The Weak Axiom and Comparative Statics

Summary

This paper examines conditions which guarantee that the excess demand function of an exchange economy will satisfy the weak axiom in an open neighborhood of a given equilibrium price. This property ensures that the equilibrium is locally stable with respect to Walras' tatonnement. A related issue is the possibility of local comparative statics; in particular, the paper examines conditions which guarantee that when an economy's endowment is perturbed, the equilibrium price will move in a direction opposite to that of the perturbation.

A distinguishing feature of this paper's approach is the heavy use of the indirect utility function, though we also provide results that allow for the translation of conditions imposed on indirect utility functions to conditions imposed on direct utility functions. Indeed we apply this to the special case of exchange economies where all agents have directly additive utilities - essentially a complete markets finance model with agents having von Neumann-Morgenstern utility functions. We show that the structural properties of demand near an equilibrium price depend on variations in the coefficient of relative risk averison.

Keywords: general equilibrium, demand, indirect utility, stability, risk aversion.

Author: John K.-H. Quah, Department of Economics, Oxford University.

Address: St Hugh's College, Oxford, OX2 6LE, U.K.

Email: john.quah@economics.oxford.ac.uk

THE WEAK AXIOM

AND COMPARATIVE STATICS

By John K.-H. Quah

June 21, 1999

1. Introduction

The aim of this paper is to study the structural properties of demand, firstly in the case of an individual agent, and then in the case of an exchange economy. This work could be thought of as a response to the indeterminacy theorems of Sonnenschein (1973, 1974), Debreu (1974) and Mantel (1976). In those papers, it was shown that utility maximization alone gives no structure to the excess demand function of an exchange economy. So while utility maximization will guarantee that each agent's demand satisfies the weak axiom, there is no guarantee that this property be preserved with aggregation. This in turn means that while we know that a general equilibrium exists under fairly mild conditions, there is no guarantee that the equilibrium is unique, or that it is stable with respect to the Walras' tatonnement.

While there are some well-known conditions which guarantee that aggregate demand is well behaved (for examples, see Shafer and Sonnenschein (1982)), they also tend to be far too strong. This problem at the heart of general equilibrium theory, has led to considerable effort in finding find more plausible conditions that one can impose, either on preferences or endowments or both, that will guarantee that aggregate demand retains or acquires some useful structure. Results of this nature can be found in Hildenbrand (1983), Mas-Colell (1991), Grandmont (1992) and Quah (1997), amongst many others.

The first structural property of demand we investigate in this paper is *monotonicity*. Assuming that the commodity space is R^l_{++}, we denote the demand vector of an agent with an income of w and facing the price vector p by $f(p, w)$. This demand function satisfies monotonicity or *the law of demand* when for any pair of distinct prices p and q,

$$(p - q)^T (f(p, w) - f(q, w)) < 0,$$

i.e., the inner product of a price change and its corresponding demand change is less than zero (the superscript "T" denotes the transpose; p and f are understood as column vectors). This means, in particular, that the demand curve for any good is downward sloping with respect to its own price.

Monotonicity has a particularly attractive characteristic: unlike the weak axiom it aggregates easily, at least in the case when incomes are exogenous. In other words, aggregate demand will be monotonic if all individuals have monotonic demand (the weak axiom also follows (see Mas-Colell (1985))). The much cited example of Giffen goods show quite clearly that violations of monotonicity are certainly compatible with utility maximization. So for a preference to generate a monotonic demand it must have some non-trivial structural properties. Section 2 of this paper is devoted precisely to establishing these properties. The instrument for that purpose is the indirect utility function.

We show in this paper that individual demand will satisfy monotonicity if it is generated by an indirect utility function v that is *convex in prices* and satisfies

$$\epsilon(p,w) \equiv \frac{wv_{ww}(p,w)}{v_w(p,w)} < 2.$$

The expression $\epsilon(p,w)$ is the elasticity of the marginal utility of income with respect to income, so we shall refer to it as the *elasticity coefficient* and the inequality as the *elasticity condition.* The indirect utility approach was also studied by Milleron (1974), who arrived at a condition closely related to ours. The conditions we have found are the natural analogue to the direct utility conditions found by Milleron (1974) and Mitjuschin and Polterovich (1978).

While monotonicity is an ordinal property, solely dependent on the structure of an agent's preference, our conditions are formulated in terms of the indirect utility function, which is certainly non-unique. So the crux of the matter is whether the agent's preference over price-income combinations *can be* represented by an indirect utility function that is both convex in prices and satisfies the elasticity condition. If it does, we know that the demand it generates is monotonic.

We know from standard consumer theory that indirect utility functions are quasi-convex in prices (see Varian (1992) or Mas-Colell et al (1995)). What we require here is something stronger: that it be convex in prices. However, this requirement is not restrictive, in the sense that an indirect utility function that is convex in prices can typically be found for a preference over price-income combinations (see Mas-Colell (1985, Propositions 2.6.4 and 2.6.5)). Similarly, a preference would typically admit an indirect utility function that satisfies the elasticity condition. Indeed, it is straightforward to check that if an indirect utility function does not satisfy the elasticity condition, then its transformation with a suitably concave function certainly would. These observations show that either of our two conditions *alone* says little about the underlying structure of the preference and nothing about monotonicity; but if a preference admits an indirect utility function that satisfies these conditions *simultaneously* then we know that it generates a monotonic demand.

The next question that naturally arises is whether our conditions are necessary. Given that indirect utility functions are non-unique, there is little hope for this in the usual sense. However, if we choose the indirect utility function to be *least convex* (which, again, will exist under mild assumptions), then the elasticity condition is also necessary.[1]

In econometric studies, consumer demand is often derived from an indirect utility function using Roy's identity. Our theorem allows us to match its conditions against the functional forms of the indirect utility function employed in these studies. The paper checks these conditions against a few of the more common functional forms. Without prejudicing the need for a more extensive and careful examination, it is fair to say that one gets the impression that these conditions are often satisfied.

In Section 3 of the paper, we extend our results on individual demand to the case of exchange economies. The extension is not completely straightforward. This is because in the individual demand so far considered, an agent's income is assumed to be independent of price, while in exchange economies an agent's income depends on the value of his endowment, which is certainly price-dependent.

Our results on individual demand *do* permit a straightforward extension to exchange economies in the special case where all agents have endowments which are collinear with one another. In that case, one can easily show that price changes which preserve mean income also preserve the income of *every* agent. If, in addition, all agents have indirect utility functions that are convex in prices and satisfy the elasticity condition, then we have monotonicity of the demand or excess demand functions, when price changes preserve mean income. One can then go on to establish that there is just one equilibrium price, that the excess demand function satisfies the weak axiom and that it is globally stable with respect to Walras' tatonnement.

The situation gets a lot more complicated when we move away from this extreme case. It will be ideal if we can still find conditions that guarantee a *global* structure to excess demand. What we succeed in doing is a little more modest: beginning at a given equilibrium price, we identify conditions on

(i) the distribution of agents' endowments and demand at that price, and

(ii) the local behavior of preferences

which guarantee that the weak axiom holds in some neighborhood of the equilibrium price. This will guarantee local stability and the possibility of some comparative statics. The local behavior of preferences is measured by conditions on the elasticity coefficient, assuming, as in Section 2, that agents have indirect utilities that are convex in prices. The relevent feature with regard to the distribution of demand and endowments is, loosely speaking, their departure from the collinear case. The less collinear is the endowment distribution relative to the demand distribution, the more stringent are the conditions on the elasticity coefficient: when endowments are collinear, the elasticity coefficient must be less than 2; when the endowments are not necessarily collinear, but are still more collinear than demand (in some precise sense) then the elasticity coefficient must be less than 0; etc.

We go on from there to examine the possibility of comparative statics. We identity conditions (of the types (i) and (ii) discussed above) which guarantee that when an economy's endowment is perturbed, the equilibrium price will move in a direction opposite to that of the perturbation.

In the final section of the paper, we first prove some results about the connection between direct and indirect utility functions. We then employ them to translate all the conditions of our earlier results, from conditions imposed on the indirect utility function to conditions imposed on the direct utility function. We apply this to an exchange economy where all agents have additive utilities: this can be interpreted as a finance model with finitely many states and complete markets, inhabited by agents with von Neumann-Morgenstern utility functions. When income is independent of price, we show that an agent's demand for consumption in different states of the world is monotonic if his coefficient of relative risk aversion does not vary by more than 4. As usual, this result extends to the whole economy when endowments are collinear. When they are not collinear, we determine precise bounds on the variation of the coefficient of relative risk aversion which guarantee the weak axiom and the valdity of certain comparative static statements. As one would expect, these bounds become more stringent as the endowment distribution becomes less collinear relative to the demand distribution.

2. Individual Demand

In this section, we will examine conditions on an agent's indirect utility function that will guarantee the monotonicity of his demand function. We assume that the commodity space is in R^l_{++}, and that the consumer has a preference over it that in turn generates a preference over the price-income combinations (p, w) in $R^l_{++} \times R_{++}$. This preference can be represented by an indirect utility function $v : R^l_{++} \times R_+ \to R$. We will typically assume v to be *regular*.

DEFINITION. The indirect utility function $v : R^l_{++} \times R_+ \to R$ is *regular* if it is homogeneous of degree zero, it is C^2, its partial derivative with respect to the price of any good is strictly negative, and it is strictly quasi-convex in prices.

Utility maximization guarantees that the indirect utility function is homogeneous of degree zero, non-increasing with respect to the price of any good, and quasi-convex in prices (see Varian (1992)). Our assumptions here are only slightly stronger. By *strict quasi-convexity* we mean that the Hessian $\partial^2_p v(p, w)$ satisfies $z^T \partial^2_p v(p, w) z < 0$ for any $\partial_p v(p, w) z = 0$ (p and z are understood as column vectors in R^l and z^T is the transpose of z). This assumption is made principally so that we can obtain results in the form of strict rather than weak inequalities (see, for example, the proof of Theorem 2.2).

It is well-known that our regularity assumptions on v guarantee that the demand function $f : R^l_{++} \times R_+ \to R^l_+$ is homogeneous of degree zero, satisfies the budget identity, $p^T f(p, w) = w$ and can be formulated by Roy's identity: $f(p, w)^T = -\partial_p v(p, w)/v_w(p, w)$. Notice that because we assume that $\partial_p v(p, w) \ll 0$, we also have $f(p, w) \gg 0$, and since v is C^2, Roy's identity guarantees that f is C^1.

