A rescaled range statistics approach to unit root tests

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Abstract

In the framework of integrated processes, the problem of testing the presence of unknown boundaries which constrain the sample path to lie within a closed interval is considered. To discuss this inferential problem, the concept of nearly-bounded integrated process is introduced, thus allowing to define formally the concept of boundary conditions within I(1) processes. When used to detect unknown boundaries, stardard unit root tests do not maintain the usual power properties and new methods need developing. Therefore a new class of tests, which are based on the rescaled range of the process, are introduced. The limiting distribution of the proposed tests can be expressed in terms of the distribution of the range of particular Brownian functionals, while the power properties are obtained through the derivation of the limiting Brownian functional of a $I(1)$ process with boundary conditions, which is done by referring to a new invariance principles for nonstationary time series with limited sample paths. Both theoretical and simulation exercises show that range-based tests outperform standard unit root tests significantly when used to detect the presence of boundary conditions.

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1 Introduction

Since the first studies about the order of integration, the presence of unit root in economic and financial time series has been mainly investigated by comparing the difference-stationary, unit root process with processes which are stationary around a deterministic trend. Tests initially proposed by Dickey (1976), Dickey and Fuller (1979, 1981), and afterwards generalized by several authors (see, *e.g.*, Phillips, 1987; Phillips and Perron, 1988; Schmidt and Phillips, 1991; Kwiatkowski *et al.*, 1992) are generally found to have suitably asymptotic properties in terms of power. The various test procedures differ in what is assumed to be the null hypothesis (unit root or trend stationarity), in the testing strategy which is applied (likelihood ratio, Lagrange multiplier or Wald approach) and, more generally, in the researcher's belief on available *a priori* information (Bayesian or classical procedures)[1]. However, all these methodologies can lose their power features as far as under the alternative hypothesis data are not generated by the common trend stationary process with a zero mean noise term, but follow a different dependence structure.

In particular, suppose that the time series has a unit root behavior but it is exogenously constrained to lie inside a specific "band". When the edges of such band are sufficiently far away, the process has locally a random walk behavior; on the other side, near the boundaries the process reverts in order to avoid crossing such limits. Hence, the presence of two boundaries which limit the variation of the process can induce mean reversion: in other words, stationarity is achieved through the effect of two "reflecting" barriers.

Empirically, *bounded* I(1) dynamics can arise in those cases where one or more agents "regulate" the trajectory of an economic variate only when its values exceed (or fall below) a specific target value: for example, in the context of exchange rate dynamics, if the monetary Authorities intervene on the foreign market in order to maintain the exchange rate inside a specific "band", namely a *target zone*, and if such interventions take place only close to the boundary of such a band, then the resulting dynamic can be well described by means of a bounded I(1) process[2].

Bounded dynamics can also be a matter of data definition. In fact, many economic variables, such as unemployment rate, nominal interest rates, market quotes, option prices can exhibit a strong persistence but cannot take values on all the real set, as they are constructed - or defined - in order to vary inside specific ranges.

In bounded I(1) framework, testing the standard unit root hypothesis means testing the absence of boundary conditions, *i.e.* the absence of a mean reversion component which depend on the effects of two (possibly unobservable) boundaries[3]. Clearly, against this kind of alternatives, standard unit root analysis cannot be proved to preserve the usual power properties, and different testing

[1] See, *e.g.*, the survey of Phillips and Xiao (1998) for a complete classification on the different approaches to unit root testing.

[2] Note that the target zone can be both observable (e.g. EMS targets) and unobservable (implicit or undeclared target zones); in the latter case, testing for the existence of boundary parameters (and therefore estimation) can be an interesting inferenctial problem (see, e.g., Cavaliere, 1998; Gardini *et al.*, 1998).

[3] Therefore, in the context of exchange rate analysis testing for a standard I(1) process against bounded variation means to test for the absence of Central Banks intervention and/or stabilyzing speculation.

techniques can be found to have higher power.

Unfortunately, in the economic and statistical literature, no inferential techniques which is directly developed to test for boundary conditions in I(1) systems is available. To fill this gap, this paper deals with the analysis of statistical procedures which can be used to verify empirically the presence and the effect of such boundaries. More precisely, a new class of unit root tests which has specifically power against I(1) processes with reflecting conditions is developed. The tests are based on special transformations of the range scale statistic of Hurst (1951), Mandelbrot (1971, 1972, 1975) and generalized by Lo (1991), which are here used to synthesize the spatial excursion of the process. The basic motivation behind these tests is that, against bounded alternatives, the sample excursion of the process can be more informative about the presence of mean reversion than other commonly used statistics, like the autocorrelation coefficient. It will in fact be proved that, even if range statistics are generally highly volatile, when used to test against the appropriate alternative they can provide very suitable power properties.

Finally, a special case in the class of bounded $I(1)$ processes is obtained by approximating the position of the barriers in terms of the sample size. In this situation, the process can - up to specific conditions - be stationary as far as T is fixed, but is nonstationary as T goes to infinite; we call these processes *nearly-bounded I(1).* Expressing the boundary parameters in terms of the sample size permits to develop a specific asymptotic theory for I(1) processes in the presence of barriers, particularly useful to local power analysis.

The paper is structured as follows. In the next paragraph we briefly reconsider the standard unit root process and we define formally the concept of bounded and nearly bounded processes; therefore, range statistics are introduced as synthetic measures of the sample excursion of the time series of interest. In paragraph 3 these statistics are used in order to develop the test procedures and in paragraph 4 the behavior of the tests under the null hypothesis is derived. Paragraph 5 analyzes the power of the test against nearly-bounded alternatives, and some power comparisons help understand the possible advantage of range-based unit root tests. Paragraph 6 reports some concluding remarks.

2 Unit roots, nearly-bounded processes and range statistics

In order to define formally the concept of bounded and nearly bounded variation, we move from the framework of the "semi-parametric" approaches to unit root modeling, based on the work of Phillips (1987) and relying on $I(1)$ processes with a general heteroskedastic, weakly dependent, zero-mean error term (*i.e.* a mixing process with suitable regularity conditions, see paragraph 4); in this case unit root tests simply are mostly based on the ordinary least squares estimation of regression equations like $x_t = \alpha + \mu t + \rho x_{t-1} + u_t$ as far as the null hypothesis is the unit root condition $\rho = 1$ (see Phillips and Perron, 1988), or like $x_t = \alpha + \mu t + u_t$ if the null is the stationary hypothesis (see Kwiatkowski *et al.*; 1991). The non parametric approach has the great advantage that most of the stochastic processes usually adopted in the econometric analyses satisfy the time-series conditions which are required for u_t.

We now introduce at a greater level of details what we are going to consider as the null hypothesis, against which we will test the presence of boundary conditions on the domain of the process of interest. Then, we generalize this definition in order to introduce boundary conditions and, finally, range statistics are introduced as test statistics to detect such boundaries.

2.1 The null hypothesis

The results of this paper are derived in the spirit of the non parametric approach. Following Phillips (1987), we suppose that x_t is generated by the following stochastic difference equation with a unit root

$$x_t = x_{t-1} + u_t \ , \ t = 1, ..., T \tag{1}$$

where x_0 is bounded almost surely and u_t is a sequence of zero mean random variables. Note that we do not need any assumption on the initial condition but requiring x_0 to be bounded with probability one. Instead of the usual weak white noise conditions

A1. $E(u_t) = 0$

A2. $E(u_t^2) = \sigma^2 < \infty, E(u_t u_{t-j}) = 0, \ j \neq 0.$

i.e. that x_t is a random walk with uncorrelated increments, following Phillips (1987), Phillips and Perron (1988) and Lo (1991) we relax assumption A1-A2 in order to allow for weak dependence and heterogeneity over time. This case can be obtained *e.g.* by referring to the following set of conditions $\mathcal{B}$ for the error term in (1) (see Phillips, 1987):

B1. $E(u_t) = 0$;

B2. $\sup_t E(|u_t|^p) < C < \infty$ for some $p > 2$;

B3. $0 < \lambda^2 \equiv \lim_{T\to\infty} E\left(T^{-1}(x_T - x_0)^2\right) < \infty$;

B4. $\{u_t\}$ is strong mixing with mixing coefficients α_m satisfying $\sum_{m=1}^{\infty} \alpha_m^{2(1/\beta - 1/p)} < \infty$ for some β such that $\beta \geq 2$ and $\beta < p$.

These conditions ensure that the error term is a zero-mean (B1), possibly heteroskedastic (B2) weakly dependent (B4) process. The mixing condition B4 implies that while there can be dependence between recent events, observations which are separate by a sufficiently long interval of time are almost independent. Condition B3 is a convergence condition on the average variance of x_t; note that λ^2 is the limit of the quantity

$$\begin{aligned}
\lambda_T^2 &= E\left(T^{-1}(x_T - x_0)^2\right) = T^{-1} E\left(\left(\sum_{t=1}^{T} u_t\right)^2\right) \\
&= T^{-1} \sum_{t=1}^{T} E\left(u_t^2\right) + 2T^{-1} \sum_{j=1}^{T-1} \sum_{t=j+1}^{T} E(u_t u_{t-j})
\end{aligned}$$

which is an alternative expression for the variance of the cumulated error process. Furthermore, B3 rules out degenerate cases like the so-called $I(-1)$ processes, *i.e.* processes whose cumulate sum is an $I(0)$ process. Alternative conditions which for our purposes can be assumed without changing substantially the subsequent analyses, can be found *e.g.* in Vogelsan (1998), who allows for a more general autocorrelation structure but only under homoskedasticity restrictions, in Phillips and Solo (1992), within the framework of linear processes, in Chan and Wei (1987), for unit root models with a martingale difference error term.

Process (1) can be generalized in order to take the presence of deterministic trends into account. By re-writing the recursive relation (1) as

$$x_t = x_{t-1} + \mu + u_t$$

process x_t is characterized by a linear trend in levels. Note that this generalization does not require to modify conditions $\mathcal{B}$.

2.2 Bounded I(1) processes

Opposite to the unit root process defined through equation (1) and conditions B1-B4, we are interested in mean reverting stochastic processes whose long run equilibrium around a deterministic component is determined by the presence of barriers, or boundary conditions, which constrain their sample paths to lie inside a specified interval.

Specifically, as an alternative to the unit root process $x_t = x_{t-1} + u_t$ we consider a general stochastic process which has locally a unit-root behavior but which satisfies the constrain

$$\underline{b} \leq x_t \leq \overline{b}$$

for all t. We formalize these concepts in the following definition:

Definition 1 (bounded $I(1)$ process) *A stochastic process* $\{x_t\}$, $0 \leq t \leq T$, *is said to be* bounded integrated of order one*, with boundaries* $\underline{b}$ *and* $\overline{b}$*, if the following set of conditions* $\mathcal{C}$ *holds:*

C1. $\underline{b} \leq x_t \leq \overline{b}$ *almost surely, all* t*;*

C2. it satisfies the recursive relation

$$x_t = x_{t-1} + u_t \ ; \tag{2}$$

C3. as $\underline{b} \to -\infty$ *and* $\overline{b} \to \infty$*, the process* $\{u_t\}$ *satisfies conditions* $\mathcal{B}$*;*

C4. the boundaries $\underline{b}$ *and* $\overline{b}$ *are deterministic and do not depend on* T*.*

Condition C1 ensures that the process lies inside the real interval $\left[\underline{b}; \overline{b}\right]$ with probability one while condition C2 allows - like in the unit root case - x_t to be recursively defined as $x_t = x_{t-1} + u_t$; implicitly, this means that the error term u_t, for each t, must satisfy the boundary condition $\underline{b} - \mu - x_{t-1} \leq u_t \leq \overline{b} - \mu - x_{t-1}$, which warranties bounded variation. Therefore, the conditional distribution of u_t (and, consequently, of x_t) is truncated, in the sense that is not defined on all the real axis, but its domain is constrained and is a function of the previous level of the process. C3 is a condition on the generating process of u_t, which has to satisfy the zero-mean, weak autocorrelation and heteroskedasticity assumptions $\mathcal{B}$ as the boundaries explode. This allows to nest the unit root process as a special case which is obtained by simply letting $\underline{b} \to -\infty$ and $\overline{b} \to \infty$. Finally, condition C4 states that the boundaries are deterministic and do not depend on T. The meaning of this requirement, which allows to discriminate between bounded and *nearly* bounded variation will become clearer later.

