, implies that at node 2 the continuation payoff is bounded below by $50 + 40\epsilon_2 = \frac{590}{11} < 60$. This shows that despite the boundedness of uncertainty aversion the increasing payoffs alone do not lead player P_2 to choose 'Across' at node 2. If he does so, then because he expects a rational opponent also to be willing to go ‘Across’. In equilibrium, these beliefs are self-fulfilling.

We now show that an equilibrium is given by

$$p_2^* = 1,$$

$$p_3^* = \frac{36}{1000},$$

and

$$p_4^* = \frac{41}{800}.$$

First, $p_2^* = 1$ is optimal because, from (5) with $\epsilon_2 = \frac{1}{11}$ and $p_3^* = \frac{36}{1000}$,

$$\begin{aligned} & (1-\epsilon_2)[p_3^* 250 + (1-p_3^*)50] + \epsilon_2 90 \\ = & \frac{590}{11} + 200\frac{10}{11}\frac{36}{1000} \\ = & \frac{662}{11} \\ > & 60. \end{aligned}$$

From (4) with $p_2^* = 1$ we have

$$\begin{aligned} \epsilon_3 &= \frac{6\epsilon}{10-4\epsilon} \\ &= \frac{3}{13}. \end{aligned}$$

Secondly, given p_4^* and ϵ_3, player P_1 is indifferent between A_3 and D_3 at node 3, and so is willing to mix. From (3) his continuation payoff is given by

$$\begin{aligned} & (1-\epsilon_3)[p_4^* 500 + (1-p_4^*)100] + \epsilon_3 140 \\ = & \frac{1}{13}[(1000 + 205 + 420] \\ = & 125. \end{aligned}$$

From (2), with $p_3^* = \frac{36}{1000}$ and $\epsilon_2 = \frac{1}{11}$ we have

$$\begin{aligned} \epsilon_4 &= \frac{6\epsilon_2}{6\epsilon_2 + 10p_3^*(1-\epsilon_2)} \\ &= \frac{6}{6+\frac{36}{10}} \\ &= \frac{5}{8}. \end{aligned}$$

Finally, given $\epsilon_4 = \frac{5}{8}$, player P_2 is indifferent between A_4 and D_4, because his continuation payoff is, from (1),

$$
\begin{aligned}
& (1-\epsilon_4)200 + \epsilon_4 280 \\
= & \frac{2000}{8} = 250.
\end{aligned}
$$

To summarize, the equilibrium is given by:

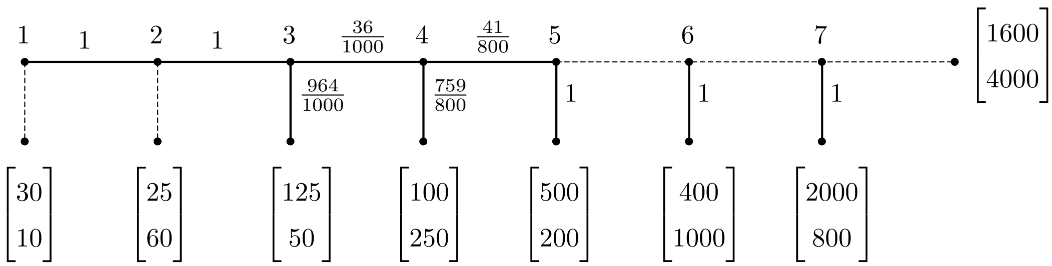

Figure 3

We end this section with some remarks:

(1) In this example, no pure strategy equilibrium exists. This can be seen from equations (1) and (2): If $p_3^* = 0$ then $\epsilon_4 = 1$, thus A_4 is optimal, which leads to $p_3^* = 1$, a contradiction. Conversely, if $p_3^* = 1$ then (5) implies $p_2^* = 1$. But then $\epsilon_3 = \frac{3}{13}$ and the continuation payoff (3) at node 3 is $(1-\epsilon_3)100 + \epsilon_3 140 < 125$, which leads to $p_3^* = 0$, another contradiction. In general, however, a pure strategy equilibrium may exist.