The important non-standard assumption that we make throughout this paper is that the indirect utility function is *convex in prices.* Preferences over prices (with income held fixed at some level), like preferences over goods, can have many indirect utility representations, but utility maximization will only guarantee that these representations are *quasi*-convex in prices. Fortunately, we know that under mild assumptions, a representation that is convex in prices always exists: when the preference over *prices*, with income held fixed at 1, are "differentiably strictly concave", C^2 and monotone (in the sense of Mas-Colell (1985), mutatis mutandis) one can guarantee that on any compact and convex set of prices P in R^l_{++}, there is an indirect utility function, defined on $R^l_{++} \times R_{++}$ and satisfying all our standing assumptions, for which $v : (\cdot, 1) \to R$ is a convex function (see Mas-Colell (1985, Propositions 2.6.4 and 2.6.5)).

We now give some precise definitions of monotonicity.

DEFINITION: A function $f : R^l_{++} \times R_{++} \to R^l_{++}$ satisfies *local monotonicity at* $(\bar{p}, \bar{w})$ if there is an open neighborhood around $(\bar{p}, \bar{w})$ such that for (p, w) and (q, w) in that neighborhood, with $p \neq q$, we have $(p - q)^T (f(p, w) - f(q, w)) < 0$.

DEFINITION: A function $f : R^l_{++} \times R_{++} \to R^l_{++}$ satisfies *monotonicity in* $\mathcal{S} \subset R^l_{++} \times R_{++}$ if for any (p, w) and (q, w) in $\mathcal{S}$ with $p \neq q$, we have $(p - q)^T (f(p, w) - f(q, w)) < 0$. (If the last inequality is weak, we say that f satisfies *weak monotonicity in* $\mathcal{S}$.)

Note that when f is homogeneous of degree zero (as a demand function certainly will be) monotonicity in a set $\mathcal{S}$ will guarantee monotonicity in the cone generated by $\mathcal{S}$, i.e., the set $\mathcal{S}' = \{(p, w) : \lambda(p, w) \in \mathcal{S} \text{ for some } \lambda \in R_{++}\}$. In virtually all the proofs in this paper we make use of the following facts (see and adapt the proofs in Hildenbrand (1994, Appendix 1)):

LEMMA 2.1: *Let* $f : R^l_{++} \times R_{++} \to R^l_{++}$ *be a* C^1 *function.*

(i) Suppose that for some $z \in R^l$ *we have* $z^T \partial_p f(\bar{p}, \bar{w}) z < (>) 0$. *Then for* $t \in R$ *sufficiently close to zero,* $\bar{q}(t) = \bar{p} + tz$ *will satisfy* $(\bar{q}(t) - \bar{p}) \cdot (f(\bar{q}(t), \bar{w}) - f(\bar{p}, \bar{w})) < (>) 0$.

(ii) The function f *is locally monotonic at* $(\bar{p}, \bar{w})$ *if* $\partial_p f(\bar{p}, \bar{w})$ *is negative definite.*

(iii) The function f *is (weakly) monotonic in the convex set* $\mathcal{S} \subset R^l_{++} \times R_{++}$ *if* $\partial_p f(p, w)$ *is negative (semi-)definite for any* $(p, w) \in S$.

We are now ready to state our first major theorem, which identifies conditions on the indirect utility function which guarantee that the demand it generates is monotonic. Our proof is analogous to that employed by Mitjuschin and Polterovich (1978) to establish their conditions on the direct utility function.

THEOREM 2.2: *Suppose* v *is a regular indirect utility function that is convex in prices. Then* v *generates a monotonic demand function if the following Elasticity Condition is satisfied:*

(1) $$\phi(p, w) \equiv -\frac{p^T \partial_p^2 v(p, w) p}{\partial_p v(p, w) p} \quad < \quad 4 \qquad \textit{for all} \qquad (p, w) \in R^l_{++} \times R_{++}.$$

or equivalently,

(2) $$\epsilon(p, w) \quad \equiv \quad \frac{w v_{ww}(p, w)}{v_w(p, w)} < 2 \qquad \textit{for all} \qquad (p, w) \in R^l_{++} \times R_{++}.$$

Proof: By Roy's identity, the demand generated by v is $f(p, y) = -\partial_p v(p, w)^T / v_w(p, w)$. Without loss of generality we fix w at 1. Differentiating this expression and omitting the arguments to save space, we get

(3) $$\partial_p f \quad = \quad \frac{-v_w \partial_p^2 v + (\partial_p v)^T A}{v_w^2}$$

where the row vector

$$A = \left(\frac{\partial^2 v}{\partial w \partial p_1}, \frac{\partial^2 v}{\partial w \partial p_2}, ..., \frac{\partial^2 v}{\partial w \partial p_l}. \right). \tag{4}$$

By Lemma 2.1 (iii) we need only prove that $\partial_p f$ is negative definite. By Euler's identity,

$$v_w = -\partial_p v p = -\sum_i p_i \frac{\partial v}{\partial p_i}. \tag{5}$$

Differentiating this expression by p_j, we see that

$$A = -p^T \partial_p^2 v - \partial_p v. \tag{6}$$

Substituting this equation into (3), we find that

$$\partial_p f = -\frac{\partial_p^2 v}{v_w} - \frac{(\partial_p v)^T p^T \partial_p^2 v}{v_w^2} - \frac{(\partial_p v)^T (\partial_p v)}{v_w^2}. \tag{7}$$

Then for any vector z in R^l,

$$z^T \partial_p f z = -\frac{z^T \partial_p^2 v z}{v_w} - \frac{(\partial_p v z)(z^T \partial_p^2 v p)}{v_w^2} - \frac{(\partial_p v z)^2}{v_w^2}.$$

If $z^T f = 0$, by Roy's identity and our assumption of strict quasi-convexity,

$$z^T \partial_p f z = -\frac{z^T \partial_p^2 v z}{v_w} < 0. \tag{8}$$

So we assume $z^T f \neq 0$. Without loss of generality, let it equal 1, so $\partial_p v z = -v_w$ by Roy's identity. We now have,

$$z^T \partial_p f z = -\frac{1}{v_w}[z^T \partial_p^2 v z - z^T \partial_p^2 v p] - 1 \tag{9}$$

$$= -\frac{1}{v_w}(z - \frac{1}{2}p)^T \partial_p^2 v (z - \frac{1}{2}p) + \frac{1}{4}\frac{p^T \partial_p^2 v p}{v_w} - 1. \tag{10}$$

By the fact that v is convex, and using equation (5), we know that $z^T \partial_p f z < 0$ provided $\phi(p, 1) < 4$. One can check easily that ϕ is homogeneous of degree zero, and so we get condition (1). Condition (2) will be correct once we establish that

$$\phi(p, w) = 2 + \epsilon(p, w). \tag{11}$$

Using (5) and (6), we have $p^T \partial_p^2 v p = -Ap - (\partial_p v)p = -Ap + v_w$. Since v_w is homogeneous of degree -1, with Euler's identity again, we obtain

$$-\frac{\partial v}{\partial w} = \sum_i p_i \frac{\partial^2 v}{\partial w \partial p_i} + \frac{\partial^2 v}{\partial w^2} = Ap + \frac{\partial^2 v}{\partial w^2}.$$

Therefore, $p^T \partial_p^2 v p = 2v_w + v_{ww}$, and so $\phi(p, 1) = 2 + v_{ww}/v_w$. Since ϕ, v_{ww} and v_w are homogeneous of degree zero, -2 and -1 respectively, we obtain (11). *QED*

Remark 1. It is clear from the proof of Theorem 2.2 and Lemma 2.1 that we also have a local result: if v is convex in prices and $\epsilon(\bar{p}, \bar{w}) < 2$ then f is locally monotonic at $(\bar{p}, \bar{w})$; and if the assumptions of the theorem hold only on a convex subset of all price-income combinations, then monotonicity will be true in that subset, again following from Lemma 2.1.

Remark 2. If the Hessian $\partial_p^2 v$ is invertible, weak monotonicity holds if and only if

$$\frac{\partial_p v(p, w) p}{\partial_p v(p, w)[\partial_p^2 v(p, w)]^{-1} \partial_p v(p, w)^T} - \frac{p^T \partial_p^2 v(p, w) p}{\partial_p v(p, w) p} \leq 4.$$

This becomes quite clear once we notice that, by a standard result, the first term on the left of this inequality is just the maximum of the first term on the right hand side of equation (10), when $\partial_p^2 v$ is invertible.[2] Milleron (1974) establishes this same condition with a somewhat different proof. Both Milleron (1974) and Mitjuschin and Polterovich (1978) have the direct analog of this result.

Remark 3 Note that equation (11) means that the convexity of indirect utility with respect to prices imposes a lower bound on $\epsilon(p, w)$: it cannot be less than -2.

The elasticity condition we have found in Theorem 2.2 is simple and appealing, but it is not in general a necessary condition. Since monotonicity is an ordinal property, dependent solely on the preference of the agent, a transformation of the indirect utility function will generate the same demand, yet this transformed indirect utility might well satisfy the conditions of Theorem 2.2, while the original one did not. Indeed, suppose one begins with an indirect utility function v that is convex in prices but does not satisfy the elasticity condition. Then we may consider $\tilde{v} = h \circ v$ where h is increasing and concave; it is easy to show that the elasticity coefficient for $\tilde{v}$ (denote it by $\epsilon_{\tilde{v}}(p, w)$) will be less than $\epsilon_v(p, w)$. By choosing a sufficiently concave h, we can guarantee that the elasticity condition will be satisfied on any compact set of prices and income. Of course, $\tilde{v}$ may no longer be convex in prices, but if it is, we know that the demand generated is monotonic.

Given this observation, we may guess that the elasticity condition is necessary if we *cannot find* a concave h for which $\tilde{v} = h \circ v$ remains convex in prices, in other words, if v is a least convex function in prices. This turns out to be true.

DEFINITION: A convex function $g : R^l_{++} \to R$ is least convex on the convex set $\mathcal{S} \subset R^l_{++}$ if there does not exist a real-valued, increasing, and concave function h such that $h \circ g$ is convex in $\mathcal{S}$.