Briefly, this wide class of processes behaves like a unit root when x_t is sufficiently far away from the barriers, while displays mean reversion only in order to remain within the band for each t. These definitions can be easily extended in order to include processes with linear trends. In particular, we can replace (2) with the stochastic difference equation

$$x_t = x_{t-1} + \mu + u_t \ ; \tag{3}$$

if all the other conditions are unchanged, as the boundaries diverge the process converges to an $I(1)$ process with a linear trend.

The boundaries are assumed to be deterministic, but not necessarily constant. In particular, when the long run equilibrium is represented by a linear trend, then the assumption of constant boundaries can become unrealistic; in such cases it is more convenient to generalize the previous boundary conditions as

$$\underline{b}_t \leq x_t \leq \overline{b}_t \ , t = 1, ...T$$

where $\underline{b}_t = \underline{\alpha} + \mu t$ and $\overline{b}_t = \overline{\alpha} + \mu t$, $\underline{\alpha} < \overline{\alpha}$. The process $\{x_t\}$ is therefore constrained to lie within two straight parallels lines; we refer to this case with the term *linear boundaries*. This kind of boundaries greatly suit differential equations involving linear trend, see (3).

In order to develop the necessary asymptotic theory, we have to introduce some definitions which quantify the degree of persistence of the process on the boundaries. In particular, we give the following definitions:

Definition 2 *A boundary b is said to be* sticky *if* $\Pr(x_t = b) > 0$ *for some t. A boundary is said to be* regular *if* $\Pr(x_t = b) = 0$ *for all t.*

In the former case we allow the process to lie *on* the boundaries; this case, even if it is the simplest from a theoretical point of view, it is less interesting for what concerns statistical inference. A regular boundary, on the other side, is never reached by the process even if the sample size goes to infinite; this means that the boundaries are unobservable and, if any, must be estimated from data.

We now have all the elements to extend the definition of bounded integration to the case of nearly-bounded integrated process:

Definition 3 (nearly bounded $I(1)$ process) *A stochastic process $\{x_t\}$, $0 \leq t \leq T$, is said to be* nearly bounded integrated of order one*, with boundaries $\underline{b}$, $\overline{b}$ if the following set of conditions $\mathcal{C}'$ holds:*

C1. $\underline{b} \leq x_t \leq \overline{b}$ *almost surely, all t;*

C2'. it satisfies the recursive relation

$$x_t = x_{t-1} + u_t$$

with the initial condition $x_0 = c\sqrt{T}$, $-\infty < c < \infty$;

C3. as $\underline{b} \to -\infty$ and $\overline{b} \to \infty$, $\{u_t\}$ satisfied the conditions $\mathcal{B}$;

C4'. the constants $\underline{b}$, $\overline{b}$ *satisfies the equalities*

$$\begin{aligned} \underline{b} &= \underline{c}\sqrt{T} \\ \overline{b} &= \overline{c}\sqrt{T} \end{aligned}$$

where $\underline{c}$ *and* $\overline{c}$, $\underline{c} < c < \overline{c}$, *are deterministic and do not depend on* T.

Conditions C1 to C3 are the same given in definition 1, and state that the process has bounded variation and that it can be built up recursively by means of a noise term which satisfies the weak heterogeneity and memory assumption of Phillips as the boundaries are infinitely far; condition C2' is slightly different from C2, as it assumes that the initial value of the process can be expressed in terms of the number of observation T. Condition C4' characterizes the concept of "nearly bounded" process: the positions of the boundaries are proportional (in the square root) to the considered sample size T. As $T \to \infty$, clearly the distance between the boundaries $\underline{b}$ and $\overline{b}$ (*i.e.* $\overline{b} - \underline{b} = (\overline{c} - \underline{c})T^{1/2}$) becomes unbounded but, as the random walk has variance proportional to T, the effect of the boundaries can still be highly significant[4]. Another way of stating the concept of nearly bounded process is by saying that its theoretical range, instead of being constant (as in definition 1) is a *constant* proportion of $T^{1/2}$.

Note that, for a fixed value of T, a nearly bounded process can be stationary[5] while, as $T \to \infty$, nonstationarity is obtained. In other words, nearly bounded variation implies asymptotic nonstationarity.

Finally, it has to be stressed that the process is generated for a given value of T: drawing the $T + 1$ observation implies re-generating also the observations from time 0 to time T. This feature has a long tradition in econometrics: for example, in the context of unit root processes, nearly integrated time series satisfies recursive equations of the form $x_t = \left(1 - cT^{-1}\right) x_{t-1} + u_t$, $c > 0$; in the environment of structural break analysis, it is often assumed - in order to develop large sample asymptotics - that the breaks appear at given fractions of the sample size, *i.e.* at times $\alpha_1 T, \alpha_2 T, ...$, with $\alpha_j \in]0, 1[$ for $j = 1, 2, ...$. In all cases, the process has to be re-generated as far as one new observation has to be included in the sample.

Formally, since the band boundary parameters and the first observation depend on T, the process generated by C1 under conditions C2'-C4 constitutes a triangular array $\{x_{t,T},\ t = 0, ..., T,\ T = 0, 1, ...\}$. Anyway, for our purposes we can simplify this notation by dropping the T index.

The class of nearly-bounded integrated processes, as defined through the previous equations, usually displays strong first-order autocorrelation, as they exhibit mean reversion only near the boundaries. Thus, basing a test of the unit root hypothesis against this kind of alternative on the first-order autocorrelation coefficient cannot ensure the usual power properties to hold, and other sample

[4] To illustrate this point, consider a simple random walk $x_t = x_{t-1} + u_t$, $u_t \sim NID(0, \sigma^2)$, where the initial condition is $x_0 = cT^{1/2}$. By considering the last observation only, we have $x_T \sim N(cT^{1/2}, T\sigma^2)$, which implies that $\Pr\left\{\underline{b} \le x_T \le \overline{b}\right\} = \Pr\left\{\underline{c}T^{1/2} \le N\left(cT^{1/2}, T\sigma^2\right) \le \overline{c}T^{1/2}\right\} = \Pr\left\{\underline{c} \le N\left(c, \sigma^2\right) \le \overline{c}\right\}$. The latter quantity is simply the probability that a standard normal random variable lies in the interval $[\frac{\underline{c}-c}{\sigma}, \frac{\overline{c}-c}{\sigma}]$; such probability can be substantially less that 1 for suitably choices of the parameters, which automatically implies that the boundaries influence the dynamics of the process.

[5] To get stationarity, apart from T to be fixed, further restrictions on the starting value of the process and on the dynamic mechanism near the boundaries are necessary.

statistics should be used in order to build a powerful test procedure. In particular, as the alternative is characterized by the presence of bounded variation for the process, it is intuitive to build a test procedure on the sample excursion of the process, *i.e.* on *range* statistics.

2.3 Range statistics

Given T observations of a stochastic process $x_1, ..., x_T$, the simplest way to synthesize the amplitude of the variations with respect to its average is to define a statistic which is based on the "range" of the sample observations. The most immediate possibility is to refer to the simple *range statistic*, which is defined as

$$r_{\mu}(T) = \max_{t=1,...,T}\{x_t\} - \min_{t=1,...,T}\{x_t\} \tag{4}$$

Range statistics as in (4) are highly recurrent in the probability literature: their applications to the analysis of the distributions of partial sums and to empirical processes are well-known.

This statistic is particularly useful when data do not exhibit a deterministic trend component. However, in the presence of a trend, a first possibility is to eliminate its influence, *i.e.* to perform a preliminary detrendization of the time series, and therefore to compute the range statistic on the corresponding residuals.

When the time series is $I(1)$ with a linear trend, then it is sufficient to define the detrended series as $\widehat{x}_t = x_t - x_1 - \widetilde{\mu}(t-1)$, where the estimated drift $\widetilde{\mu}$ is obtained as $(T-1)^{-1}(x_T - x_1)$. In this way, the efficiently estimated (linear) trend is forced to pass through x_1 and x_T. The corresponding range statistic is therefore given by

$$\begin{aligned} r_{\tau_a}(T) &= \max_{t=1,...,T}\{\widehat{x}_t\} - \min_{t=1,...,T}\{\widehat{x}_t\} \\ &= \max_{t=1,...,T}\{x_t - x_1 - \widetilde{\mu}(t-1)\} - \min_{t=1,...,T}\{x_t - x_1 - \widetilde{\mu}(t-1)\} \end{aligned}$$

Furthermore, as x_1 and $\widetilde{\mu}$ do not depend on t, the range statistic can be simplified and equivalently defined as

$$r_{\tau_a}(T) = \max_{t=1,...,T}\{x_t - \widetilde{\mu}t\} - \min_{t=1,...,T}\{x_t - \widetilde{\mu}t\} \tag{5}$$

When the time series of interest is not $I(1)$, then the previous procedure does not necessarily lead to an "efficient" detrendization. A more general solution is to interpolate the data by means of a first-order polynomial: thus, the detrended series is obtained as $\widehat{x}_t = x_t - \widehat{\alpha} - \widehat{\mu}t$, where $\widehat{\alpha}$ and $\widehat{\mu}$ are the ordinary least square estimators of the intercept and the slope term respectively. Again, the corresponding range statistic is given by

$$\begin{aligned} r_{\tau_b}(T) &= \max_{t=1,...,T}\{\widehat{x}_t\} - \min_{t=1,...,T}\{\widehat{x}_t\} \qquad (6) \\ &= \max_{t=1,...,T}\{x_t - \widehat{\alpha} - \widehat{\mu}t\} - \min_{t=1,...,T}\{x_t - \widehat{\alpha} - \widehat{\mu}t\} \end{aligned}$$

Note that in equation (6) the intercept estimate $\widehat{\alpha}$ does not depend on t and, consequently, the range statistic can be equivalently written as

$$r_{\tau_b}(T) = \max_{t=1,...,T}\{x_t - \widehat{\mu}t\} - \min_{t=1,...,T}\{x_t - \widehat{\mu}t\} \tag{7}$$

which depends on the slope estimate $\widehat{\mu}$ only. Thus, the range statistics (5) and (6) differ in the way the trend slope μ is estimated: a "ratio" method for the former, a "regression" method for the latter.

3 Testing the I(1) hypothesis

We now have all the elements which we need to test the null hypothesis of a unit root process against the alternative of a bounded unit root processes. As nearly-bounded variation is the main characteristic of the alternative hypothesis, we base the test strategy on the range statistics previously introduced , which - as it will be shown later - can lead to a serious improvement of the test power with respect to standard unit root tests.

Let us consider, as a starting point, the simple range statistic (10):

$$r_{\mu}(T) = \max_{t=1,\ldots,T}\{x_t\} - \min_{t=1,\ldots,T}\{x_t\}$$

It is easy to detect which source of variation can influence the distribution of $r_{\mu}(T)$. Firstly, in the presence of a non-stationary $I(1)$ behavior of the process $\{x_t\}$, this statistic grows as the number of observations increases, at the rate $T^{1/2}$: this is an immediate consequence of the variance of the process which, in the $I(1)$ case, is proportional to T. Secondly, it depends on the size of the conditional increments Δx_t. Following Hurst (1951) and Mandelbrot (1975), it is possible to eliminate these nuisance quantities by normalizing (4) in an appropriate way: we can in fact refer to the following standardization

$$\widehat{r}_{\mu}(T) = \frac{\max_{t=1,\ldots,T}\{x_t\} - \min_{t=1,\ldots,T}\{x_t\}}{\widehat{\sigma}T^{1/2}} \tag{8}$$

where $\widehat{\sigma}^2 = (T-1)^{-1}\sum_{t=1}^{T}\left(\Delta x_t - \overline{\Delta x_t}\right)^2$ is the sample variance of the differenced process; (8) is here denoted as *standardized range statistic*. The $T^{1/2}$ normalization is necessary as $\mathrm{Var}(x_t) = O(T)$, while $\widehat{\sigma}^2$ is an estimator of the asymptotic variance of the differenced process, *i.e.* of $\lim_{T\to\infty} T^{-1}\sum_{t=1}^{T}\mathrm{Var}(\Delta x_t)$, given that this quantity exists. The proposed standardization, on one side, eliminates the influence of the sample size while, on the other side, cancels the dependence of the statistic on the magnitude of the process increments.

