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Modeling Matched Job-Worker Flows

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Abstract: What can one infer about labor market flows from matched employer-employee panel data? The purpose of this paper is to sketch possible answers to this question. A general but simple labor market equilibrium model of hire and separation flows is developed in the paper. The model embodies the hypothesis that worker productivity differs across employers and that worker and employer flows reflect responses to these differences in a labor market characterized by friction. In the modeled market, each agent acts optimally taking as given the wage offer distribution and market tightness and these in turn are determined by their collective action. The existence of a labor market equilibrium is established under two different wage determination models: rent sharing and wage posting. A demonstration that market flow parameters, search and recruiting effort functions, and the equilibrium wage distribution can be estimated with matched job-work flow data is the principal contribution of the paper.

Keywords: Job Flows, Worker Flows, Labor Market Equilibrium, Wage Distribution

JEL-Code: J21, J23, J63, J64, J65, E24

1 Introduction

What can one infer about the role of job and worker flows from panel data on the labor force dynamics of individual establishments matched to worker identifiers? Preliminary answers to this question are raised in this paper. Essentially the idea proposed here is to use a synthesis of the theoretical constructs found in the recent literature on the 'flows approach' to equilibrium labor market analysis as developed in Pissarides (1990), Mortensen and Pissarides (1994), Bertola and Caballero (1994), Burdett and Mortensen (1998), and Bontemps et al. (1998) as a tool for analyzing the recently available matched employer-worker panel data described by Abowd and Kramerz (1999).

A simple framework for the analysis of the relationships between hire, layoff, and quit flows and their determinants designed to explain the evolution of establishment size is presented in the paper and set within a synthesis of the search and matching equilibrium frameworks. Following the argument of Idson and Oi (1999), the explanation for labor force size differences studied in the paper is heterogeneity in employer productivity. Heterogeneity of this form is also needed to understand the behavior of job flows as documented by Davis, Haltiwanger and Schuh (1996). An implied positive correlation between the wage paid and plant size for both a bilateral bargaining rent sharing model of wage determination as assumed by Pissarides (1990) and a dynamic monopsony model in which each employer posts a wage offer as in Burdett and Mortensen (1998) is an implication of the hypothesis.[1] Within the framework studied, more productive establishments hire at greater rates, pay better, experience lower turnover rates, and, as a consequence, grow to a larger size in both cases.

The behavioral relationships of the model, appropriately aggregated, determine the distributions of wage offers as well as market tightness and the level of employment under either wage determination hypothesis. But, market tightness, unemployment, and the wage offer distribution provide the information that agents use to make search and recruiting decisions. A proof that a rational expectations market equilibrium exists for both the rent sharing and wage posting models of wage determination is the principal theoretical result of the paper. A demonstration that the equilibrium behavior relationships and aggregates can be estimated using matched available matched job-worker flow data is the more original and interesting contribution. In sum, the goal of the paper is the development of a theoretical market equilibrium approach for the empirical analysis of matched employer-employee data.

In section 2, a dynamic theory of an establishment's labor force size is created by merging the theory of optimal worker search effort with the theory of vacancy creation. The result is a characterization of establishment labor force size, n; as a simple 'birth-death' Markov process characterized by a hiring frequency,

[1] The existence of the relationship is documented by Brown and Medoff (1989) for the U.S. Bontemps et al. (1998) and Abowd et al. (1999) demonstrate that the relationship exists in France, Blanchflower et al. (1996) provides evidence for the relationship in England, and Abraek (1998) and Bingley and Westergaard-Nielsen (1996) show that plant size effects on wages are of similar magnitude in Denmark.

h; and separation rate, d. In the special case of constant marginal product per worker both are independent of n and the invariant distribution for this process is Poisson with mean equal to h/d. The expected frequencies with which a worker receives offers and an employer receives job applicants as well as the distribution of wage offers are regarded as given by each individual agent but are determined by their collective activities. An equilibrium is a set of optimal agent actions that generate the market flow parameters and wage offer distribution that all the agents expect. Equilibrium is defined and existence proofs are provided in section 3 for both the rent sharing and wage posting cases. As noted above, the model implies that an employer's labor force size is a known stochastic process. A recursive maximum likelihood estimation strategy based on this fact is devised in section 4 of the paper. Under the strategy, search and recruiting levels for each employer type and the equilibrium distribution of wage offers are identified.

