CK-Equilibria and Informational E±ciency in a Competitive Economy

Gabriel Desgranges[¤]

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Abstract

We consider a very simple competitive economy with in¯nitesimal agents and asymmetric information. We de¯ne a Common Knowledge (CK hereafter) Equilibrium as a price distribution compatible with CK of market clearing and rationality. At equilibrium, expectational mistakes and incorrect information revelation by price are possible. But, whenever unique, the CK equilibrium is a fully revealing Rational Expectations Equilibrium. Hence uniqueness of equilibrium means market informational e±ciency. We give di®erent conditions of uniqueness of equilibrium bearing on the information structure. The ¯rst ones emphasize that many informed agents are required for market e±ciency. Agents need not be perfectly informed, but each "piece" of information has to be known by a large enough proportion of the population. The main result is a characterization of the information structures allowing for local uniqueness: multiplicity of equilibria obtains when all the agents have to extract information from the price to obtain information about the same event. We show that this result holds in an exchange economy with ¯nitely many goods and generic preferences. Finally, we provide a simple market game in which the CK-equilibria obtain through in¯nitely repeated elimination of weakly dominated strategies.

1. Introduction

This paper is an attempt to discuss the informational e±ciency of the price (the so-called E±cient Market Hypothesis) using a weaker solution concept than the

¤THEMA, université de Cergy-Pontoise. gabriel.desgranges@eco.u-cergy.fr. I thank Bruno Biais, Françoise Forges and my thesis advisor, Roger Guesnerie, for comments. I began this work while I was completing my doctoral thesis at DELTA (Joint Reseach Unit CNRS EHESS ENS). Remaining errors are mine.

usual Rational Expectations Equilibrium (REE hereafter). We call this solution concept Common Knowledge Equilibrium (CK equilibrium hereafter) because it is defined as a price distribution compatible with common knowledge (CK hereafter) of rationality and market clearing. The REE is a specific CK equilibrium: it requires also CK of the expected price distribution.

In a competitive economy with incomplete and asymmetric information, the traditional solution concept is REE. The most striking results in this REE literature are among the oldest ones. Grossman (1976) and Radner (1979) show that REE is generically fully revealing, i.e. every agent learns from the price all the relevant information, public and private, available among the whole population. This result sustains the Efficient Market Hypothesis in its strong form. However, it looks paradoxical because, at a fully revealing REE, there is no incentive to use private information and therefore no reason for which prices should aggregate private information (Grossman and Stiglitz 1980).

This paper considers the influence of expectational coordination on market efficiency. It applies to a setting with asymmetric information a method defined in Guesnerie (1992) in a case with complete information.[1] The point of view adopted in this paper is then the following. The important question is not existence of CK equilibrium (following from existence of REE) but uniqueness of CK equilibrium. A unique CK equilibrium means that CK of expectations and actions is the consequence of the two assumptions of CK of rationality and market clearing only. Notice then that the unique CK equilibrium is a fully revealing REE. Hence, the conditions of uniqueness are conditions of market informational efficiency: if there is a unique CK equilibrium, then market efficiency follows from CK of rationality and market clearing only. No other *a priori* knowledge about agents'actions is required.

Otherwise, there are multiple equilibria, meaning failure of the expectational coordination triggered by CK assumptions. When they are not REE, CK equilibria namely involve mistakes by agents. Hence, with multiple equilibria, incorrect learning from prices may be a plausible outcome of competitive markets.

A motivation for this CK equilibrium concept can be found in Guesnerie (1992). In this line, CK equilibria are the consequence of a so-called "eductive" learning in virtual time. Eductive learning assumes that every agent computes the set of outcomes compatible with the CK assumptions and, whenever he predicts the only possible outcome is a REE, he plays a REE strategy. This is an

[1] The present work about asymmetric information originates in Desgranges (1994) and Desgranges and Guesnerie (1996). Desgranges, Guesnerie and Geoffard (1998) consider the simplest version of the model used in this paper and they add some exogenous random supply. Desgranges (1999b) studies the model of Grossman (1976) with a random supply. Some of the results in this paper comes from my doctoral thesis (Desgranges 1999a).

individual learning with no communication, no observation of past prices and no tâtonnement across times. But an interesting question would be to find other motivations for CK equilibria, namely a class of real time learning algorithms converging to the CK equilibria.[2]

The most related literature is Dutta and Morris (1997), Mac Allister (1990) and Morris (1995). In these papers, similar solution concepts relying on CK assumptions are defined. Examples of competitive economies are given in which the REEs are, or are not the only equilibria. Hence, these papers show that, when information is incomplete and asymmetric, the REE is not the consequence of CK of rationality and market clearing (i.e. rules of the game) only, it also requires a stronger assumption like CK of every agent's expectations. However, they do not give conditions implying that the REE is the unique equilibrium. This is exactly the question that we address.

In other contexts than competitive economies with asymmetric information, this question has been already considered. The solution concepts can then be called rationalizable solutions, dominant solvable solutions, correlated equilibria. They all rely on the same idea: there is no CK of actions played by agents and this implies that suboptimal decisions can be taken. This is what we call a coordination failure. Among (not so) many examples are Battigalli (1996), Guesnerie (1992), Moulin (1979), Watson (1993). The case of games with strategic complementarities is examined in Milgrom and Roberts (1990). This is the broadest class of games (which does not encompass the model in this paper) for which conditions for uniqueness of the CK-solution are known (namely, this condition is uniqueness of Nash equilibrium). Notice also that the idea of agents ignoring others'actions can be taken into account with Bayesian Nash equilibrium also. This is the approach considered in Morris and Shin (1998a) and Morris and Shin (1998b).

There are alternative approaches of the Efficient Market Hypothesis. Recent papers include Blume and Easley (1999), Routledge (1999) and Vives (1993). They consider learning in a repeated game (either Bayesian or by boundedly rational agents). Their results are more optimistic than those in the present paper. Further research is needed to understand how they relate to the CK approach.

Section 2 presents the model. This is a simple model somewhat inspired from the inventory model in Guesnerie and Rochet (1993). It looks like a 2 goods exchange economy with asymmetric information. Uncertainty bears on a single

[2] About this point, notice that the replicator dynamics (see Weibull 1995) or the evolutionary stochastic stability of Young (1998) eliminates every iteratively (strictly, at least) dominated strategies in the long run. See also the special issue (vol. 29, 1999) of Games and Economic Behavior showing possible learning of correlated equilibria.

parameter taking ¯nitely many values. In Section 3, we give a technical characterization of uniqueness of equilibrium. We then derive some explicit su±cient conditions of uniqueness. They show that uniqueness of equilibrium is related to many agents being perfectly informed. However, no perfectly informed agent is required as soon as every one is well informed enough.

In Section 4, we show the main result in the paper: The fully revealing (Rational Expectations) equilibrium is locally unique if and only if the information structure is "sharp". A sharp information structure means that, in each state and for each value of the unknown parameter, some agents learn from their private information that this value has not occurred with probability 1 (whenever it is the case). When information is not sharp, it is said to be di®use and it is such that, at some states, all the agents need to extract information from the price to obtain information about the same event. Di®use information structures include many usual settings. For example, an information structure where every agent observes the true value of µ with a given probability and another wrong value with the complementary probability is di®use. Hence this result suggests that, with di®use information, it is very demanding to obtain the REE as a plausible outcome of a competitive market.

In Section 5, we show that CK equilibria can have well de¯ned game-theoretical foundations. We de¯ne a market game in which agents submit simultaneously a demand curve to a "walrassian" auctioneer. This auctioneer observes the aggregate demand curve only and he chooses the price among the set of market clearing prices (according to a well de¯ned stochastic rule). This implementation mechanism is relatively poor, it mimics competitive market clearing. We show that CK equilibria are the price distributions compatible with in¯nitely iterated elimination of all the weakly dominated strategies. The fully revealing REE coincides with a Nash equilibrium of the game if and only if the information structure is "di®use".[3]

In section 6, we show that the characterization of local uniqueness extends to a generic competitive exchange economy. Section 7 concludes.

2. A ¯rst one dimensional model and the equilibrium concepts

2.1. A model with two goods and a simple information structure

We consider a static exchange economy with two goods and a continuum of in¯nitesimal agents. The reduced form of the model is very simple. It is inspired of

[3] This last result makes precise a kind of impossibility of REE implementation that was noticed in Hellwig (1980).

the inventory model in Guesnerie and Rochet (1993). It can be also interpreted as a model of exchange of a risky asset. In any case, this model is a simple case of the competitive exchange economies with asymmetric information that are considered in the REE literature. In particular, the results of Radner (1979) apply (see below).

