ON THE EVOLUTIONARY EMERGENCE OF OPTIMISM

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Abstract

Successful individuals were frequently found to be overly optimistic. These findings are puzzling, as one could expect that realists would perform best in the long run. We show, however, that in a large class of strategic interactions of either cooperation or competition, the equilibrium payoffs of optimists may be higher than those of realists. This is because the very fact of being optimistic changes the game, and drives the adversary to change her equilibrium behavior, possibly to the benefit of the optimist. Suppose, then, that a population consists initially of individuals with various perceptional tendencies – pessimists and optimists to various extents, as well as of realists. Individuals meet in pairs to interact, and more successful tendencies proliferate faster. We show that as time goes by, some moderate degree of optimism will take over, and outnumber all other tendencies.

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1 Introduction

There is by now a considerable body of evidence, that in many kinds of circumstances, successful individuals are overly optimistic regarding the return to their own investment or effort. Taylor and Brown (1988) found that most mentally healthy people have somewhat unrealistically positive self-views, while the less mentally healthy perceive themselves more accurately. In peer reviews, for example, non-depressed individuals' self-ratings were considerably more favorable than those given to them by others (Lewinsohn, Mischel, Chaplin and Barton 1980). Non-depressed individuals exhibited an illusion of control in a dice-throwing experiment (Fleming and Darley 1986), and most individuals believe that their driving ability is above average (Svenson 1981). In the Economic arena, excess entry of new businesses that fail within several years is common in the US, and recent experimental work (Camerer and Lovallo 1999) suggests that this phenomenon may be due to entrepreneurs being overconfident regarding their own ability in comparison with other entrepreneurs.

These observations naturally lead the following puzzle: Given that human beings are technically capable of evaluating their stakes in a cold-blooded fashion, how come that consistent biases in estimation survive evolutionary pressures? If success pays off in wealth, which translates to more supportable descendants and more imitators, one might have guessed that individuals whose estimations are not biased would perform best on average, and thus would outnumber the biased types in the long run.

What we aim at offering here is an insight based on the consequences of optimism in strategic interactions. These consequences are twofold. On the down side, optimistic individuals do not react optimally to their environment as shaped by the objective setting and the actions of others. But on the up side, the very fact of being optimistic changes the environment, as it drives others to change their behavior. When biased individuals misperceive their payoffs in the interaction, the resulting equilibrium behavior and true payoffs are different than those that would appear if the interacting parties were both realistic. Given a specific way the other party perceives her utility, it may very well "pay" to misperceive one's own utility, so that the resulting equilibrium would be better according to the true payoffs. For moderate levels of optimism, this beneficial effect outweighs the losses due to the biases in judgement, in a large class of interactions.

Cautious optimism may pay off in games with strategic substitutes as well as in games with strategic complements. This is because overestimation of the return to one's effort translates to a more aggressive reaction curve - i.e. a higher level of investment for any given level of investment by the adversary. When the two parties are competing with one another and the reaction curves are downward-sloping, aggressiveness would "intimidate" the adversary and induce her to invest less, possibly to the benefit of the aggressor. And when the two parties are cooperating, with upward-sloping reaction curves, aggressiveness would encourage the other party to invest more, potentially benefiting the aggressor in this case as well. A "hard-wired" optimism implies that the individual is committed to react more strongly, and this commitment may thus yield strategic advantages.

The benefits of commitment to non-optimal reactions in strategic interactions is well established in the literature – in remuneration schemes for the managers of firms in oligopoly (Fershtman and Judd 1987), or in domestic subsidy schemes for exporters that compete (imperfectly) with foreign firms in international markets (Brander and Spencer 1985, Eaton and Grossman 1986). In a similar vein, the positive return to being fair (Güth and Yaari 1992, Huck and Oechssler 1998), socially minded (Fershtman and Weiss 1997, 1998), altruistic (Bester and Güth 1998), spiteful (Possajennikov 1999), concerned with relative success (Koçkesen, Ok and Sethi 1998), or overconfident in financial investments (Kyle and Wang 1997), may render a population with such non-selfish or irrational individuals evolutionary stable, i.e. immune to the appearance of few selfish or rational "mutants".[1]

In the current work, we take these ideas one step further. We show under what conditions optimism would evolve in a full-fledged, dynamic evolutionary context. Our main result is as follows. Let there be any initial distribution[2] of types in the population – realistic, pessimistic to various degrees or optimistic to various degrees. If individuals are frequently and repeatedly matched in pairs at random, get the (true) payoffs in the equilibrium of the interaction (as they perceive it according to their types), and proliferate in a speed which is monotonically increasing in their true payoffs, then in the long run the cautious optimists would reign the population. The exact degree of

[1] Related ideas appear already in the works of Frank (1987, 1988). The indirect evolutionary approach, where the preferences rather than the strategies are the subject of evolutionary pressures, is employed also by Dekel and Scotchmer (1999), Rogers (1994), Robson (1996a,b), Waldman (1994) and Vega-Redondo (1997). See also further references in the sequel.

[2] with an adequate support -see more below

optimism that would wipe out all other types depends, of course, on the true payoffs of the game. But for a large family of two-player games, a positive, moderate degree of optimism would emerge.[3]

To prove our result, we consider an artificial, preliminary game, in which two players can commit simultaneously to their degree of optimism, knowing that consequently they will be playing the equilibrium of the game defined by the types to which they committed, but get the true payoffs that result from their behavior. If the reaction curves in this preliminary game (where, again, the choice variables are optimism levels) has a slope with absolute value between 0 and 1, then the preliminary game will have a unique Nash equilibrium, to which a cob-web process of myopic best responses would converge. Under some mild assumptions, this cob-web process can also be interpreted as a process of iterated elimination of strictly dominated strategies, and strategies that are thus eliminated are wiped off by the replicator dynamics or any other regular, payoff monotonic dynamics. This last result is known for a distribution of strategies or types with a finite support (Samuelson and Zhang 1992), and is proved in the appendix for general distributions when the payoff function is continuous.

The paper is organized as follows. Section 2 brings a simple example that exhibits most of the properties of the model. Section 3 discusses several interpretational issues. Section 4 continues with a family of quadratic payoff functions, where the resulting optimism level has a closed form. Section 5 explores general conditions on the payoff functions which are sufficient for our results to hold. Section 6 concludes. Proofs are relegated to the appendix.

2 A Simple Example

Consider a bilateral Cournot game, with no production costs, and where the price is some positive constant α minus total quantity. Profits as a function of produced quantities $q_1; q_2$ are therefore

$$\begin{aligned} \pi_1(q_1;q_2) &= (\alpha - q_1 - q_2)q_1 \\ \pi_2(q_1;q_2) &= (\alpha - q_1 - q_2)q_2 \end{aligned} \tag{2.1}$$

[3] In a similar dynamic setting, Huck, Kirchsteiger and Oechssler (1997) examine the emergence of an endowment effect – an excess valuation of one's own endowment in bargaining. They show that the proportion of realists with no such effect will shrink to zero with time, as will types with a very high endowment effect. However, unlike in our case, the dynamics is not shown to converge in the long run.