In order to test for bounded variation, the main characteristic of this statistics is that, if the process x_t has a unit root and is not bounded, then the numerator is asymptotically unbounded, it increases at the rate $T^{1/2}$ as well as the denominator, and the statistic has - under general conditions - a well defined asymptotic distribution. On the other side, if x_t is bounded around a constant term, with barriers sufficiently close to the mean value of the process, then the numerator should be significantly lower than the range of a standard, unbounded $I(1)$ process; in particular, as $T \to \infty$ it converges to a finite constant and the test statistic goes to 0. Hence, in order to test for a unit root against the alternative of a bounded process, it is necessary to consider left-sided tests based on statistics (8).

Unfortunately, as Lo (1991) points out, normalizing the range statistics by the sample standard deviation $\widehat{\sigma}$ can be misleading when the first differenced process is not independent but is possibly autocorrelated. If, *e.g.*, $x_t - x_{t-1}$ is the stationary first-order autoregressive process $x_t - x_{t-1} = \phi(x_t - x_{t-1}) + \sigma\varepsilon_t$,

$|\phi| < 1$, $\varepsilon_t \sim IID(0,1)$, then $\widehat{\sigma}^2 \xrightarrow{p} \sigma^2/\left(1-\phi^2\right)$ and the distribution of the rescaled range statistic depends on the nuisance parameter ϕ.

Following Phillips (1987), a more useful solution is to normalize the statistic with an estimator of the "long-run" variance of the differenced process, $\lambda^2 = \lim_{T\to\infty} \text{Var}\left(T^{-1}\left(x_t - x_0\right)\right)$. As first possibility is to choose an estimator of the form

$$\widehat{\lambda}_T^2 = \sum_{j=-q_T}^{q_T} \omega\left(\frac{j}{q_T}\right)\widehat{\gamma}(j) \quad (9)$$

$$\widehat{\gamma}(j) = \frac{1}{T}\sum_{t=|j|+1}^{T}\left(\Delta x_t - \overline{\Delta x_t}\right)\left(\Delta x_{t-j} - \overline{\Delta x_t}\right)$$

where $q_T \leq T$ and $\{\omega(x), |x| \leq 1\}$ is a system of weights satisfying the regularity condition

K1. For all x such that $|x| \leq 1$, $|\omega(x)| \leq 1$ with $\omega(0) = 1$, $\omega(-x) = \omega(x)$ and $\int_{-1}^{1} |\omega(x)|\, dx < \infty$.

For $\omega(x) = 1$, all x, the White covariance estimator is obtained while, setting $\omega(x) = 1 - |x|$ gives the Newey-West (1987) positive semidefined estimator, with truncation lag $q_T - 1$.

Under some general assumptions, the use of $\widehat{\lambda}_T$ allows the range statistic to have an asymptotic distribution which is free of unknown nuisance parameters under the null. Unfortunately, while this estimator can be consistent under the null hypothesis and in the presence of nearly bounded variation[6], it does not converge to a positive constant under the alternative of bounded variation, as the boundary conditions imply a moving average unit root in the differenced process, thus violating condition B3 and consistency of the test. To avoid this problem it is convenient to estimate λ^2 without imposing the unit root, *i.e.* by referring to the residuals of the auxiliary OLS regression

$$x_t = \alpha + \phi x_{t-1} + \eta_t$$

and therefore by considering the corresponding long-run variance estimator $\widetilde{\lambda}_T^2 = \sum_{j=-q_T}^{q_T} \omega(j/q_T)\,\widetilde{\gamma}(j)$, $\widetilde{\gamma}(j) = \frac{1}{T}\sum_{t=|j|+1}^{T} \widehat{\eta}_t\widehat{\eta}_{t-j}$, which is proved to be consistent under the null and strictly positive under the alternative (see Phillips, 1987).[7] The test statistic is therefore given by

$$\widehat{r}_\mu(T) = \frac{\max_{t=1,\dots,T}\{x_t\} - \min_{t=1,\dots,T}\{x_t\}}{\widetilde{\lambda}_T T^{1/2}} \quad (10)$$

[6] See paragraph 5 for a proof.

[7] A different solution is to introduce a stopping rule in the estimator of $\widehat{\lambda}_T$:

$$\widehat{\lambda}_T^2(c_T) = \begin{cases} \widehat{\lambda}_T^2 & \text{if } \widehat{\lambda}_T^2 > c_T \cdot \widehat{\gamma}(0) \\ c_T \cdot \widehat{\gamma}(0) & \text{otherwise} \end{cases} \quad c_T < 1$$

with $c_T = o\left(T^{-1/2}\right)$. This choice does not affect the asymptotic limit of $\widehat{\lambda}_T^2(c)$ under the null hypothesis (as, for condition B3, $\lambda > 0$) while, under the alternative hypothesis, it allows the denominator of the range statistics to grow to infinite.

In the case of trended time series, the following statistics can be used:

$$\widehat{r}_{\tau_a}(T) = \frac{\max_{t=1,\ldots,T}\{x_t - \widetilde{\mu}t\} - \min_{t=1,\ldots,T}\{x_t - \widetilde{\mu}t\}}{\widetilde{\lambda}_T T^{1/2}} \tag{11}$$

$$\widehat{r}_{\tau_b}(T) = \frac{\max_{t=1,\ldots,T}\{x_t - \widehat{\mu}t\} - \min_{t=1,\ldots,T}\{x_t - \widehat{\mu}t\}}{\widetilde{\lambda}_T T^{1/2}} \tag{12}$$

where the long run variance estimator $\widetilde{\lambda}_T^2$ is computed on the residual of the regression of $\widehat{x}_t = x_t - \widetilde{\mu}(t-1)$ on $\widehat{x}_{t-1}$ and a constant term for the $\widehat{r}_{\tau_a}(T)$ statistic, on the residual of the regression of $\widehat{x}_t = x_t - \widehat{\alpha} - \widehat{\mu}t$ on $\widetilde{x}_{t-1}$ and a constant term for the $\widehat{r}_{\tau_b}(T)$ statistic. The unit root hypothesis is tested by referring to left-sided tests based on statistics (10)-(12).

Before analyzing the behavior of these test statistics, it is interesting to notice that there is a strict connection between (10)-(12) and the rescaled range statistics of Hurst, Mandelbrot and Lo. Given a stochastic process $\{u_t\}$, the rescaled range statistic R/S is defined, up to a normalization factor, as

$$R/S = \max_{t=1,\ldots,T}\left\{\sum\nolimits_{j=1}^{t}(u_t - \overline{u})\right\} - \min_{t=1,\ldots,T}\left\{\sum\nolimits_{j=1}^{t}(u_t - \overline{u})\right\} \tag{13}$$

The idea behind this statistic is that the higher the degree of persistence of the process is, the wider the cumulate deviations from the sample mean should be. By appropriately standardizing this statistic, Lo (1991) shows that a test based on (13) has power against long memory processes like fractionally integrated, nearly-integrated and processes with unit roots.

In order to compare the R/S statistic with the range statistics (10)-(12), it is sufficient to expand the summation in (13) as

$$\sum\nolimits_{j=1}^{t}(u_j - \overline{u}) = \sum\nolimits_{j=1}^{t} u_j - \frac{t}{T}\sum\nolimits_{j=1}^{T} u_j \tag{14}$$

Now, if we interpret the cumulated process $\sum_{j=1}^{t} u_j$ as the increment of an integrated process x_t, *i.e.* $x_t = x_0 + \sum_{j=1}^{t} u_j$, then (14) can be written as

$$x_t - x_0 - \frac{t}{T}(x_T - x_0) = x_t - x_0 - \widetilde{\mu}t$$

with $\widetilde{\mu} = T^{-1}(x_T - x_0)$. By eliminating the terms which do not depend on t, then the R/S statistic (13) becomes

$$R/S = \max_{t=0,1,\ldots,T}\{x_t - \widetilde{\mu}t\} - \min_{t=0,1,\ldots,T}\{x_t - \widetilde{\mu}t\}$$

which is equal to statistic (11) computed on a sample of $T+1$ observation. Therefore, the R/S statistic corresponds to a special case of our proposed test statistics.

4 The behavior of the test statistics under the null hypothesis

Our next goal is to provide the asymptotic distributions of the range statistics under the null hypothesis of a unit root, *i.e.* under the assumption of absence of boundary conditions; this problem will be analyzed in the present paragraph. Through the rest of the paper, with "$\xrightarrow{w}$" we indicate weak convergence with respect to the uniform metric on the real set $[0,1]$; furthermore, with $B(s)$, $s \in [0,1]$ we indicate a standard (zero-mean, unit conditional variance) Brownian motion defined on $[0,1]$.

Accordingly to the previous paragraph, let us assume that x_t is generated by the following stochastic difference equation with a unit root

$$x_t = x_{t-1} + u_t \ , \ t = 1, ..., T \tag{15}$$

where x_0 is bounded almost surely and $\{u_t\}$ is a sequence of zero mean random variables which satisfies the weak heterogeneity and autocorrelation conditions $\mathcal{B}$. We therefore assume that the process is $I(1)$ but does not contain any deterministic trend.

In order to obtain the asymptotic distribution of the range statistics, it is convenient to assume that the truncation lag of q_T of the long-run variance estimator $\widehat{\lambda}_T$ asymptotically satisfies K1 and the following property:

K2. as $T \to \infty$, $q_T \to \infty$ at a rate such that $q_T = o(T^{\alpha})$, with $0 < \alpha < 1/2$

Furthermore, it is necessary the modify the mixing condition B4 with the stronger assumption on the mixing coefficients:

B4' $\{u_t\}$ is strong mixing with mixing coefficients α_m satisfying $\sum_{m=1}^{\infty} \alpha_m^{2(1/\beta - 1/p)} < \infty$ for some $\beta \in (2,4]$ satisfying $\beta > 2(1 + \alpha/(1-\alpha))$, and $\beta < p$.

and the condition B2 with the stronger moment condition[8]:

B2'. $\sup_t E[|u_t|^p] < C < \infty$ for some $p > \beta$;

which requires existence of $2(1 + \alpha/(1-\alpha))^+$ moments for all u_t's, where $\alpha/(1-\alpha)$ depends on the long-variance estimator lag truncation rule.

By referring to the range statistic $r_\mu(T)$, which is build up for the case of the absence of nonstochastic trends, under the set of conditions $\mathcal{B}' = \{B1, B2', B3, B4'\}$ and $\mathcal{K} = \{K1, K2\}$ the following proposition holds:

Proposition 1 *Under the assumptions $\mathcal{B}'$ and $\mathcal{K}$, as $T \to \infty$ the following weak convergence holds*

$$r_\mu(T) \xrightarrow{w} \sup_{s\in[0,1]} \{B(s)\} - \inf_{s\in[0,1]} \{B(s)\}$$

Proof. See Appendix A.

This proposition states that, by referring to the property that the unit root process (15) - appropriately scaled - converges weakly to a standard Brownian motion, the sample range statistic converges weakly to the range of a standard

[8]Again, it should be stressed that other systems of conditions could be used. In particular, what is needed is (i) a set of conditions which warranties the process to converge weakly to a standard Wiener process; (ii) a set of conditions which warranties the consistency of the estimator $\widehat{\lambda}_T^2$ of the long run variance. Thus, the results of the paper remain unchanged by replacing conditions $\mathcal{B}'$ with any other set of assumptions which maintains unchanged these two asymptotic results.