- There is one good whose future value θ is unknown. Its current price is denoted p. The parameter θ describes all the intrinsic uncertainty (i.e. the uncertainty on fundamentals, not the uncertainty on others'actions). The other good is the numéraire, its price and future value are normalized to 1 and this good can be omitted.

- Agents have identical preferences and endowments (that could be normalized to 0).[4] They differ in their private information only. Every agent's private information about θ is described by a privately observed signal s_i. We assume that individual demand for the risky good has the following form:[5]

  $$x(s_i; p) = E(\theta | s_i; p) - p$$

  where $E(\theta | s_i; p)$ is the mean of the future value θ conditionally to the price being p and the private signal being s_i. The beliefs of an agent at $(s_i; p)$, i.e. the conditional distribution $P(\theta | s_i; p)$, can be arbitrary at this stage. Defining conditions on these beliefs consists precisely in defining an equilibrium concept.

  This demand is derived from the maximization of the expected objective function $(\theta - p)x - x^2/2$ ($x^2/2$ is interpreted as a quadratic inventory cost). Notice that demand is not linear because $E(\theta | s_i; p)$ is a function of p. This is precisely this functional dependence of $E(\theta | s_i; p)$ that embodies the ability of agents to extract information from the price.

- The population is identified to the interval $[0; 1]$ (endowed with Lebesgue measure). It is divided into finitely many I groups of size α_i ($\sum_i \alpha_i = 1$). Agents being infinitesimal, an individual demand has no influence on aggregate demand and no agent can manipulate the information revealed by price. In particular, agents can rationally consider the price distribution

[4] Notice that, due to the very specific form of preferences, only aggregate endowment has to be certain. Idiosyncratic shocks on individual endowment are possible as they do not influence aggregate demand.

[5] We assume linear demand for simplicity. It must be clear that every argument in the paper holds true as long as demand is monotonic. Only some computational details would be modified. The last section consider a N goods economy and it gives some insights about possible generalizations of the results.

as given when computing their optimal demand. Therefore, the problem we consider is only a problem of coordination of expectations, not one of credible revelation of private information. Let $Z(s;p) = \int_{[0;1]} x_k(s_k;p)\,dk$ the aggregate demand when price is p and the pooled signal is s. $Z(s;p)$ depends on every agent's beliefs on θ.[6]

The information structure is the following. The private information of an agent consists in a signal s_i in a set S_i. Every agent in the same group i observes the same signal s_i. The vector $s = (s_1;\ldots;s_I)$ is the pooled signal and S denotes the Cartesian product of the S_i. Let $\theta(s)$ the unique value[7] of θ revealed by the pooled signal s and Θ the finite set of the possible values of θ. Let π the common prior distribution of the $(\theta;s)$ (notice $\pi(\theta|s) > 0$ if and only if $\theta = \theta(s)$).[8] Let $\theta(s_i)$ the subset of Θ consisting in all the θ compatible with signal s_i ($\theta \in \theta(s_i)$ if and only if $\pi(\theta|s_i) > 0$).

We call the information structure "simple" in this section's title because we assume the following axiom:

Axiom 1. Every pooled signal s in S reveals a different value $\theta(s)$ of θ ($s \neq s' \Rightarrow \theta(s) \neq \theta(s')$). Hence the two sets Θ and S can be fully identified.

This axiom will be relaxed in Section 4. Notice that the private information of group i defines a partition on the set S (or, equivalently, the set Θ), namely s and s' belongs to the same element in the partition if and only if $s_i = s'_i$.

2.2. Common Knowledge Equilibrium and Rational Expectations Equilibrium

We define an equilibrium concept relying on CK assumptions and we compare it to the usual REE. Analogous equilibria are defined in Mc Allister (1990) and Dutta and Morris (1997). The two properties that will be CK at equilibrium are:

Definition 2.1. i) (rationality) every agent i in $[0;1]$ is rational at a price p if and only if, for every i and every s_i in S_i, there is a conditional probability distribution $P_i(\theta|p)$ such that agent i's demand satisfies:

$$x_i(s_i;p) = \sum_{\theta} \theta P_i(\theta|s_i;p) - p$$

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[6] This definition assumes some measurability properties of the individual demands. We do not enter into technical details.

[7] This is without loss of generality: $\theta(s)$ is simply defined as being all the relevant information revealed by the signal s.

[8] With heterogenous prior distributions π_i, the results still hold true as long as $\pi_i(\theta) > 0$ for every i and θ (for every argument, only the support of the beliefs matters).

where the conditional distribution $P_i(\theta|s_i;p)$ is obtained by bayesian updating:[9]

$$P_i(\theta|s_i;p) = \frac{\pi(s_i|\theta)\,P_i(\theta|p)}{\sum_{\theta' \in \Theta} \pi(s_i|\theta')\,P_i(\theta'|p)}$$

ii) (market clearing) Market clearing obtains at a price p in state s if and only if aggregate demand:

$$Z(s;p) = 0$$

The first "rationality" assumption means that bayesian rationality of every agent at p, i.e. every individual demand maximizes expected utility for an arbitrary probability distribution on θ. The important point is that this probability distributions $P_i(s|p)$ and $P_i(\theta|s_i;p)$ can differ across agents (even if they observe the same private signal s_i). Notice that $P_i(\theta|s_i;p)$ can be any probability distribution with support in $\theta(s_i)$ (the set of values of θ to which signal s_i gives positive probability).

The second "market clearing" assumption is self understanding.

We now give a formal content to the consequences of CK of rationality and market clearing at a given price p. We define iteratively a decreasing sequence of sets $\theta^n(p)$ of values of θ compatible with n levels of knowledge of rationality and market clearing at p.

- Let $\theta^0(p) = \Theta$ the set of all possible values of θ.
- $\theta^1(p)$ is defined in the following way: $\theta \in \theta^1(p)$ if and only if, for every agent i, there is a conditional probability distribution $P_i(\theta|p)$ with support in $\theta^0(p)$ such that the aggregate demand computed with these $P_i(\theta|p)$ clears market, i.e.

$$Z(s;p) = \int_{[0;1]} \left(\sum_{\theta} \theta P_i(\theta|s_i;p) - p \right) di = 0$$

  where the conditional distributions $P_i(\theta|s_i;p)$ are obtained by bayesian updating. $\theta^1(p)$ is the set of values of θ compatible with rationality and market clearing at p.
- $\theta^2(p)$ is defined in the same way: for every $\theta \in \theta^1(p)$, $\theta \in \theta^2(p)$ if and only if, for every agent i, there is a conditional probability distribution $P_i(\theta|p)$ with support in $\theta^1(p)$ such that the aggregate demand computed with these $P_i(\theta|p)$ clears market. $\theta^2(p)$ is the set of values of θ compatible with rationality and market clearing at p and knowledge of these facts.

[9] Remind π is the common prior distribution and the pooled signal s and the value of θ revealed by s can be identified without confusion.

- ...
- $\theta^{n+1}(p)$ is defined in the same way: for every $\theta \in \theta^n(p)$, $\theta \in \theta^{n+1}(p)$ if and only if, for every agent i, there is a conditional probability distribution $P_i(\theta|p)$ with support in $\theta^n(p)$ such that the aggregate demand computed with these $P_i(\theta|p)$ clears market. $\theta^n(p)$ is the set of values of θ compatible with rationality and market clearing at p, knowledge of these facts, ..., knowledge of knowledge of ... (repeated $n-1$ times) ... of these facts.
- ...

As the sequence $\theta^{n+1}(p)$ is decreasing, it converges one can define the limit set:

$$\theta^{\infty}(p) = \cap_n \theta^n(p)$$

Definition 2.2. i) $\theta^{\infty}(p)$ is the set of values of θ that are compatible with common knowledge of rationality and market clearing at p.

ii) An individual beliefs $P_i(\theta|s_i;p)$ is compatible with common knowledge of rationality and market clearing at p

if and only if there is a distribution $P_i(\theta|p)$ with support in $\theta^{\infty}(p)$ such that $P_i(\theta|s_i;p)$ is obtained by bayesian updating:[10]

$$P_i(\theta|s_i;p) = \frac{\pi(s_i|\theta)P_i(\theta|p)}{\sum_{\theta' \in \Theta}\pi(s_i|\theta')P_i(\theta'|p)}$$

if and only if the support of $P_i(\theta|s_i;p)$ is in $\theta^{\infty}(p) \cap \theta(s_i)$.