It is known that any preference over prices (with income normalized at 1) that admits a convex representation also admits, on any given convex set of prices, a least convex representation (see Debreu (1976) and Kannai (1977)).[3] If v is the least convex representation, then any other indirect utility $\tilde{v}$ representing the same preference will be related to v by $\tilde{v} = h \circ v$, where h is an increasing function. If $\tilde{v}$ is convex in prices, then h is also convex (so, $\tilde{v}$ is "more convex" than v). This least convex representation is unique up to affine transformations, and affine transformations leave $\epsilon(p, w)$ unchanged. The next proposition shows that with this (essentially) unique least convex representation, the elasticity condition is necessary for monotonicity.

PROPOSITION 2.3: *Suppose the regular indirect utility function v has the following properties on the compact and convex set $\mathcal{P} \subset R^l_{++}$: $v(\cdot, 1) : \mathcal{P} \to R$ is least convex and $\epsilon(p, 1) > 2$ for all p in $\mathcal{P}$. Then there must be a price $\bar{p}$ in $\mathcal{P}$, for which f is not locally monotonic at $(\bar{p}, 1)$.*[4]

Proof: See Appendix.

While indirect utility functions are not always convex in prices, nor do they always satisfy the elasticity condition, the indirect utility functions employed in econometric applications commonly satisfy these conditions for some if not all price-income combinations; at the least, they are often of the form that allow the conditions to be easily checked. The following are a few applications of Theorem 2.2.

Example 1. When a preference defined over commodity space is homothetic, it induces a homothetic preference over price space, with income held fixed. This means, in turn, that this prefernce is representable by an indirect utility function v which is convex in prices and homogeneous of degree one, and so $\epsilon \equiv -2$, achieving the lower bound on ϵ implied by the convexity of v in prices (see *Remark 3*).

Example 2. A generalization of homothetic preferences is the class of PIGL (price-independent generalized linearity) preferences introduced by Muellbauer (1975, 1976). These preferences have indirect utility functions given by

$$v(p,w) = \frac{\alpha w^{\alpha}}{b(p)^{\alpha}} - \frac{a(p)^{\alpha}}{b(p)^{\alpha}},$$

where $a(p)$ and $b(p)$ are concave and homogeneous of degree one and α is a positive number. If $a(p) = 0$ the preference is homothetic; if $\alpha = 1$, the indirect utility function is of the Gorman form (1953, 1961).

To apply Theorem 2.2, we require v to be convex in prices and to satisfy the elasticity condition. Obviously the latter is satisfied if $\alpha < 3$; we assume this and also that $b(p)$ is *strictly* quasi-concave. Then it can be shown quite easily that $B(p) = 1/b(p)^{\alpha}$ is strictly convex, i.e., $z^T \partial^2 B(p) z > 0$ for all $z \neq 0$. Since $\partial_p^2 v(p,w) = \alpha w^{\alpha} \partial_p^2 B(p) - \partial_p^2 [a(p)^{\alpha}/b(p)^{\alpha}]$, on any compact set P in R_{++}^l, there is $\bar{w} > 0$ such that the indirect utility function will be convex in prices in $P \times (\bar{w}, \infty)$, and demand will be monotonic in this set.

Example 3. Consider the class of additive indirect utility functions, i.e., functions of the form $v(p,w) = \sum_{i=1}^{l} v_i \left(p_i/w\right)$. We want v to be a decreasing function of price and to be convex in prices. This holds if $v_i' < 0$ and $v_i'' > 0$. Interestingly, these conditions are also sufficient to guarantee the law of demand when the price of only one good changes. This can be easily checked using Roy's identity. Monotonicity when the price of more than one good changes will be guaranteed by the elasticity condition, which in the form of inequality (1) is just $\sum p_i^2 v_i''(p_i) < -4(\sum p_i v_i'(p_i))$. This condition will also be *necessary* if v is least concave, which will be true if some v_j is linear in p_j. The condition will certainly hold if $p_i v_i''(p_i)/v_i'(p_i) > -4$ for all i.

3. The Weak Axiom

We will now employ the methods developed in the last section to study the monotonicity of demand in exchange economies. This extension is not completely straightforward: the main reason for this is that, unlike the setting of the last section, agents in an exchange economy generally have incomes that are dependent on price.

The only exchange economy setting where we can apply Theorem 2.2 in a straightforward way is when all agents in the economy have endowments that are collinear with one another. In such an economy, each agent will have an endowment of the form $y\bar{\omega}$, where $\bar{\omega}$ is the mean endowment, and so price changes that preserve mean income also preserve the income of *every* agent. Now if all these agents also have indirect utility functions that are convex in prices and that satisfy the elasticity condition of Theorem 2.2, then each of them will satisfy monotonicity for price changes that preserve mean income. More formally, the aggregate demand function $F : R^l_{++} \to R^l_{++}$ will satisfy

$$(p-q)^T(F(p)-F(q)) \;<\; 0, \tag{12}$$

whenever $p \neq q$ and $p^T\bar{\omega} = q^T\bar{\omega}$. Clearly, this implies that there must be only one price for which the markets are in equilibrium, i.e., when $F(\bar{p}) = \bar{\omega}$. Less obviously, this aggregate property of the demand function is also sufficient to guarantee the global stability of the equilibrium price with respect to the Walras' tatonnement (see Hildenbrand and Kirman (1988)).

We now move away from this special case to consider an exchange economy $\mathcal{E}$, not necessarily with collinear endowments. The economic agents come from a set of "types", represented by the separable metric space $\mathcal{A}$. The agent a has an endowment ω_a in R^l_+, which we assume is a continuous function of a. The agent also has a C^1 demand function f_a, understood (as in Section 2) to be a function of both price and income. We assume that it is homogeneous of degree zero and satisfies the budget identity $p^T f_a(p, w) = w$. We also assume that both f_a and its derivatives are continuous in (a, p, w). The agent in an exchange economy derives his income from his endowment; we denote a's demand, as a function of price alone, by

$$\tilde{f}_a(p) = f_a(p, p^T \omega_a).$$

The distribution of types in the economy $\mathcal{E}$ is given by the probability measure μ on $\mathcal{A}$, which we assume has a compact support. Given these assumptions we may meaningfully write the economy's mean endowment as $\bar{\omega} = \int_{\mathcal{A}} \omega_a d\mu$, which we assume is in R^l_{++}. Similarly, we may write mean demand $F : R^l_{++} \to R^l_{++}$ as $F(p) = \int_{\mathcal{A}} \tilde{f}_a(p) d\mu$. Our assumptions guarantee that this is a C^1 function with its derivative

$$\partial_p F(p) = \int_{\mathcal{A}} \partial_p \tilde{f}_a(p) d\mu. \tag{13}$$

The economy's excess demand at price p is defined as $\zeta(p) = F(p) - \bar{\omega}$. The function ζ is C^1, bounded below, homogeneous of degree zero, and satisfies Walras' Law, i.e., $p^T \zeta(p) = 0$ at all prices p. It is well-known that these conditions, plus a boundary condition, are sufficient to guarantee the existence of an equilibrium price (see Hildenbrand and Kirman (1988)). We let $\bar{p}$ be such an equilbrium price, so $\zeta(\bar{p}) = 0$. We normalize the price to satisfy $\bar{p}^T \bar{\omega} = 1$. For the rest of this section, we shall study the properties of ζ in an open neighborhood around $\bar{p}$.

DEFINITION. The excess demand function ζ satisfies *the weak axiom at the equilibrium price* $\bar{p}$, if there is some open neighborhood around $\bar{p}$ such that whenever two non-collinear prices p and q in that neighborhood satisfy $p^T\zeta(q) \leq 0$, we have $q^T\zeta(p) > 0$.

Clearly, if ζ satisfies the weak axiom at $\bar{p}$, we must also have $\bar{p}^T\zeta(p) > 0$, for all p in some neighborhood of $\bar{p}$, p not collinear with $\bar{p}$. This property is sufficient to guarantee the local stability of the equilibrium price with respect to Walras' tatonnement (see Hildenbrand and Kirman (1988)). Instead of directly establishing the weak axiom, we will be find conditions for a stronger, differentiable version of the weak axiom (see Mas-Colell et al (1995)).

DEFINITION. The excess demand function ζ satisfies the *differentiable weak axiom at the equilibrium price* $\bar{p}$ if $z^T\partial_p\zeta(\bar{p})z < 0$ whenever the vector z is not zero and not collinear with $\bar{p}$.

The conditions we find for the weak axiom at $\bar{p}$ will depend on the distribution of endowments and demand at that price, as well as on the local behavior of demand as price varies, with the latter captured by specifications on the indirect utility function. We assume that f_a can be generated by some regular indirect utility function v_a which is convex in prices in some neighborhood of $(\bar{p}^T, \bar{p}^T\omega_a)$. We assume that the elasticity coefficients of all agents are jointly bounded above. Denoting the coefficient of agent a by ϵ_a, we define

$$\bar{\epsilon} \equiv \sup_{a\in\mathcal{A}} \epsilon_a(\bar{p}, \bar{p}^T\omega_a). \tag{14}$$

This completes our specification of the model.

In trying to establish the weak axiom, it is natural, given equation (13), to look at $\partial_p\tilde{f}_a(p)$, and find some way of estimating $z^T\partial_p\tilde{f}_a(p)z$. This is the object of the next proposition. Note that the subscript a has been dropped for convenience.

PROPOSITION 3.1: *The demand function $\tilde{f}$ is generated from the endowment ω and a regular utility function, v, which is convex in prices. For any $z \in R^l$,*

$$(15) \qquad 4z^T \partial_p^2 \tilde{f}(p) z \leq (-2 + \epsilon(p, p^T\omega)) \frac{(z^T \tilde{f}(p))^2}{p^T\omega} - 2\epsilon(p, p^T\omega) \frac{(z^T \tilde{f}(p))(z^T\omega)}{p^T\omega} + (2 + \epsilon(p, p^T\omega)) \frac{(z^T\omega)^2}{p^T\omega}$$

where ϵ is the elasticity coefficient of the indirect utility v.

Proof: See Appendix.