T	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
50	0.771	0.832	0.893	0.974	1.396	2.136	2.410	2.670	2.995
100	0.770	0.835	0.900	0.985	1.422	2.153	2.418	2.664	2.971
250	0.781	0.850	0.917	1.005	1.451	2.178	2.438	2.679	2.975
500	0.792	0.861	0.929	1.019	1.467	2.194	2.452	2.691	2.983
1000	0.801	0.871	0.940	1.030	1.480	2.206	2.463	2.701	2.991
10000	0.821	0.890	0.960	1.051	1.502	2.228	2.484	2.719	3.006
∞	0.833	0.903	0.921	1.063	1.515	2.241	2.498	2.734	3.023

Table 1: Fractiles of the range statistic $\widehat{r}_{\mu}(T)$

Brownian motion. More specifically, through assumption $\mathcal{B}$ it is straightforward to prove that the sample range, as far as the time and space axes are rescaled, goes - apart for a scale factor - to the range of a Brownian motion, *i.e.*

$$T^{-1/2}\left(\max_{t=1,\ldots,T}\{x_t\}-\min_{t=1,\ldots,T}\{x_t\}\right)\xrightarrow{w}\lambda\left(\sup_{s\in[0,1]}\{B(s)\}-\inf_{s\in[0,1]}\{B(s)\}\right) \tag{16}$$

while assumptions B2', B5 and $\mathcal{K}$ allow to standardize the l.h.s. of 16 by $\widehat{\lambda}_T$ in order to get an asymptotic distribution which is free of nuisance parameters.

Table 1 contains the fractiles of $\widehat{r}_{\mu}(T)$. The finite sample values are obtained through simulation of x_t with gaussian $IID(0,1)$ error component[9]; the truncation lag q_T in the computation of $\widehat{\lambda}_T$ is set to 0; this choice implies that the empirical range is normalized by the standard deviation of the residual of a regression of x_t on x_{t-1} and a constant term. The asymptotic critical values are obtained by using the following representation of the distribution function of the range of a Brownian motion on $[0,1]$ (see Borodin and Solminen, 1996, or Feller, 1957, in terms of probability density function):

$$F(x)=1+4\sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}(-1)^k k\exp\left(-\frac{k^2x^2}{2}\right)$$

It can be noticed that the range statistic has increasing variance as T grows (see also Figure 1): consequently, as the test is left-tailed the use of the asymptotic critical values can lead to over-reject the null hypothesis, *i.e.* the size of the test can be greater than the selected significance level.

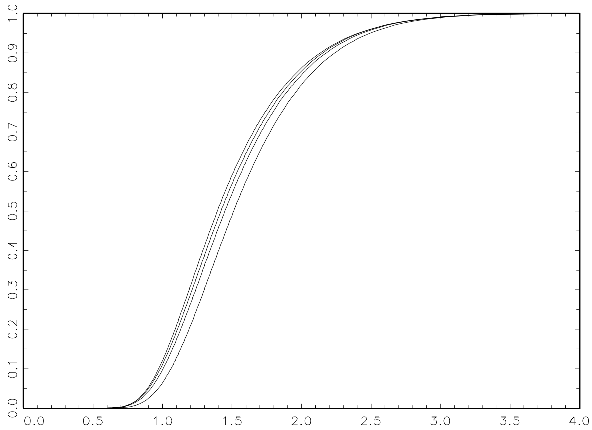
We can now pay attention to the more general case of a linear trend in data. It is clear that in this case the range statistic $\widehat{r}_{\mu}(T)$ is useless, as the linear trend "kills" the stochastic trend of the process, and the test statistic simply grows at the rate $T^{1/2}$ while its variance goes to 0: consequently, the null is never rejected, even in the presence of nearly-boundary conditions. Therefore it is necessary to refer to the range statistics $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$, which are computed after a preliminary deterministic detrendization. By assuming that the data generating process is the random walk with drift

$$x_t=x_{t-1}+\mu+u_t$$

and assuming that conditions $\mathcal{B}'$ and $\mathcal{K}$ hold, the following proposition can be proved:

[9] See note 11.

Figure 1: Cumulative distribution function of $\widehat{r}_{\mu}(T)$ for (from left to right) $T = 50, 100, 250, +\infty$.



T	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
50	0.705	0.751	0.794	0.851	1.109	1.456	1.564	1.660	1.773
100	0.699	0.748	0.796	0.856	1.133	1.504	1.621	1.726	1.851
250	0.707	0.759	0.809	0.873	1.160	1.546	1.670	1.781	1.914
500	0.716	0.769	0.820	0.885	1.177	1.568	1.693	1.807	1.942
1000	0.725	0.779	0.830	0.896	1.889	1.583	1.709	1.823	1.961
10000	0.744	0.798	0.849	0.916	1.212	1.607	1.734	1.849	1.986
∞	0.755	0.809	0.861	0.927	1.223	1.620	1.747	1.862	2.001

Table 2: Fractiles of the range statistic $\widehat{r}_{\tau_a}(T)$

Proposition 2 *Under the assumptions $\mathcal{B}'$ and $\mathcal{K}$, as $T \to \infty$ the following weak convergence holds*

$$\widehat{r}_{\tau_a}(T) \stackrel{w}{\to} \sup_{s\in[0,1]} \{V(s)\} - \inf_{s\in[0,1]} \{V(s)\}$$

$$\widehat{r}_{\tau_b}(T) \stackrel{w}{\to} \sup_{s\in[0,1]} \{V^*(s)\} - \inf_{s\in[0,1]} \{V^*(s)\}$$

where $\{V(s), s \in [0,1]\}$ is a standard Brownian bridge

$$V(s) = B(s) - sB(1)$$

and $\{V^*(s), s \in [0,1]\}$ is the stochastic process

$$V^*(s) = B(s) + 6s\left(\int_0^1 B(r)\,dr - 2\int_0^1 rB(r)\,dr\right)$$

which is, up to a level-shift factor, a detrended Brownian motion.

Proof. See Appendix A.

T	0.01	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.99
50	0.682	0.723	0.763	0.815	1.055	1.409	1.530	1.641	1.776
100	0.671	0.716	0.760	0.815	1.073	1.445	1.572	1.689	1.831
250	0.676	0.724	0.770	0.829	1.097	1.482	1.614	1.735	1.883
500	0.685	0.734	0.780	0.840	1.112	1.501	1.635	1.757	1.907
1000	0.693	0.742	0.790	0.850	1.124	1.515	1.649	1.772	1.923
10000	0.711	0.761	0.809	0.869	1.146	1.539	1.672	1.796	1.947
∞	0.722	0.772	0.819	0.880	1.157	1.550	1.682	1.806	1.956

Table 3: Fractiles of the range statistic $\widehat{r}_{\tau_b}(T)$

The proposition shows that, if the data are characterized by a linear trend, then it is possible to compute the range statistic on the detrended data and the corresponding normalized statistic has a nondegenerate asymptotic distribution which can be expressed in terms of the range of Brownian functional. Obviously, the range statistics $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$ have different asymptotic distributions, and this is consequence of which method is used to detrend the data: in the former case, the "efficient" estimate of the trend parameter leads to the range of a Brownian motion, detrended by the straight line which goes through its first $(B(0)=0)$ and last $(B(1))$ values while, in the latter case, the "OLS" detrendization leads to the range of a Brownian motion, detrended by the line which has the minimum distance (in terms of the integral of the squared differences) from the path of the process[10].

The critical values of the range statistics $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$ are reported in tables 2 and 3 respectively. For the finite sample critical values, we again set $q_T = 0$; figures 2 and 3 report the simulated distributions for different values of the sample size.

For the $\widehat{r}_{\tau_a}(T)$ statistics it is possible to provide also exact asymptotic critical values. The random variable $\sup_{s\in[0,1]}\{V_s\} - \inf_{s\in[0,1]}\{V_s\}$, *i.e.* the range of a Brownian bridge, can in fact be equivalently expressed as $K = \sup_{s\in[0,1]}\{|V_t - V_s|\}$, which is known as the *Kuiper statistic* (see Shorack and Wellner, 1987). Its distribution function is known and is given by (Kuiper, 1960; Kennedy, 1976)

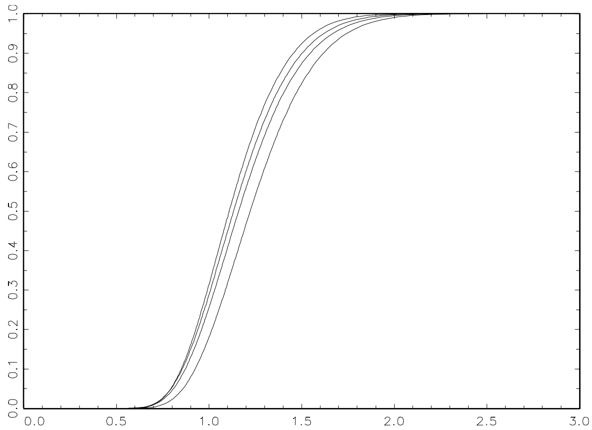
$$\Pr(K \leq k) = 1 - 2\sum_{j=1}^{\infty}\left(4j^2k^2 - 1\right)\exp\left(-2j^2k^2\right)$$

To our knowledge, the range of the detrended Brownian motion has not been expressed in a closed form, therefore the asymptotic critical values reported in table 3 are obtained through simulation[11].

[10] The Brownian Bridge is in fact given by $V(s) = B(s) - B(0) - \frac{B(1)-B(0)}{1-0}s = B(s) - sB(1)$, while the detrended Wiener process is given by (see Schmidt and Phillips, 1992) $W(s) = B(s) - \widehat{\alpha}_0 - \widehat{\alpha}_1 s$, where $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ minimise the least squares criterion in the L_2 norm, namely $\int_0^1 (B(r) - \widehat{\alpha}_0 - \widehat{\alpha}_1 r)^2 dr$. Solving for $\widehat{\alpha}_0$ and $\widehat{\alpha}_1$ gives $\widehat{\alpha}_1 = 12\int_0^1 rB(r)\,dr - 6\int_0^1 B(r)\,dr$, $\widehat{\alpha}_0 = \int_0^1 B(r)\,dr - \widehat{\alpha}_1/2$, which leads to the following expression for the detrended Wiener process: $W(s) = B(s) + (6s-4)\int_0^1 B(r)\,dr + (6-12s)\int_0^1 rB(r)\,dr$; this expression differs from V_s^* only for the quantity $6\int_0^1 rB(r)\,dr - 4\int_0^1 B(r)\,dr$, which is independent of s and consequently represents just a level shift.

[11] We estimate small sample and asymptotic critical values by means of response surfaces.

Figure 2: Cumulative distribution function of $\widehat{r}_{\tau_a}(T)$ for (from left to right) $T = 50, 100, 250, +\infty$.



Finally, in figure 4 the asymptotic cumulative distribution functions of $\widehat{r}_{\mu}(T)$, $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$. As expected, the standard range statistic tends to take, on average, higher values with respect to the detrendization-based statistics; furthermore, compared with efficient detrending, OLS detrending produces, on average, values closer to the origin and - consequently - leads, on average, to a lower range.