2 Labor Force Dynamics

2.1 Search Strategy and Reservation Wage

Workers choose acceptance and search strategies to maximize the expected present value of their own future income stream. Let $r \geq 0$ represent the common discount rate and let $F(w)$ represent the fraction of vacancies that offer wage w or less.. The worker's maximal value of a match with that pays wage w solves

$$rW(w) = \max_{s \geq 0} \left\{ \begin{array}{c} w + \delta(U - W) \\ + \lambda s \int \max\langle W(\tilde{w}); W(w)\rangle - W(w)] dF(\tilde{w}) - c_w(s) \end{array} \right\} \quad (1)$$

where λs is the Poisson rate at which outside offers arrive given search intensity s, $c_w(s)$ denotes the cost search effort, δ is the exogenous job destruction rate, and U represents the value of unemployment. The relationship reflects the fact that the worker only quits when an outside offer has a higher value than the current job and the fact that worker's expected present value of future income falls to U when destruction occurs. Similarly, the value of unemployment solves

$$rU = \max_{s \geq 0} \left\{ b + \lambda s \int [\max\langle W(\tilde{w}); U\rangle - U] dF(w) - c_w(s) \right\} \quad (2)$$

where b represents income when unemployed. Given that the offer distribution has a bounded support, the existence and uniqueness of a increasing value function $W(w)$ and an associated value of unemployment U is consequence of the fact that both can be represented as fixed points of contraction maps. (See Mortensen and Pissarides (1999b).)

One can rewrite the Bellman equation (1) as

$$rW(w) = \max_{s \geq 0} \left(w - c(s) + \lambda s \int_{w}^{\overline{w}} [W(\tilde{w}) - W(w)] dF(\tilde{w}) - \delta[W(w) - U] \right)$$

where $\bar{w}$ denotes the upper support of the wage offer distribution. Consequently,

$$W'(w) = \frac{1}{r+\delta+\lambda s(w)[1-F(w)]}$$

by the envelope theorem, where

$$s(w) = \arg\max_{s \ge 0}\left\{\lambda s \int_w^{\bar{w}} [W(\tilde{w}) - W(w)]dF(\tilde{w}) - c_w(s)\right\}$$

is the optimal choice of search effort.

Integration by parts implies

$$\begin{aligned} s(w) &= \max_{s \ge 0}\left\{\lambda s \int_w^{\bar{w}} W'(\tilde{w})[1-F(\tilde{w})]d\tilde{w} - c_w(s)\right\} \\ &= \max_{s \ge 0}\left\{\lambda s \int_w^{\bar{w}} \frac{[1-F(\tilde{w})]d\tilde{w}}{r+\delta+\lambda s(\tilde{w})[1-F(\tilde{w})]} - c_w(s)\right\}. \end{aligned} \quad (3)$$

Hence, optimal search effort is the unique particular solution to the ordinary differential equation

$$c_w''(s(w))s'(w) = \frac{-\lambda[1-F(w)]}{r+\delta+\lambda s(w)[1-F(w)]}$$

consistent with the boundary condition $c_w'(s(\bar{w})) = 0$. As an implication, $s(w)$ is decreasing and differentiable if $F(w)$ is continuous.

When unemployed, the typical worker accepts an offer only if it is no smaller than the reservation wage. As the reservation wage equates the value of employment to the value of unemployed search, denoted by U, acceptance requires that $w \ge R$ where the reservation wage, R, solves $U = W(R)$. Given that a wage offer arrives at rate λs and that wage is a random draw from the distribution of offers F, U solves the Bellman equation

$$\begin{aligned} rU &= \max_{s \ge 0}\left\{b - c_w(s) + \lambda s \int [\max\langle W(\tilde{w}), U\rangle - U]dF(w)\right\} \\ & \max_{s \ge 0}\left\{b - c_w(s) + \lambda s \int_R^{\bar{w}} [W(\tilde{w}) - W(R)]dF(w)\right\} \end{aligned}$$