(2) In our example, $\epsilon = \frac{1}{3}$ is larger than in Kreps et al. (1982) and McKelvey & Palfrey (1992). It can be shown, however, that for no $\epsilon > 0$ will D_1 be chosen with probability one. More generally, here ϵ refers to a player's belief that the opponent is rational, and reasons in the same way as the player himself. This makes a high ϵ a plausible parameter value.

(3) Players adjust the belief ϵ about the opponent's rationality both upward and downward, and not just in one direction. An action that is taken by a rational player with high probability is taken as evidence of rationality and ϵ is adjusted downward. Conversely, an action that a rational player only chooses with low probability is considered as evidence of non-rationality and ϵ is adjusted upward.

(4) It is interesting to note that the taking probability does not increase monotonically. Also, in contrast to the sequential equilibrium in McKelvey & Palfrey (1992) the taking probability may be 1 not only at the last two nodes of the game.

(5) The analysis does not give a bell-shaped distribution over the terminal nodes. In McKelvey & Palfrey (1992), the sequential equilibrium alone does not either, however, they are able to show that the incorporation of learning can explain the empirical data.

4 Results

We now state and prove the results formally.

Definition.

A *centipede game* $\Gamma = (n, (\{D_i, A_i\})_{i=1,\ldots,n}, (a_i, b_i)_{i=1,\ldots,n+1})$ is given by a set N of n nodes $i \in N$, for each node two actions D_i and A_i, and for each action D_i and for A_n two payoffs a_i and b_i such that

(1) a_i and b_i are strictly increasing in i,

(2) $a_i > b_{i+1}$,

(3) $\eta_i := \frac{a_i - b_{i+1}}{a_{i+2} - b_{i+1}}$ is weakly increasing in i,

(4) $\eta_i \leq \frac{1}{8}$ for all $i \in N$.

For pure strategies s_1 and s_2 let $u_j(s_1, s_2)$ be a_k or b_k, where $k := \min\{k' | s_j(k') = D_{k'}$ for some player j $\}$, depending on whether $k \in N_j$ or not. Let σ_j be a behavior strategy of player j, where $\sigma_j(i)$ specifies the probability of 'Across' at node i under σ_j. Let $u_j(\sigma_1, \sigma_2)$ be the expected utility[8] of player j under the behavior strategies σ_1, σ_2.

Let θ be the common degree of uncertainty aversion of the two players, where[9] $0 \leq \theta < \overline{\theta} := \frac{1}{2^{\frac{n+1}{2}}}$. For given n, $\overline{\theta}$ is the upper bound on θ to ensure uncertainty aversion.

[8] Behavior strategies define additive probabilities over the pure strategy sets, so this is the usual expectation.

[9] The upper bound on θ preserves uncertainty aversion. If it is violated, both propositions still hold, but proposition 2 is due to uncertainty love alone.

Note that $\theta < \frac{1}{2^{\frac{n+1}{2}}}$ is equivalent[10] to $n \leq \overline{n} := [-(2\ (\mathrm{ld}\ \theta) + 1)]$, where ld denotes the logarithm to the base 2. For given θ, $\overline{n}$ is the upper bound on n that ensures that players are uncertainty averse even at the beginning of the game.

Definition.

Let Γ be a centipede game. Let N_1 and N_2 be the set of player 1's and 2's decision nodes i. Let θ be the degree of the players' attitude towards uncertainty aversion, and let ϵ be the common prior about rationality. let $\epsilon_0 = \epsilon_1 := \epsilon$ Then a *θ-perfect Choquet-Nash equilibrium* is a pair of behavior strategies (σ_1^*, σ_2^*) such that if s_1^* and s_2^* are in the support of σ_1^* and σ_2^*

(1) $s_1^* \in \arg\max_{s_1}(1-\epsilon_i)u_1(s_1, \sigma_2^*) + \epsilon_i u_1(s_1, v_i), \quad \forall i \in N_1,$

$s_2^* \in \arg\max_{s_2}(1-\epsilon_i)u_2(\sigma_1^*, s_2) + \epsilon_i u_2(v_i, s_2), \quad \forall i \in N_2,$