We observe that a tranformation of v_a to $\hat{v}_a = h \circ v_a$ by a convex function h will preserve convexity in prices while *raising* the elasticity coefficient; indeed one can always choose a transformation so that the new $\epsilon_a(\bar{p}, \bar{p}^T\omega_a)$ (corresponding to the function $\hat{v}_a$ and not v_a) is exactly equal to $\bar{\epsilon}$. Of course, such a transformation will not change the preference, and therefore, neither the demand. With this observation, Proposition 3.1 and equation (15) guarantee that

$$(16) \qquad 4z^T \partial_p F(\bar{p}) z \leq (-2 + \bar{\epsilon}) \int_{\mathcal{A}} \frac{(z^T \tilde{f}_a(\bar{p}))^2}{\bar{p}^T\omega_a} d\mu - 2\bar{\epsilon} \int_{\mathcal{A}} \frac{(z^T \tilde{f}_a(\bar{p}))(z^T\omega_a)}{\bar{p}^T\omega_a} d\mu + (2 + \bar{\epsilon}) \int_A \frac{(z^T\omega_a)^2}{\bar{p}^T\omega_a} d\mu.$$

The three integrals on the right hand side of this inequality can be re-formulated in a way that will make its significance clear. We first define a new probability measure $\hat{\mu}$: for any measurable subset S of $\mathcal{A}$, we define $\hat{\mu}(S) = \int_S p^T\omega_a d\mu$. Since we assume that $p^T\bar{\omega} = 1$, we also have $\hat{\mu}(\mathcal{A}) = 1$. The effect of $\hat{\mu}$ is to "re-weigh" agents according to their contribution to mean income at the price $\bar{p}$. In this case, the first integral

$$\int_{\mathcal{A}} \frac{(z^T \tilde{f}_a(\bar{p}))^2}{\bar{p}^T\omega_a} d\mu = \int_{\mathcal{A}} (z^T \hat{f}_a(\bar{p}))^2 d\hat{\mu}$$

where, by definition, $\hat{f}_a(p) = \tilde{f}_a(p)/p^T\omega_a$. The function $\hat{f}_a$ is simply the projection of $\tilde{f}_a$ onto the plane $B = \{x \in R^l_{++} : \bar{p}^T x = \bar{p}^T \bar{\omega}\}$, so the integral is just the expected value of $(z^T \hat{f}_a)^2$ when it is distributed according to the probability measure $\hat{\mu}$. We can apply a similar transformation to the other integrals on the right hand side of (16). This gives us

$$4z^T\partial_p F(\bar{p})z \quad \leq \quad (-2+\bar{\epsilon})\int_{\mathcal{A}}(z^T\hat{f}_a(\bar{p}))^2 d\hat{\mu} - 2\bar{\epsilon}\int_{\mathcal{A}}(z^T\hat{f}_a(\bar{p}))(z^T\hat{\omega}_a)d\hat{\mu} + (2+\bar{\epsilon})\int_{A}(z^T\hat{\omega}_a)^2 d\hat{\mu},$$

where we define $\hat{\omega}_a = \omega_a/\bar{p}^T\omega_a$.

We now make two more observations. Firstly, the sum of the coefficients of the integrals on the right hand side add up to zero, and secondly, because $\bar{p}$ is the equilibrium price,

$$\int_{\mathcal{A}} z^T\hat{f}_a(p)d\hat{\mu} = \int_{\mathcal{A}} z^T f_a(\bar{p})d\mu = \int_{\mathcal{A}} z^T\omega_a d\mu = \int_{\mathcal{A}} z^T\hat{\omega}_a d\hat{\mu}.$$

It follows that we may replace the the first integral with the variance of $z^T\hat{f}$, the second with the covariance of $z^T\hat{f}$ and $z^T\hat{\omega}$, and the third integral with the variance of $z^T\hat{\omega}$. All this is summarized in the next theorem.

THEOREM 3.2: *The excess demand function of the economy $\mathcal{E}$ satisfies the differentiable weak axiom at the equilibrium price $\bar{p}$ if for all z in R^l, z not collinear with $\bar{p}$,*

$$(17) \qquad L(z) \equiv (-2+\bar{\epsilon})\mathrm{Var}(z^T\hat{f}) - 2\bar{\epsilon}\mathrm{Cov}(z^T\hat{f}, z^T\hat{\omega}) + (2+\bar{\epsilon})\mathrm{Var}(z^T\hat{\omega}) < 0.$$

[Recall the definition of $\bar{\epsilon}$ in equation (14) and note that the variances and covariance are calculated according to the distribution given by $\hat{\mu}$.]

This theorem gives sufficient conditions for ζ to satisfy the weak axiom at $\bar{p}$ in terms of the distribution of demand and endowments (as measured by their variances and covariance) and the local behavior of demand as measured by the elasticity coefficient. As stated it may seem a little opaque, but its corollaries will make it clearer. We first note that it contains the result for collinear endowments as a special case. If ω_a are collinear for all a, $\mathrm{Cov}(z^T\hat{f}, z^T\hat{\omega}) = \mathrm{Var}(z^T\hat{\omega}) = 0$ for all z, and so the weak axiom will hold if $\mathrm{Var}(z^T\hat{f}) \neq 0$ for all z non-zero and not collinear with $\bar{p}$, and $\bar{\epsilon} < 2$. To go beyond this special case, and to obtain more informative bounds on $\bar{\epsilon}$, we need to measure the sizes of the variance and covariance terms relative to each other.

LEMMA 3.3: *Suppose that* $\mathrm{Var}(z^T\hat{f}) \neq 0$ *for all* z *non-zero and orthogonal to* $\bar{p}$*. Then for all* z *non-zero and not collinear with* $\bar{p}$*,*

(i) $\mathrm{Var}(z^T\hat{f}) \neq 0$,

(ii) there exists θ *satisfying* $\mathrm{Var}(z^T\hat{\omega}) < \theta\mathrm{Var}(z^T\hat{f})$*, and*

(iii) there exist positive numbers K_1 *and* K_2 *satisfying*

$$-K_1\mathrm{Var}(z^T\hat{f}) \leq \mathrm{Cov}(z^T\hat{f}, z^T\hat{\omega}) \leq K_2\mathrm{Var}(z^T\hat{f}).$$

Proof: See Appendix.

This lemma's assumption - essentially that demand has a non-zero variance - is of course very mild. The next corollary makes the same assumption.

COROLLARY 3.4: *Suppose that* $\mathrm{Var}(z^T\hat{f}) \neq 0$ *for all* z *non-zero and orthogonal to* $\bar{p}$ *and define* θ *and* K *as in Lemma 3.3. The excess demand of* $\mathcal{E}$ *satisfies the differentiable weak axiom at* $\bar{p}$ *if any of the following situations hold:*

(i) $\theta = 1$ *and* $\bar{\epsilon} < 0$,

(ii) $K_1 = K_2 = 0$ *and*

$$\bar{\epsilon} < 2\left(\frac{1-\theta}{1+\theta}\right),$$

(iii) $\theta \leq 1$ *and*

$$\bar{\epsilon} < \frac{2(1-\theta)}{(1+\theta+2K_1)},$$

(iv)$\theta > 1$, $K_2 \leq 1$, *and*

$$\bar{\epsilon} < -\frac{2(\theta-1)}{[(\theta-1)+2(1-K_2)]}.$$

Proof: Straightforward and will be omitted.

Note firstly that all the bounds on $\bar{\epsilon}$ are numbers less than 2, as one would expect, and greater than -2, as one would require, in order to be compatible with the convexity of the indirect utility functions in prices (see Remark 3 following Theorem 2.2). In all cases of the corollary, the set of permissible preferences include homothetic preferences (see Example 1 in Section 2). Part (i) of this corollary tells us that provided endowment is less dispersed than demand, in the sense that it has a smaller variance, a sufficient condition for the weak axiom is that $\bar{\epsilon} < 0$. This is stronger than the condition $\bar{\epsilon} < 2$ that is required when the endowments are collinear. More generally the bound is either positive or negative depending on whether demand is more or less dispersed than the endowment distribution; furthermore the bound becomes more stringent as endowment becomes more dispersed relative to demand. This is clear from (ii) (where demand and endowment are assumed to be essentially independent), and also in (iii) and (iv), if the covariance term is held constant. This pattern is, in retrospect, unsurprising, and is certainly reminiscent of a notion that has manifested itself in various forms in the literature; namely, that a dispersed demand distribution (variously defined) makes it more likely that *mean* demand is well behaved as a function of price (see, for example, Grandmont (1992), Hildenbrand (1994) and Quah

(1997)).

The distributional conditions on demand and endowments in Corollary 3.4 cover all the possible cases except the case of $\theta > 1$ and $K_2 \geq 1$. The reason for this is quite evident in the light of Mantel's (1976) result. Mantel has shown that simply assuming that all agents have homothetic preferences need place no restriction on the excess demand function. Yet, as we had pointed out above, our conditions on $\bar{\epsilon}$ permit all homothetic preferences, so clearly a distributional restriction is needed. That K_2 is greater than 1 is certainly possible, but there is no reason to believe it is especially likely. Indeed, while much empirical work must still be done, the overall impression given by Corollary 3.4 is that the conditions needed for the weak axiom to hold at an equilibrium price are well within the range of the plausible.

4. Comparative Statics

In the last section, we established conditions for the excess demand function of an exchange economy to satisfy the differentiable weak axiom at the equilibrium price $\bar{p}$. In addition to ensuring the local stability of Walras' tatonnement, the weak axiom also allows us to make a statement about comparative statics. Imagine that the economy is subjected to a small perturbation of its parameters which causes excess demand to change; the weak axiom guarantees that the change in excess demand at the original equilibrium price $\bar{p}$ will be in the same direction (in some precise sense) as the change in the economy's equilibrium price (see Mas-Colell et al (1995)).

The local comparative statics we discuss in this section is of a stronger variety. Again the case of collinear endowments provides the motivation. Suppose the economy's mean endowment changes from $\bar{\omega}$ to $\bar{\omega}'$ while all agents continue to have the same preference and the same (collinear) shares of the new mean endowment. The new equilibrium price is $\bar{p}'$, which we choose to satisfy $\bar{p}'^T\bar{\omega}' = \bar{p}^T\bar{\omega}$. Since mean income is preserved, so is the income of every agent; it follows that when each agent's demand is monotonic (for example, if they satisfy the conditions of Theorem 2.2), so is the demand of the economy as a whole. Since $\bar{p}$ and $\bar{p}'$ are equilibrium prices, market demand is $\bar{\omega}$ and $\bar{\omega}'$ respectively, and we have

$$(\bar{p}' - \bar{p})^T(\bar{\omega}' - \bar{\omega}) < 0.$$

This means, in particular, that the economy's mean endowment and its associated equilibrium price vary in a way that obeys the weak axiom (the proof is straightforward and can be found in Mas-Colell (1985)).