where b is the unemployment benefit forgone when employed and $W(R) = U$ is the definition of the reservation wage. Hence, equations (1) and (3) imply that the search intensity of an unemployed worker is the same as that of a worker employed at the reservation wage provided that the reservation wage is the unemployment benefit,

$$s_0 = s(R) \quad (4)$$

which in turn implies that the reservation wage is equal to the unemployment benefit if the lowest wage offered is the unemployment benefit, i.e.,

$$R = b \text{ if } \underline{w} = b. \quad (5)$$

2.2 Recruiting Strategy and Wage Offer

In this subsection, a theory of recruiting behavior along the lines developed by Pissarides (1990) and Bertola and Caballero (1994) is used to derive an optimal establishment hire flow. Each employer recruits new workers by posting job vacancies, a number denoted by v.[2]

Employers differ in productive efficiency. Let p represent the productivity of a job-worker match and $\mu(p)$ represent the measure of employers with productivity equal to or less than p. All workers are perfect substitutes and all have the same outside option to employment characterized by a common reservation wage $R = b$ as established above. For the moment, let the pair $(p; w)$ characterize and employer where w represents her wager offer.

Let $\eta_0 s_0$ and $\eta_1 s(w)$ denote the frequencies with which unemployed workers and employed worker respectively apply for a vacancy. As all workers are identical, all wages offered in the market by participating employer must be acceptable to unemployed workers. However, because an employed worker accepts only when offered a higher paying job, the rate at which a vacancy offering wage w is matched with some worker is $\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)$ where $G(w)$ represents the fraction of workers earn w or less, i.e., the wage distribution. Each employer posts the number of vacancies that maximizes the expected present value of the firm's future cash flow attributable to recruiting activity. Formally, the number of vacancies posted solves

$$\pi(p; w) = \max_{v \geq 0} \left\{ v[\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)]J(p; w) - c_f(v) \right\} \quad (6)$$

where $J(p; w)$ denotes the expected present value of an employed worker given the employer's productivity p and wage w and $c_f(v)$ is an increasing, twice differentiable, and strictly convex cost of recruiting function.

Because an employed worker quits when a higher outside offer is located, the employer's value of a continuing match solves the Bellman equation

$$rJ(p; w) = p - w - \delta J(p) - \lambda s(w)[1 - F(w)]J(p).$$

Equivalently,

$$J(p; w) = \frac{p - w}{r + \delta + \lambda s(w)[1 - F(w)]}. \quad (7)$$

After substitution for the value of a filled job from (7), equation (6) implies that the optimal number of vacancies posted by an employer of productivity p is

$$v(p; w) = \arg\max_{v \geq 0} \left\{ \left(\frac{[\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)]}{r + \delta + \lambda s(w)[1 - F(w)]} \right) (p - w)v - c_f(v) \right\}. \quad (8)$$

[2] Vacancies here are better measured by an indicator of recruiting effort rather than a count of job openings; e.g., the volume of want ads will do.

The solution to the sufficient first order condition

$$c_f'(v(p;w)) = \frac{[\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)](p-w)}{r+\delta+\lambda s(w)[1-F(w)]}. \tag{9}$$

In a labor market characterized by friction, the wage rate offered by any employer is essentially indeterminate.[3] Following Pissarides (1990) among others, one might suppose that the wage is the outcome of a simple bilateral bargaining problem. For example, suppose that when an unemployed worker and employer meet, the worker makes a wage demand with probability β and the employer accepts or rejects. Conversely, the employer sets the wage with complementary probability $1-\beta$ and the worker accepts or rejects. If negotiation ends after the first round, the expected outcome yields the simple linear rent sharing rule

$$w(p) = \beta p + (1-\beta)R \tag{10}$$

because each agent will demand all the rent when he or she has the power to set the wage. Note that the recruiting effort under the rent sharing wage rule $v(p;w(p))$; increases with employer productivity p from (9) given that the employer's expected share of match rent β is a fraction.