(2) $\epsilon_{i+2} = \frac{\epsilon_i \cdot (1-|S_{j,i+1}|\theta)}{1-\epsilon_i|S_{j,i+1}|\theta-(1-\epsilon_i)(1-\sigma_j^*(i+1))},$

(3) $u_1(s_1, v_i) := \int_{S_{2,i+1}} u_1(s_1, s_2) dv_i,$

$u_2(v_i, s_2) := \int_{S_{1,i+1}} u_2(s_1, s_2) dv_i,$

(4) $v_i(E) = \begin{cases} 1 & , \quad E = S_{j,i} \\ \theta|E| & , \quad E \subset S_{j,i}, \end{cases}$

where $S_{j,i}$ is the strategy set of the opponent in the subgame beginning at node i.

Proposition 1.

For all ϵ and all θ, there exists a θ-perfect Choquet-Nash equilibrium.

Proof.[11]

Let $u_1(s_1, v_i)$ and $u_2(v_i, s_2)$ be defined as in (3) and (4), and let $u_1(\sigma_1, v_i)$ and $u_2(v_i, \sigma_2)$ be the (additive) expectations of $u_1(s_1, v_i)$ and $u_2(v_i, s_2)$ under the behavior strategies σ_1 and σ_2. Consider the correspondence[12] $\varphi : [0,1]^n \times [0,1]^{n-1} \longrightarrow [0,1]^n \times [0,1]^{n-1}$, $(\sigma_1(i), \sigma_2(i), \epsilon_i) \mapsto (\sigma_1'(i), \sigma_2'(i), \epsilon_i')$:

$$\sigma_1'(i) := \arg\max_p (1-\epsilon)u_1(p, \sigma_2) + \epsilon_i u_1(p, v_i) \quad \forall i \in N_1, \tag{5}$$

[10] Following Kolmogorov & Fomin (1954), we denote for $a \in \mathbb{R}$ the integral part by $[a]$ (the largest integer smaller than a), and the fractional part by $< a >$ ($< a >= a - [a]$).

[11] The only difference to the standard existence proof is that we apply fixed point arguments directly to the extensive form. The reason for this is that there is no agent normal form, since non-rational players cannot be modelled as players, who would choose additive behavior strategies. On the other hand, applying non-additive equilibrium concepts to the normal form game between rational agents only would require a model of independent choices by more than two players with heterogeneous priors about the rationality of the opponents.

[12] Note that equation (7) is well-defined for $\epsilon_i = 0$. Given our assumption that $\epsilon > 0$, ϵ_i will not assume this value, yet it must be included in order to have a compact domain.

$$(6) \qquad \sigma_2'(i) := \arg\max_p (1-\epsilon) u_2(\sigma_1, p) + \epsilon_i u_2(v_i, p) \quad \forall i \in N_2,$$

$$(7) \qquad \epsilon'_{i+2} \; := \; \frac{\epsilon_i \cdot (1 - |S_{j,i+1}|\theta)}{1 - \epsilon_i |S_{j,i+1}|\theta - (1-\epsilon_i)(1-\sigma_j(i+1))}.$$

We first show that a fixed point of this correspondence is a θ-perfect Choquet-Nash equilibrium:

Let $(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i) \; \in \; \varphi(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i)$. This means

$$(8) \qquad \hat{\sigma}_1(i) \in \arg\max_p (1-\hat{\epsilon}) u_1(p, \hat{\sigma}_2) + \hat{\epsilon}_i u_1(p, v_i) \quad \forall i \in N_1,$$

$$(9) \qquad \hat{\sigma}_2(i) \in \arg\max_p (1-\hat{\epsilon}) u_2(\hat{\sigma}_1, p) + \hat{\epsilon}_i u_2(v_i, p)$$

$$(10) \qquad \hat{\epsilon}_{i+2} \; := \; \frac{\hat{\epsilon}_i \cdot (1 - |S_{j,i+1}|\theta)}{1 - \hat{\epsilon}_i |S_{j,i+1}|\theta - (1-\hat{\epsilon}_i)(1-\hat{\sigma}_j(i+1))}.$$

By their definitions, $u_1(p, \sigma_2)$, $u_2(\sigma_1, p)$, $u_1(p, v_i)$ and $u_2(v_i, p)$ are linear in p, so if $\hat{\sigma}_1(i)$ and $\hat{\sigma}_2(i)$ are maximisers then so are the pure strategies $\hat{s}_1(i)$ and $\hat{s}_2(i)$ in their support[13]. Consequently $(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i)$ satisfy (1) — (4) for any given $\epsilon \equiv \epsilon_0 \equiv \epsilon_1$.