$\theta^{\infty}(p)$ can be interpreted as the set of values of θ revealed by p. Remind that $\theta(s_i)$ is the set of values of θ revealed by s_i ($\pi(\theta|s_i) > 0$). Hence a beliefs is compatible with the CK assumptions as soon as it gives positive probability to values of θ revealed by both s_i and p.

We adopt the following terminology: a function $p(.) : S \to \mathbb{R}$ that maps each pooled signal into a price is called a price function; a distribution $\nu(s;p)$ on $S \times \mathbb{R}$ such that the marginal distribution on S is the common prior distribution π is called a price distribution.

Definition 2.3. A CK-equilibrium is a price distribution $\nu(s;p)$ such that, for every price p and pooled signal s, $\nu(s;p) > 0$ implies that there exists a collection of individual beliefs $P_i(\theta|s_i;p)$ compatible with CK of rationality and market clearing at p, i.e. such that the support of every $P_i(\theta|s_i;p)$is in $\theta^{\infty}(p)$ and the aggregate demand computed with the $P_i(\theta|s_i;p)$ clears market.

[10] Remind π is the common prior distribution and the pooled signal s and the value of θ revealed by s can be identified without confusion.

At a CK equilibrium, there is no reason that every $P_i(\theta|s_i;p)$ coincides with the conditional distribution on θ obtained by bayesian updating of $\pi(\theta|s_i)$ using the price distribution $\mu(s;p)$. This means that, at a CK equilibrium, some agents may expect a mean value $E(\theta|s_i;p)$ which is different from its true value. Moreover, if every agent always expects the true value $E(\theta|s_i;p)$, then the CK equilibrium is a REE (as defined below). Hence, we say that price is not informationally efficient at a CK equilibrium unless it is a REE.

Notice also that this equilibrium concept does not specify any price formation process (like the REE). An example of price formation process is given in Section 5.

The definition of the REE is the usual one:

Definition 2.4. A Rational Expectations Equilibrium is a price function $p(.) : S \to \mathbb{R}$ such that, for every s,

The Radner (1979) result applies:

Result 1 (Radner 1979). Let us call the price function $p^*(s) = \theta(s)$ the fully revealing price function. This price function is a REE. It is generically the unique REE (with respect to a topology on S and π that could be made precise).

In the present model, this result is straightforward. As we assume that the values of θ are all distinct, every price $p^*(s)$ reveals the unique value $\theta(s)$ and, at the price $p^*(s)$, every agent demands:

$$x(s_i;p^*(s)) = E(\theta|s_i;p^*(s)) - p^*(s) = \theta(s) - p^*(s) = 0$$

and market clears.

The link between CK equilibrium and REE is clear:

Result 2. Every REE is a CK equilibrium. Moreover, every REE is compatible with CK of every agent's expectations and decisions.

A REE is a very specific CK equilibrium: it requires that everyone correctly understands the informational content of price and it is compatible.

3. A unique equilibrium: the requirement of many, well enough informed, agents

In this section, we give conditions of uniqueness of CK equilibrium. It follows from the above definitions that, whenever there is a unique CK equilibrium, this equilibrium is the fully revealing REE. Following the above definitions, the conditions of uniqueness are computed through an iterative argument at every price (bearing on the set of possible values of θ).

A motivation for these results of uniqueness is that they can be interpreted as conditions of market informational efficiency. Namely, when there are many CK equilibria, none of them (but the REE) is fully revealing and one even knows that some agents at least do not expect the true price distribution and they make mistakes when extracting information from price. At these equilibria, market is then inefficient in a very strong sense: every agent does not only learn every relevant information from the price but some agents learn some incorrect information.

We first describe two examples (with 2 and 3 values of θ respectively). We then give some results in the general case. They emphasize the role played by the existence of a large proportion of well informed agents (if not perfectly informed).

3.1. Uniqueness with many perfectly informed agents: an example

This is the case considered in Desgranges (1994) and further analyzed in Desgranges and Guesnerie (1996). It is the most simple example of the model. There is two values of θ denoted B and G (let $\Delta = G - B > 0$). There are only two groups of agents: a proportion α of agents observes the true value of θ, the remaining proportion $(1-\alpha)$ of agents receives no private signal.

The result in Desgranges (1994) can be restated in terms of CK equilibrium.

Result 3. There is a unique CK equilibrium if and only if the proportion α of perfectly informed agents satisfies $\alpha > 1/2$. If $\alpha \leq 1/2$, every price in the interval $[R-(1-\alpha)\Delta; V+(1-\alpha)\Delta]$ is the price of some CK equilibrium in states B and G.

Proof. The result is proved iteratively. Notice non informed agents only needs to extract information from the price. Informed agents have a dominant strategy (see below step 1).

- Step 1 (rationality and market clearing): every informed agent observes θ and he demands $x_{Inf}(\theta; p) = \theta - p$. Every non informed agents demands a quantity $x_i(p)$ in $[B-p; G-p]$. Hence aggregate demand in state θ is in an interval $[\alpha\theta + (1-\alpha)V - p; \alpha\theta + (1-\alpha)R - p]$:

$$Z(\theta; p) \in [\alpha\theta + (1-\alpha)V - p; \alpha\theta + (1-\alpha)R - p]$$

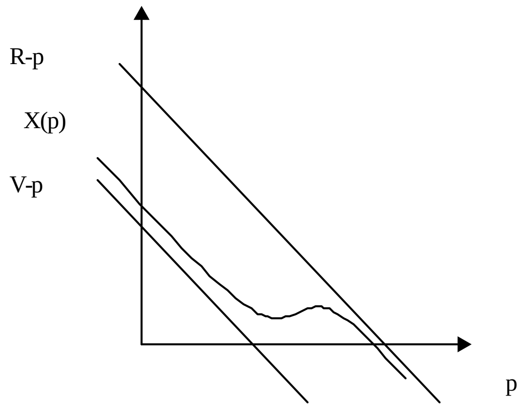

Figure 3.1: Examples of individual demands by a rational agent.

and the price p can clear market in state θ if and only if p is in an interval I_θ:

$$p \in I_\theta \overset{def}{=} [\alpha\theta + (1-\alpha)V; \alpha\theta + (1-\alpha)R]$$

With the notations of Section 2, the values of θ revealed by the price p are:

$$\begin{aligned}
\theta^{1}(p) &= \{B\} \text{ if } p \in I_B - I_G \\
\theta^{1}(p) &= \{G\} \text{ if } p \in I_G - I_B \\
\theta^{1}(p) &= \{B; G\} \text{ if } p \in I_B \cap I_G \\
\theta^{1}(p) &= \emptyset \text{ otherwise}
\end{aligned}$$

Only the prices in $I_B \cup I_G = [B; G]$ are compatible with rationality and market clearing.

- Step 2 (knowledge of rationality and market clearing): every non informed agent learns from step 1 the values of θ revealed by p. Hence 2 cases are possible:

  If $\alpha > 1/2$, then $I_B \cap I_G = \emptyset$ and every price compatible with market clearing reveals either B or G, but never both states. Hence, every non informed agent demands:

  $$\begin{aligned}
  x(p) &= B - p \text{ if } p \in I_B - I_G \\
  x(p) &= G - p \text{ if } p \in I_G - I_B
  \end{aligned}$$

  It follows that the only market clearing price in state θ after this second step is the fully revealing REE price $p^*(\theta) = \theta$. The REE is then the only CK equilibrium in this case.

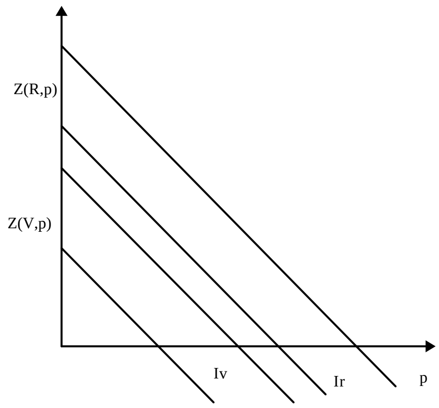

Figure 3.2: Aggregate demands and prices after step 1.