Instead of sharing the rent, Burdett and Mortensen (1998) assume that employers post the wage they pay all employees once and for all and that each worker only have the power to accept or reject. Because the employer sets the wage as well as vacancies to maximize profit from recruiting activities as defined in equation (6), the wage posting rule is defined by

$$w(p) = \arg\max_{w \geq R} \left\{ \left(\frac{[\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)]}{r+\delta+\lambda s(w)[1-F(w)]} \right) (p-w) \right\}. \tag{11}$$

Because the first term, the product of the acceptance probability divided by the rate at which match rent is discounted, is increasing in the wage offer, the wage rule is well defined. By implication, recruiting effort $v(p;w(p))$ also increases with employer productivity from equation (9) and the envelope theorem under wage posting.

Finally, it is useful to note that the productivity of the marginal employer, the smallest wage offered and the reservation wage all equal the unemployment benefit b under either wage setting rule if employers exist with productivity slightly higher than the benefit. Formally,

$$\underline{w} = \underline{p} = R = b \text{ if } \mu(b+\varepsilon) > 0 \text{ for all small } \varepsilon > 0. \tag{12}$$

This result is implied by (8), (5), and the wage rule in each case. Namely, because $v(p;w) > (=)\,0$ as $p > (=)\,w$ in both cases, $v(\underline{p};w(\underline{p})) = 0 \Rightarrow \underline{p} = w(\underline{p}) = \underline{w} = R = b$ under both the rent sharing rules, equation (10) and the wage posting rule (11). The qualification is needed in the rent sharing case because $\underline{p} = p_0$ if $\mu(p_0) = 0$ for some $p_0 > b$ and, therefore, $\underline{w} = w(p_0) > R > b$. Although $\underline{w} = R = b$ always in the case of wage posting, the qualification it is still necessary for $\underline{p} = b$.

[3] See Mortensen and Pissarides (1999b) for a more extensive development of this point.

2.3 Labor Force Size

That employer productivity, wage and labor size are all positively correlated is a well documented stylized fact. In this section we demonstrate that positive associations between any two of these three variable is implied under either wage determination hypothesis.

Because in any sufficiently small time interval either a worker is hired, one separates or neither event occurs, labor force size is a special Markov chain on the integers known as a birth-death process. Formally, an employer's labor force size $\{n(t)\}$ evolves as a stochastic process characterized by the 'birth' frequency

$$h(p,w) = \left[\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)\right]v(p,w) \tag{13}$$

and a 'death' rate per employed worker

$$d(w) = \delta + \lambda s(w)[1 - F(w)]. \tag{14}$$

It is well known the distribution of the future value of $n(t)$ converges to a Poisson with mean

$$E\{n\} = n(p) = \frac{h(p,w(p))}{d(w(p))} = \frac{\left[\eta_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)\right]v(p,w(p))}{\delta + \lambda s(w)[1 - F(w(p))]}. \tag{15}$$

at $t \to \infty$ for any initial values (See, Feller (1968)). Because the acceptance probability increases with w, the quit rate decreases with w; and recruiting effort $v(p,w(p))$ increase with p under either wage determination hypothesis imply that the labor force size $n(w)$ increases with both productivity p and the wage $w(p)$. Of course, the positive association is induced by the fact that more productive employers both pay a higher wage and employ more workers in the long run.

For future reference, it useful to derive the laws of motion for the probability that the labor force size will take on any particular integer value in the future. The argument is based on the fact that either a worker is hired, a worker separates, or neither happens in any sufficiently short period of time. Any other change in size would involve some combination of independent hire and separation events, the probability of which vanishes relative to the length of the period as the period length itself converges to zero.

Indeed, the probability of a transition from a labor force size $n-1$ to one of n in any sufficiently short time interval of length Δ is approximate equal to $h\Delta$; the probability of transition from size $n+1$ to n is approximately $d(n+1)\Delta$ and the probability of making a transition from any other labor force size to n is of order Δ^2 or higher. Hence, the probability that the labor force is of size n at date $t+\Delta$, denoted as $p_n(t+\Delta)$ conditional on the distribution over n at time t satisfies

$$\begin{aligned} p_n(t+\Delta) &= p_{n-1}(t)h\Delta + p_{n+1}(t)d(n+1)\Delta \\ &\quad + [1 - h\Delta - dn\Delta]p_n(t) + o(\Delta) \end{aligned}$$