It remains to be shown that such a fixed point exists. Since φ maps a closed, bounded and convex subset of a finite-dimensional Euclidean space into itself, Kakutani's Theorem (1941) implies that a fixed point exists if φ is non-empty, convex-valued and has a closed graph. Since the maximands are linear in p, they are continuous over a compact domain and, by Weierstraß' Theorem, the maxima in (5) and (6) exist. Moreover, (7) uniquely determines ϵ'_{i+2}. So φ is non-empty. Also, from the linearity of (5) and (6) and the uniqueness of (7), φ is convex-valued. Finally, by Berge's Maximum Theorem (1959), φ is closed-valued and upper hemi-continuous. This implies that φ has a closed graph (Border 1985, p.56, Theorem 11.9 (a)). This completes the proof.

Note that the equilibrium is not unique. Intuitively, a rational player will go across if his expected utility from a non-rational opponent — determined by his uncertainty aversion — and the expected utility from a rational opponent — weighted by his belief about the likelihood of non-rationality — is higher or equal than his payoff from going down. His belief at this node is his update given his initial beliefs and the rational strategies. It may be that the initial belief is exactly such to make him indifferent. Generically, however, this will not be the case.

[13]Note that for $u_1(p, v_i)$ and $u_2(v_i, p)$ this is due to the order of integration.

We refer to player P_i as the player who moves at node i, and denote by S_i the set of pure strategies of player P_i in the subgame starting at node i. We denote by $\sigma^*(i)$ the equilibrium probability with which player P_i plays A_i at node i.

Proposition 2.
$\forall \theta > 0\ \forall \epsilon > 0\ \exists N\ \forall n :\quad$ *If* $N \leq n \leq \overline{n}$ *then* $\sigma^*(1) \neq 0$.

Proof. Indirect. Suppose $\sigma^*(1) = 0$. Then $\epsilon_2 = 1$ and P_2 will choose A_2 if

$$\begin{aligned} a_2 \quad \leq \quad & (1-\epsilon_2)\ [\sigma^*(3)a_4 + (1-\sigma^*(3))b_3] \\ & +\epsilon_2\ [(1-\theta|S_3|)b_3 + \theta\frac{|S_3|}{2}b_3 + \theta\frac{|S_3|}{2}a_4], \end{aligned}$$

which is equivalent to

$$\eta_2 \leq \theta\frac{|S_3|}{2}.$$

Now define N as the smallest integer bigger than $4 + 2(\text{ld}\ \eta_2) - 2(\text{ld}\ \theta)$. Note that

$$\begin{aligned} & 4 + 2(\text{ld}\ \eta_2) - 2(\text{ld}\ \theta) && \leq -2(\text{ld}\ \theta) - 2 \\ \Longleftrightarrow\quad & \text{ld}\ \eta_2 && \leq -3 \\ \Longleftrightarrow\quad & \eta_2 && \leq \frac{1}{8} \end{aligned}$$

so that $N \leq \overline{n}$. Finally consider n with $N \leq n \leq \overline{n}$: Note that

$$\begin{aligned} & 4 + 2(\text{ld}\ \eta_2) - 2(\text{ld}\ \theta) && \leq n \\ \Longleftrightarrow\quad & (\text{ld}\ \tfrac{\eta_2}{\theta}) && \leq \frac{n-2}{2} - 1 \\ \Longleftrightarrow\quad & \eta_2 && \leq \theta\frac{2^{\frac{n-2}{2}}}{2}. \end{aligned}$$

But $|S_3| \geq 2^{\frac{n-2}{2}}$, and thus $\sigma^*(2) = 1$. But since

$$1 > \eta_1,$$

$$|S_2| \geq |S_3|,$$
$$\theta\frac{|S_3|}{2} \geq \eta_2$$

and

$$\eta_2 \geq \eta_1,$$

we have independently of ϵ

$$\eta_1 < (1-\epsilon) + \epsilon\theta\frac{|S_2|}{2}.$$

This would imply $\sigma^*(1) \neq 0$, a contradiction. So indeed $\sigma^*(1) > 0$. This completes the proof.