5 The behavior of the test statistics under the alternative hypothesis

Under the null hypothesis, each of the range statistics proposed in paragraphs 2.3 and 3 has an asymptotic distribution which is nondegenerate, nonexplosive, free of nuisance parameters and, moreover, with fractiles which can be computed through simulation or (in some special cases) through exact formulas. We are now able to concentrate our attention on the last step of our analysis, that

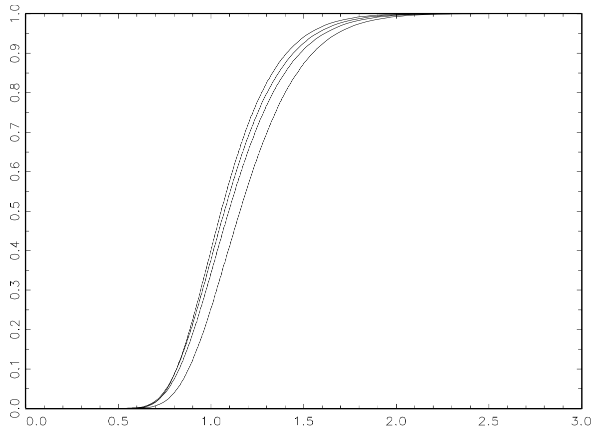
Firstly, for 18 values of T (*i.e.* 50, 60, 75, 100, 125, 150, 200, 250, 300, 400, 500, 750, 1000, 1250, 1500, 2000, 5000 and 10000), 100000 standard gaussian random walks are generated and the test statistics are computed; this simulation is repeated 50 times and the fractiles of interest are stored. For each fractile, small sample critical values are interpolated by means of the response surface regression

$$q_{\alpha}(T_i) = q_{\alpha}(\infty) + \gamma_1 T_i^{1/2} + \gamma_2 T_i + \gamma_3 T_i^{3/2} + \nu_i, \; T_i = 50, 60, ..., 10000. \qquad (17)$$

where the intercept term, $q_{\alpha}(\infty)$, is the a fractile of the asymptotic distribution and is therefore treated here as an unknown parameter. As in MacKinnon *et al.* (2000), regression equation (17) is estimated by means of GMM, which furthermore allows to test the selected specification. It is interesting to notice that, while in the estimation of unit root and cointegration test statistics the terms T_i^2 and T_i^3 mostly need being included (while $T_i^{1/2}$ and $T_i^{3/2}$ can be drop), for range statistics $T_i^{1/2}$ and $T_i^{3/2}$ need being included (while T_i^2 and T_i^3 do not).

Finally, in the case of the $\widehat{r}_{\mu}(T)$ and $\widehat{r}_{\tau_a}(T)$ statistics (whose asymptotic distributions are known) the maximum estimation error of the adopted response surface methodology is in modulus less than 0.01.

Figure 3: Cumulative distribution function of $\widehat{r}_{\tau_a}(T)$ for (from left to right) $T = 50, 100, 250, +\infty$.



is to obtain the distribution of the proposed statistics under the alternative hypothesis and to evaluate their local power properties.

Nearly bounded $I(1)$ processes, as in definitions 1 and 3, do not explicitly model the reflection mechanism which keep the process within the boundaries. Nevertheless, power evaluation needs explicitly a precise formulation of the data generating process under the alternative. Without loss of generality[12], the following process is here considered: given a sequence $\{u_t\}_{t=1}^{\infty}$ which satisfies the heterogeneity and weak dependence conditions $\mathcal{B}'$ and whose long-run variance exists and is equal to the positive constant λ, we define the process $\{x_t\}$ through the recursion

$$x_t = \begin{cases} \overline{b}_T & \text{if } x_{t-1} + u_t > \overline{b}_T \\ \underline{b}_T & \text{if } x_{t-1} + u_t < \underline{b}_T \\ x_{t-1} + u_t & \text{elsewhere} \end{cases} \quad (18)$$

with the band parameters defined as

$$\overline{b}_T = \overline{c}\lambda T^{1/2} \quad (19)$$

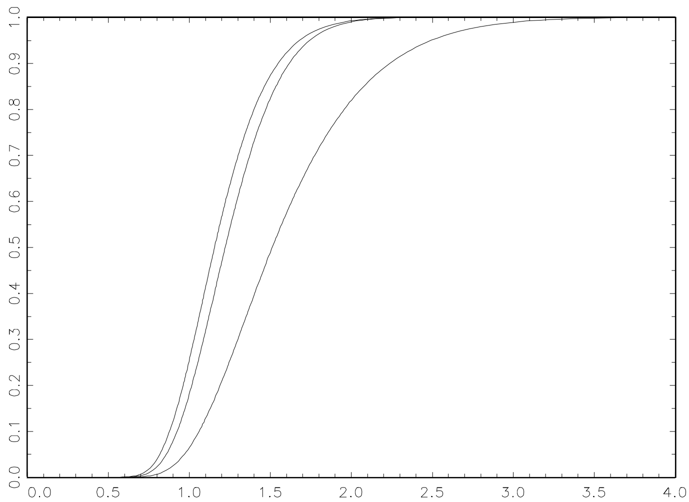
$$\underline{b}_T = \underline{c}\lambda T^{1/2} \quad (20)$$

with $-\infty < \underline{c} < \overline{c} < \infty$; furthermore, without loss of generality, we can set the initial condition $x_0 = 0$, which implies $\underline{c} < 0 < \overline{c}$. It is easy to show that $\{x_t\}$ is a nearly bounded integrated process, as it satisfies all the requirements of definition 3. Furthermore, it can be noticed that for finite T the boundaries are *sticky* in the sense of definition 2, as the probability that the process stops exactly on one boundary is not zero almost surely; this assumption, together with the behavior of the process near the boundaries (as in formula (18))[13], even

[12] It can be proved that local power remains unchanged within a wide class of reflection mechanisms.

[13] This process is known as random walk with reflecting barriers (see Cox and Miller, 1965; Rose, 1997). At each time t, if the process exceeds the boundary, say b, of an amount ε, then it is forced to lie in b, independently of the value of ε; the reflection mechanism is therefore sticky. Alternatively, it is possible to define a regular reflection in several ways. As an example, it

Figure 4: Asymptotic cumulative distribution functions of (from left to right) $\widehat{r}_{\tau_b}(T), \widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\mu}(T)$.



if not necessary, is particularly useful as it allows to obtain closed expressions for the distributions of the test statistics.

The null hypothesis of no boundary conditions is a limit case of (18) and it can be obtained by simply letting the constants $\underline{c}$ and $\overline{c}$ go to infinite. This property allows us to evaluate the power of the tests as the distance of the boundaries from the initial value of the process increase.

Let us firstly concentrate our attention on the range statistic $\widehat{r}_{\mu}(T)$. In order to evaluate its distribution, we can prove the following generalization of Donsker's invariance principle:

Theorem 1 *Let $\{x_t\}_{t=0}^{T}$ be defined as in (18), with the boundaries satisfying (19)-(20), $-\infty < \underline{c} < \overline{c} < \infty$, and let $B_{\underline{c}}^{\overline{c}}(s)$, $s \in [0,1]$ be a* regulated Brownian motion*, with barriers in $\underline{c}$ and $\overline{c}$, starting in $B_{\underline{c}}^{\overline{c}}(0) = 0$. Then, in the space of the cadlag functions $D[0,1]$, the following weak convergence holds*

$$\frac{x_{[sT]}}{\lambda T^{1/2}} \xrightarrow{w} B_{\underline{c}}^{\overline{c}}(s)\,,\ \textit{all } s \in [0,1]$$

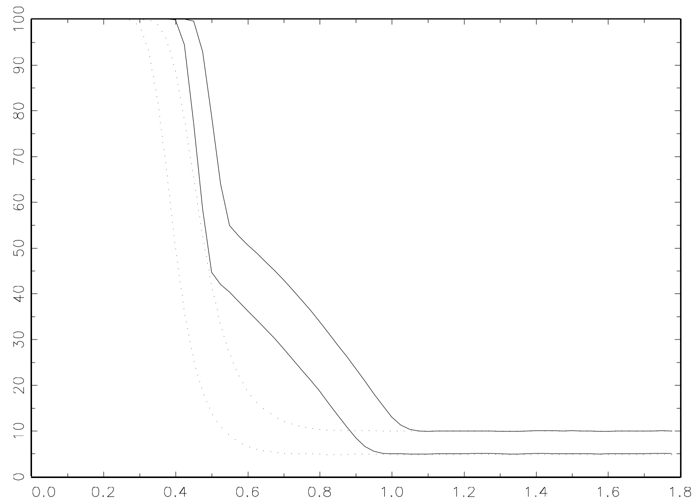
Proof. See Appendix A.

This version of the functional central limit theorem states that the nearly-bounded integrated process $\{x_t\}$, well-normalized, converges to a regulated Brownian motion[14]. Thus, the presence of two nearly-boundaries induces the

can be assumed that if the boundary is exceeded of ε, then the process is "reflected" to the position $b - \varepsilon$: consequently, the process does not stop on the boundary unless $\varepsilon = 0$ (and this case can be ruled out by assuming that ε is a continuous random variable)and hence the boundary is *regular*.

[14] We will not give here a deep inspection of the properties of the regulated Wiener process, for which we refer the reader to the existing literature (see, *e.g.*, the excellent book of Harrison, 1985, in the framework of flow systems; Dixit, 1993, for some general economic applications; Bertola, 1994, within target zone modelling; Cavaliere, 1997, for some inferential properties).

Figure 5: Asymptotic power function of the $\widehat{r}_{\mu}(T)$ test (solid line) against the alternative of a reflected random walk, for different values of the band parameter c and significance levels $\alpha = 0.05$ and $\alpha = 0.10$ ($T = 1000$), compared with the asymptotic power of the Phillips-Perron $Z_{nc}(\rho)$ test (dotted line).



presence of two reflecting barriers in the limiting Brownian motion. We can now analyze how this property is transferred onto the behavior of the range statistics:

Corollary 1 *Under the assumption of Theorem 1, if the long-run variance estimator $\widehat{\lambda}_T$ satisfies condition $\mathcal{K}$ then $\widehat{\lambda}_T \xrightarrow{p} \lambda$, which implies the following convergence:*

$$r_{\mu}(T) \xrightarrow{w} \sup_{s\in[0,1]} \left\{ B_{\underline{c}}^{\overline{c}}(s) \right\} - \inf_{s\in[0,1]} \left\{ B_{\underline{c}}^{\overline{c}}(s) \right\} \tag{21}$$

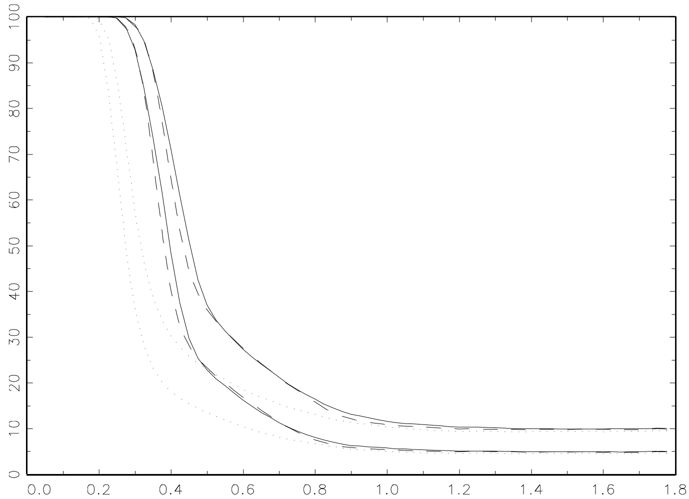
Proof. See Appendix A.

This corollary states that $\widehat{\lambda}_T \to \lambda$ in probability, even if the data generating process is nearly bounded. This surprising result derives from one if the main characteristic of the regulated Brownian motion (see Harrison, 1985, prop. 1.6): the set of points where the sample path is "regulated" in order to lie between the two boundaries has zero Lebesgue measure. Consequently, the bias induced by the boundaries is asymptotically negligible and does not affect the estimator of the long-run variance.

As a consequence, the sample range converges to the range of a regulated Brownian motion while, in the absence of boundaries, it converges to the range of a standard Brownian motion. The distribution of the random variable on the r.h.s. of (21) is not defined on all the nonnegative real set, but is truncated, as it cannot take values greater than $\overline{c} - \underline{c}$. Thus, as $\overline{c} - \underline{c}$ decreases, the power of the test increases.