for all $n \geq 1$ and

$$p_0(t+\Delta) = p_1(t)d\Delta + [1 - h\Delta]p_0(t) + o(\Delta)$$

where $o(\Delta)/\Delta \to 0$ as $\Delta \to 0$: In continuous time the distribution evolves according to the laws of motion

$$\dot{p}_n = \lim_{\Delta \to 0} \left\{ \frac{p_n(t+\Delta) - p_n(t)}{\Delta} \right\} = hp_{n-1} + (n+1)dp_{n+1} - (h+nd)p_n \quad (16)$$

for $n \geq 1$ and

$$\dot{p}_0 = dp_1 - hp_0: \quad (17)$$

The invariant distribution is the steady state solution to these differential equations. The steady state conditions imply

$$p_1 = \left(\frac{h}{d} \right) p_0;$$

and

$$p_2 = \frac{(h+d)p_1 - hp_0}{2d} = \frac{1}{2} \left(\frac{h}{d} \right)^2 p_0:$$

Hence, by induction

$$\begin{aligned} p_n &= \frac{(h+(n-1)d)p_{n-1} - hp_{n-2}}{nd} \\ &= \frac{1}{n} \left(\frac{h}{d} \right) p_{n-1} = \frac{1}{n!} \left(\frac{h}{d} \right)^n p_0 \quad \text{for } n = 1, 2, \ldots \end{aligned}$$

Finally, because

$$1 = \sum_0^\infty p_n = p_0 \sum_0^\infty \frac{1}{n!} \left(\frac{h}{d} \right)^n = p_0 e^{\frac{h}{d}}$$

the invariant distribution of labor force size given hiring and separation rates is the Poisson distribution with mean h/d, i.e.,

$$p_n(h;d) = \frac{e^{-(h/d)}}{n!} \left(\frac{h}{d} \right)^n: \quad (18)$$

3 Labor Market Equilibrium

3.1 The Matching Process

The various meeting rate parameters, λ, η_0 and η_1; are interrelated through the process by which workers and job vacancies meet. Suppose that this process can be represented by a matching function that transforms matching inputs, search and recruiting effort, into an aggregate meeting flow. We made two assumptions that simplify the analysis that follows: (1) The search effort of unemployed and employed workers are perfect substitutes for one another. (2) The matching function exhibits constant returns. Both of these assumptions are standard in the search and matching equilibrium literature. (See Mortensen and Pissarides (1999a, 1999b).)

The market meeting rate is the value of the linearly homogenous matching function $M(V;S)$ where

$$V = \int_{\underline{p}}^{\overline{p}} v(p; w(p))\,d\Gamma(p) \tag{19}$$

represents the aggregate number of vacancies posted,

$$S = us_0 + (1-u)\int_{\underline{w}}^{\overline{w}} s(w)\,dG(w): \tag{20}$$

Under the usual assumption of random matching at rates proportional to own search (recruiting) effort, the meeting frequencies per unit search effort and per vacancy respectively are

$$\begin{aligned}
\lambda &= \frac{M(V;S)}{S} = M\left(\frac{V}{S};1\right) = m(\theta) \\
s_0\eta_0 &= \frac{M(V;S)}{V}\left(\frac{u}{S}\right) = \left(\frac{m(\theta)}{\theta}\right)\left(\frac{s_0 u}{S}\right) \\
s(w)\eta_1 &= \frac{M(V;S)}{V}\left(\frac{(1-u)s(w)}{S}\right) = \left(\frac{m(\theta)}{\theta}\right)\left(\frac{(1-u)s(w)}{S}\right)
\end{aligned} \tag{21}$$

where market tightness is represented by the ratio of recruiting to search effort,

$$\theta \equiv \frac{V}{S} = \frac{\int_{\underline{w}}^{\overline{w}} v(p;w(p))\,d\Gamma(p)}{us_0 + (1-u)\int_{\underline{w}}^{\overline{w}} s(w)\,dG(w)}: \tag{22}$$