Think of this game more abstractly, though: interpret $q_1; q_2$ as levels of effort or investment, with increasing marginal costs to effort. The interacting parties are competing with each other – the higher the effort exerted by the other party, the lower is the return to every unit of one's own effort.

Individuals often meet and interact in this fashion with each other. However, not all individuals conceive the interaction in the same way: Pessimist types underestimate the parameter α; optimists overestimate it, and only realists assess it correctly. If players 1 and 2 in the interaction believe α to be α_1 and α_2; respectively, they conceive their utility functions to be

$$\begin{aligned} U_1(q_1; q_2) &= (\alpha_1 - q_1 - q_2)q_1 \\ U_2(q_1; q_2) &= (\alpha_2 - q_1 - q_2)q_2 \end{aligned} \tag{2.2}$$

Let us assume that the players play the unique Nash equilibrium of the game with these utility functions[4]

$$\begin{aligned} q_1^* &= \frac{2\alpha_1 - \alpha_2}{3} \\ q_2^* &= \frac{2\alpha_2 - \alpha_1}{3} \end{aligned} \tag{2.3}$$

This is either because the assessments $\alpha_1; \alpha_2$ are "written on the players' foreheads" and are thus mutually recognized immediately, or alternatively because the players approach pretty quickly this equilibrium behavior after several rounds in which they each play their best response to the other's previous action (or some average of the other's previous actions).

The true payoffs to these players from this behavior are

$$\begin{aligned} f_1(\alpha_1; \alpha_2) &= \left(\alpha - \frac{\alpha_1 + \alpha_2}{3}\right)\frac{2\alpha_1 - \alpha_2}{3} \\ f_2(\alpha_1; \alpha_2) &= \left(\alpha - \frac{\alpha_1 + \alpha_2}{3}\right)\frac{2\alpha_2 - \alpha_1}{3} \end{aligned} \tag{2.4}$$

Imagine now that these payoffs translate into fitness terms, and reflect fertility rates: the number of descendants of each individual is monotonically increasing in the payoffs of her interactions during her lifetime, and the descendants inherit the individual's assessment of α: The way this assessment

[4] given that α_1 and α_2 are not too far below or above α, so that the Nash equilibrium is interior. In section 4 we show that the analysis below is valid also when corner Nash equilibria are possible.

is inherited can be interpreted as either purely biological, or as a process in which parents transmit their "optimism level" to their children via education, or as a process where more successful "approaches to life" are imitated more often and become more popular in the next generation.

Suppose that the population of individuals is large. At each moment of time, individuals are matched in pairs to interact. This matching occurs at random, according to the current distribution of optimism levels in the population. Reproduction takes place instantaneously, where reproduction rates of individuals are increasing in their success in the interactions, as explained above. In short, the distribution of optimism levels in the population evolves according to a regular, payoff monotonic dynamics.

Where would this process lead the population? What optimism levels would perform best and survive in the long run? To analyze this problem, consider first a two-player game with payoffs as specified in (2.4) above. This is a preliminary, artificial game, in which the players have to commit simultaneously to their assessments α_1, α_2 of the parameter α, knowing that consequently they will be bound to play the Nash equilibrium strategies (2.3) of the perceived game (2.2), but where the true payoffs are indicated in the original game (2.1).

In this preliminary game of commitment, where the choice variables are the assessments α_1, α_2; the best-reply reaction functions of the players are

$$\begin{aligned} \alpha_1(\alpha_2) &= \frac{6\alpha - \alpha_2}{4} \\ \alpha_2(\alpha_1) &= \frac{6\alpha - \alpha_1}{4} \end{aligned} \tag{2.5}$$

whose intersection is the unique Nash equilibrium

$$\alpha_1^* = \alpha_2^* = \frac{6}{5}\alpha. \tag{2.6}$$

In other words, at equilibrium the players will commit to overestimate α by 20%, and therefore to behave as if the return to each unit of their effort is larger by $\frac{1}{5}\alpha$ than it actually is[5].

[5] In the preliminary game, the players effectively commit to their reaction functions in the effort game. At equilibrium, they would not choose their reaction function differently had they been Stackelberg leaders in the effort game – each chooses her reaction function so as to intersect that of her opponent in the most prefered point for her along the opponent's reaction function.

Notice further that the slope of the reaction functions (2.5) is ¡ $\frac{1}{4}$, smaller than 1 in absolute value. Consequently, the cob-web process of myopic best responses converges to the Nash equilibrium (2.6). If we start this process with both players choosing $\text{®}_1 = \text{®}_2 = 0$; the process can be immediately read as iterative elimination of dominated strategies: If player 1 commits to a non-negative assessed ®; it is better for player 2 to commit to $\frac{3}{2}$® than to commit any higher assessment. Understanding this, committing to $\frac{9}{8}$® is better for player 1 than committing to any lower assessment. But with this in mind, player 2 is better o¤ committing to $\frac{39}{32}$® than to any higher value, and so forth.

It turns out that when we come back to our original population dynamics, assessments that do not survive the iterative elimination of dominated strategies are wiped o¤ by regular, payo¤ monotonic dynamics[6]. That is, as time goes by, the cautious optimists who overestimate ® by 20% will gradually take over the population. Whatever is the initial distribution of assessments in the population, it will converge in distribution to the point mass[7] $\frac{6}{5}$® – provided only that the support of the initial population is an interval that contains $\frac{6}{5}$®:

3 Discussion

Before we continue with more general results, we sidestep to discuss several interpretational issues of the model.

3.1 Learning about ® with time

The above example raises immediately the following question: How come that optimists do not come to realize that they overestimate ® once they observe their true payo¤s (2.4)? Two possible interpretations are in order here.

Suppose, …rstly, that the nature of interaction changes over time: the value of ® ‡uctuates at random, so that past realizations of ® provide little or no information regarding its current value. Individuals are technically

[6] Samuelson and Zang (1992) proved this result for distributions with a …nite support. We prove it for general distributions and continuous payo¤ functions in the appendix.

[7] In other words, for any " > 0; the proportion of types in the population who overestimate ® by 20% ¡ " to 20% + " will eventually become larger than 1 ¡ ":

able to recognize this current value at once. However, our result implies that those who systematically tend to moderately overestimate the true value, whatever is its realization, are those who prosper and proliferate.

A second potential interpretation is that individuals do face uncertainty regarding the true current value of α. To illustrate, suppose there are two possible values of α – a high value α_h and a low value α_ℓ. The realizations are independent across periods, where the high value α_h appears with probability p. Let p_t be the realized frequency α_h by period t. By the strong law of large numbers, p_t converges to p almost surely.

Suppose that individuals start with an initial, prior distribution in mind regarding the probability of α_h; a distribution that contains the true p in its support. Individuals form their posteriors using Bayes rule. Realistic individuals use the observed p_t in this updating process, and therefore their posteriors will converge in distribution to a point mass on p almost surely. Since the utility functions (2.1) are quadratic, these individuals act in period $t + 1$ as they would in case there was no uncertainty, and the value of α is the average α_t of their posterior in that period. Thus, α_t will converge to $p\alpha_h + (1 - p)\alpha_\ell$ almost surely.