If α $1/2$, then $I_B \cap I_G \neq \emptyset$ and every non informed agent demands:

$$
\begin{aligned}
x(p) &= B - p \text{ if } p \in I_B - I_G \\
x(p) &= G - p \text{ if } p \in I_G - I_B \\
x(p) &\in [B - p; G - p] \text{ otherwise}
\end{aligned}
$$

if p reveals both B and G after the first step, non informed agents learn nothing. In this case, the market clearing prices in state θ are $p^*(\theta) = \theta$ and the prices in $I_G \cap I_B$.

With the notations of Section 2, the values of θ revealed by the price p are:

$$
\begin{aligned}
\theta^2(p) &= \{\theta\} \text{ if } p = p^*(\theta) \\
\theta^2(p) &= \{B; G\} \text{ if } p \in I_B \cap I_G \\
\theta^2(p) &= \emptyset \text{ otherwise}
\end{aligned}
$$

At every following step, the argument is the same and non informed agents do not learn anything more ($\theta^1(p) = \theta^2(p)$ for every p). This proves the Result. ■

Some comments are in order.

- The proof shows that uniqueness obtains after 2 steps (when there are many informed agents). This means that the result follows from a much weaker assumption than CK (namely, rationality and market clearing and knowledge of this facts).

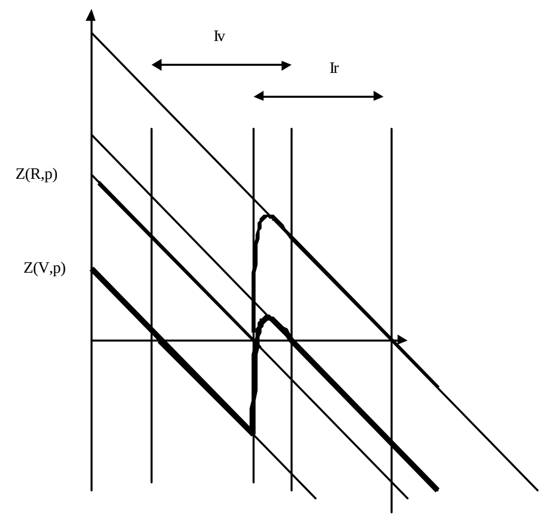

Figure 3.3: Example of market clearing prices distinct from the REE prices when ® 1=2.

² The iterative argument determines the set of prices that are compatible with the CK assumptions in every state µ. But it does not specify what the "optimal" demand of a non informed agent would be at other prices, whenever those prices were to appear for some reason. The argument just says that, if those prices appeared, this would mean that market is not cleared or some agents are not rational or some agents have not taken account of the CK assumptions when choosing their demand.

² When ® > 1=2, uniqueness of CK equilibrium obtains because: Step 1) there are enough informed agents so that di®erent states µ lead to disjoint sets of market clearing prices (intervals $I_µ$ in proof); Step 2) non informed agents can then predict aggregate demand precisely enough so that they restrict their set of possible behaviors in a way that makes the REE the only possible price distribution.

² When ® 1=2, a price p in I_B \ I_G corresponds to 2 polar cases of mistakes: 1) a certain proportion of non informed agents demands B ¡ p and the others demand G ¡ p, some agents make therefore a correct guess; 2) every non informed agent has the same beliefs E (µjp), beliefs are homogenous but certainly wrong (because p cannot clear market in both states).

3.2. Uniqueness with few perfectly informed agents: an example

This example shows that the proportion of perfectly informed agents can be arbitrary small and, still, there can be a unique CK equilibrium if the non perfectly informed agents have "enough" private information. It then gives an example of more intricate situations than the preceding very sharp example.

There are 3 values B(ad), M(iddle), G(ood) of θ ($B < M < G$). For simplicity, we normalize:

$$B = -1, M = 0, G = 1$$

There are 3 groups of agents whose size are respectively α, β et γ ($\alpha + \beta + \gamma = 1$). We assume that, thanks to their private signal, agents in group α distinguish M from the 2 other states, agents in group β distinguish B and agents in group γ are perfectly informed. Therefore, the partitions of $\{B, M, G\}$ defining private information are:

- $\{B, G\}$ and $\{M\}$ for group α,
- $\{B\}$ and $\{G, M\}$ for group β,
- $\{B\}$, $\{M\}$ and $\{G\}$ for group γ.

Agents α (resp. β) must then learn to distinguish B (resp. M) from G using the price.

Result 4. There is a unique CK equilibrium if and only if $\alpha < 1/2$ and $\beta < 1/2$. Therefore, whatever the proportion γ of perfectly informed agents, there can be a unique CK equilibrium.

Uniqueness obtains after 3 steps at most. Hence assuming 2 levels of knowledge of rationality and market clearing is enough.

Proof. After step 1, the sets of possible prices are: $P_B = [0, 2\alpha]$, $P_M = [1, 1 + \beta]$ and $P_G = [\beta + 2\gamma, 2]$ in states B, M and G respectively. Hence the sets $\theta^{1}(p)$ of values of θ revealed by p are:

$$\begin{aligned}
\theta^{1}(p) &= \{B\} \text{ if } p \in P_B - P_M \cap P_G \\
\theta^{1}(p) &= \{M\} \text{ if } p \in P_M - P_B \cap P_G \\
\theta^{1}(p) &= \{G\} \text{ if } p \in P_G - P_M \cap P_B \\
\theta^{1}(p) &= \{G, M\} \text{ if } p \in P_M \cap P_G - P_B \\
\theta^{1}(p) &= \{G, B\} \text{ if } p \in P_B \cap P_G - P_M \\
\theta^{1}(p) &= \{B, M\} \text{ if } p \in P_M \cap P_B - P_G \\
\theta^{1}(p) &= \{B, G, M\} \text{ if } p \in P_M \cap P_B \cap P_G \\
\theta^{1}(p) &= \emptyset \text{ otherwise}
\end{aligned}$$

Case of multiplicity. If $\beta \geq 1/2$, the argument proving multiplicity of equilibrium is the same as in the preceding section: the set $P_M \cap P_G - P_B$ is not empty. For prices in this set, agents α learn G at step 2. But, still, some prices p in this set clear market in both states G and M after step 2, implying $\mu^2(p) = \{G; M\}$. Agents β do not learn anything at the following steps. This is namely the mistakes of agents β that allow these prices to be compatible with both states G and M (agents α and γ are perfectly informed at these prices).

The case $\alpha \geq 1/2$ is similar.

If $\alpha < 1/2$ and $\beta < 1/2$, then it is interesting to distinguish between 3 different cases.

Case with many perfectly informed agents. If $\gamma > 2/3$, the argument is the same as in the preceding section. The 3 sets are disjoint and uniqueness is proved after step 2.

Case with simultaneous learning by both groups α and β. If $2\alpha > \beta + 2\gamma$, uniqueness obtains after 3 steps. One sees that

$$\begin{aligned} P_B \cap P_G &\neq \emptyset \\ P_M \cap P_G &\neq \emptyset \\ P_B \cap P_M \cap P_G &= \emptyset \end{aligned}$$

Step 2 allows agents α to learn that prices in $P_B - P_G$ reveals B and that prices in $P_G - P_B$ reveals G. But they learn nothing at a price in $P_B \cap P_G$. In the same way, Step 2 allows agents β to learn that prices in $P_M - P_G$ reveals M and that prices in $P_G - P_M$ reveals G. But they learn nothing at a price in $P_M \cap P_G$.

The important point is that $P_B \cap P_M \cap P_G = \emptyset$ implies that one group at least learns something at every price. For example, for prices in $P_B \cap P_G$, agents β act as they were perfectly informed: they demand $B - p$ (resp. $G - p$) when they observe the signal $\{B\}$ (resp. $\{M; G\}$).

Therefore, Step 3 imposes further restrictions. One checks that prices in $P_B \cap P_G$ (resp. $P_M \cap P_G$) reveals G (resp. M) and no more B (resp. G). Agents α and β learn all the information they still needed after step 2.

Case of successive learning of the 2 groups α and β. If $2\alpha < \beta + 2\gamma$, uniqueness obtains after 3 steps. One sees that:

$$\begin{aligned} P_B \cap P_G &= \emptyset \\ P_M \cap P_G &\neq \emptyset \\ P_B \cap P_M \cap P_G &= \emptyset \end{aligned}$$

Step 2 allows agents α to distinguish B from G at every price ($P_B \cap P_G = \emptyset$) but agents β do not learn anything at prices in $P_M \cap P_H \neq \emptyset$ that reveal $\{M; H\}$.