3.2 Steady State Unemployment and Wage Distribution

Because all offers are acceptable to an unemployed worker, the unemployment rate that equates flow in and out of unemployment, the steady state, solves

$$\frac{u}{1-u} = \frac{\delta}{\lambda s_0} = \frac{\delta}{s_0 m(\theta)}:$$

from (4) and (21). By substitution for u into (22), θ solves

$$\frac{\theta\left[\delta s_0 + m(\theta)\int_{\underline{w}}^{\overline{w}} s(w)\,dG(w)\right]}{\delta + s_0 m(\theta)} = \int_{\underline{w}}^{\overline{w}} v(p;w(p))\,d\Gamma(p) \tag{23}$$

given the search effort and vacancy function. Because $m(\theta)$ is increasing and concave, the left side is monotone increasing in θ and is zero if and only if $\theta = 0$. Hence, there is a unique solution for θ given any search effort and vacancy function pair $(s(w); v(p;w))$.

As the steady state measure of workers earning w or less, $G(w)$, balances the flow of workers into and out of the category, the steady state wage distribution function solves

$$\delta G(w) + \lambda[1 - F(w)]\int_R^w s(z)dG(w) = \frac{\lambda s(R)F(w)u}{(1-u)} = \delta F(w) \qquad (24)$$

where the left side accounts for the fraction of employment is the category who are laid off plus the fraction who quit to take higher paying jobs and the right side is the flow into the category from unemployment expressed as a fraction of employment. Finally, the wage offer distribution, the fraction of vacancies that offer wage w or less, is induced by the vacancy function $v(p;w)$ and the measure of firms that pay wage $w(p)$ or less. Indeed,

$$F(w(p)) = \frac{\int_{\underline{w}}^{w(p)} v(\tilde p; w(\tilde p))d\Gamma(\tilde p)}{\int_{\underline{w}}^{\overline{w}} v(\tilde p; w(\tilde p))d\Gamma(\tilde p)}. \qquad (25)$$

Definition 1 Given a vacancy function $v(p)$, and a search intensity function $s(w)$, the associated labor market steady state is the tightness parameter θ, offer distribution $F(w)$, and wage distribution $G(w)$ defined by equations (23), (24), and (25).

3.3 Existence

In a steady state equilibrium, each worker searches and each employer recruits at optimal rates taking as given the labor market steady state induced by their collective actions.

Definition 2 A steady state rent sharing equilibrium is a pair of functions $(s(w); v(p))$ and the associated labor market steady state that satisfy the optimality conditions, equations (2) and (8), given that the wage is determined by the linear rent sharing rule (9).

If search and recruiting effort are both essential inputs in the matching process, i.e., $M(0;S) = M(V;0) = 0$ for all S and V, then a trivial no-trade equilibrium always exists. Since the application rate per vacancy is $\eta_0 + \eta_1 = M(V;0)/V = 0$ when no workers search and because the offer arrival rate per unit of search effort is $\lambda = M(0;S)/S = 0$ when no employer recruits, no trade in the sense that $V = S = 0$ is an equilibrium because every individual agent on either side of the market has no incentive to make the effort to find a match. Below we demonstrate that a non-trivial equilibrium also exists if a little search effort is costless. For technical reason, we also need to assume an uniform upper bound on vacancies.

Assumption 1 The cost of search $c_w(s)$ is strictly increasing, strictly convex, and twice differentiable on $[\underline{s};\infty)$ for some arbitrarily small number $\underline{s} > 0$ and $c_w(s) = c_w'(s) = 0$ for all $s \in [0;\underline{s}]$.

Assumption 2 The cost of recruiting $c_f(v)$ is strictly increasing, strictly convex, and twice differentiable on $[0;\bar{v}]$ for some arbitrarily large number $\bar{v} < \infty$, $c_f(0) = c_f'(0) = 0$ and $c_f(v) = \infty$ for $v > \bar{v}$.