In contrast, optimistic types do not use p_t when they update their beliefs: In some of the periods where α_ℓ appears and thus the return to their effort is low, they discard the observation, attributing it to exceptional, non-systematic bad circumstances, which render the outcome irrelevant for updating. As they keep discarding the same proportion of the α_ℓ realizations along the periods, their posterior will converge almost surely to a point mass on some $p' > p$. Therefore, they will almost surely tend to act as if they believed that the parameter in their utility function (2.1) is $\alpha' = p'\alpha_h + (1 - p')\alpha_\ell$, which is higher than the true average $\alpha = p\alpha_h + (1 - p)\alpha_\ell$. The higher the optimism level, the higher the proportion of discarded α_ℓ realizations, and hence the higher α' becomes.

Similarly, pessimistic types consistently discard some proportion of the α_h realizations, assuming that the high return to their effort in those periods was due to exceptionally favorable circumstances, which should exclude those observations from the sample. Consequently, they will almost surely tend to behave as if they believed that the parameter in their utility function (2.1) is lower than the true average $\alpha = p\alpha_h + (1 - p)\alpha_\ell$.

3.2 Optimism regarding what?

In our model, pessimists believe that the return to their e¤ort is low, and indeed at equilibrium they end up investing relatively little e¤ort in interactions. Couldn't the pessimists be regarded, though, as believing that their utility from leisure is high, and therefore as optimistic about their "return from leisure"? More broadly, isn't it the case that the limited resources of an individual are always distributed between several lanes, and hence relative optimism regarding the return in some of them immediately implies relative pessimism regarding the return in the others?

This view is certainly a legitimate one. Still, there is clearly room for a distinction between leisure wilfully invested in recreation or resting, and time waisted in idleness by default rather than by choice. It is not that uncommon for individuals to spend part of their time "not doing anything", without consciously preferring whatever they are in fact doing over potential alternatives. It is then meaningless to discuss what they perceive to be their return from idleness: If the individual is unaware of her revealed preference for idleness, it is absurd to assume that this preference is further distorted unconsciously in her mind.

In our model, individuals choose the amount of e¤ort to be invested in interactions, implicitly implying that unused individual resources are left idle. With the above interpretation of idleness, it is indeed sensible to use unambiguously the term "optimism" to mean overestimation of the return to one's e¤ort: Such an overestimation does not imply an (absolute or relative) underestimation or pessimism regarding the return to idleness – a phrase which is simply a contradiction in terms.

3.3 Larger families of perceived utility functions

We have thus far considered a one-dimensional family of distortions in the perception of one's utility function - those that result from di¤erent evaluations of the parameter ®: What would happen if we were to consider every possible distortion of the utility function?

Dekel, Ely and Yilankaya (1998) show that if the distribution of types (i.e. perceived utilities) ever reaches a stable state in which all the individuals take the same action, then this action must be e¢cient. In our Cournot example, this action is the cartel quantity $\frac{®}{4}$: A population of types who all take another action $q \neq \frac{®}{4}$ is not immune to an invasion of mutants who perceive their utility di¤erently. To see why, assume for simplicity that all the individuals are of the same type t: Consider a mutant t^0 that di¤ers from t only by the fact that it perceives its utility from forming a cartel to be extremely high (much more than the true payo¤). Thus, when a mutant t^0 meets an incumbent t; she will play q, just as an incumbent t would, so that both t and t^0 get the same true payo¤ when matched with t. But when a mutant t^0 meets another mutant t^0 they form a cartel, and fare better. Hence, such mutants would not tend to disappear.

This result is static in nature. In particular, it does not predict if and when a payo¤ monotonic dynamics would ever converge. Moreover, it does not preclude stable states with a polymorphic distribution of types. But in any case, it does imply that our result need not hold with an extended family of utility distortions, and it leads to ask what biases are relevant for consideration and in which contexts.

For example, it might be relevant to consider a type that, for some peculiar reason, likes extremely to take part in a cartel ($\frac{®}{4}; \frac{®}{4}$) for one speci…c value ®: However, it looks much less natural to consider a type that extremely likes to form a cartel ($\frac{®}{4}; \frac{®}{4}$) for di¤erent values of ®; i.e. if and when the environment changes. This is because the cartel strategies are the result of a logical, non-trivial calculation, that necessarily goes through a computation using the true payo¤ function. Therefore, it is rather arti…cial to consider somebody who is conscious of her true payo¤s function, and is yet committed to distort it in her mind.

In contrast, our optimistic types who unconsciously but systematically overestimate the objective parameter ® (or its average) do not go through the correct computation and then ignore it. Rather, they tend to misperceive the environment before any computation is carried out. In this respect, the family of types that we consider seems to be a natural abstraction of relevant and frequently observed human biases.

It will certainly be of interest to model more families of biases, and confront the theoretical predictions with empirical …ndings. The results of Dekel et al. (1997) show that the triumph of biased types is not a trivial result in such an exercise, and thus open the door for a challenging process of modeling relevant biases.

4 A family of games with quadratic payoffs

In this section we provide a full analysis of our argument for a class of two-player games with quadratic payoffs. The next section will explore how the argument can be extended to an even larger class of games.

The payoff functions we consider here take the form

$$\Pi_i(q_i, q_j) = (\alpha - bq_j - cq_i)q_i \qquad \text{for } i = 1, 2 \text{ and } j = 2 - i \tag{4.1}$$

where the parameters satisfy $\alpha > 0$, $-1 \leq b \leq 1$ and $c \geq 1$. As in section 2 above, the actions $q_1, q_2 \geq 0$ are to be interpreted as the amount of effort the players invest in the interaction. When b is positive, the game exhibits strategic substitutes – the higher the investment of player j, the lower the return to each unit of effort of player i. With a negative b, the game has strategic complements – there is a positive, linear correlation of the return to one's effort with the effort level of the other player. The marginal cost of effort is increasing with rate $2c$. In section 2 we had $b = 1$, $c = 1$.

Types of individuals are distinguished by the way they perceive the parameter α. Thus, type α_i of player i perceives his utility function to be

$$U_i(q_i, q_j) = (\alpha_i - bq_j - cq_i)q_i \tag{4.2}$$

A type is optimistic (realistic, pessimistic) regarding the return to his input of effort when α_i is bigger (equal, smaller) than α. Types α_i belong to an interval $[\underline{\alpha}, \overline{\alpha}]$. The reaction function of type α_i is thus

$$q_i(q_j) = \max\left\{\frac{\alpha_i - bq_j}{2c}, 0\right\} \tag{4.3}$$

As explained in section 2, we assume that individuals recognize immediately[8] each other's types α_i, α_j, and play the Nash equilibrium actions that result

[8] or alternatively after a short learning phase which we do not model here. Our results on the emergence of moderate optimism would not be valid if individuals were unable to identify the types with which they interact: If all the individuals plan against the same, true distribution of types in the population, realistic types fare best by definition. Hence, as Dekel, Ely and Yilankaya (1998) and Ok and Vega-Redondo (1999) show (see also an example in Possajennikov 1999), in any stable distribution of such a population dynamics, all the individuals play a (true) Nash equilibrium strategy.

from (4.3). These are

$$q_i^* = \begin{cases} 0 & 2c\alpha_i < b\alpha_j \\ \frac{\alpha_i}{2c} & 2c\alpha_j < b\alpha_i \\ \frac{2c\alpha_i - b\alpha_j}{4c^2 - b^2} & \text{otherwise} \end{cases} \tag{4.4}$$

The last case of (4.4) corresponds to the usual circumstance of an interior Nash equilibrium, where both players invest positive amounts of effort. When b is positive (strategic substitutes), however, corner equilibria may appear, where one of the players exerts zero effort. These are reflected in the first two cases of (4.4).