In order to evaluate the power properties we compute the asymptotic power function for the $\widehat{r}_{\mu}(T)$ test, *i.e.* the percentage of rejections under the alternative hypothesis . The boundary coefficients $\overline{c}$ and $\underline{c}$ are chosen equal in absolute

Figure 6: Asymptotic power function of the $\widehat{r}_{\tau_a}(T)$ (solid line) and $\widehat{r}_{\tau b}(T)$ test (dashed line) against the alternative of a reflected random walk, for different values of the band parameter c and significance levels $\alpha = 0.05$ and $\alpha = 0.10$ ($T = 1000$), compared with the asymptotic power of the Phillips-Perron $Z_{c\tau}(\rho)$ test (dotted line).



value and opposite in signs, *i.e.* $\overline{c} = -\underline{c} = c > 0$, with c taking values in the interval $(0, 1.8]$. Furthermore, the process $\{u_t\}$ is chosen gaussian $IID(0,1)$, in order to eliminate the influence of the long-run variance estimation procedure; consequently, the truncation lag q_T is set to 0. Small sample critical values are used, with the significance level set to 0.05 and 0.10. Results obtained for $T = 1000$ are illustrated in figure 5.

As expected, power decreases as the distance between the boundaries grows. For c lower than 0.4 (0.45), the test has unit asymptotic power for the significance level $\alpha = 0.05$ (0.10). Unfortunately, there are values of c [15] which lead to an undetectable boundary structure: this evidence is meaningful as, when the boundaries are relatively too far (*i.e.* $c > 0.95$ for $\alpha = 0.05$, $c > 1.05$ for $\alpha = 0.10$), then the probability of the process to reach the boundaries is not significant and, on the real set $[0, 1]$, the limiting regulated Brownian motion has mostly the same dynamic properties of the standard Brownian motion.

We can now turn the attention to the $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$. As such modified range statistics are introduced just to eliminate the presence of linear trends, instead of simulating a nearly bounded process with trend and linear boundaries power analysis can be conducted by simulating the same process used for the previous test. Results obtained by applying small sample critical values, with significance levels 0.05 and 0.10 and $T = 1000$ are plotted in figure 6. It can be noted that, with respect to the $\widehat{r}_{\tau_a}(T)$ test, for $\alpha = 0.05$ (0.10) the power function starts growing as far as c gets lower than 1.2 (1.4); for $\alpha = 0.05$ (0.10) unit power is now achieved for $c < 0.225$ ($c < 0.20$) for the $\widehat{r}_{\tau_a}(T)$ test, and is achieved for $c < 0.275$ ($c < 0.225$) for the $\widehat{r}_{\tau_b}(T)$ test. At a 0.10 significance

[15] That approximately are $c \geq 1$ for $\alpha = 0.05$ and $c \geq 1.1$ for $\alpha = 0.10$.

level, the $\widehat{r}_{\tau_a}(T)$ test seems dominating the OLS detrended test, even if for a 0.05 level such result is less clear.

It is interesting to observe that, even in the absence of a linear trend, the $\widehat{r}_{\tau_a}(T)$ test does not dominate the other two tests, in the sense that for certain values of the band parameter c the detrended range tests are able to detect the presence of reflecting condition whereas the standard range test can not.

Finally, asymptotic power of the range tests is compared with the power of a standard Phillips Perron tests, which is not explicitly defined in order to test the presence of reflecting conditions, but it can have power also against bounded alternatives. Asymptotic power of the Phillips Perron $Z_{nc}(\rho)$ test and $Z_{c\tau}(\rho)$ test for the trend case are also reported in the previous figures.

Basically, the performed simulation shows that also standard unit root tests can be useful to detect nearly-bounded variation, even if their power is substantially lower than the power achieved by referring to range-based statistics.

6 An application

The proposed test are illustrated with an application on the dynamics of the Deutsche Mark (DM) /U.S. Dollar (USD) exchange rate. Although in principle this exchange rate can freely float, it is know that Central Banks occasionally intervene in the foreign market in order to stabilize the exchange rates around implicit targets. Moreover, international agreements like the *Plaza Agreement* of September 1985 or the *Louvre Accord* of February 1987 require Monetary Authorities to intervene in order to increase the stability of the international monetary system and maintain the exchange rate around specific levels.

The present analysis concerns the DM/USD exchange rate dynamics, after the Louvre Accord and until the end of 1998. In particular, we are interested in understanding if the range of variability of the DM/USD exchange rate has been affected by Central Banks intervention or if it behaves like a free floating rate: it is in fact well documented that during this period both the Federal Reserve and the Bundesbank frequently intervene on the foreign market[16] to limit the exchange rates variability.

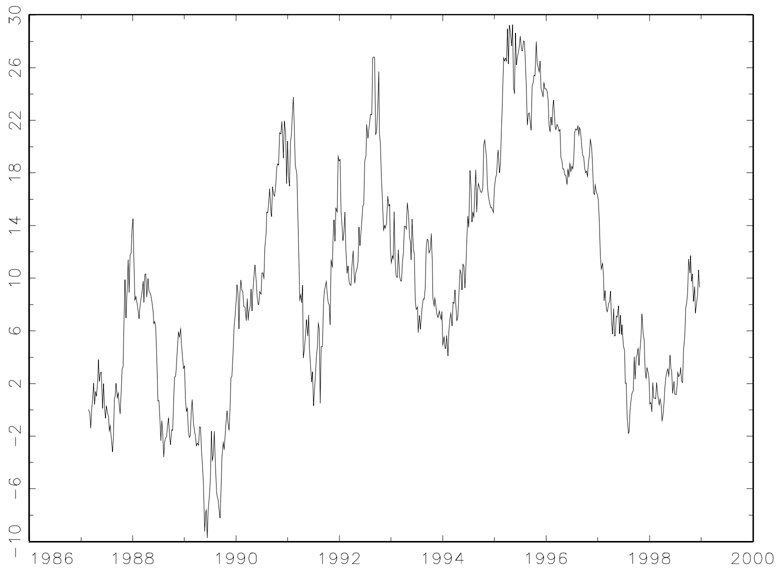
We consider (100 times) the logarithms of the weekly spot rates from 23 February 1987 to 31 December 1998 (618 observations), which are plotted (as deviations from the initial value of the series) in Figure 7. With respect to the first trading day after the Louvre Accord, during this period the exchange rate reached is minimum, -9.75, in June 1989 and its maximum, 29.27, in May 1995, which corresponds to an excursion of 39.03.

By using the proposed tests, we can now investigate if such a range is compatible with the standard $I(1)$ hypothesis or, otherwise, if is significant of the presence of mean reversion, due to the presence of stabilizing intervention. Results, for different values of the truncation parameter q_T are reported in table 4; note that, for weekly data, $q_T = 9$ corresponds to two months, $q_T = 13$ approximately to three months and $q_T = 17$ to a four-month period.

In the case of the range statistic $\widehat{r}_{\mu}(T)$, on the basis of the asymptotic critical values the null hypothesis is rejected at the 5% significance level for $q_T = 9$, $q_T = 13$ and $q_T = 17$. These values of the truncation lag seem to allow to catch short-term swings of the exchange rate which cannot be taken into account by

[16] See, *e.g.*, Dominguez (1998).

Figure 7: DM/USD spot exchange rate (log form) - 100 times the deviation from the 23 February 1987 rate.



		Truncation lag q_T					
		$q_T = 0$		$q_T = 1$		$q_T = 4$	
	range	$\widetilde{\lambda}$	test statistic	$\widetilde{\lambda}$	test statistic	$\widetilde{\lambda}$	test statistic
$r_\mu(T)$	29.271	1.539	1.020	1.498	1.048	1.566	1.002
$r_{\tau_a}(T)$	22.831	1.539	0.899	1.499	0.923	1.569	0.881
$r_{\tau_b}(T)$	17.168	1.539	0.887	1.499	0.912	1.570	0.871

		Truncation lag q_T					
		$q_T = 9$		$q_T = 13$		$q_T = 17$	
	range	$\widetilde{\lambda}$	test statistic	$\widetilde{\lambda}$	test statistic	$\widetilde{\lambda}$	test statistic
$r_\mu(T)$	29.271	1.637	0.959	1.646	0.953	1.670	0.940
$r_{\tau_a}(T)$	22.831	1.644	0.842	1.656	0.835	1.682	0.822
$r_{\tau_b}(T)$	17.168	1.644	0.831	1.656	0.825	1.682	0.812

Table 4: Range tests for the I(1) hypothesis

considering lower values of q_T. By referring to the small-sample critical values (we have already proved that asymptotic values can lead to an over-rejection of the null hypothesis), the null hypothesis of absence of bounded variation is rejected at the 10% significance level for q_T greater then 4. Moreover, for a 4 month truncation lag the p-value associated to the test statistic is very close to 5%. These results state that, at a 10% significance level, the exchange rate exhibits mean reverting dynamics and the $I(1)$ hypothesis must be therefore rejected.

Similar results are obtained by considering the $\widehat{r}_{\tau_a}(T)$ and $\widehat{r}_{\tau_b}(T)$ statistics, which implicitly eliminate the influence of the observed depreciation rate of the Deutsche Mark with respect to the US Dollar. The $\widehat{r}_{\tau_a}(T)$ test allow to reject the null hypothesis for $q_T \geq 4$ at a 10% significance level; for $q_T = 13$ and, in particular, $q_T = 17$, the associated p-values are extremely close to 5%. Finally, statistic $\widehat{r}_{\tau_b}(T)$ rejects at a 10% rate for $q_T \geq 9$. On the same data, the nonparametric $Z_{c\tau}(\rho)$ unit root test of Phillips and Perron (1988) do not reject the null of a unit root even at the 10% significance level.

These results suggests that, after the Louvre Accord of February 1987, the DM/USD spot exchange rate seems to exhibit a mean reversion component which is particularly effective when such rate reach strong deviation from its average value. Such dynamic is consistent with the hypothesis that Central Banks intervene infrequently and, in particular, only when the exchange rate deviate "too much" from its implicit target - or equilibrium - value.

7 Conclusions

The analyses carried on in this paper move from the following question: given that a time series exhibits a $I(1)$ behavior, can we claim that there are no restrictions on the wideness of the movements the process has around its average value? In other words: how sure are we that there does not exists a nonobservable mean reversion component which is induced from the presence of boundary conditions on the trajectory of the process?

In order to give the statistical tools which enable to make inference about unobservable boundaries, and hence to answer the previous question, we define a new class of models, which we call nearly bounded integrated processes, which display a unit root behavior but are constrained to lie within a particular interval. Unit root processes are clearly nested within this class of processes model, as they can be obtained by letting the distance of the boundaries from the starting value of the process going to infinite at a proper rate; therefore, it is relatively easy to develop statistical methods to test the hypothesis of no boundaries conditions, *i.e.* the classic unit root hypothesis.

In this context, standard unit root inference cannot be proved to have the usual power properties. Therefore, we choose to define new tests which are based on the sample *range*, or *excursion*, of the process: the main idea is that, if there are boundary conditions, then the range of the process should be significantly lower (at least on average) than the range of an unrestricted integrated process. One important characteristic of such tests is that they do not require to estimate parameters under the alternative hypothesis, but can simply rely on the estimation under the null of no boundaries conditions.

The proposed tests seem to have improved power with respect to standard autocorrelation-based unit root inference. Even if range statistics are generally highly variable, the results obtained in a simulation exercise shows that these new methodologies can detect the presence of mean reversion even when standard unit root tests are not able to discriminate between the null and the alternative hypotheses.

Finally, it must be stressed that these procedures cannot show significant power whenever the alternative hypothesis does not belong to the class of bounded and nearly-bounded processes. To avoid such problems it could be convenient to combine the proposed range tests with standard unit root inference into a two-stages procedure, where, in the first step, nonstationarity is evaluated by means of classical unit root tests (*e.g.*, accordingly to the previous definition $\mathcal{B}$ of the null hypothesis, Phillips-Perron tests), while, in case the null is not rejected, the presence of boundary conditions is evaluated by means of the rescaled range tests. This solution allows to use the proposed range tests in a wider setting, where the alternative hypothesis can be far from the nearly bounded $I(1)$ class.