Step 3 is required for agents β at prices in $P_M \cap P_H$. At a price in $P_M \cap P_H$, agents α have learned from step 1 that the true state is H. One checks that, after step 3, every price in $P_M \cap P_H$ reveals one value of at most. This allows agents β to restrict their set of possible demands in a way that makes the REE prices the only market clearing prices. ■

3.3. The general case

We now consider the general information structure defined in Section 2. We give a technical lemma providing a necessary and sufficient condition of uniqueness of CK equilibrium. We then apply this lemma to give sufficient conditions of uniqueness in some (hopefully) explicit cases.

Lemma 3.1. There exists a unique CK equilibrium if and only if, for every subset $\hat{S} \subset S$ of cardinal $\#\hat{S} \geq 2$,

$$\bigcap_{s \in \hat{S}} \left[\underline{B}\left(s; \hat{S}\right); \overline{B}\left(s; \hat{S}\right) \right] = \emptyset$$

if and only if, for every $\hat{S} \subset S$ with $\#\hat{S} \geq 2$,

$$\max_{s \in \hat{S}} \underline{B}\left(s; \hat{S}\right) < \min_{s \in \hat{S}} \overline{B}\left(s; \hat{S}\right)$$

where $\underline{B}\left(s; \hat{S}\right)$ (resp. $\overline{B}\left(s; \hat{S}\right)$) is the smallest (largest) price in state s compatible with rationality, market clearing and CK of $s \in \hat{S}$, i.e.

$$\underline{B}\left(s; \hat{S}\right) = \sum_{i \in I} \alpha_i \min_{\theta \in \theta(s_i) \cap \hat{S}} \theta$$

$$\overline{B}\left(s; \hat{S}\right) = \sum_{i \in I} \alpha_i \max_{\theta \in \theta(s_i) \cap \hat{S}} \theta$$

The condition in the lemma means that, when it is CK that the true state s is in a given subset $\hat{S}$, no price compatible with rationality clears market in every state of $\hat{S}$. The intuitive idea is that, whatever the information already revealed by a price (summarized by $s \in \hat{S}$), every price reveals a set of values of θ that is strictly smaller than $\hat{S}$ and this implies that some agents at least learn something from the price. This means that a further step of learning is always useful. Therefore, at every price, every agent successfully achieves complete learning of information revealed by the price. The only market clearing prices are then the REE prices.

Proof. We first prove iteratively the "if" part. At the first step, $\bigcap_{s\in S}\left[\underline{B}(s;S),\overline{B}(s;S)\right]=\emptyset$ implies that for every price, $\theta^1(p)\subsetneq\theta^0(p)=\Theta$ (no price reveals all the values of θ). Hence, either p reveals one state or zero (i.e. price does not clear market) and learning successfully takes place at p; or p reveals at least two states and we examine the next steps.

Assume, for a given price, $\theta^n(p)\subsetneq ... \subsetneq\theta^0(p)$, and $\theta^n(p)$ contains at least two states. Then $\bigcap_{s\in\theta^n(p)}\left[\underline{B}(s;\theta^n(p)),\overline{B}(s;\theta^n(p))\right]=\emptyset$ implies that $\theta^{n+1}(p)\subsetneq\theta^n(p)$. Either $\theta^{n+1}(p)$ contains at most one state and the argument comes to its end at p; or $\theta^{n+1}(p)$ contains two or more states and we examine the next step.

This shows that the argument comes to its end at every p in $\#\Theta$ at most -1.

We prove the "only if" part. If there is $\hat{S}$ such that $\bigcap_{s\in\hat{S}}\left[\underline{B}\left(s;\hat{S}\right),\overline{B}\left(s;\hat{S}\right)\right]\neq\emptyset$, then consider p in this intersection. One has necessarily $\hat{S}\subseteq\theta^n(p)$ for every n. This ends the proof. ■

A few remarks about the lemma:

- The proof of the lemma proceeds iteratively and the set of CK equilibrium obtains after $\#\Theta$ only. This shows that the CK assumptions can be relaxed into assumptions of $\#\Theta$ levels of knowledge.

- The condition in the lemma illustrates the fact that, for each "piece of information", many agents, who are informed of this piece, are required to predict aggregate demand precisely enough. Namely, uniqueness of CK equilibrium obtains when agents can predict aggregate demand and they can therefore correctly learn information from prices. For example, uniqueness of CK does not obtain if only few agents knows a given piece of information. But uniqueness obtains if few agents are not well informed, implying that this lack of information has a small influence on aggregate information.

- The necessity of the condition is illustrated in the first example with many perfectly informed agents. When there are two values θ and θ' ($\Delta=\theta'-\theta>0$) distinguished by a proportion α $1/2$ of agents only, then some prices clear market in both states because non informed agents can make mistakes. These mistakes can be rationalized precisely because market is cleared in both states (i.e. they are rationalized by the expectation of mistakes by others).

This lemma allows for some more explicit conditions of uniqueness of CK-equilibrium in some specific cases. The first result is a direct generalization of a result in Desgranges and Guesnerie (1996).

Proposition 3.2. Assume there is a proportion α of perfectly informed agents and the remaining proportion $(1-\alpha)$ is non informed (i.e. receives the same private signal in every state). There is a unique CK equilibrium if and only if $\alpha > 1 - 1/\#\Theta$ with Θ the number of possible values of θ.

Proof. Consider there are N values of θ denoted $\theta_1 < \theta_2 < ... < \theta_N$. At step 1, informed agents demand $\theta_n - p$ and non informed agents demand a quantity in $[\theta_1 - p; \theta_N - p]$. Hence aggregate demand in state θ_n is in $D_n = [\alpha\theta_n + (1-\alpha)\theta_1 - p; \alpha\theta_n + (1-\alpha)\theta_N - p]$.

If $\alpha > 1 - 1/N$, then a given price cannot clear market in more than one state because the sets D_n do not intersect ($n \neq m \Rightarrow D_n \cap D_m = \emptyset$). This proves the "if" part.

If $\alpha \leq 1 - 1/N$, then consider $n_0 = \arg\min_n (\theta_{n+1} - \theta_n)$. Then $D_{n_0} \cap D_{n_0+1} \neq \emptyset$ holds true because of the following inequality holds true (notice $(\theta_N - \theta_1) \geq (N-1)(\theta_{n_0+1} - \theta_{n_0})$):

$$
\begin{aligned}
\alpha\theta_{n_0} + (1-\alpha)\theta_N - p &\geq \alpha\theta_{n_0+1} + (1-\alpha)\theta_1 - p \\
&\Leftrightarrow \left(\frac{1}{\alpha} - 1\right)(\theta_N - \theta_1) \geq \theta_{n_0+1} - \theta_{n_0}
\end{aligned}
$$

Hence non informed agents do not learn anything at a price in $D_{n_0} \cap D_{n_0+1}$. The further steps do not help learning because of the same argument. This proves the "only if" part. ■

The next proposition shows that perfectly informed agents are, in some sense, necessary for uniqueness.

Proposition 3.3. Assume that in a given state s, no agent learns from their private signal that the true value of θ is $\theta(s)$ (i.e. $\{\theta(s)\} \neq \theta(s_i)$ for every i), then there are multiple CK equilibria.

If there are multiple equilibria, then some of them have prices around $\theta(s)$ in states different from s.

Proof. Consider first $\hat{S} = \{\theta(s); \theta'\}$ and, without loss of generality, $\theta(s) > \theta'$. The lemma shows that stability requires:

$$
\underline{B}\left(\theta(s); \hat{S}\right) > \overline{B}\left(\theta'; \hat{S}\right) \Leftrightarrow \theta' > \alpha\theta' + (1-\alpha)\theta(s)
$$

with α the proportion of agents distinguishing the two values of θ in state θ'. This does not hold true. ■

The following result shows, that no perfectly informed agents are required when every agent is well informed enough in some sense (knowing that θ is below a given threshold, or above).

Proposition 3.4. Assume that, in every state s, there is at least half of the population learning from its private signal that μ ¸ $\mu(s)$ and there is at least half of the population learning from its private signal that μ $\mu(s)$.

There is a unique CK equilibrium.