There is an alternative presentation of this property that is independent of how equilibrium prices are normalized. It is straightforward to show that the inequality could be re-written in the form

$$\sum_{i=1}^{l}\left(\frac{x_i'}{\bar{\omega}_i'} - \frac{x_i}{\bar{\omega}_i}\right)(\bar{\omega}_i' - \bar{\omega}_i) < 0,$$

where $x_i = \bar{p}_i\bar{\omega}_i/\bar{p}^T\bar{\omega}$. So x_i is the share of the economy's expenditure devoted to good i at equilibrium when the endowment is $\bar{\omega}$ (and analogously for x_i' and $\bar{\omega}_i'$). The property says in particular, that if we vary the endowment of some good j, while holding the endowments of all other goods fixed, then the *share of aggregate expenditure devoted to good j per unit of good j*, i.e., $x_j/\bar{\omega}_j$ decreases as $\bar{\omega}_j$ increases. Put another way, if good j's endowment goes up by $K\%$, then the share of aggregate expenditure devoted to good j at equilibrium might rise or fall, but it *cannot rise* by $K\%$ or more. This is a natural generalization, in a multi-good economy, of the notion that "the price" of a good falls when supply increases. The rest of this section is devoted to showing how this property might hold in economies with non-collinear endowments.

Our starting point is the economy $\mathcal{E}$, as constructed in Section 3, with an equilibrium price $\bar{p}$ satisfying $\bar{p}^T\bar{\omega} = 1$. We now perturb the economy slightly in the following way. Let $\mathcal{B}$ denote the compact set of unit vectors in R^l. The economy $\mathcal{E}(t, b)$ has exactly the same characteristics as the economy $\mathcal{E}$ except that the agent a now has an endowment $\omega_a + \gamma_a tb$, where t is a positive number, b is in $\mathcal{B}$, and the function $\gamma : \mathcal{A} \to R$ satisfies $\int_{\mathcal{A}} \gamma_a d\mu = 1$. So the economy $\mathcal{E}(t, b)$ has a mean endowment of $\bar{\omega} + tb$ and $\mathcal{E}(0, b) = \mathcal{E}$. We assume that $\bar{p}$ is *a regular equilibrium price of* $\mathcal{E}$. By that we mean the following:

(i) there is an open ball $\mathcal{O}$ around $\bar{p}$ and a positive number T such that all the economies $\mathcal{E}(t, b)$, with (t, b) in $(0, T) \times \mathcal{B}$ have a unique equilibrium price $\bar{p}(t, b)$ in $\mathcal{O}$ that satisfies $\bar{p}(t, b)^T(\bar{\omega} + tb) = 1$; and

(ii) the price $\bar{p}(t, b)$ is distinct from $\bar{p}$ and varies continuously with (t, b), with $\bar{p}(t, b)$ converging to $\bar{p}$ uniformly on $\mathcal{B}$ as t tends to 0.

The next definition states precisely the property of $\bar{p}$ we wish to investigate.

DEFINITION: The equilibrium price $\bar{p}$ of $\mathcal{E}$ *varies monotonically with mean endowment* (or "is monotonic" for short) if it is a regular equilibrium price of $\mathcal{E}$, and there is a $t' > 0$ such that for all (t, b) in $(0, t') \times \mathcal{B}$

$$b^T(\bar{p}(t, b) - \bar{p}) < 0$$

for all b in $\mathcal{B}$.

Note that while the economies $\mathcal{E}(t, b)$ do not have collinear endowments, we *are* confining ourselves to collinear endowment *changes*: in the economy $\mathcal{E}(t,b)$, all agents have had their endowments perturbed in the same direction, that of the vector b, albeit with different (and possibly zero or negative) magnitudes. This is a restrictive assumption, which cannot be straightforwardly weakened, but it does include the special case where all agents experience a small perturbation in their endowment of a single good.

As in Section 3, the conditions we find will depend on the local distribution of demand and endowments, and on the local behavior of preferences, as measured by $\bar{\epsilon}$. In addition, we need to consider the distribution of the endowment change, as captured by γ. Actually, what is directly relevant is not γ, but the function $\eta : \mathcal{A} \to R$ where $\eta_a = \gamma_a / \bar{p}^T \omega_a$. If η_a is identically 1, each agent's share of the mean endowment change tb, exactly equals his share of the mean income at the equilibrium price $\bar{p}$. Since $\int_{\mathcal{A}} \eta_a d\hat{\mu} = \int_{\mathcal{A}} \gamma_a d\mu = 1$, the mean of η when it is weighted by $\hat{\mu}$ is also 1. Weighted according to $\hat{\mu}$, we may sensibly speak of the variance of η, as well as its covariance with $z^T \hat{f}$ and $z^T \hat{\omega}$ for any non-zero vector z. The next theorem sets forth conditions for local comparative statics in terms of these parameters.

THEOREM 4.1: *The equilibrium price vector $\bar{p}$ of $\mathcal{E}$ varies monotonically with the mean endowment provided it is regular and satisfies the following condition: for any non-zero vector z in R^l, we have $L(z) + M(z) < 0$, where*

$L(z) = (\bar{\epsilon} - 2)\mathrm{Var}(z^T \hat{f}) - 2\bar{\epsilon}\mathrm{Cov}(z^T \hat{f}, z^T \hat{\omega}) + (\bar{\epsilon} + 2)\mathrm{Var}(z^T \hat{\omega})$ *and*

$M(z) = (2+\bar{\epsilon})(z^T \bar{\omega})^2 \mathrm{Var}\eta + (\bar{\epsilon}-2)(z^T \bar{\omega})^2 + 2\bar{\epsilon}(z^T \bar{\omega})\mathrm{Cov}(z^T \hat{f}, \eta) - 2(2+\bar{\epsilon})(z^T \bar{\omega})\mathrm{Cov}(z^T \hat{\omega}, \eta)$.

Proof: See Appendix.

The next two corollaries will help clarify the meaning of this theorem. The first considers the case where η is independent of $\hat{f}$ and $\hat{\omega}$.

COROLLARY 4.2: *Suppose that* $\text{Cov}(z^T\hat{f},\eta) = \text{Cov}(z^T\hat{\omega},\eta) = 0$ *for all* z *in* R^l. *Then the regular equilibrium price* $\bar{p}$ *of* $\mathcal{E}$ *varies monotonically with the mean endowment if*

(i) $L(z) < 0$ *for all* z*, non-zero and not collinear with* $\bar{p}$*, and*

(ii)

$$\bar{\epsilon} < 2\left[\frac{1-\text{Var}\eta}{1+\text{Var}\eta}\right].$$

Proof: Condition (ii) guarantees that $M(z) \leq 0$. $M(z) = 0$ if and only if z is orthogonal to $\bar{\omega}$; in this case, z is not collinear with $\bar{p}$ and $L(z) < 0$ by (i) provided z is non-zero. Therefore $L(z) + M(z) < 0$ for all non-zero z. Conclusion follows from Theorem 4.1. *QED*

Condition (i) is simply the condition we have found for the local weak axiom to hold (see Theorem 3.2); condition (ii) imposes an additional bound on $\bar{\epsilon}$, which becomes more stringent as Varη increases. An interesting special case is where $\eta \equiv 1$, then we certainly have $\text{Cov}(z^T\hat{f},\eta) = \text{Cov}(z^T\hat{\omega},\eta) = \text{Var}\eta = 0$ so condition (ii) is simply $\bar{\epsilon} < 2$. When this happens, the conditions on $\bar{\epsilon}$ in Corollary 3.4, which guarantee the weak axiom, also guarantee the monotonicity of the equilibrium price since they are all require $\bar{\epsilon}$ to be less than 2 anyway.

We will now deal with the case where η is not independent of $\hat{f}$ and $\hat{\omega}$. As in the previous section, we need some way of measuring the covariance of these variables relative to the standard deviation of $\hat{f}$.

LEMMA 4.3: *Suppose that* $\text{Var}(z^T\hat{f}) \neq 0$ *for all non-zero* z *orthogonal to* $\bar{p}$. *There exist numbers* L_1 *and* L_2 *such that whenever* z *is non-zero and not collinear with* $\bar{p}$*,* *(i)* $|\text{Cov}(z^T\hat{f},\eta)| \leq L_1\sqrt{\text{Var}(z^T\hat{f})}$ *and*

(ii) $|\text{Cov}(z^T\hat{\omega},\eta)| \leq<< L_2\sqrt{\text{Var}(z^T\hat{f})}$.

Proof: Similar to proof of Lemma 3.3 and will be omitted.

Using L_1 and L_2 we will be able to construct a precise bound on the size of $\bar{\epsilon}$ that will guarantee the monotonicity of the equilibrium price.

COROLLARY 4.4: *Suppose that* $\text{Var}(z^T\hat{f}) \neq 0$ *for all non-zero* z *orthogonal to* $\bar{p}$*, and define* θ *and* K_2 *as in Lemma 3.3, and* L_1 *and* L_2 *as in Lemma 4.3. Suppose also that* $\theta > 1$ *and that* $L_1^2/4 + K_2 < 1$*. Then the quadratic equation in* ϵ*,*

$$[2\epsilon(L_2 - L_1) + 4L_2]^2 = 4[\epsilon(1 - 2K_2 + \theta) - 2 + 2\theta][\epsilon(\text{Var}\eta + 1) + 2(\text{Var}\eta - 1)]$$

has exactly one root ϵ^* *in the interval* $(-2, \min\{M_1, M_2\}\,]$*, where*

$$M_1 = -\frac{2(\theta - 1)}{[(\theta - 1) + 2(1 - K_2)]}$$

and

$$M_2 = 2\left[\frac{1 - \text{Var}\eta}{1 + \text{Var}\eta}\right].$$

The equilibrium price $\bar{p}$ *varies monotonically with the mean endowment provided* $\bar{\epsilon} < \epsilon^*$*.*

While the corollary allows η to co-vary with $\hat{f}$ and $\hat{e}$, this is restricted by the condition $L_1^2/4 + K_2 < 1$. The corollary gives a condition for the monotonicity of the equilibrium price that is clearly stronger than what we obtained for the weak axiom, which is that $\bar{\epsilon}$ be less than M_1 (see Corollary 3.4 (iv)). Quite naturally it is also stronger than the condition we obtained in the case where $L_1 = L_2 = 0$; that required $\bar{\epsilon}$ to be less than M_2 (see Corollary 4.2).