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A Mathematical Appendix

In this section we give the proofs for the asymptotic results previously reported. Before, we introduce the following lemmas (see also Lo, 1991). The first is a generalization of the well known Donsker's invariance principle, and is adapted from Herrndorf (1984). The second is the continuous mapping theorem and can be found *e.g.* in Billinglsey in the framework of general Skorohod's topology. The third lemma shows the conditions for the weak convergence of the sample range to the range of the limiting process. Finally, the last lemma shows under which conditions a consistent estimator of the long-run variance λ^2 can be obtained.

Lemma 1 (Herrndorf's invariance principle) *If $\{\varepsilon_t\}_{t=1}^{\infty}$ is random sequence satisfying assumptions $B1-B4$, then in the space of the cadlag functions $D[0,1]$, as $T \to \infty$:*

$$x_T(s) = \frac{1}{\lambda\sqrt{T}} \sum\nolimits_{t=1}^{[sT]} \varepsilon_t \xrightarrow{w} B(s)$$

where the convergence holds jointly for all $s \in [0,1]$.

Proof. See Herrndorf (1984). □

Lemma 2 (Continuous mapping theorem) *If $g()$ is a function defined from $D[0,1]$ to itself and almost-surely continuous (i.e. with 0 Lebesgue-measure discontinuity set), and if $x_T(\cdot) \in D[0,1]$ is such $x_T(\cdot) \xrightarrow{w} B(\cdot)$, then $g(x_T(\cdot)) \xrightarrow{w} g(B(\cdot))$.*

Proof. See Billingsley (1968). □

Lemma 3 *In the cadlag space* $D[0,1]$*, if* $x_T(\cdot) \xrightarrow{w} \widetilde{B}(\cdot)$*, where* $\widetilde{B}(\cdot) = g(B(\cdot))$*, with* g *satisfying the conditions of the previous lemma, then*

$$\max_{s\in[0,1]}\{x_T(s)\} - \min_{s\in[0,1]}\{x_T(s)\} \xrightarrow{w} \sup_{s\in[0,1]}\left\{\widetilde{B}(s)\right\} - \inf_{s\in[0,1]}\left\{\widetilde{B}(s)\right\} \tag{22}$$

Proof. The proof is obtained by referring to the continuity of the range function and the application of Lemma 2. □

Lemma 4 *Under conditions* $\mathcal{B}'$ *and* $\mathcal{K}$*,* $\widehat{\lambda}_T \xrightarrow{p} \lambda$*.*

Proof. See Hansen (1992). □

A.1 Proof of Proposition 1

Under the null hypothesis, x_t is recursively defined as

$$x_t = x_0 + \sum_{j=1}^{t} \varepsilon_j \ ;$$

which implies the equality:

$$\frac{1}{\lambda\sqrt{T}}\left(\max_{t=0,1,\dots,T}\{x_t\} - \min_{t=0,1,\dots,T}\{x_t\}\right) = \sup_{s\in[0,1]}\left\{\frac{\sum_{j=1}^{[sT]}\varepsilon_j}{\lambda\sqrt{T}}\right\} - \inf_{s\in[0,1]}\left\{\frac{\sum_{j=1}^{[sT]}\varepsilon_j}{\lambda\sqrt{T}}\right\} \tag{23}$$

By Lemma 1, in $D[0,1]$:

$$x_T(s) = \frac{\sum_{j=1}^{[sT]}\varepsilon_j}{\lambda\sqrt{T}} \xrightarrow{w} B(s)$$

where λ is defined as in B3, $B(\cdot)$ is a standard Brownian motion and convergence holds jointly for all $s \in [0,1]$; furthermore, Lemma 3 ensures that (23) converges weakly to the range of the limiting Brownian motion $B(\cdot)$, *i.e.* to $\sup_{s\in[0,1]}\{B(s)\} - \inf_{s\in[0,1]}\{B(s)\}$. The proof is completed by noting that

$$r_\mu(T) = \frac{\lambda}{\widehat{\lambda}_T}\frac{1}{\lambda\sqrt{T}}\left(\sup_{s\in[0,1]}\{\widehat{x}_T(s)\} - \inf_{s\in[0,1]}\{\widehat{x}_T(s)\}\right)$$

and that $\widehat{\lambda}_T \xrightarrow{p} \lambda$ by Lemma 4. □

A.2 Proof of Proposition 2

The proof follows the same lines of the previous case. In particular, it is enough to prove that the detrended processes converge weakly to well-defined functionals of a Brownian motion; then, by Lemma 3, the ranges of the detrended processes converge weakly to the ranges of their corresponding limiting processes and, finally, Lemma 4 ensures that substituting λ with its consistent estimator $\widehat{\lambda}_T$ does not affect the convergence result.

Consider the detrended process $\widetilde{x}_t = x_t - x_1 - \widetilde{\mu} t$, where $\widetilde{\mu} = (T-1)^{-1}(x_T - x_1)$. Under the null hypothesis it follows that

$$\begin{aligned} \widehat{x}_t &= x_1 + \widetilde{\mu}(t-1) + \sum_{j=2}^{t} \varepsilon_j - (T-1)^{-1} \left(x_1 + \mu(T-1) + \sum_{j=2}^{T} \varepsilon_j - x_1 \right) t \\ &= \sum_{j=2}^{t} \varepsilon_j - \frac{t-1}{T-1} \sum_{j=1}^{T} \varepsilon_j = \sum_{j=1}^{t} \varepsilon_j - \frac{t}{T} \sum_{j=1}^{T} \varepsilon_j + O_p(1) \end{aligned}$$

In the $D[0,1]$ space we can define the approximant $\widehat{x}_T(s) = \lambda^{-1} T^{-1/2} \widehat{x}_{[sT]} = \lambda^{-1} T^{-1/2} \left(\sum_{j=1}^{[sT]} \varepsilon_j - \frac{[sT]}{T} \sum_{j=1}^{T} \varepsilon_j \right)$, $s \in [0,1]$, and by lemmas 1 and 2

$$\widehat{x}_T(s) = \frac{\widehat{x}_{[sT]}}{\lambda T^{1/2}} \xrightarrow{w} B(s) - sB(1)$$

where the convergence holds jointly for all s and λ is defined as in B3; the limiting process $V(s) = B(s) - sB(1)$ is the well-known Brownian bridge.

Now consider the second detrendization, *i.e.* $\widehat{x}_t = x_t - \widehat{\alpha} - \widehat{\mu} t$, where $\widehat{\alpha}$ and $\widehat{\mu}$ are the OLS estimators. The Waugh-Frisch theorem allows to express the residuals as $\widehat{x}_t = (x_t - \overline{x}) - \widehat{\mu}\left(t - \overline{t}\right)$, with $\widehat{\mu} = \sum_t (x_t - \overline{x})\left(t - \overline{t}\right) / \sum_t \left(t - \overline{t}\right)^2$. Again, in $D[0,1]$ we can define the following continuous-time approximant associated to $\widehat{x}_t$:

$$\widehat{x}_T(s) = \frac{1}{\lambda T^{1/2}} \left(x_{[sT]} - \overline{x} \right) - \frac{1}{\lambda} \frac{T^{-5/2} \sum \left(x_{[sT]} - \overline{x} \right) \left([sT] - \overline{t} \right)}{T^{-3} \sum \left([sT] - \overline{t} \right)^2} \frac{1}{T} \left([sT] - \overline{t} \right)$$

It is easy to prove that jointly for all $s \in [0,1]$:

$$\widehat{x}_T(s) \xrightarrow{w} B(s) - \overline{B} - 12 \left(s - \frac{1}{2} \right) \left(\int_0^1 sB(s)\, ds - \frac{1}{2} \overline{B} \right) \tag{24}$$

where $\overline{B} = \int_0^1 B(s)\, ds$; the process defined in (24) is the *detrended Brownian motion* (see Schmidt and Phillips, 1991). By using lemmas 3 and 4 and by eliminating the components of the limiting Brownian functional which do not depend on s it follows that

$$\begin{aligned} \widehat{r}_{\tau_b}(T) \xrightarrow{w} & \sup_{s \in [0,1]} \left\{ B(s) - 12s \left(\int_0^1 sB(s)\, ds - \frac{1}{2} \overline{B} \right) \right\} \\ & - \inf_{s \in [0,1]} \left\{ B(s) - 12s \left(\int_0^1 sB(s)\, ds - \frac{1}{2} \overline{B} \right) \right\} \end{aligned}$$

where the process between brackets is a detrended Brownian motion, translated in order to satisfy $B(0) = 0$. □

A.3 Proof of Theorem 1

We firstly set $\overline{b}_T = \infty$, *i.e.* the process has a single barrier in $\underline{b}_T < 0$. It is useful to translate the process of the quantity $-\overline{b}_T$, such that the process starts in $x_0 = -\underline{b}_T$ and has a reflecting barrier is 0. According to the theorem's assumptions, x_0 can be expressed as $x_0 = c\lambda T^{1/2}$, $c > 0$.

We firstly need to proof the following proposition:

Proposition 3 *An alternative representation of (18) is given by the recursive equation*

$$\begin{aligned} x_t &= x_0 + S_t + L_t \\ S_t &= \sum\nolimits_{i=1}^{t} u_t \\ L_t &= -\min_{t' \leq t} \{x_0 + S_{t'}; 0\} \end{aligned} \tag{25}$$

Proof. Firstly, note that at each t the process can be written as $x_t = x_{t-1} + u_t + l_t$, where $l_t = -(x_{t-1} + u_t)\, \mathbf{1}\{x_{t-1} + u_t < 0\}$; so, in case the process falls under the barrier in 0, l_t "regulates" its trajectory, bringing x_t back onto the barrier 0. Recursive substitutions allow to express x_t as $x_t = x_0 + \sum_{i=1}^{t} u_i + \sum_{i=1}^{t} l_i = x_0 + S_t + L_t$, where S_t is the ordinary partial sum process and $L_t = \sum_{i=1}^{t} l_i$. We need to show relation (25), which can be proved by induction. For $t = 0$, $S_0 = 0$, $L_0 = \min\{x_0; 0\} = 0$ and the relation is therefore satisfied. Assume that the relation is satisfied at time t, *i.e.* $L_t = -\min_{t' \leq t}\{x_0 + S_{t'}; 0\}$. At time $t+1$, if $x_t + u_t \geq 0$, then $l_{t+1} = 0$, $L_{t+1} = L_t = -\min_{t' \leq t}\{x_0 + S_{t'}; 0\} = -\min_{t' \leq t+1}\{x_0 + S_{t'}; 0\}$ and the relation (25) is satisfied. On the other hand, if $x_t + u_{t+1} < 0$, then $l_{t+1} = -(x_t + u_{t+1}) = -(x_0 + S_t + L_t + u_{t+1})$, which implies $-L_{t+1} = x_0 + S_{t+1}$. But as $l_{t+1} > 0$, $-L_{t+1} = x_0 + S_{t+1} < -L_t = \min_{t' \leq t}\{x_0 + S_{t'}; 0\}$, which gives $\min_{t' \leq t+1}\{x_0 + S_{t'}; 0\} = x_0 + S_{t+1} = -L_{t+1}$ and the proposition is proved. □

We can now consider the $D[0,1]$ element $x_T(s) = \lambda^{-1} T^{-1/2} x_{[sT]}$:

$$x_T(s) = c + \frac{x_{[sT]}}{\lambda\sqrt{T}} = c + \frac{S_{[sT]}}{\lambda\sqrt{T}} + \frac{L_{[sT]}}{\lambda\sqrt{T}} = c + \frac{S_{[sT]}}{\lambda\sqrt{T}} - \frac{1}{\lambda\sqrt{T}} \inf_{t' \leq [sT]} \{x_0 + S_{t'}, 0\}$$

By Lemma 1, $\lambda^{-1} T^{-1/2} x_{[sT]} \xrightarrow{w} B(s)$ and, by applying the continuous mapping theorem it follows that

$$\begin{aligned} -\frac{1}{\lambda\sqrt{T}} \inf_{t' \leq [sT]} \{x_0 + S_{t'}\} &= -\frac{1}{\lambda\sqrt{T}} \inf_{r \leq s} \{x_0 + S_{[rT]}\} = -\inf_{r \leq s} \left\{c + \frac{S_{[rT]}}{\lambda\sqrt{T}}, 0\right\} \\ &\xrightarrow{w} -\inf_{r \leq s} \{c + B(r), 0\} \end{aligned}$$

where both convergencies hold jointly for all s. Continuity of the two limiting functionals implies that process (18) converges weakly to $B_0(s) = c + B(s) - \min_{r \leq 1}\{B(r)\}$ which is a regulated Brownian motion on $[0,1]$, starting in c and with a reflecting barrier in 0 (Harrison, 1985). □

We can now extend the proof to the two-barrier case. Without loss of generality we set $\underline{b}_T = 0$, $\overline{b}_T = \overline{c}\lambda T^{1/2}$, $x_0 = cT^{1/2}$. Unfortunately, when two barriers are present, it is not possible to define the limiting regulated Brownian motion as directly as before; still, we prove that the definition of nearly-bounded random walk (18) corresponds to Harrison's (1985) construction of regulated Brownian motion as $T \to \infty$.