Proof. Consider a subset $\hat{S}$ and let $\mu_{min} = \min_{s2\hat{S}} \mu(s)$ and $\mu_{max} = \max_{s2\hat{S}} \mu(s)$. One computes:

$$\underline{B}\left(\mu_{max};\hat{S}\right) > \frac{1}{2}\mu_{max} + \frac{1}{2}\mu_{min}$$
$$\overline{B}\left(\mu_{min};\hat{S}\right) < \frac{1}{2}\mu_{min} + \frac{1}{2}\mu_{max}$$

This shows that $\bigcap_{s2\hat{S}} \left[\underline{B}\left(s;\hat{S}\right);\overline{B}\left(s;\hat{S}\right)\right] = ;$ and uniqueness follows from the lemma. ■

The next result shows that, if the information structure is more complex than the simple case with many perfectly informed agents, then a Non Exclusive Information property is required for uniqueness.[11] The underlying idea is that if only one group knows a given piece of information in some state s, then 2 cases are possible: 1) this group is small and other agents cannot predict aggregate demand in state s; 2) this group is large and there is some state s^0 where this group is not perfectly informed and agents in this group cannot predict aggregate demand in state s^0.

Proposition 3.5. Assume the following property of Exclusive Information holds: there exist a state s and a value $\mu^0 \neq \mu(s)$ such that, in state s, only one group i_0 of agents knows the true value of μ is not μ^0, i.e.:

$$9 i_0 = i \neq i_0 \;) \; \mu^0 \; 2 \; \mu(s_i)$$

If there is a unique CK equilibrium, then group i_0 is perfectly informed and its size satis¯es $®_{i_0} > 1=2$.

There is no su±cient condition of uniqueness in the proposition because we make no assumption on other groups'information structure.

[11] A similar property plays an important role for implementation of the walrassian correspondence (see Palfrey and Srivastava 1986, Postlewaite and Schmeidler 1986). But our NEI property is de¯ned on groups and not agents (only one group knows a given piece of information but every agent is the group knows this piece). The NEI property of implementation theory is always satis¯ed and there is no problem of credible revelation here.

Proof. Assume without loss of generality $\theta(s) > \theta'$. Consider first $\hat{S} = \{\theta(s);\theta'\}$. The lemma shows that stability requires:

$$\underline{B}\left(\theta(s);\hat{S}\right) > \overline{B}\left(\theta';\hat{S}\right) \Leftrightarrow \alpha_{i_0}\theta(s) + (1-\alpha_{i_0})\theta' > \alpha_{i_0}\theta' + (1-\alpha_{i_0})\theta(s)$$
$$\Leftrightarrow \alpha_{i_0} > 1/2$$

If the group i_0 is not perfectly informed, then there are two states s^1 and s^2 such that $s^1_{i_0} = s^2_{i_0}$ and $\theta(s_1) > \theta(s_2)$. Then $\alpha_{i_0} > 1/2$ $\underline{B}(\theta(s_1);\{\theta(s_1);\theta(s_2)\})$ $\overline{B}(\theta(s_2);\{\theta(s_1);\theta(s_2)\})$. There are multiple equilibria. Hence group must i_0 be perfectly informed. ■

4. Multiple equilibria: a "diffuse" information structure

In this section, we consider a more general information structure than above. This allows us to define a condition of the information structure implying multiplicity of CK-equilibria, whatever the proportion of informed agents. If the information structure satisfies this condition, it is said to be "diffuse". If it does not, it is said to be "sharp", according to the terminology of Desgranges and Guesnerie (1996). The intuition for "diffuse" information is that, at some states, no agent is confident enough in its private signal so that extracting information from price is useless. More precisely, "diffuseness" of information occurs when, at some states, all the agents need to extract information from the price to obtain information about the same event.

In the present model, the price can fully reveal the underlying state θ with probability 1 (in the case of a Rational Expectations Equilibria, that is generically fully revealing). Hence, the formal definition of "sharp" information will be very demanding: it will require that, at each state and for each event, some agents at least knows with probability 1 that this event has not occurred whenever it is the case.

4.1. An example

We first present an example of "diffuse" information. It relies on the same spirit as the one in Desgranges and Guesnerie (1996). There are two values $\theta = B;G$ $(B < G)$ of the asset return and three groups of agents of identical size $1/3$ (group sizes could be different, it will modify the computations only, not the result). Every agent receives one of the two following signals: a pessimistic one (denoted -) and an optimistic one (denoted +). Every agent in the same group receives the same signal. Therefore there are eight pooled signals, depending on which groups observe - or +. These pooled signals are: $(-\ -\ -)$, $(+\ -\ -)$, $(-\ +\ -)$, $(-\ -\ +)$,

(¡ + +), (+ ¡ +), (+ + ¡) and (+ + +) (the i-th coordinate is the signal observed by the i-th group).

Assume that 1) the two states (¡ ¡ ¡) and (+ + +) have probability 0, we forget them from now on;[12] 2) the three states (+ ¡ ¡), (¡ + ¡), (¡ ¡ +) where one group only observes + have positive probability and reveal µ = B; 3) the three states (+ + ¡), (+ + ¡), (¡ + +) where two groups observes + have positive probability and reveal µ = G.

A ¯rst comment is that, contrarily to the example with perfectly informed agents, everyone's information is incomplete, everyone has the same set of signals and the value of µ depends on how many agents observe the good signal +. One easily convinces himself that this example does not correspond to the model in the preceding sections.

In the preceding sections, we already saw examples where no agent is perfectly informed and therefore everyone has to learn something from others through the price in some states at least. But a crucial feature of this example, that is not satis¯ed by these preceding examples, is that, in some states, all the agents have to learn from others about the same event. Precisely, in every state where µ = B, no agent knows with probability 1 that the wrong µ = G has not occurred (and the similar statement is true in states with µ = G). As µ can take two values only, this statement is equivalent to no agent knowing the true value of µ, but we will see in the following section that the generalization to many values of µ corresponds to the statement "no one knows a given value of µ has not occurred".

We now show that CK equilibria are multiple.

First remind that there is a unique REE that is fully revealing. This is:

p¤(+ ¡ ¡) = p¤(¡ + ¡) = p¤(¡ ¡ +) = B
p¤(¡ + +) = p¤(+ ¡ +) = p¤(+ + ¡) = G

At the REE, the price is equal to the true value of µ. Notice that every agent learns something from the REE price in every state and this feature of the REE exactly coincides with the just mentioned fact that no agent knows from his private signal which value of µ has occurred, or not.

Consider a given price p. Individual Rationality implies that aggregate demand can take every value in [B ¡ p; G ¡ p] in every state. Hence, in every state, the prices compatible with market clearing and rationality are exactly the prices in [B; G].

For a given price p in [B; G], knowledge of market clearing and rationality does not restrict further the set of possible states. Namely, p can reveal both µ = B

[12] Taking account of these two states does not change the result and makes the argmument less straight.

and $\theta = G$ with an arbitrary probability. Hence, in every state, every demand in $[B - p, G - p]$ is still a best reply to some beliefs.

In a similar way, taking account of the full CK assumptions of market clearing and rationality does not restrict further the set of possible prices. We have therefore shown the following fact:

Result 5. In every state, every price in $[B, G]$ is a CK-equilibrium price.

In this example, every possible price can reveal any value of θ and no agent can be certain that the price transmits some information. This is an extreme example of informational inefficiency.

4.2. The general case: a more general information structure

This more general information structure simply include the cases where two pooled signals can reveal the same value of θ (which was impossible in section 2 and 3). This requires the introduction of some more notations. This case corresponds to many usual settings that were not included in the model of the preceding sections. For example, consider a information structure where every agent observes the true value of θ with a given probability and another wrong value with the complementary probability.

Remind agents belong to a finite number I of groups i of size α_i ($\Sigma\alpha_i = 1$). The private information of an agent consists in a signal s_i in a set S_i. Every agent in the same group i observes the same signal s_i. The vector $s = (s_1, \ldots, s_I)$ is the pooled signal and S denotes the Cartesian product of the S_i. The optimal demand of the unique asset at price p and observing s_i is $x_i(s_i, p) = E(\theta|s_i, p) - p$. The conditional mean operator $E(\theta|s_i, p)$ is associated to the agent's beliefs on θ which is not yet determined at this stage.

Let $\theta(s)$ the unique value[13] of θ revealed by the pooled signal s and Θ the finite set of the possible values of θ. Let π the distribution of the (θ, s) (notice $\pi(\theta|s) > 0$ if and only if $\theta = \theta(s)$). Let $\theta(s_i)$ the subset of Θ consisting in all the θ compatible with signal s_i ($\theta \in \theta(s_i)$ if and only if $\pi(\theta|s_i) > 0$).