When an α_i type meets an α_j type, what is the true payoff of i? Substituting the equilibrium actions (4.4) into the payoff function (4.1) yields the result:

$$f_i(\alpha_i;\alpha_j) = \begin{cases} 0 & 2c\alpha_i < b\alpha_j \\ \left(\alpha - \frac{\alpha_i}{2}\right)\frac{\alpha_i}{2c} & 2c\alpha_j < b\alpha_i \\ \left(\alpha - \frac{(2c^2-b^2)\alpha_i + bc\alpha_j}{4c^2-b^2}\right)\frac{2c\alpha_i - b\alpha_j}{4c^2-b^2} & \text{otherwise} \end{cases} \tag{4.5}$$

Assume now that there is a large population of individuals. The distribution of types in the population is expressed by a probability measure Q on the set $[\underline{\alpha};\bar{\alpha}]$ of possible types, endowed with the Borel σ-field. We denote

$$f_i(\alpha_i;Q) \equiv E_Q f_i(\alpha_i;\cdot) \tag{4.6}$$

the average true payoff of an individual of type α_i when matched with an individual j drawn at random from the population. When both i and j are drawn at random from the population, the average true payoff to i is

$$f_i(Q;Q) \equiv E_Q f_i(\cdot;Q) \tag{4.7}$$

As a first step, we assume now that the instantaneous rate with which the probability distribution Q evolves is determined by the difference between (4.6) and (4.7):

$$\sigma_i(\alpha_i;Q) = f_i(\alpha_i;Q) - f_i(Q;Q) \tag{4.8}$$

That is, if $Q(t)$ is the probability distribution at time t; then for every (Borel) set of types $A \subseteq [\underline{\alpha};\bar{\alpha}]$

$$Q'(t)(A) = \int_A \sigma_i(\alpha_i;Q(t))dQ(t)(\alpha_i) = \tag{4.9}$$

$$\int_A f_i(\alpha_i;Q(t))dQ(t)(\alpha_i) - Q(t)(A)f_i(Q(t);Q(t))$$

This differential equation defines the replicator dynamics, which was introduced by Taylor and Jonker (1978) for distributions with a finite support, and by Oechssler and Riedel (1998) for general distributions. Qualitatively, if the average performance of a set of types A is better than the average population performance $f_-(Q;Q)$; the probability of A will increase, at the expense of other sets of types whose average performance is worse than the overall average. The specific quantitative form (4.9) is implied if we assume that the absolute growth rate of an α_--type individual is its true average payoff (4.6), possibly plus some constant[9].

More generally, we are also interested in other population dynamics in which higher payoff implies higher growth-rate. Such more general dynamics may be appropriate when the reproduction process of types is not purely biological, but rather relies on education or imitation (see e.g. Weibull 1995, section 4.4). Technically, the payoff differential $\sigma_-(\alpha_-;Q)$ in (4.9) should then be replaced by another growth-rate function $g:[\underline{\alpha};\bar{\alpha}]\times\Delta([\underline{\alpha};\bar{\alpha}])\to\mathbb{R}$ which

[9] Indeed, let $N(t)$ be a positive, finite Borel measure on the set of types $[\underline{\alpha};\bar{\alpha}]$; where $N(t)(A)$ expresses the "number" of individuals with types in the set A: Suppose that the growth rate of the A population at time t is

$$N'(t)(A)=\int_A [k(t)+f_-(\alpha_-;Q(t))]\,dN(t)(\alpha_-)$$

for some constant $k(t)$: Dividing by $N(t)([\underline{\alpha};\bar{\alpha}])$ (and supressing, for brevity, the t-s in the expressions) yields

$$\frac{N'(A)}{N([\underline{\alpha};\bar{\alpha}])}=\int_A [k+f_-(\alpha_-;Q)]\,dQ(\alpha_-)=kQ(A)+\int_A f_-(\alpha_-;Q)dQ(\alpha_-)$$

Hence, differentiating $N(A)=N([\underline{\alpha};\bar{\alpha}])Q(A)$ and isolating $Q'(A)$ gives

$$\begin{aligned}Q'(A)&=\frac{[N([\underline{\alpha};\bar{\alpha}])Q(A)]'-N'([\underline{\alpha};\bar{\alpha}])Q(A)}{N([\underline{\alpha};\bar{\alpha}])}=\\&\frac{N'(A)}{N([\underline{\alpha};\bar{\alpha}])}-\frac{N'([\underline{\alpha};\bar{\alpha}])}{N([\underline{\alpha};\bar{\alpha}])}Q(A)=\\ \left[kQ(A)+\int_A f_-(\alpha_-;Q)dQ(\alpha_-)\right]&-\left[k+\int_{\underline{\alpha}}^{\bar{\alpha}} f_-(\alpha_-;Q)dQ(\alpha_-)\right]Q(A)=\\&\int_A f_-(\alpha_-;Q)dQ(\alpha_-)-Q(A)f_-(Q;Q)\end{aligned}$$

which is indeed (4.9).

always preserves the population size –

$$\int_{\underline{\alpha}}^{\bar{\alpha}} g(\cdot; Q)\, dQ = 0; \tag{4.10}$$

is payoff monotonic –

$$g(\alpha_i; Q) > g(\alpha_i'; Q) \iff f_i(\alpha_i; Q) > f_i(\alpha_i'; Q) \tag{4.11}$$

for every $Q \in \Delta([\underline{\alpha}; \bar{\alpha}])$; and is regular – bounded and K-Lipschitz continuous for some finite constant K

$$\sup_{\alpha \in [\underline{\alpha};\bar{\alpha}];\, Q \in \Delta([\underline{\alpha};\bar{\alpha}])} |g(\alpha; Q)| < \infty \tag{4.12}$$

$$\sup_{\alpha \in [\underline{\alpha};\bar{\alpha}]} |g(\alpha; Q_1) - g(\alpha; Q_2)| < K \|Q_1 - Q_2\|; \qquad Q_1; Q_2 \in \Delta([\underline{\alpha}; \bar{\alpha}])$$

where $\|Q\| = \sup_{|f| \leq 1} \left| \int_{\underline{\alpha}}^{\bar{\alpha}} f\, dQ \right|$ is the variational norm on signed measures. Oechssler and Riedel (1998, lemma 1) proved that the regularity of g guarantees that the mapping $Q \to \int_{\underline{\alpha}}^{\bar{\alpha}} g(\cdot; Q)\, dQ$ is bounded and Lipschitz continuous in the variational norm, which implies that the differential equation in the space of distributions $\Delta([\underline{\alpha}; \bar{\alpha}])$ defined by

$$Q'(t)(A) = \int_A g(\cdot; Q(t))\, dQ(t); \qquad A \subseteq [\underline{\alpha}; \bar{\alpha}] \tag{4.13}$$

has a unique solution for any initial distribution $Q(0)$:

We are now interested in the following question. Suppose that in the dawn of time, the distribution of types in the population was $Q(0)$: Suppose further that both pessimistic types and optimistic types are well represented in this initial population, by assuming that the support $[\underline{\alpha}; \bar{\alpha}]$ of $Q(0)$ contains α: As time goes by, how will the distribution of types evolve with a regular, payoff monotonic dynamics?