The argument for the existence of an equilibrium is based on the fact that the equilibrium conditions define a mapping, T; from the space of real valued vector function pairs $f(w) = (s(w); v(w))$ defined on the compact interval $[R;\bar{w}]$, call it z; to itself where $R = b$ under the assumption that there are potential entrant who would pay less than the reservation wage, i.e., $\mu(b) > 0$. By construction, any fixed point of the map is an equilibrium. To derive T, note that for any $f \in z$ the steady state labor market equations (23) - (25) uniquely determine a vector $(\theta; F) \in \Re_+ \times \Phi$; where Φ is the space of distribution functions. Let T_1 denote this transformation. The optimality conditions, equations (2) and (8), generate a unique $f \in z$ for any choice of $(\theta; F) \in \Re_+ \times \Phi$. Let T_2 represent this map. To sum up then, a transformation $T : z \to z$ exists, where $Tf = T_1 T_2 f$ and any fixed point $f = Tf$ is a rent sharing equilibrium.

In the case of a measure of employers offering wage w or less, $\mu(w)$, which is continuous, z is a subset of the space of continuous bounded real valued function pairs defined on $[R;\bar{w}]$; with the sup norm, under Assumptions 1 and 2. Formally, equations (2) and (21) imply that

$$c_w'(s(w)) = m(\theta) \int_w^{\bar{w}} \frac{[1 - F(\tilde{w})]d\tilde{w}}{r + \delta + m(\theta)s(\tilde{w})[1 - F(\tilde{w})]} \tag{26}$$

for any θ. If $\mu(w)$ is continuous then $F(w)$ is also continuous given any continuous function $v : [R;\bar{w}] \to \Re_+$ from equation (24). Hence, the solution $s(w)$ to the equation above is a continuous map from the interval $[R;\bar{w}]$ to $[\underline{s}; s(R)]$. Furthermore, $s(w) \to \underline{s}$ as $m(\theta) \to 0$ and

$$\begin{aligned} s(R) &\le c_f'^{-1}\left(m(\bar{\theta}) \int_R^{\bar{w}} \frac{[1 - F(\tilde{w})]d\tilde{w}}{r + \delta + m(\theta)s(\tilde{w})[1 - F(\tilde{w})]} \right) \\ &\le \bar{s} \equiv c_f'^{-1}\left(\frac{m(\bar{\theta})(\bar{w} - R)}{r + \delta} \right) \end{aligned} \tag{27}$$

where $\bar{\theta}$ is a uniform upper bound on possible equilibrium values of market tightness implied by Assumptions 1 and 2 and derived below. Of course, the first inequality is implied by the fact that $\theta \le \bar{\theta}$ and that $m(\theta)$ is increasing and the second by the fact that $F(w) \in [0;1]$, $s(w) \ge 0$. In sum, $s(w)$ is a continuous bounded decreasing function if a uniform upper bound on market tightness $\bar{\theta}$ exists.

We proceed to show that a bound on market tightness exists under Assumptions 1 and 2. First, note that the market equations of (21) imply that the first order condition for and interior solution, equation (9), can be written as

$$c_f'(v(p;w(p))) = \left(\frac{m(\theta)}{\theta}\right)\left[\frac{\delta + m(\theta)\int_{\underline{w}}^{w(p)} s(z)dG(z)}{\delta + m(\theta)\int_{\underline{w}}^{\overline{w}} s(w)dG(w)}\right] \times \left\{\frac{p - w(p)}{r + \delta + s(w(p))m(\theta)[1 - F(w(p))]}\right\} \tag{28}$$

by equation (8). The solution is unique, continuous and increasing in p and $v(R) = 0$ for any continuous distribution function $F(w)$, a condition already established. Of course, for sufficiently small values of θ, $v(p)$ is at the corner $\overline{v}$ specified in Assumption 2.

Given that (26) and Assumption 1 imply that $s(w)$ is uniformly bounded below by $\underline{s}$ and that (28) and Assumption 2 imply $v(x)$ is uniformly bounded above by $\overline{v}$;

$$\overline{\theta} \equiv \frac{\overline{v}[\mu(\overline{p}) - \mu(\underline{p})]}{\underline{s}} \tag{29}$$

represents a uniform upper bound on θ by equation (22) given any $f \in \mathcal{Z}$. Consequently, we have completed the demonstration that $T : \mathcal{Z} \to \mathcal{Z}$ where $\mathcal{Z}$ is a subset of the space of continuous bounded real valued function pairs defined on $[R, \overline{w}]$.