5. The Direct Utility Approach

In the previous sections, we had studied the structure of demand using the indirect utility function. We will now explore the connection between direct and indirect utility, so that we may translate the indirect utility conditions that we have employed in various contexts into conditions imposed on the direct utility functions. We first make a definition of regularity for direct utility functions. The restrictions are standard (see Mas-Colell (1985)) and innocuous.

DEFINITION: The utility function $u : R^l_{++} \to R$ is *regular* if it is C^2, its partial derivatives are strictly positive, it is differentiably strictly quasi-concave, and the sets $\mathcal{C}_{\bar{x}} = \{x \in R^l_{++} : u(x) \geq u(\bar{x})\}$ are closed in R^l for any $\bar{x}$ in R^l_{++}.

Milleron (1974) and Mitjuschin and Polterovich (1978) show that a regular utility function will generate a demand satisfying monotonicity (in the sense defined in Section 2, i.e., with income fixed as price changes) if it is *concave* and satisfies the condition

$$\psi(x) \equiv -\frac{x^T \partial^2 u(x) x}{\partial u(x) x} \; < \; 4. \qquad \text{for all} \quad x \in R^l_{++}. \tag{18}$$

Condition (18) is really a statement about the behavior of the utility function along rays emanating from the origin. For any $\bar{x}$ in R^l_{++} we define the function $G_{\bar{x}}(t) = u(t\bar{x})$, where t is a positive real number, and the function $g_{\bar{x}}(t) = -tG''_{\bar{x}}(t)/G'_{\bar{x}}(t)$. So $g_{\bar{x}}$ is a measure of the curvature of the utilty function in the direction of $\bar{x}$. Since $G'_{\bar{x}}(t) = x^T \partial_x u(t\bar{x})$ and $G''_{\bar{x}}(t) = x^T \partial^2_x u(t\bar{x})\bar{x}$ we have $g_{\bar{x}}(t) = \psi(t\bar{x})$ and so condition (18) is equivalent to having $g_x < 4$ for all x.

It turns out that ψ is also relevant in determing whether or not the indirect utility function generated by u, i.e., $v(p,w) = u(f(p,w))$, is convex in prices.

PROPOSITION 5.1: *Suppose that the regular indirect utility function v is generated by the regular utility function u. The following are equivalent:*

(i) v is convex in prices,

(ii) the functions μ_x are convex for all x, where $\mu_x : R_{++} \to R$ is defined by $\mu_x(s) = u(x/s)$.

(iii) $\psi(x) \leq 2$ for all x.

Proof: See Appendix.

The proposition essentially says that a direct utility will generate an indirect utility that is convex in prices if and only if its $\psi \leq 2$. The next proposition is analogous and says that a utility function is concave if and only if it generates an indirect utility that is a concave function of income. Its proof is also similar to that of the last proposition and will be omitted.

PROPOSITION 5.2: *Suppose that the regular indirect utility function v is generated by the regular utility function u. The following are equivalent:*

(i) u is concave

(ii) v is a concave function of income (equivalently, $\epsilon(p,w) \leq 0$ for all (p,w)).

(iii) $\phi(p,w) \leq 2$ for all (p,w).

[Recall the definition of ϕ in Theorem 2.2.]

Propositions 5.1 and 5.2 together tell us that u is concave and $\psi(x) \leq 2$ for all x, if and only if the indirect utility it generates is convex in prices and concave in income. This statement is also true locally when suitably modified. In particular, we will need the following result, whose proof is omitted because it is similar to that of the previous propositions.

LEMMA 5.3: *Suppose that a preference generates the demand function f, and that in an open and convex neighborhood around $f(p^*, 1)$, the preference admits a regular utility function u, which in turn generates a regular indirect utility function v in an open neighborhood of $(p^*, 1)$.*

(i) If u is concave, $\epsilon(p^*, 1) \leq 0$.

(ii) If u satisfies $\psi(f(p^*, 1)) < 2$, then there is an open and convex neighborhood of prices around p^* (with income fixed at 1) such that v is convex in prices.

Using this lemma, we can prove the next result.

THEOREM 5.4: *Suppose that a preference generates the demand function f and that in an open and convex neighborhood around $f(p^*, 1)$, the preference admits a utility function u which is concave, with $\psi(f(p^*, 1)) < M$. Then there is an open and convex neighborhood of prices around p^* such that the indirect preference over prices in that neighborhood (with income fixed at 1) admits an indirect utility function (not necessarily the one generated by u) which is convex in prices, with $\epsilon(p^*, 1) < M - 2$.*

Proof: See Appendix.

The impact of this theorem is far reaching. Firstly, it allows us to re-establish the direct utility conditions for individual monotonicity which we already know (see condition (18)) via Theorem 2.2: if a utility function u is concave, with $\psi(x) < 4$ for all x, then Theorem 4.4 says that around the supporting price of every commodity bundle x there is, locally, an indirect utility function (not necessarily generated by u) which is convex in prices and with the elasticity condition of Theorem 2.2 satisfied. This guarantees monotonicity.

Similarly, the conditions required in Corollaries 3.4, 4.2 and 4.4 can be translated into conditions on the utility function. In those results, we assumed that each agent had an indirect utility representation v_a that, at least in a neighborhood of $(\bar{p}, \bar{p}^T\omega_a)$ is convex in prices. Conditions of the form

$$\bar{\epsilon} \equiv \sup_{a \in \mathcal{A}} \epsilon_a(\bar{p}, \bar{p}^T\omega_a) < J, \tag{19}$$

were then imposed. Instead of this, we can assume that each agent has a regular utility representation u_a that is concave in an open and convex neighborhood of $\tilde{f}_a(\bar{p})$. Theorem 5.4 guarantees that any condition of the form (19) can be replaced with the condition

$$\bar{\psi} \equiv \sup_{a \in \mathcal{A}} \psi_a(\tilde{f}_a(\bar{p})) < J + 2, \tag{20}$$

where ψ_a is just the formula in (18) applied to u_a.

One special situation in which such a translation is particularly useful is when all agents in the economy $\mathcal{E}$ have utility functions that are regular and directly additive, i.e., $u(x) = \sum_{i=1}^{l} \pi_i u_i(x_i)$, where the π_is are positive, and the u_is are concave. This is essentially a finance model with complete markets, where agents choose consumption across l states of the world according to their von Neumann-Morgenstern utility functions. In this case, the monotonicity conditions are expressible in terms of the coefficients of relative risk aversion, and in particular, on its *variation.* We define the function $B : R^l_{++} \to R$ by

$$B(x) = \max_{1 \le i,j \le l} \left| \frac{x_i u_i''(x_i)}{u_i'(x_i)} - \frac{x_j u_j''(x_j)}{u_j'(x_j)} \right|. \tag{21}$$

LEMMA 5.5: *Suppose that a preference admits a regular and additive utility function u, with $B(x^*) < M$ at some commodity bundle x^*. Then in an open and convex neighborhood* *around x^*, the preference also admits a (possibly different) concave utility function with its $\psi(x^*) < M$.*

Proof: See Appendix.

The next result on individual demand follows immediately from this lemma and the direct utility conditions for monotonicity (see condition (18)).

THEOREM 5.6: *The regular and additive utility u will generate a monotonic demand if $B(x) < 4$ for all $x \in R^l_{++}$.*

Suppose that all agents in the economy $\mathcal{E}$ have directly additive utility functions. Then for each agent, we may define B_a (as in (21)). Using Theorem 5.4, the constant

$$\bar{B} \equiv \sup_{a \in \mathcal{A}} B_a(\tilde{f}_a(\bar{p}))$$

then takes the place of $\bar{\epsilon}$, in Corollaries 3.4, 4.2 and 4.4. The next theorem simply restates Corollary 3.4 in this context; Corollaries 4.2 and 4.4 can be similarly restated.

THEOREM 5.7: *Suppose that all the agents in* $\mathcal{E}$ *have directly additive utility functions. We also assume that* $\text{Var}(z^T \hat{f}) \neq 0$ *for all* z *non-zero and orthogonal to* $\bar{p}$ *and define* θ *and* K *as in Lemma 3.3. The excess demand of* $\mathcal{E}$ *satisfies the differentiable weak axiom at* $\bar{p}$ *if any of the following situations hold:*

(i) $\theta = 1$ *and* $\bar{B} < 2$,

(ii) $K_1 = K_2 = 0$ *and*

$$\bar{B} < 2 + 2\left(\frac{1-\theta}{1+\theta}\right),$$

(iii) $\theta \leq 1$ *and*

$$\bar{B} < 2 + \frac{2(1-\theta)}{(1+\theta+2K_1)},$$

(iv)$\theta > 1$, $K_2 \leq 1$, *and*

$$\bar{B} < 2 - \frac{2(\theta-1)}{((\theta-1)+2(1-K))}.$$

St Hugh's College, Oxford, OX2 6LE, UNITED KINGDOM; john.quah@economics.ox.ac.uk

APPENDIX

Proof of Proposition 2.3: To each p in $\mathcal{P}$ we associate the set

$$Z(p) = \{z \in R^l : \|z\| = 1 \text{ and } p + tz \in \mathcal{P} \text{ for some } t \neq 0\}.$$

Then $v(\cdot, 1)$ is convex in $\mathcal{P}$ if and only if $z^T \partial_p v(p,1) z \geq 0$ for all (p, z) in $C = \cup_{p \in \mathcal{P}} \{p\} \times Z(p)$. By taking limits, $z^T \partial_p v(p,1) z \geq 0$ for all (p,z) in $\bar{C}$, the closure of C.

For a positive α, the function $h(y) = -\exp(-\alpha y)$ is convex and increasing in $y \in R$, and non-affine in $v(\mathcal{P})$ if $\mathcal{P}$ is non-singleton. For z in R^l, the composition $h \circ v = -\exp(-\alpha v)$ satisfies

$$z^T \partial_p^2 [-\exp(-\alpha v)] z = \alpha \exp(-\alpha v) [z^T \partial_p^2 v z - \alpha (\partial_p v z)^2].$$

Since $z^T \partial_p^2 v(p,1) z$ can be thought of as the image of a continuous map from the compact set $\bar{C}$ to R, if $z^T \partial_p^2 v(p,1) z$ is always strictly positve in $\bar{C}$, it must also be uniformly bounded away from zero in $\bar{C}$, and there will be an $\hat{\alpha}$ sufficiently small, such that $z^T \partial_p^2 [-\exp(-\hat{\alpha} v)] z > 0$ for all (p,z) in $\bar{C}$. This means that $-\exp(-\hat{\alpha} v)$ is convex in $\mathcal{P}$, contradicting our assumption that v is least convex.