To answer this question, we consider, as in section 2, the artificial two-player game where the players have to choose and commit to their types simultaneously, and their payoffs are as derived in (4.5) from the induced Nash equilibrium of the effort game. The best response for player i is then

$$\alpha_i^*(\alpha_j) = \max \left\{ \frac{2c(4c^2 - b^2)\alpha - b^3\alpha_j}{4c(2c^2 - b^2)}; 0 \right\} \tag{4.14}$$

whenever maximizing (4.5) given α_j implies that both players exert positive levels of effort at that Nash equilibrium, so that the last case of (4.5) is applicable. When b is positive, the best reply of player i is not unique for big enough and for small enough values of α_j; and for these values (4.14) is one of the best replies.

In the unique symmetric[10] Nash equilibrium of this game, both players commit to

$$\alpha^* = \frac{8c^3 - 2cb^2}{b^3 - 4cb^2 + 8c^3}\alpha \tag{4.15}$$

which is strictly greater than α for $b \neq 0$ (when $b = 0$; one can see from (4.1) that individuals face a decision problem, not a genuine game with strategic interaction).

Suppose then that the optimistic type (4.15) is in the support $[\underline{\alpha}, \overline{\alpha}]$ of the initial population $Q(0)$ (and therefore also in the support of all the distributions $Q(t)$, as our dynamics preserve the support of the initial distribution). Assume now that at some point t in time, the proportion of types smaller than α' in the population becomes very small, for some $\alpha' < \alpha^*$. If b is positive (negative), then by (4.5) for every $\tilde{\alpha}_i$ greater (smaller) than $\alpha_i^*(\alpha')$ it is the case that $f_i(\alpha_i^*(\alpha'); \alpha_j) \geq f_i(\tilde{\alpha}_i; \alpha_j)$ for $\alpha_j > \alpha'$; and $f_i(\alpha_i^*(\alpha'); \alpha^*)$ is strictly greater than $f_i(\tilde{\alpha}_i; \alpha^*)$: Hence on average

$$f_i(\alpha_i^*(\alpha'); Q(t)) > f_i(\tilde{\alpha}_i; Q(t)) \tag{4.16}$$

and therefore in a payoff monotonic dynamics, $\alpha_i^*(\alpha')$ will reproduce more quickly than any larger (smaller) type $\tilde{\alpha}_i$: By a similar argument, if the proportion of types larger than α'' in the population becomes very small, then $\alpha_i^*(\alpha'')$ will reproduce more quickly than any smaller (larger) type.

Starting with $\alpha' = \underline{\alpha}$; it then follows that the proportion of types bigger (smaller) than $\alpha_i^*(\underline{\alpha})$ will eventually become very small, as they will replicate more slowly than $\alpha_i^*(\underline{\alpha})$: Repeating the argument implies that the proportion of types smaller than $\alpha_i^*(\alpha_i^*(\underline{\alpha}))$ will eventually become very small, because they will replicate more slowly than $\alpha_i^*(\alpha_i^*(\underline{\alpha}))$: Since the slope of (4.14) is between 0 and 1 in absolute value, the sequence

$$\alpha_i^*(\underline{\alpha});\ \alpha_i^*(\alpha_i^*(\underline{\alpha}));\ \alpha_i^*(\alpha_i^*(\alpha_i^*(\underline{\alpha})));\ \ldots \tag{4.17}$$

[10] When b is positive, there are also Nash equilibria where i commits to a high α_i and j commits to $\alpha_j = 0$: These equilibria are irrelevent for our analysis, which is only auxiliary to that of the population dynamics.

converges to α^*. The same applies if we start the process with $\alpha'' = \bar{\alpha}$. Hence α^* will eventually replicate much faster than all other types, and "take over" the population. This is the intuition for our main theorem below, which we prove formally in the appendix.

Theorem 1 Let there be a large population of individuals with an initial distribution $Q(0)$ of types α_i, characterized by their perceived utilities (4.2). Individuals in this population are continuously matched in pairs, play the Nash equilibrium (4.2) and thus get the payoff according to (4.5). Let (α^*, α^*) be the unique Nash equilibrium (4.15) of the symmetric two-player game where each player chooses a type and gets the payoff (4.5). Then any regular, payoff monotonic dynamics on the types in this population will converge in distribution to a unit mass on α^*, provided only that the support of the initial distribution $Q(0)$ is an interval $[\underline{\alpha}, \bar{\alpha}]$ that contains α^*.

Thus, in the limit population, individuals are optimistic ($\alpha^* > \alpha$), and at the Nash equilibrium (4.4) they exert more effort then realistic players would exert at the Nash equilibrium of (4.1). In the case of strategic substitutes ($b > 0$), these optimists are worse off than realists, since the optimists compete more fiercely. In the case of strategic complements ($b < 0$), the optimists fare better than the realists, because here the bigger investment of optimists leads to enhanced cooperation.

5 More General Interactions

In the previous section, the relation between the true payoff function $\pi_i(q_i, q_j)$ and the utility $U_i(q_i, q_j)$ as perceived by some type had the structure

$$U_i(q_i, q_j) = \pi_i(q_i, q_j) + \tau_i q_i \qquad (5.1)$$

where the type was characterized by the parameter τ_i. Optimistic types had positive τ_i, and thus overestimated the return to each unit of their effort, while the converse was the case for pessimistic types. In this formulation, the types ranged in an interval $[\underline{\tau}, \bar{\tau}]$, where $\underline{\tau} < 0$ and $\bar{\tau} > 0$. We will now try to explore how general can the true payoff function be taken to be, in order for our results to be sustained when types are characterized by their parameter τ_i in (5.1).

Observe that our argument in the previous section hinged on the following ingredients:

1. The slope of the reaction function of each type τ_i is between 0 and 1 in absolute value. Consequently, for every pair of types $(\tau_i; \tau_j)$ with utilities as in (5.1), the game has a unique Nash equilibrium $(q_i(\tau_i; \tau_j); q_j(\tau_i; \tau_j))$. We denote

$$f_i(\tau_i; \tau_j) = \pi_i(q_i(\tau_i; \tau_j); q_j(\tau_i; \tau_j)) \tag{5.2}$$

   the true payoff of i at this equilibrium.