Notice that the private information of group i defines a partition on the set S of the pooled signals (s and s' belongs to the same element in the partition if and only if $s_i = s'_i$). But, contrarily to the preceding model, it does not define a partition on the set of possible values of θ (the sets $\theta(s_i)$ and $\theta(s'_i)$ can intersect). Precisely, private information of some group does not define a partition of Θ if the cardinal of Θ is strictly smaller than the cardinal of S, a situation that could not happen in the preceding model.

---

[13] This is without loss of generality: $\theta(s)$ is simply defined as being all the relevant information revealed by the signal s.

The price function $p^*(s) = \theta(s)$ is a fully revealing REE. It is generically (in a sense that could be made precise, see Radner 1979) the unique REE.

We now give the definition of sharp and diffuse information. We use the expression "information structure" rather than "information" to emphasize the fact that this is a requirement on the combinations of the private signals that is needed for equilibrium to be unique (even if we later write "sharp/diffuse information" for short). The information structure in the preceding model is always sharp (this is why we introduce the second model).

Definition 4.1. The information structure is "sharp" if and only if, for every pooled signal s in S and every value $\theta' \neq \theta(s)$ in Θ, there exists a group i such that $\theta' \notin \theta(s_i)$, i.e.

$$\forall s \in S, \cap_i \theta(s_i) = \{\theta(s)\}$$

Otherwise, the information structure is "diffuse".

When information is sharp, in every state s, there are agents who learn from their private signal that a given wrong value $\theta' \neq \theta(s)$ has not occurred with probability 1. When information is diffuse, there are at least a state s and a value $\theta' \neq \theta(s)$ such that no agent learns from his private signal that θ' has not occurred. This means that all the agents have to learn a same given piece of information.

With diffuse information, informational efficiency is intuitively very demanding. For instance, in the above example, information is diffuse and multiplicity of CK equilibria obtains. We show that this result is general: when information is diffuse, coordination of all the agents on the REE fully revealing prices cannot be a consequence of CK of rationality and market clearing only.

We give two propositions completing the preceding results.

Proposition 4.2. If the information structure is diffuse, then there are multiple CK-equilibria.

Remind that the preceding model is a particular case of sharp information and, in this model, uniqueness obtains if and only if there are "enough correctly informed" agents (one could show that a similar result holds for an arbitrary sharp information). Notice that the CK equilibrium exhibited in the proof is in the spirit of a remark in Hellwig (1980) (noticing that, for his model, the REE cannot be implemented by a walrassian auctioneer; these cases are made precise in the next section).

Proof. Consider for example that, in state s^0, $\theta' \notin \theta(s^0_i)$ for every i. Consider the following permutation $p(s)$ of the REE $p^*(s) = \theta(s)$: $p(s) = \theta(s)$ if $s \neq s^0$

and $p(s^0) = \theta'$. If every agents expects the REE, then the price function $p(.)$ clears market in every state (in state s^0, everyone learns from $p(s^0)$ that θ' has occurred). It is then a CK equilibrium. ■

We now introduce a concept of local CK-equilibrium. Here, "local" means that the price is always in the neighborhood of a fully revealing price $p^*(s) = \theta(s)$, but this does not mean that prices around $p^*(s) = \theta(s)$ only occurs in state s. Typically, prices around a given value of θ appears in states s associated with a different value of θ. But the word "local" also means that every agent believes that a price near $p^*(s) = \theta(s)$ reveals s with probability 1 unless he learns from his private signal that s has not occurred. Therefore, at a local CK equilibrium, price is not informationally efficient.

Definition 4.3. A local CK-equilibrium is a CK-equilibrium $\tilde{p}(.)$ (i.e. a price distribution compatible with CK of individual rationality market clearing) such that the two following assumptions are CK:

i) the price is in the neighborhood of a REE price with probability 1, i.e.:

$$\exists \varepsilon > 0 / \forall s \in S, P\left[\exists s' \in S / |\tilde{p}(s) - p^*(s')| < \varepsilon\right] = 1$$

ii) every agent expects that a price near the fully revealing REE price $p^*(s) = \theta(s)$ reveals the state s with probability 1, whenever this beliefs is compatible with his private signal s_i i.e.

$$\exists \varepsilon > 0 / \forall \theta \in \Theta, \forall s_i \in S_i, \forall p; \qquad \theta \in \theta(s_i) \text{ and } |p - \theta| < \varepsilon \Rightarrow P_i(\theta | s_i, p) = 1$$

with $P_i(.|s_i, p)$ the beliefs of an agent observing s_i and p.

The motivation for this definition is that such price distributions (whenever they exists) are good candidates for destabilizing the fully revealing REE. Namely, these equilibria are compatible with CK of rationality, market clearing and expectations of the REE but, still, they are not necessarily REE, as shown in the following proposition.

Proposition 4.4. There exists a unique local CK-equilibrium if and only if the information structure is sharp. This unique CK-equilibrium is the Rational Expectations equilibrium.

Notice that uniqueness of local CK equilibrium in the sharp case follows from two steps of the iterative argument only.

Remind also the CK characterization of the REE is stronger the one of the local CK equilibrium: this is a price distribution compatible with CK of rationality, market clearing and also compatible with the fact that it is itself CK(hence it is CK that a given price reveals a given value of θ).

Proof. The "only if" part is proved by the proof of the preceding proposition because the CK equilibrium is local.

We prove the "if" part. Consider a sharp information and a given price p in the neighborhood of a given REE price $p^*(s) = \theta(s)$. In a state s', a group i observing s'_i such that $\theta(s) \in \theta(s_i)$ learns from the price that $\theta(s)$ has occurred. Every agent in this group demands $\theta(s) - p$. Let α be the proportion of agents in this case. Every other agent observes s'_i such that $\theta(s) \notin \theta(s_i)$ and his demand is in $[\inf\theta(s_i) - p; \sup\theta(s_i) - p]$.

If $s' \neq s$ then $\alpha < 1$. One checks that $\theta(s) \notin \theta^{-1}(p)$, i.e. p cannot clear market when taking account of the above restrictions on demand. Namely aggregate demand is in $[\alpha\theta + (1-\alpha)\inf\theta(s_i) - p; \alpha\theta + (1-\alpha)\sup\theta(s_i) - p]$ and a price p around $p^*(s) = \theta(s)$ can clear market if

$$\begin{aligned} \theta(s) &\in [\alpha\theta(s) + (1-\alpha)\inf\theta(s_i); \alpha\theta(s) + (1-\alpha)\sup\theta(s_i)] \\ &\Leftrightarrow \inf\theta(s_i) \leq \theta(s) \leq \sup\theta(s_i) \end{aligned}$$

which is impossible. ■

5. Implementation by a specific market mechanism

In this section, we define a simple static market game and we show that the above CK-equilibria are the price distributions resulting from iterative elimination of weakly dominated strategies. In our specific game, the elimination process is not sensitive to the order of elimination of weakly dominated strategies, a drawback that this process is well known to suffer in many games.

The aim of this section consists only in providing an example of implementation of CK-equilibria. The implementation mechanism relies on aggregate demand functions only and it is therefore relatively poor. There is a continuum of agents of every type. Therefore there exist mechanisms implementing the rational expectations equilibrium (see Laffont 1985, Palfrey and Srivastava 1986, Postlewaite and Schmeidler 1986). We rather try to examine what outcomes can obtain with a mechanism mimicking competitive market clearing.[14]

The chronology of the game is the following: 1) a state s is randomly drawn by nature. Every agent observes his private signal only; 2) Every agent submits a

[14] This game is the one used in Desgranges and Guesnerie (1996), Desgranges, Geoffard and Guesnerie (1998), Desgranges (1999a) (1999b). Notice this method of looking at "reasonable" mechanisms only is advocated in Blume and Easley (1989). Alternative game form can be found in Dubey, geanakoplos and Shubik (1987).

demand curve $x(s_i; p)$ to a "walrassian" auctioneer; 3) The auctioneer determines the price: it is randomly drawn among the set of all market clearing prices. Each market clearing price is equiprobable ; 4) the transactions are implemented at the price p.