So there exists $(\bar{p}, \bar{x})$ in $\bar{C}$ such that $\bar{x}^T \partial_p^2 v(\bar{p}, 1) \bar{x} = 0$. Since $\mathcal{P}$ is compact, $\bar{p}$ is in $\mathcal{P}$; because v is strictly quasi-convex, $x^T f(\bar{p}, 1) \neq 0$, so there is a number λ such that $\lambda \bar{x} = x = \bar{p}/2 + x'$, where x' is perpendicular to f. Of course, we also have $x^T \partial_p^2 v(\bar{p}, 1) x = 0$. If we choose $z = x + \bar{p}/2$, then $z^T f(\bar{p}, 1) = 1$, and from equations (10) and (11), we see that $z^T \partial_p f(\bar{p}, 1) z = [\epsilon(\bar{p}, 1) - 1]/2$, which is positive by assumption (ii). By Lemma 2.1 (i), f is not locally monotonic at $(\bar{p}, 1)$. *QED*

Proof of Proposition 3.1: Let us first assume that $p^T\omega = 1$. From equation (7) and Roy's identity we find that

$$\partial_p f = -\frac{\partial_p^2 v}{v_w} + \frac{f p^T \partial_p^2 v}{v_w} - f f^T \tag{22}$$

Differentiating Roy's identity with respect to w, we obtain

$$\frac{\partial f}{\partial w} = \frac{v_{ww}(\partial_p v)^T - v_w A^T}{v_w^2},$$

with A defined by (4). Using equation (6) and Roy's identity, we obtain

$$\left[\frac{\partial f}{\partial w}\right]\omega^T = \frac{1}{v_w}\partial_p^2 v p[\omega^T] + \left(-1 - \frac{v_{ww}}{v_w}\right) f\omega^T. \tag{23}$$

Then, using (22), (23) and (11), and denoting $s = z^T(f+\omega)$, one can check that

$$\begin{aligned} z^T\partial_p \tilde{f}(p) z &= z^T \partial_p f(p,1) z + (z^T\omega)\left(z^T\left[\frac{\partial f}{\partial w}\right]\right) \\ &= -\frac{1}{v_w}\left(z - \frac{s}{2}p\right)^T \partial_p^2 v \left(z - \frac{s}{2}p\right) + \frac{s^2}{4}(2+\epsilon) - (v^T f)^2 + (-1-\epsilon)(v^T f)(v^T\omega) \\ &\leq \frac{s^2}{4}(2+\epsilon) - (v^T f)^2 + (-1-\epsilon)(v^T f)(v^T\omega) \end{aligned}$$

Re-substituting $s = z^T(f+\omega)$ and re-arranging terms, we obtain inequality (14), with $w = p^T\omega = 1$. When $w \neq 1$, we need only note that $\partial_p \tilde{f}$ is homogeneous of degree -1, to get inequality (14) again. *QED*

Proof of Lemma 3.3: If z is not collinear with $\bar{p}$, we may write it as $a\bar{p} + bz'$, where $b \neq 0$, and z' is a unit vector orthogonal to p. Since $\bar{p}^T\hat{f} \equiv 1$, we have $\mathrm{Var}(z^T\hat{f}) = \mathrm{Var}(a\bar{p}^T\hat{f} + bz'^T\hat{f}) = b^2\mathrm{Var}(z'^T\hat{f}) \neq 0$. This proves (i). To prove (ii), we note that θ certainly exists if we confine ourselves to z that are unit vectors orthogonal to $\bar{p}$. For any other z, we may decompose it as we did above, and we have $\mathrm{Var}(z^T\hat{f}) = b^2\mathrm{Var}(z'^T\hat{f})$, and

$\mathrm{Var}(z^T\hat{\omega}) = b^2\mathrm{Var}(z'^T\hat{\omega})$. The inequality follows. Part (iii) can proved in a similar way.

. *QED*

Proof of Theorem 4.1: The economy $\mathcal{E}(\bar{t}, b)$, $(\bar{t}, b) \in (0, T) \times \mathcal{B}$ has the equilibrium price $\bar{p}(\bar{t}, b)$, which is not equal to $\bar{p}$. So we may write $\bar{p}(\bar{t}, b) = \bar{p} + k\bar{t}z$, where z is a unit vector and k is a positive number. We define the functions g_a and G from $[0, \bar{t}]$ to R by

$$g_a(t) = z^T f_a(\bar{p} + ktz, (\bar{p} + ktz)^T(\omega_a + \gamma_a tb)) - z^T f_a(\bar{p}, \bar{p}^T\omega_a) \text{ and}$$

$$G(t) = z^T F'(\bar{p} + ktz) - z^T F(\bar{p})$$

where F' is the market demand of the economy $\mathcal{E}(t, b)$ at the price $\bar{p} + ktz$, so G is just the integral of g_a over $\mathcal{A}$. Crucially, $G(\bar{t}) = z^T(\bar{t}b)$, precisely the object which we are trying to show is negative. Differentiating g_a and dropping the subscript a for convenience we obtain

$$g'(t) = kz^T\partial_p fz + z^T\left[\frac{\partial f}{\partial w}\right]\left(\gamma\bar{p}^T b + kz^T\omega + 2tk\gamma z^T b\right).$$

Note also that

$$\begin{aligned}
\bar{p}^T b &= \frac{1}{\bar{t}}\left\{\bar{p}^T(\bar{\omega} + \bar{t}b) - p^T\bar{\omega}\right\} \\
&= \frac{1}{\bar{t}}\left\{(\bar{p} + k\bar{t}z)^T(\bar{\omega} + \bar{t}b) - k\bar{t}z(\bar{\omega} + \bar{t}b) - \bar{p}^T\bar{\omega}\right\} \\
&= -kz^T(\bar{\omega} + \bar{t}b).
\end{aligned}$$

This means that

$$g'(t) = k\left\{z^T\partial_p fz + \left[z^T\frac{\partial f}{\partial w}\right](z^T\omega) - \gamma\left[z^T\frac{\partial f}{\partial w}\right](z^T\bar{\omega}) - \gamma\bar{t}\left[z^T\frac{\partial f}{\partial w}\right](z^T b) + 2\gamma t\left[z^T\frac{\partial f}{\partial w}\right](z^T b)\right\}.$$

When $\bar{t}$ is small, we can essentially ignore the last two terms. Therefore, we conclude that if the expression

$$J = \int_{\mathcal{A}}\left\{z^T\partial_p f_a(\bar{p}, \bar{p}^T\omega_a)z + \left[z^T\frac{\partial f_a}{\partial w}(\bar{p}, \bar{p}^T\omega_a)\right](z^T\omega_a) - \gamma_a\left[z^T\frac{\partial f_a}{\partial w}(\bar{p}, \bar{p}^T\omega_a)\right](z^T\bar{\omega})\right\}d\mu < 0$$

for all unit vectors z, then by the regularity of $\bar{p}$, there exists a t'' (applicable to any b in $\mathcal{B}$) such that provided $\bar{t}$ is less than t'', we have $G'(t) < 0$ for all t in the interval $[0, \bar{t}]$. The integral J (or rather its integrand) we have to estimate bears a remarkable resemblance to the expression for $z^T \partial_p \tilde{f}(p) z$ in the proof of Proposition 3.1. We can essentially repeat its arguments, having $s = z^T(f + \omega - \gamma\bar{\omega})$ instead; then following the same arguments as those which follow the statement of Proposition 3.1, we obtain the expression

$$(-2+\bar{\epsilon})\int_{\mathcal{A}} \frac{(z^T \tilde{f}_a(\bar{p}))^2}{\bar{p}^T \omega_a} d\mu - 2\bar{\epsilon}\int_{\mathcal{A}} \frac{(z^T \tilde{f}_a(\bar{p}))[z^T(\omega_a - \gamma_a\bar{\omega})]}{\bar{p}^T \omega_a} d\mu + (2+\bar{\epsilon})\int_{A} \frac{[z^T(\omega_a - \gamma_a\bar{\omega})^2]}{\bar{p}^T \omega_a} d\mu$$

which we require to be negative. Note that this is expression is identical to the right hand side of (16), except that $z^T\omega_a$ has be replaced with $z^T(\omega_a - \gamma_a\bar{\omega})$. We can then modify this expression, as we did with (16) to obtain

$$(-2+\bar{\epsilon})\mathrm{Var}(z^T\hat{f}) - 2\bar{\epsilon}\mathrm{Cov}(z^T\hat{f}, z^T(\hat{\omega} - \eta\bar{\omega})) + (2+\bar{\epsilon})\ \mathrm{Var}(z^T(\hat{\omega} - \eta\bar{\omega})) < 0.$$

This is identical to the expression in (17), except that $\hat{\omega}$ has been replaced with $\hat{\omega} - \eta\bar{\omega}$. Expanding this expression gives us the condition we need. *QED*

Proof of Corollary 4.4: Provided $\bar{\epsilon} < 0$, we have $L(z) + M(z) \leq Ax^2 + Bxy + Cy^2$, where

$x = \sqrt{\mathrm{Var}(z^T\hat{f})}$,

$y = z^T\bar{\omega}$,

$A = \bar{\epsilon} - 2 - 2\bar{\epsilon}K_2 + (\bar{\epsilon} + 2)\theta$,

$B = -2\bar{\epsilon}L_1 + 2(2 + \bar{\epsilon})L_2$ and

$C = (2 + \epsilon)\,Var\eta + (\bar{\epsilon} - 2)$.