2. When the reaction function of each type τ_i are downward sloping (strategic substitutes), the level curves of π_i are concave, and $\pi_i(q_i; \cdot)$ is decreasing. Similarly, when the reaction function of each type τ_i are upward sloping (strategic complements), the level curves of π_i are convex, and $\pi_i(q_i; \cdot)$ is increasing. As a result, in either case the level curves of $f_i(\cdot; \tau_j)$ are single peaked. (To see this, suppose that $\tau_i(\tau_j)$ maximizes $f_i(\cdot; \tau_j)$: Augmenting τ_i beyond $\tau_i(\tau_j)$ yields a more aggressive reaction function of i; and thus the Nash equilibrium $(q_i(\cdot; \tau_j); q_j(\cdot; \tau_j))$ moves rightward along the reaction function of j; and π_i is weakly lowered. In like fashion, diminishing τ_i from $\tau_i(\tau_j)$ yields a less aggressive reaction function of i; so the Nash equilibrium $(q_i(\cdot; \tau_j); q_j(\cdot; \tau_j))$ moves leftward along the reaction function of j; and π_i gets weakly lower.) Therefore, in the preliminary, artificial game of commitment to types with payoffs (5.2), the cob-web processes of myopic best replies

$$\tau_j(\underline{\tau}); \; \tau_i(\tau_j(\underline{\tau})); \; \tau_j(\tau_i(\tau_j(\underline{\tau}))); \ldots$$

   and

$$\tau_j(\overline{\tau}); \; \tau_i(\tau_j(\overline{\tau})); \; \tau_j(\tau_i(\tau_j(\overline{\tau}))); \ldots$$

   can be read as iterative elimination of strongly dominated strategies.

3. In the preliminary game of commitment to types, the reaction functions $\tau_i(\tau_j)$ have a slope with absolute value between 0 and 1. (As a result, that game had a unique Nash equilibrium, which is the limit of the cob-web process of myopic best replies.)

It turns out that these three properties hold if some expressions that involve the …rst, second and third derivatives of $¦_i$ obey certain strict inequalities. This yields the following theorem, which we prove in the appendix.

Theorem 2 In $C^3(R^2_+)$ there is an open set[11] of payo¤ functions $¦_i(¢; ¢)$ for which the following holds true: Let there be an initial population of individuals with di¤erent types $¿_i$; characterized by perceived utilities (5.1). The support of the initial distribution is an interval $[\underline{¿}; \overline{¿}]$: Individuals in this population are continuously matched in pairs, play the Nash equilibrium of (5.1) and thus get the payo¤ according to (5.2). Let $(¿^¤; ¿^¤)$ be the unique Nash equilibrium of the symmetric two-player game where each player chooses a type and gets the payo¤ (5.2). Then the replicator dynamics on the types in this population will converge in distribution to a unit mass on $¿^¤$, provided only that $¿^¤$ is in the support of the initial distribution of types.

6 Conclusion

We have shown how the pressures of explicit, dynamic evolutionary processes select for moderate optimism rather than for realism, when …tness is gained through interactions of either competition or cooperation in a large class of games. According to this insight, the phenomenon of overcon…dence and unrealistic self-esteem of individuals who face objective circumstances (or a competitive market) may just be due to a bias that "pays" well in many kinds of strategic settings.

Clearly, the way humans evaluate their capabilities has evolved along the generations via con‡icts with both natural hazards and strategic adversaries or parties. The premises of our model are therefore far from being all-encompassing. And on the implications side, our heterogenous society is far from exhibiting the single approach or type, as in the long run prediction of the model. Thus, our modest aim was to point at one possible source for the apparently unreasonable optimism which is so frequently observed in

[11] $C^3(R^2_+)$ is the space of thrice continuously di¤erentiable functions $¦ : R^2_+ ! R$; with the minimal topology in which $¦_n$ converges to $¦$ i¤ $¦_n ¡ ¦$ and each of its …rst, second and third derivatives converge to zero uniformly on compact sets on R^2_+.

the process of decision making. Searching for competing and complementing evolutionary insights for this and similar behavioral puzzles may prove to be both challenging and rewarding.

7 Appendix

Theorem 1 is a corollary to the following theorem, which may be of separate interest also on its own. It was proved by Samuelson and Zhang (1992) for the particular case of games with finitely many strategies. For notational simplicity, we state it for symmetric two-player games with a compact one-dimensional strategy space, but the method of proof works just as well for more general compact strategy spaces and for asymmetric games.

Theorem 3 Let $[\underline{\alpha},\bar{\alpha}] \subset \mathbb{R}$ be a space of strategies, $f : [\underline{\alpha},\bar{\alpha}] \times [\underline{\alpha},\bar{\alpha}] \to \mathbb{R}$ a continuous payoff function of a symmetric two-player game, and $g : [\underline{\alpha},\bar{\alpha}] \times \Delta([\underline{\alpha},\bar{\alpha}]) \to \mathbb{R}$ a regular, payoff monotonic growth-rate function, satisfying (4.10)-(4.12). Let $Q(t)$ be the population dynamics defined by the differential equation (4.13) with an initial distribution of strategies $Q(0)$. Suppose that $D \subset [\underline{\alpha},\bar{\alpha}]$ is the subset of strategies which do not survive the process of iterated elimination of strictly dominated strategies. Then the strategies in D are asymptotically eliminated from the population: Every iteratively dominated startegy $d \in D$ has an open neighborhood W_d for which $\lim_{t\to\infty} Q(t)(W_d) = 0$. In particular, if there is only one non-eliminated strategy $u \in [\underline{\alpha},\bar{\alpha}] \setminus D$, then $Q(t)$ converges in distribution to the unit mass δ_u.

Proof of Theorem 3: Let D_n be the set of strategies that do not survive n or less rounds of iterated elimination of dominated strategies, so $D = \cup_{n=0}^{\infty} D_n$. Denote also by $U_n = [\underline{\alpha},\bar{\alpha}] \setminus D_n$ the set of strategies that do survive n rounds of iterated elimination of dominated strategies. We prove by induction on n that U_n is compact, and every eliminated strategy $d \in D_n$ has an open neighborhood W_d for which $\lim_{t\to\infty} Q(t)(W_d) = 0$.

Since $D_0 = \emptyset$ and $U_0 = [\underline{\alpha},\bar{\alpha}]$, the claim holds for $n = 0$. Suppose the claim holds for $n < k$. Let $d \in D_k$ be round-k dominated by the strategy

$x \in [\underline{\alpha}; \bar{\alpha}]$; that is for every $y \in U_{k-1}$

$$f(x;y) - f(d;y) > 0.$$

By the induction hypothesis U_{k-1} is compact, and since f is continuous, the function $[f(x;y) - f(d;y)]$ attains its minimum on U_{k-1}:

$$\min_{y \in U_{k-1}} [f(x;y) - f(d;y)] = \rho > 0.$$

Notice further that $\min_{y \in U_{k-1}} [f(x;y) - f(d;y)]$ is a continuous function of x and d.[12] It therefore follows that

$$D_k = D_{k-1} \cup \bigcup_{x \in U_{k-1}} \bigcup_{\ell=1}^{\infty} \left\{ d \in U_{k-1} : \min_{y \in U_{k-1}} [f(x;y) - f(d;y)] > \frac{1}{\ell} \right\};$$

is an open set, so $U_k = [\underline{\alpha}; \bar{\alpha}] \setminus D_k$ is compact, as required.