This is bayesian game with simultaneous actions. A strategy is a demand function $x(s_i; p)$. The selection rule of the auctioneer among the market clearing prices can be arbitrary as long as it gives positive probability to every market clearing price.[15]

One can naturally associate to every profile of strategy the price distribution that it generates. In particular, the price distribution associated to a Nash equilibrium is a stochastic REE (we did not define stochastic REEs properly but the definition of the REE extends to the case of a price distribution quite easily). The first proposition shows that implementation in Nash equilibrium of the fully revealing REE is not always possible.

Proposition 5.1. The fully revealing REE $p^*(s) = \theta(s)$ is the price distribution associated to a Nash equilibrium of the game if and only if the information structure is sharp.

We do not know what a Nash equilibrium looks like when information is diffuse. There may be multiple Nash equilibria only if there are multiple REE, which is not the case generically.

Proof. The proof of the "if" part is tedious but not difficult. It consists in checking that demand can be chosen at every REE price $p^*(s)$ such that it clears market in state s only. We do not give it here (see Desgranges 1999a).

We prove the "only if" part. If information is diffuse, then there is a state s such that $\theta' \in \cap_i \theta(s_i)$ for some $\theta' \neq \theta(s)$. Assume there is a Nash equilibrium sustaining the fully revealing REE. At this Nash equilibrium, in state s and at price θ', every agent believes that the price reveals θ'. Hence aggregate demand is 0 at this price and the auctioneer sometimes chooses price θ' in state s. This contradicts this outcome being part of a Nash equilibrium. ■

This result can be compared with the analogous result about local CK equilibrium in Section 4. They both suggest that, when information is diffuse, it is very demanding to obtain the REE as a plausible outcome of a competitive market.

We now show that CK equilibrium have a precise counterpart in the market game.

[15] Desgranges and Guesnerie (1996), Desgranges (1999a) show that selection of the smallest market clearing price partially change the results.

Proposition 5.2. Consider the set of strategies surviving to in¯nitely repeated elimination of all weakly dominate strategies. The price distributions associated with strategies in this set are the CK equilibria.

The proof is omitted. It is given in Desgranges (1999a).

Weakly dominated strategies (not only strictly dominated) have to be deleted in order to restrict out of equilibrium beliefs. This is required to meet the set of CK equilibrium. Namely, if a CK equilibrium appears as an outcome of the game, then this means that agents have submitted demand curves chosen in accordance with the consequences of the CK assumptions, even at prices with probability 0.

In this game, elimination of weakly dominated strategies is not sensitive to order of elimination as shown in the following proposition. We say that a strategy X id dominated by a strategy Y on the set P of strategy pro¯les if and only if Y gives a higher or equal payo® than X against every strategy pro¯le in P and it a strictly higher payo® for at least one strategy pro¯le in P.

Proposition 5.3. Consider a sequence of strategy pro¯les (i.e. demand functions $x(s_i;p)$) E^N satisfying:

i) E^0 is the set of all the strategies,

ii) for every N, E^{N+1} contains every strategy in E^N that are not weakly dominated by another strategy in E^N on the set E^N,

iii) for every N, si E^N contains a weakly dominated strategy on E^N, then E^{N+1} Ã E^N.

Then the sequence E^N converges to a limit set E^1 and the price distributions associated to the strategy pro¯les in E^1 are the CK equilibria.

The proof is omitted. It is in Desgranges (1999a).

At every step of the sequence E^N, some weakly dominated strategies are deleted, but not necessarily all of them. Point iii) means that elimination of strategies stops when there is no dominated strategy anymore only.

This result means that the prices distributions does not depend on the order of elimination. However, the proof shows that the set of strategies does depend on the order of elimination (but not at prices appearing with positive probability at a CK equilibrium).

6. Extensions: the case of many goods

In this section, we argue that the main result of Section 4 generalizes to a generic economy with ¯nitely many goods and arbitrary utility functions.

The model is the same as above except there are many goods and the (state dependant) utility function only satisfies the usual concavity and smoothness assumptions. Precisely:

- the continuum $[0;1]$ of infinitesimal agents exchange N goods. $p = \left(p^1; \ldots; p^N\right)$ is the price vector and $x_j = \left(x_j^1; \ldots; x_j^N\right)$ denotes a commodity bundle.
- an agent j maximizes a state dependant utility $u_j(\theta; x_j)$ the state dependent utility. $e_j = \left(e_j^1; \ldots; e_j^N\right)$ is his initial endowment (e_j is type independent). Every function $u_j(\theta; .)$ is C^2, increasing with respect to every x_j^n and strictly concave.

We first give standard definitions an results about REE.

A REE is a price function $p^*(.)$ such that the allocation $x_j[p^*(s)]$ is feasible in every state s (i.e. $\int_{[0;1]} x_j[p^*(s)]\,dj = \int_{[0;1]} e_j\,dj$) where the optimal demand $x_j[p^*(s)]$ in state s is defined as:

$$x_j[p^*(s)] = \arg\max_{p^*(s).x \leq p^*(s).e_j} \sum_{\theta} u_j(\theta; x) P(\theta|s_i; p^*(s))$$

The conditional probability $P(\theta|s_i; p^*(s))$ takes account of the information revealed by the price when agent j expects the REE:

$$P(\theta|s_i; p^*(s)) = \frac{\pi(\theta|s_i)\,\pi(\theta|s_i)}{\sum_{s'=p^*(s')=p^*(s)} \pi(\theta|s_i')\,\pi(\theta|s_i)}$$

A fully revealing equilibrium is a REE $p^*(.)$ such that, for every s and s', $\theta(s) \neq \theta(s')$ implies $p^*(s) \neq p^*(s')$.

The next result is well known. To prove existence, it is enough to consider the modified economy obtained by assuming that the pooled signal s is public. One checks that the equilibrium of this economy with complete information is a separating one (and therefore it is a fully revealing REE of the initial economy). The genericity of the result obtains with transversality argument. We do not give a formal definition of the genericity.

Result 6 (Radner 1979, Pietra and Siconolfi 1998). For generic specifications of utility and endowments, there exists a REE and every REE is fully revealing. A fully revealing REE depends on preferences and endowments but not on the information structure.

We now generalize the proposition in section 4 stating when uniqueness of local CK equilibrium obtains. A careful examination of the proof shows that the same conditions of uniqueness holds under mild assumptions. One only needs that, at every REE price, the aggregate demand of the modi¯ed economy with complete information takes non singular values (a usual assumption in general equilibrium theory).

Notice that the set of CK equilibria depends on the information structure, whereas existence of fully revealing equilibrium does not.

Proposition 6.1. For generic speci¯cations of utility and endowments, when the economy admits a fully revealing REE, the following equivalence holds true: a fully revealing REE is the only local CK equilibrium if and only if the information structure is sharp.

7. Conclusion

In a competitive setting with asymmetric information, we have de¯ned a CK equilibrium. This equilibrium concept is similar to the traditional REE except the price distribution has not to be perfectly expected. We only require that expectations are compatible with CK of rationality and market clearing. REE appears as a speci¯c CK equilibrium. The main consequence of this de¯nition is that equilibrium behavior is compatible with incorrect learning from the price.

We give two kinds of results. The ¯rst results (Section 3) emphasize that many informed agents are required for an e±cient market. These agents need not be perfectly informed, but each "piece" of information has to be known by a large enough proportion of the population. The second result is a necessary and su±cient condition for local uniqueness of equilibrium, with an appropriate de¯nition of a local CK equilibrium (prices near the REE prices and CK of expectations of e±cient prices). This uniqueness condition of "sharp" information holds true in a very general context. When the information structure is not "sharp" but "di®use", then, in some states, all the agents have to learn a same given piece of information. In those states, the CK assumptions are compatible with expectational mistakes.

We give an example of implementation of CK equilibrium. This is implementation in iteratively undominated strategies in a static market game. Hence a natural interpretation of this solution concept is the so-called "eductive" learning in virtual time (see Guesnerie 1992). It assumes that every agent computes the set of outcomes compatible with the CK assumptions and, whenever he predicts the only possible outcome is a REE, he plays a REE strategy. But an interesting question would be to ¯nd a class of learning algorithms converging to the CK equilibria.

This approach sheds light on the role played by expectations coordination for market e±ciency. This point is forgotten by the REE approach (including repeated Bayesian learning, Vives 1993). However, learning by bounded rational agents considers possible expectational mistakes. At ¯rst sight, this recent literature (see Blume and Easley 1999, Routledge 1999, this line of work originates in Bray 1982) suggests a more e±cient market than the CK approach in this paper does. An important point would then be to relate these approaches more precisely.

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