The quadratic equation in ϵ in the Corollary is obtained by setting $B^2 = 4AC$. It is easy to check that if $\epsilon = -2$, $B^2 < 4AC$ provided $L_1^2/4 + K_2 < 1$. $A = 0$ when

$$\epsilon = \frac{-2(\theta - 1)}{[(\theta - 1) + 2(1 - K)]}$$

and $C = 0$ when

$$\epsilon = 2\left[\frac{1 - Var\eta}{1 + Var\eta}\right].$$

Since $\theta > 1$, the minimum of these two terms, ϵ' must be negative. Between -2 and ϵ', $4AC$ is a positive and decreasing function. On the other hand B^2 is always non-negative, so by the intermediate value theorem, there exists ϵ^* in $(-2, \epsilon']$ with $B^2 = 4AC$. Because B^2 and $4AC$ are both quadratic functions of ϵ, there is only one such ϵ^*. Provided $\bar{\epsilon} < \epsilon^*$, we have $A < 0$, $C < 0$ and $B^2 < 4AC$ so $Ax^2 + Bxy + Cy^2 < 0$ unless $x = y = 0$. But $x = y = 0$ cannot happen unless $z = 0$: if $y = 0$, x is orthogonal to $\bar{\omega}$, which means that it is not collinear with p, and therefore cannot be zero by Lemma 3.3. *QED*

Proof of Proposition 5.1: We first assume that v is convex in p and show that μ_x is convex. Let p and p' be supporting prices of x/s_1 and x/s_2 respectively, where s_1 and s_2 are two positive numbers. Then we have

$$\begin{aligned} \alpha\mu_x(s_1) + (1-\alpha)\mu_x(s_2) &= \alpha v(p,1) + (1-\alpha)v(p',1) \\ &\geq v(\alpha p + (1-\alpha)p', 1) \\ &\geq \mu_x(\alpha s_1 + (1-\alpha)s_2). \end{aligned}$$

The first inequality follows from the convexity of v with respect to prices; the second inequality follows from the fact that the bundle $x/[\alpha s_1 + (1-\alpha)s_2]$ is valued at 1 when the price is $\alpha p + (1-\alpha)p'$.

Now we assume that μ_x is convex for all x and show that v is convex in prices. Let p and p' be two prices and let $x = f(\alpha p + (1-\alpha)p', 1)$. By the convexity of μ_x, $u(x) = \mu_x(1) \leq \alpha\mu_x(p^T x) + (1-\alpha)\mu_x(p'^T x)$ since $(\alpha p + (1-\alpha)p')^T x = 1$. Therefore,

$$\begin{aligned} v(\alpha p + (1-\alpha)p', 1) &= \mu_x(1) \\ &\leq \alpha\mu_x(p^T x) + (1-\alpha)\mu_x(p'^T x) \\ &= \alpha u\left(\frac{x}{p^T x}\right) + (1-\alpha)u\left(\frac{x}{p'^T x}\right) \\ &\leq \alpha v(p,1) + (1-\alpha)v(p',1) \end{aligned}$$

where the last inequality follows from the definition of v.

To show the equivalence of (ii) and (iii), we need only check that

$$\mu_x''(1) = x^T \partial^2 u(x) x + 2\partial u(x).$$

This is positive if and only if $g_x(1) = \psi(x) \leq 2$. *QED*

Proof of Theorem 5.4: Without loss of generality, we can assume that $x^{*T}\partial_x u(x^*) = 1$, where $x^* = f(p^*, 1)$. [If this is not satisfied, multiply u with a number - this leaves both the preference and ψ unchanged.] Transform u to $\tilde{u} = h \circ u$, where h is an increasing function. It is easy to check that the formula (18) when applied to $\tilde{u}$ (we denote it as $\tilde{\psi}$) is given by

$$\tilde{\psi}(x) = -\frac{h''(u(x))}{h'(u(x))}(x^T \partial_x u(x)) - \frac{x^T \partial_x^2 u(x) x}{x^T \partial_x u(x)}.$$

If we can find h with $h''(u(x^*))/h'(u(x^*)) = M' - 2$, where $M > M' > \psi(f(p^*, 1))$ then

$$\begin{aligned} \tilde{\psi}(x^*) &= -\frac{h''(u(x^*))}{h'(u(x^*))} - \frac{x^{*T} \partial_x^2 u(x^*) x^*}{x^{*T} \partial_x u(x^*)} \\ &< -(M'-2) + M' = 2. \end{aligned}$$

By Lemma 5.3, $\tilde{v} = h \circ v$, the indirect utility generated by $\tilde{u}$ is convex in prices in a neighborhood of p^*. Note that $v_w(p^*, 1) = x^{*T} \partial_x u(x^*) = 1$, and that $v_{ww}(p^*, 1) \leq 0$, the latter following from the concavity of u (by Lemma 5.3 again). Therefore, the elasticity coefficient corresponding to $\tilde{v}$

$$\begin{aligned} \tilde{\epsilon}(p^*, 1) &= \frac{h''(v(p^*, 1))}{h'(v(p^*, 1)} v_w(p^*, 1) + \frac{v_{ww}(p^*, 1)}{v_w(p^*, 1)} \\ &\leq M' - 2 < M - 2. \end{aligned}$$

For any number r, a function h with $h''/h' = r$ can easily be found: simply choose $h(t) = e^{rt}$ if r is positive and $h(t) = -e^{rt}$ if r is negative. *QED*

Proof of Lemma 5.5: As in the proof of Theorem 5.4, we assume without loss of generality, that $x^{*T} \partial_x u(x^*) = 1$. Subsuming the π_i into u_i, we can write u as $u(x) = \sum u_i(x_i)$. This also leaves the function B unchanged. Suppose also that

$$k < -\frac{x_i^* u_i(x_i^*)}{u_i(x_i^*)} < K,$$

for $1 \leq i \leq l$. Given that $B(x^*) < M$, we can certainly choose K and k such that $K - k < M$. It is straightforward to check that $\psi(x^*) < K$. If we choose an increasing function h with $h''/h' = k$, then formula (18) when applied to $\tilde{u} = h \circ u$ (denote this by $\tilde{\psi}$) satisfies

$$\tilde{\psi}(x^*) = -\frac{h''(u^*)}{h'(u^*)} + \psi(x^*) < K - k < M.$$

Now we need only show that $\tilde{u}$ is concave. We have

$$z^T \partial_x^2 \tilde{u}(x^*) z = h'(u(x^*)) \left[\frac{h''(u(x^*))}{h'(u(x^*))} [z^T \partial_x u(x^*)]^2 + z^T \partial_x^2 u(x^*) z \right]. \tag{24}$$

If $z^T \partial_x u(x^*) = 0$, then, by strong quasi-concavity $z^T \partial_x^2 u(x^*) z < 0$, so $z^T \partial_x^2 \tilde{u}(x^*) z < 0$. Otherwise, without loss of generality we assume $z^T \partial_x u(x^*) = 1$. In that case, by a standard formula (see Footnote 2),

$$\begin{aligned}\max z^T \partial_x^2 u(x^*) z &= \frac{1}{(\partial_x u(x^*))^T (\partial_x^2 u(x^*))^{-1} (\partial_x u(x^*))} \\ &= \frac{1}{\sum_{i=1}^{l} u_i'(x_i^*)^2 / u_i''(x_i^*)}.\end{aligned}$$

It is not difficult to check that this expression is less than $-k$. Therefore, when $z^T \partial_x u(x^*) = 1$, equation (24) tells us that

$$z^T \partial_x^2 \tilde{u}(x^*) z < h'(u(x^*)) \left[\frac{h''(u(x^*))}{h'(u(x^*))} - k \right] = 0.$$

Therefore, we have $z^T \partial_x^2 \tilde{u}(x^*) z < 0$ for all $z \neq 0$, which guarantees the local concavity of $\tilde{u}$.

. *QED*

Footnotes:

1. Least convex functions are the convex analog to the better known least concave functions. The definition is in Section 2.

2. For a symmetric and positive definite $l \times l$ matrix A, a column vector $b \in R^l$, and a number r, $\min_{x^T b = r} x^T A x = r^2 / b^T A^{-1} b$. Note that in our case, $r = 1/2$, $A = \partial_p^2 v$ and $b = f$.

3. The results cited pertain to the existence of least concave functions. It is quite obvious they are also applicable here since a function is concave if and only if its negative is convex.

4. For a similar result with a different proof in the direct utility case, see Kannai (1989).

REFERENCES

GRANDMONT, J. M. (1992): "Transformations of the commodity space, behavioral heterogeneity, and the aggregation problem," *Journal of Economic Theory*, 57, 1-35.

HILDENBRAND, W. (1983): "On the Law of Demand," *Econometrica*, 51, 997-1019.

— (1994): *Market Demand*. Princeton: Princeton University Press.

HILDENBRAND, W. AND A. KIRMAN (1988): *Equilibrium Analysis*. Amsterdam: North Holland.

KANNAI, Y. (1977): "Concavifiability and constructions of concave utility functions," *Journal of Mathematical Economics*, 4, 1-56.

KANNAI, Y. (1989): "A Characterisation of Monotone Individual Demand Functions," *Journal of Mathematical Economics*, 18, 87-94.

MANTEL, R. (1976): "Homothetic Preferences, and Community Excess Demand Functions," *Journal of Economic Theory*, 12, 197-201.

MAS-COLELL, A. (1985): *The Theory of General Economic Equilibrium: A differentiable approach*. Cambridge: Cambridge University Press.

— (1991): "On the Uniqueness of Equilibrium Once Again," in BARNETT, W., B. CORNET, C. D'ASPREMONT, J. GABSZEWICZ, AND A. MAS-COLELL (eds.),*Equilibrium Theory and Applications*, Cambridge: Cambridge University Press.

MAS-COLELL, A., M. D. WHINSTON, AND J. R. GREEN (1995): *Microeconomic Theory.* Oxford: Oxford University Press.

MILLERON, J. C. (1974): "Unicite et stabilite de l'equilibre en economie de distribution" *Seminaire d'Econometrie Roy-Malinvaud.*

MITJUSCHIN, L. G. AND W. M. POLTEROVICH (1978): "Criteria for monotonicity of demand functions," *Ekonomika i Matematicheskie Metody*, 14, 122-128 (in Russian).

MUELLBAUER, J. (1975): "Aggregation, Income Distribution and Consumer Demand," *Review of Economic Studies*, XLII (4), 525-543.

— (1976): "Community Preferences and the Representative Consumer," *Econometrica*, 44, 979-999.

QUAH, J. K.-H. (1997): "The Law of Demand When Income is Price Dependent," *Econometrica*, 65, 1421-1442.

SHAFER, W. AND H. SONNENSCHEIN (1982): "Market Demand and Excess Demand functions," in ARROW, K. J., and M. D. INTRILIGATOR (eds.), *Handbook of Mathematical Economics,* Vol. II, New York: North Holland.

SONNENSCHEIN, H. (1973): "Do Walras' identity and continuity characterize the class of community excess demand functions?" *Journal of Economic Theory*, 6, 345-354.

VARIAN, H. (1992): *Microeconomic Analysis.* New York: W. W. Norton.