The continuity of $\min_{y \in U_{k-1}} [f(x;y) - f(d;y)]$ also implies that there are open neighborhoods $V_x \ni x$ and $W_d \ni d$ such that for every $x' \in V_x$, $d' \in W_d$

$$\min_{y \in U_{k-1}} [f(x';y) - f(d';y)] > \frac{\rho}{2}.$$

[12] To see this, suppose that $x_m \to x$, $d_m \to d$, and for each $m \geq 1$, $y^*_m \in \arg\min_{y \in U_{k-1}} [f(x_m;y) - f(d_m;y)]$. Let y^* be any accumulation point of the sequence y^*_m, and y^*_r some sub-sequence of y^*_m which converges to y^*. So for any $y' \in U_{k-1}$,

$$f(x_r;y^*_r) - f(d_r;y^*_r) \leq f(x_r;y') - f(d_r;y').$$

Taking the limits of both sides as $r \to \infty$ gives

$$f(x;y^*) - f(d;y^*) \leq f(x;y') - f(d;y').$$

Thus $y^* \in \arg\min_{y \in U_{k-1}} [f(x;y) - f(d;y)]$, and

$$\min_{y \in U_{k-1}} [f(x_r;y) - f(d_r;y)] \to_{r \to \infty} \min_{y \in U_{k-1}} [f(x;y) - f(d;y)].$$

Decomposing the sequence $\{y^*_m\}$ to subsequences $\{y^*_r\}$ in such a way that each subsequence converges to an accumulation point y^*, gives

$$\min_{y \in U_{k-1}} [f(x_m;y) - f(d_m;y)] \to_{r \to \infty} \min_{y \in U_{k-1}} [f(x;y) - f(d;y)]$$

as required.

Now, since f is continuous, the set

$$B = \{y \in [\alpha,\beta] : f(x;y) - f(d;y) \leq 0\}$$

is a compact subset of the open set D_{k-1}. Hence B is a proper subset of D_{k-1}, (except when $k = 1$, in which case $D_{k-1} = \emptyset$ and $B = \emptyset$). So

$$s \equiv \begin{cases} 1 & k = 1 \\ \sup_{y \in D_{k-1}} [f(x;y) - f(d;y)] & k > 1 \end{cases}$$

is positive, and

$$C = \left\{ y \in D_{k-1} : f(x;y) - f(d;y) \leq \frac{s}{2} \right\}$$

is a compact subset of D_{k-1}. For $k = 1$ the set C is empty, and $Q(t)(C) = 0$. For $k > 1$, every $y \in C$ has an open neighborhood $W_y \ni y$ such that $\lim_{t\to\infty} Q(t)(W_y) = 0$, by the induction hypothesis. Since $C \subseteq \bigcup_{y\in C} W_y$ and C is compact, there exist $y_1,\dots,y_m \in C$ such that $C \subseteq \bigcup_{i=1}^m W_{y_i}$. Therefore, $Q(t)(C) \leq \sum_{i=1}^m Q(t)(W_{y_i})$ and hence $\lim_{t\to\infty} Q(t)(C) = 0$.

Denote

$$\varepsilon = \min \left\{ \frac{\rho}{2}, \frac{s}{2}, \frac{1}{2} \right\}.$$

Since f is continuous on the compact domain $[\alpha,\beta]$ and hence also bounded, there exists a time T such that for $t \geq T$, $x' \in V_x$ and $d' \in W_d$

$$\int_C [f(x';\cdot) - f(d';\cdot)]\, dQ(t) > -\frac{\varepsilon}{4},$$

and furthermore $Q(t)(C) < \varepsilon$. In addition, we have $\inf_{y\in[\alpha,\beta]\setminus C} [f(x';y) - f(d';y)] \geq \varepsilon$. Altogether

$$f(x';Q(t)) - f(d';Q(t)) = \int_{[\alpha,\beta]} [f(x';\cdot) - f(d';\cdot)]\, dQ(t) =$$

$$\int_C [f(x';\cdot) - f(d';\cdot)]\, dQ(t) + \int_{[\alpha,\beta]\setminus C} [f(x';\cdot) - f(d';\cdot)]\, dQ(t) >$$

$$-\frac{\varepsilon}{4} + (1-\varepsilon)\varepsilon \geq -\frac{\varepsilon}{4} + (1-\frac{1}{2})\varepsilon = \frac{\varepsilon}{4}$$

By the payoff monotonicity of the growth-rate function g, there exists a $\delta > 0$ such that for every $t \geq T$, $x' \in V_x$ and $d' \in W_d$

$$g(x'; Q(t)) - g(d'; Q(t)) > \delta.$$

A fortiori, for $t \geq T$

$$\frac{\int_{V_x} g(\cdot; Q(t))dQ(t)}{Q(t)(V_x)} - \frac{\int_{W_d} g(\cdot; Q(t))dQ(t)}{Q(t)(W_d)} > \delta.$$

Hence, by (4.13), for $t \geq T$

$$\frac{Q'(t)(V_x)}{Q(t)(V_x)} - \frac{Q'(t)(W_d)}{Q(t)(W_d)} > \delta,$$

so that

$$\frac{Q(t)(V_x)}{Q(t)(W_d)} \geq \frac{Q(T)(V_x)}{Q(T)(W_d)} \exp[\delta(t - T)] \to_{t \to \infty} \infty.$$

Therefore, $\lim_{t \to \infty} Q(t)(W_d) = 0$, as required. ∎

Proof of Theorem 2. As explained above the statement of the theorem, we have to specify sufficient conditions on the payoff function[13] π^i under which the properties 1., 2. and 3. stated there hold.

1. Denote by $b^i(q^j)$ the best reply function of an individual of type τ^i, according to her perceived utility function U^i in (5.1). At an interior best reply,

$$0 = U^i_i(b^i(q^j); q^j) = \pi^i_i(b^i(q^j); q^j) + \tau^i \tag{A.1}$$

Differentiating with respect to q^j yields that the slope of $b^i(q^j)$ is

$$\frac{db^i}{dq^j} = -\frac{\pi^i_{ij}}{\pi^i_{ii}} \tag{A.2}$$

Requiring this slope to be in the interval (-1,0) (strategic substitutes) or in the interval (0,1) (strategic complements) for every q^j defines in each case an open subset in the function space $C^3(R^2_+)$.

[13] we now resort to using superscripts for the players' indices, while subscripts will denote partial derivatives.