The proof consists of two steps. Firstly, in the $C[0,1]$ space endowed with the uniform metric, we define a continuous approximant which satisfies Harrison's construction, and weak convergence is proved. Then, it is shown that weak convergence holds for the $D[0,1]$ with Skorohod topology version too.

As in Proposition 3, the process x_t can be recursively defined as $x_t = x_0 + S_t + L_t - V_t$, where $S_t = \sum_{i=1}^{t} u_i$, $L_t = \sum_{i=1}^{t} l_i$, $V_t = \sum_{i=1}^{t} v_i$, $l_t = -(x_{t-1} + u_t)\mathbf{1}\{x_{t-1} + u_t < 0\}$ and $v_t = \left(x_{t-1} + u_t - \overline{b}_T\right)\mathbf{1}\left\{x_{t-1} + u_t > \overline{b}_T\right\}$. Clearly, $x_t \in [0, \overline{b}_T]$, all t. In order to define a $C[0,1]$ approximant of x_t, say $x_T(s)$, it is enough to define $C[0,1]$ approximations for all its components, *i.e.* S_t, L_t and V_t. For the partial sum S_t we can set

$$S_T(s) = \frac{1}{\lambda\sqrt{T}} \sum_{i=1}^{[sT]} u_i + u_{[sT]+1} \frac{(sT - [sT])}{\lambda\sqrt{T}} \tag{26}$$

which represents the process obtained by joining the points $(t/T, \lambda^{-1}T^{-1/2}S_t)$ by means of straight lines. For V_t and L_t we define this approximation in a slightly different way:

$$V_T(s) = \begin{cases} \frac{1}{\lambda\sqrt{T}} \sum\limits_{i=1}^{[sT]} v_i & \text{if } v_{[sT]+1} = 0 \\ \\ \frac{1}{\lambda\sqrt{T}} \sum\limits_{i=1}^{[sT]} v_i + \frac{1}{\lambda\sqrt{T}}((sT - [sT])u_{[sT]+1} & \\ -(u_{[sT]+1} - v_{[sT]+1}))\mathbf{1}\left\{sT > [sT] + \frac{u_{[sT]+1} - v_{[sT]+1}}{u_{[sT]+1}}\right\} & \text{if } v_{[sT]+1} > 0 \end{cases}$$

$$L_T(s) = \begin{cases} \frac{1}{\lambda\sqrt{T}} \sum\limits_{i=1}^{[sT]} l_i & \text{if } l_{[sT]+1} = 0 \\ \\ \frac{1}{\lambda\sqrt{T}} \sum\limits_{i=1}^{[sT]} l_i + \frac{1}{\lambda\sqrt{T}}((sT - [sT])u_{[sT]+1} & \\ -(u_{[sT]+1} - v_{[sT]+1}))\mathbf{1}\left\{sT > [sT] + \frac{u_{[sT]+1} - l_{[sT]+1}}{u_{[sT]+1}}\right\} & \text{if } l_{[sT]+1} > 0 \end{cases}$$

With respect to linear interpolations like (26), this construction allows to reduce the set of increasing points of V (of L) to the set of points with $S_T(s) > \overline{c}$ ($S_T(s) < 0$).

It is easy to prove that:

1. $L_T(\cdot)$ and $V_T(\cdot)$ are increasing and continuous with $L_T(0) = V_T(0) = 0$;
2. $x_T(s) = c + S_T(s) + L_T(s) - V_T(s) \in [0, \overline{c}]$, all $s \in [0,1]$;
3. $L_T(\cdot)$ and $V_T(\cdot)$ increase only when $x = 0$ and $x = \overline{c}$ respectively.

From Harrison (1985), Proposition 2.4.6, the continuous mapping $x_T(\cdot) = g_0^{\overline{c}}(S_T(\cdot)) = S_T(\cdot) + c + L_T(\cdot) - V_T(\cdot)$ is the unique functional which regulates $S_T(\cdot)$ to lie within the interval $[0, \overline{c}]$ and which satisfies properties $1-3$. This allows to obtain the limiting distribution of $x_T(\cdot)$ by applying the continuous mapping theorem, Lemma 2, to the limit of $S_T(\cdot)$. But $S_T(s)/\lambda T^{1/2} \xrightarrow{w} B(s)$, which implies that $x_T(s)/\lambda T^{1/2} = g_0^{\overline{b}}\left(S_T(s)/\lambda T^{1/2}\right) \xrightarrow{w} g_0^{\overline{b}}(B(s))$, which is a regulated Brownian motion.

To prove weak convergence on the cadlag space $D[0,1]$ it is sufficient to prove that the process

$$\begin{aligned} x_T(s) - \frac{x_{[sT]}}{\lambda\sqrt{T}} &= \left(L_T(s) - L_{[sT]}\right) - \left(V_T(s) - V_{[sT]}\right) \\ &\quad + \frac{1}{\lambda\sqrt{T}} u_{[sT]+1}(sT - [sT]) \end{aligned} \tag{27}$$

converges to 0 in probability on $C[0,1]$. As both $L_T(s)-L_{[sT]}$ and $V_T(s)-V_{[sT]}$ are in modulus less that $\lambda^{-1}T^{-1/2}\left|u_{[sT]+1}\right|$, and as the set of increasing points of $L_T(s)$ and the set of increasing points of $V_T(s)$ are disjoint, it follows that

$$\left|x_T(s)-\frac{x_{[sT]}}{\lambda\sqrt{T}}\right| \leq \frac{2}{\lambda\sqrt{T}}\left|u_{[sT]+1}\right|(sT-[sT]) \leq \frac{2}{\lambda\sqrt{T}}\left|u_{[sT]+1}\right| \ ;$$

By referring to Bonferroni inequality, Markov inequality and Condition B2 we have, for all $\varepsilon>0$:

$$\begin{aligned}
&\Pr\left\{\sup_{s\in[0,1]}\left|x_T(s)-\frac{x_{[sT]}}{\lambda T^{1/2}}\right|>2\varepsilon\right\} \leq \Pr\left\{\max_{t=1,\ldots,T}|u_t|>\varepsilon\lambda T^{1/2}\right\}\\
&\leq \sum_{t=1}^{T}\Pr\left\{|u_t|>\varepsilon\lambda T^{1/2}\right\} \leq \sum_{t=1}^{T}\frac{1}{(\lambda\varepsilon)^pT^{p/2}}E|u_t|^p\\
&\leq \frac{T}{(\lambda\varepsilon)^pT^{p/2}}\sup_{t=1,\ldots,T}E|u_t|^p = \frac{C}{(\lambda\varepsilon)^p}T^{1-p/2}
\end{aligned}$$

which is $o(1)$ since $p>2$ and convergence of the process (27) on $C[0,1]$ follows. □

A.4 Proof of Corollary 1

We prove the corollary for the case of one barrier in 0. Firstly, by Proposition 3 the long-run variance estimator can be decomposed as:

$$\begin{aligned}
\widehat{\lambda}_T^2 &= \frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}\Delta x_t\Delta x_{t-|j|} = \frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}(u_t+l_t)\left(u_{t-|j|}+l_{t-|j|}\right)\\
&= \frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}u_tu_{t-|j|}+\frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}\left(l_tu_{t-|j|}+u_tl_{t-|j|}+l_tl_{t-|j|}\right) \qquad (28)
\end{aligned}$$

Phillips (1987) has shown that under the conditions of the Corollary, then $(1/T)\sum\omega_j\sum u_tu_{t-|j|}$ converges in probability to λ; hence, to complete the proof we only need to prove that the last term on the r.h.s. of equation (28) goes to 0 in probability. Consider the following inequalities

$$\begin{aligned}
&\left|\frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}\left(l_tu_{t-|j|}+u_tl_{t-|j|}+l_tl_{t-|j|}\right)\right|\\
&\leq \frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\sum_{t=|j|+1}^{T}\left|l_tu_{t-|j|}+u_tl_{t-|j|}+l_tl_{t-|j|}\right|\\
&\leq \frac{1}{T}\sum_{j=-q_T}^{q_T}\omega_j\max_{t=1,\ldots,T}\{|u_t|\}\sum_{t=1}^{T}3l_t = \frac{3L_T}{T^{1/2}}\max_{t=1,\ldots,T}\left\{\frac{|u_t|}{T^{1/2}}\right\}\sum_{j=-q_T}^{q_T}\omega_j
\end{aligned}$$

as $l_t\geq 0$, all t. Now, $L_T/T^{1/2}$ converges weakly to the well-defined random variable $-\lambda\min_{r\leq 1}\{c+B(r),0\}$, see paragraph A.3; furthermore, writing the truncation lag as $q_T=T^{\alpha}$, $\alpha<1/2$, see condition K2, the quantity $T^{-\alpha}\sum_{j=-q_T}^{q_T}\omega_j$ converges to a finite constant $K > 0$.[17] Hence, in order to prove the corollary one could show that $\max_t\left\{|u_t|/T^{1/2}\right\}$ is $o_p(T^{-\alpha})$, i.e. $\Pr\left\{\max_t\left\{|u_t|/T^{1/2}\right\} > \varepsilon\right\} = o(T^{-\alpha})$, all $\varepsilon > 0$. Similarly to Paragraph A.3, under the moment condition B2', by applying Bonferroni's inequality and Markov's inequality it follows that

$$\begin{aligned}
\Pr\left\{\max_{t=1,\dots,T}\left\{|u_t|/T^{1/2}\right\} > \varepsilon\right\} &= \Pr\left\{\max_{t=1,\dots,T}\{|u_t|\} > \varepsilon/T^{1/2}\right\} \leq \sum_{t=1}^{T}\Pr\left\{|u_t| > \varepsilon/T^{1/2}\right\} \\
&\leq \sum_{t=1}^{T}\frac{1}{\varepsilon^p T^{p/2}}E|u_t|^p \leq \frac{T}{\varepsilon^p T^{p/2}}\sup_{t=1,\dots,T}E|u_t|^p \\
&= \frac{C}{\varepsilon^p}T^{1-p/2} \leq \frac{C}{\varepsilon^p}T^{-\alpha/(1-\alpha)}
\end{aligned}$$

since $p > 2(1+\alpha/(1-\alpha))$ for conditions B2' and B4'. As $T^{-\alpha/(1-\alpha)} = o(T^{-\alpha})$ the above probability converges to 0 and therefore it is proved that the dynamics of the regulator process L_t do not affect the estimation of the error term long-run variance[18]. Extension to the two-barrier case follows straightly. □

[17] In general (see Phillips, 1991), $K = \int_{-1}^{1}\omega(x)\,dx$, where $\omega(x)$ is the weighting function used to compute the long-run variance estimator, see paragraph K1.

[18] Note that the proof relies on condition B2', which imposes existence of $2(1+\alpha/(1-\alpha))^{+}$ moments; this is not required to prove weak convergence to regulated Brownian motion, see proof A.3, where only finite 2^{+} moments (condition B2) are necessary.