5 Conclusion

A θ-perfect Choquet-Nash equilibrium is a solution concept for the centipede game that combines subgame-perfection with uncertainty aversion. We suggest as a reason why players choose 'Across' early in the game the boundedness of uncertainty aversion. Even though players are uncertainty averse, if there is enough uncertainty from which players can profit and if they expect their rational opponents also to play 'Across' then it is indeed rational to play 'Across'.

On a conceptual level, the equilibrium concept allows the analysis of the centipede game without the assumption that rationality is mutual knowledge. It avoids several difficulties that arise in the Kreps et al. (1982) approach: First, non-rational players are not necessarily 'altruistic' and always play 'Across'. Secondly, we do not need to specify any particular belief about non-rational opponents, which in the absence of a theory of non-rational play would necessarily be ad hoc. In particular, we can avoid the difficulties associated with the uniform distribution as a model of ignorance. Thirdly, we do not need to refer to non-rational players as types, which would ascribe to them a consistent hierarchy of beliefs. Finally, the solution concept is consistent with the interpretation of equilibrium strategies as rational strategies, which implicitly defines all other strategies as non-rational. As a result, the structure of the game may be assumed to be mutual knowledge.

At the same time, our solution concept builds on existing game-theoretic concepts. First, the analysis is in the same spirit as Kreps et al. (1982), which has proved to be so useful in industrial organization. Secondly, the solution concept is an equilibrium concept, and avoids the indeterminateness associated with weaker solution concepts. Similarly, the solution concept is static, and does not rest on the specification of a dynamic learning or evolutionary process. Finally, we preserve the spirit of subgame perfection in requiring optimality at all decision nodes. Thus we extend the approach of Dow & Werlang (1994) to extensive games.

The limitations of our approach are the following: First, the actual computation of an equilibrium may be complicated, it corresponds to the computation of a fixed point, as does the sequential equilibrium in McKelvey & Palfrey (1992). Secondly, the degree of uncertainty aversion is not directly observable. How to elicit this degree from a purely decision-theoretic environment is an issue for future research.

That the degree of uncertainty aversion is bounded, however, seems a plausible hypothesis whose usefulness can only be established empirically. Finally, while our solution concept gives an 'inner' equilibrium for the centipede game, it does not replicate the distribution of actual choices. While the sequential equilibrium with 'altruistic' types in McKelvey & Palfrey (1992) alone does not give this distribution either, McKelvey & Palfrey (1992) show that additional hypotheses, both about how players make mistakes and how they learn during the game, do. To introduce such hypotheses in a consistent way is another topic for future research.

Appendix

Let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

Let ϵ be the prior probability that the opponent is not rational. Assume that the opponent has two actions A and D at the given node n. Assume that a rational opponent chooses action A with probability p. Finally, assume S is the set of the opponent's pure strategies that specify the action D at the given node.

Then the posterior belief ϵ' about the opponent's rationality after action A is given by

$$\epsilon' := \frac{\epsilon \cdot (1 - |S|\theta)}{1 - (1-\epsilon)(1-p) - |S|\theta\epsilon},$$

where $|S|$ is the number of the opponents' strategies S, and ϵ the prior belief about the opponent's rationality, with $0 < \epsilon < 1$.

This is derived as follows:

Let R be the event that the opponent is rational, let $\overline{R}$ be the event that he is non-rational.