2. A level curve $l(q^i)$ of $¦^i$ is implicitly de…ned by

$$¦^i(q^i; l(q^i)) = \text{const.} \tag{A.3}$$

Di¤erentiating with respect to q^i yields that the slope of the level curve is

$$\frac{dl}{dq^i}(q^i) = \text{i} \frac{¦^i_i(q^i; l(q^i))}{¦^i_j(q^i; l(q^i))} \tag{A.4}$$

For such level curves to be concave (convex), the following expression has to be negative (positive) for every q^i :

$$\begin{aligned} \frac{d^2l}{(dq^i)^2} &= \text{i} \frac{\left(¦^i_{ii} + ¦^i_{ij} \frac{dl}{dq^i}\right) ¦^i_j \;\text{i}\; ¦^i_i \left(¦^i_{ji} + ¦^i_{jj} \frac{dl}{dq^i}\right)}{(¦^i_j)^2} \\ &= \text{i} \frac{\left(¦^i_{ii} \;\text{i}\; ¦^i_{ij} \frac{¦^i_i}{¦^i_j}\right) ¦^i_j \;\text{i}\; ¦^i_i \left(¦^i_{ji} \;\text{i}\; ¦^i_{jj} \frac{¦^i_i}{¦^i_j}\right)}{(¦^i_j)^2} \end{aligned} \tag{A.5}$$

which de…nes in either case an open subset in $C^3(R^2_+)$: The further requirement that $¦^i(q_i; ¢)$ is decreasing (increasing), i.e. $¦^i_j$ negative (positive) for every q^i, de…nes in either case another open subset of $C^3(R^2_+)$:

3. In order to express the slope of the reaction function $¿^i(¿^j)$; we make the following observations. If $¿^i(¿^j)$ maximizes $f^i(¢; ¿^j)$; then the Nash equilibrium $(q^i(¿^i(¿^j); ¿^j); q^j(¿^i(¿^j); ¿^j))$ is the point on the best reply function $b^j(q^i)$ of type $¿^j$ which is on the highest (value-wise) of the level curves $l(q^i)$ of $¦^i$ that intersect $b^j(q^i)$: At an interior solution, this value-maximizing level curve of $¦^i$ would thus be tangent to the best reply function of $¿^j$; and their slopes at $q^i(¿^i(¿^j); ¿^j)$ would coincide:

$$\frac{dl}{dq^i}(q^i(¿^i(¿^j); ¿^j)) = \frac{db^j}{dq^i}(q^i(¿^i(¿^j); ¿^j)) \tag{A.6}$$

Substituting for these expressions from (A.4) and (A.2) above, we get

$$\text{i} \frac{¦^i_i(q^i(¿^i(¿^j); ¿^j); q^j(¿^i(¿^j); ¿^j))}{¦^i_j(q^i(¿^i(¿^j); ¿^j); q^j(¿^i(¿^j); ¿^j))} = \text{i} \frac{¦^j_{ji}(q^i(¿^i(¿^j); ¿^j); q^j(¿^i(¿^j); ¿^j))}{¦^j_{jj}(q^i(¿^i(¿^j); ¿^j); q^j(¿^i(¿^j); ¿^j))} \tag{A.7}$$

This equality should hold for each type τ^j for which $(q^i(\tau^i(\tau^j);\tau^j);q^j(\tau^i(\tau^j);\tau^j))$ is interior. Differentiating (A.7) with respect to τ^j and extracting $\frac{d\tau^i}{d\tau^j}(\tau^j)$ yields

$$\frac{d\tau^i}{d\tau^j} = \frac{(\pi^i_{ji}q^i_j + \pi^i_{jj}q^j_j)\pi^j_{ji} + \pi^i_j(\pi^j_{jii}q^i_j + \pi^j_{jij}q^j_j) - (\pi^i_{ii}q^i_j + \pi^i_{ij}q^j_j)\pi^j_{jj} - \pi^i_i(\pi^j_{jji}q^i_j + \pi^j_{jjj}q^j_j)}{(\pi^i_{ii}q^i_i + \pi^i_{ij}q^j_i)\pi^j_{jj} + \pi^i_i(\pi^j_{jji}q^i_i + \pi^j_{jjj}q^j_i) - (\pi^i_{ji}q^i_i + \pi^i_{jj}q^j_i)\pi^j_{ji} - \pi^i_j(\pi^j_{jii}q^i_i + \pi^j_{jij}q^j_i)} \tag{A.8}$$

It now remains to express the partial derivatives of the Nash equilibrium strategies $q^i(\tau^i;\tau^j)$ and $q^j(\tau^i;\tau^j)$ in terms of the partial derivatives of π^i and π^j of various orders. To this end, observe that at a Nash equilibrium, $q^i(\tau^i;\tau^j)$ maximizes $U^i(\cdot;q^j(\tau^i;\tau^j))$ for type τ^i. When the equilibrium is interior we thus have the first order condition

$$0 = U^i_i(q^i(\tau^i;\tau^j);q^j(\tau^i;\tau^j)) = \pi^i_i(q^i(\tau^i;\tau^j);q^j(\tau^i;\tau^j)) + \tau^i \tag{A.9}$$

Differentiating (A.9) with respect to τ^i and with respect to τ^j gives

$$\pi^i_{ii}q^i_i + \pi^i_{ij}q^j_i = -1 \tag{A.10}$$

$$\pi^i_{ii}q^i_j + \pi^i_{ij}q^j_j = 0 \tag{A.11}$$

Repeating the same procedure for U^j (or simply interchanging i and j in (A.10)-(A.11)) gives

$$\pi^j_{ji}q^i_j + \pi^j_{jj}q^j_j = -1 \tag{A.12}$$

$$\pi^j_{ji}q^i_i + \pi^j_{jj}q^j_i = 0 \tag{A.13}$$

Now, (A.10)-(A.13) can be regarded as 4 linear equations in the 4 variables $q^i_i;\ q^j_i;q^i_j$ and q^j_j. Solving for these variables, substituting in (A.8) above and rearranging finally yields

$$\frac{d\tau^i}{d\tau^j} = \frac{(\pi^i_{jj}\pi^i_{ii} - \pi^i_{ji}\pi^i_{ij})\pi^j_{ji} + \pi^i_j(\pi^j_{jij}\pi^i_{ii} - \pi^j_{jii}\pi^i_{ij}) - \pi^i_i(\pi^j_{jjj}\pi^i_{ii} - \pi^j_{jji}\pi^i_{ij})}{(\pi^i_{ii}\pi^j_{jj} - \pi^i_{ij}\pi^j_{ji})\pi^j_{jj} - \pi^i_i\pi^j_{jjj}\pi^j_{ji} - (\pi^i_{ji}\pi^i_{ii} - \pi^i_{jj}\pi^j_{ji})\pi^j_{ji} - \pi^i_j\pi^j_{jii}\pi^j_{jj}} \tag{A.14}$$

More explicitly, when the left-hand side is evaluated at τ^j; the right-hand side is evaluated at $(q^i(\tau^i(\tau^j);\tau^j);q^j(\tau^i(\tau^j);\tau^j))$. Requiring this expression to be in (-1,0) in the case of strategic substitutes (when (A.2) is negative), or in (0,1) in the case of strategic complements (when (A.2) is positive), defines in either case an open set in $C^3(R^2_+)$.

Taking the intersection of the finitely many open subsets of $C^3(\mathbb{R}^2_+)$ mentioned in 1., 2. and 3. above, separately for the cases of strategic substitutes and strategic complements, and then the union of these two intersections, yields an open set of payoff functions Π^i in $C^3(\mathbb{R}^2_+)$ that satisfy the sufficient conditions that make our argument work. This open set contains, in particular, the quadratic payoff functions considered in section 4 (for which all the third-order derivatives vanish), as well as many other payoff functions. ■

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