We want to calculate

$$\epsilon' \equiv v(\overline{R}|A) := \frac{v(\overline{R} \cup D) - v(D)}{1 - v(D)}. \tag{11}$$

First,

$$v(R|A) + v(\overline{R}|A) = 1, \tag{12}$$

and

(3) $v(R|A) = \frac{v(R \cup D) - v(D)}{1 - v(D)}$,

(4) $v(\overline{R}|A) = \frac{v(\overline{R} \cup D) - v(D)}{1 - v(D)}$

imply

$$v(D) = v(R \cup D) + v(\overline{R} \cup D) - 1.$$

Secondly,

(5) $v(D|R) = \frac{v(D \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})}$, and

(6) $v(D|\overline{R}) = \frac{v(D \cup R) - v(R)}{1 - v(R)}$.

We know that

(7) $v(R) = 1 - \epsilon$,

(8) $v(\overline{R}) = \epsilon$,

(9) $v(D|R) = 1 - p$, and

(10) $v(D|\overline{R}) = |S|\theta$,

so that

(11) $v(D \cup \overline{R}) = (1 - \epsilon)(1 - p) + \epsilon$, and

(12) $v(D \cup R) = |S|\theta\epsilon + (1 - \epsilon)$.

Thus

$$v(D) = (1 - \epsilon)(1 - p) + |S|\theta\epsilon.$$

Consequently,

$$\epsilon' := \frac{\epsilon \cdot (1 - |S|\theta)}{1 - (1 - \epsilon)(1 - p) - |S|\theta\epsilon}. \tag{13}$$

Note:

- The derivation is only valid under lack of mutual knowledge of rationality, i.e. for $\epsilon > 0$ and $\epsilon < 1$, otherwise $v(D|R)$ or $v(D|\overline{R})$ are not well-defined.
- With $0 < \epsilon < 1$ there are no probability zero events. Since $|S|$ strategies specify action D and there are two actions at this node, the number of strategies is $2|S|$. So uncertainty aversion means $\theta < \frac{1}{2|S|}$. It follows that

  $$v(D) = (1 - \epsilon)(1 - p) + |S|\theta\epsilon < (1 - \epsilon)(1 - p) + \frac{1}{2}\epsilon < 1.$$

  This holds for any $p \in [0, 1]$, including the boundaries.
- In particular, if $\epsilon > 0$ then $\epsilon' > 0$, independently of p. However, if $p = 0$, then $\epsilon' = 1$. Thus we also need to be able to update the belief $\epsilon = 1$. Intuitively, if

the prior belief about the opponent is that he is non-rational and beliefs about his behavior are boundedly uncertainty averse, then there are no probability zero events, and the posterior belief should also be that the opponent is non-rational.This can be justified directly from the Dempster-Shafer rule (1): From monotonicity, $v(\overline{R}) \leq v(\overline{R} \cup D)$, therefore $v(\overline{R} \cup D) = 1$. Also, (6) implies $v(D|\overline{R}) = v(D \cup R)$, so again by monotonicity, $v(D) \leq v(D \cup R) = |S|\theta < 1$. Since this result also follows if we substitute $\epsilon = 1$ into (13), we do not have to explicitly track this special case.

- The reason why $\epsilon = 0$ has to be excluded is that there is no parallel argument that $\epsilon = 0$ and $p = 0$ should give $\epsilon' = 1$. (3) implies $v(D \cup \overline{R}) = v(D|R) = 1$ and (1) gives $\epsilon' = \frac{1-v(D)}{1-v(D)}$, but $v(D) \not< 1$.
- Whether action A is interpreted as evidence of rationality or evidence of non-rationality depends on p, $|S|$ and θ:

$$
\begin{aligned}
& \epsilon' && \leq \epsilon \\
\iff & \frac{\epsilon \cdot (1-|S|\theta)}{1-(1-\epsilon)(1-p)-|S|\theta\epsilon} && \leq \epsilon \\
\iff & (1-\epsilon)(1-p) && \leq (1-\epsilon)|S|\theta \\
\iff & p && \geq 1-|S|\theta.
\end{aligned}
$$

  Other things equal, the higher the probability of A, the more likely it is that A is evidence of rationality, because A is taken with high probability by rational players. The lower θ and $|S|$, the less likely it is that A is interpreted as evidence of rationality, because the greater is the uncertainty that A is taken by a non-rational player.
- Finally, note that the argument rests heavily on (2), i.e. the the requirement about beliefs that an opponent is either rational or non-rational, so that these beliefs have to be additive.

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