Theorem 3 *If the measure of employers of productivity p or less, $\mu(p)$, is continuous and $\mu(b) > 0$, if the difference between the upper support of the productivity distribution, $\overline{x}$, and the unemployment benefit, b, is positive and finite, and if the offer arrival rate per unit of search effort, $m(\theta)$, is non-negative, increasing, continuous and concave, then a non-trivial rent sharing equilibrium exists under Assumptions 1 an 2.*

Proof. See the Appendix. ■

3.4 An Example

Pissarides' (1990) theory of equilibrium unemployment is a limiting example of the general framework. Specifically, his model corresponds to the case of only one employer type and a constant posting cost per vacancy. It is well known that a unique non-trivial equilibrium exists in this special case.

To characterize the equilibrium, first note that the lack of wage dispersion implies no search while employed, i.e., $s(w) = 0$ for $w > R$ from equation (26) because $w = \overline{w}$ is the only wage offered. As a consequence,

$$\begin{aligned} W(w) - U &= \frac{w - rU}{r + \delta} \\ R &= rU = \max_{s \geq 0}\{b - c(s) + sm(\theta)[W(w) - U]\} \\ J(p) &= \frac{\overline{p} - w}{r + \delta} \end{aligned}$$

from equation (1), equation (3), the definition of the reservation wage $U = W(R)$; and equation (7) where $p = \bar{p}$ is the productivity common to all employer. Hence, Pissarides' assumption of Nash bargaining over match surplus, i.e.,

$$w = \arg\max_{w} \left\{ (W(w) - U)^{\beta} \left[J(\bar{x}; w) \right]^{1-\beta} \right\};$$

is the equivalent to the linear rent sharing rule defined in equation (9).[4]

Equilibrium market tightness is determined by equations (28) and (9), rewritten here as the Pissarides (1990) 'free entry' condition

$$\frac{a\theta}{m(\theta)} = \frac{\bar{p} - w(\bar{p})}{r+\delta} = (1-\beta)\left[\frac{\bar{p} - R}{r+\delta}\right]$$

where $a = c_f'(v)$ is the constant cost of posting a vacancy and the aggregate number of vacancies posted by all employers is implied by $\theta = V/s_0 u$ from equation (22). Because search intensity when unemployed solves the first order condition

$$c_w'(s_0) = m(\theta)[W(w(\bar{p})) - U] = m(\theta)\left[\frac{\beta(\bar{p} - R)}{r+\delta}\right]$$

from equation (9), the reservation wage can be expressed as

$$R = b - c_w(s_0) + s_0 c_w'(s_0).$$

A non-trivial equilibrium is any triple (θ, s_0, R) that satisfies these three equations and the associated steady state unemployment rate is

$$u = \frac{\delta}{\delta + s_0 m(\theta)}. \tag{30}$$

Given that $m(\theta)$ is increasing and concave, the first equation implies a negatively sloped relation between R and θ which lies in the positive quadrant. As $s_0 c_w'(s_0) - c(s_0)$ is non-negative and increasing in s_0 under Assumption 1, the second and third equation imply an increasing relationship between the R and θ such that $R = b$ when $\theta = 0$. Consequently, a unique positive solution for (θ, s_0, R) exists if $\bar{x} - b \geq 0$.

3.5 Wage Posting Equilibrium

Instead of sharing the rent, Burdett and Mortensen (1998) assume that employers post the wage they pay all employees and that each worker only have

[4] However, this equivalence result holds only in the case of homogeneous employers. In the heterogeneous case, workers employed at less than the highest wage offered will search while employed. Because employed worker's don't account for the loss in rents suffered by their employer when a quit takes place, the Nash bargaining solution will be set to partially offset this external effect of search on the job.

the power to accept or reject. Because the employer sets the wage as well as vacancies to maximize profit from recruiting activities as defined in equation (6), the linear rent sharing rule, equation (10), is replaced by the wage posting rule, equation (11), which maximizes the employer's expected future discounted profit attributable to employing a worker. That condition can be written as

$$w(p) = \arg\max_{w \geq 0} \{(p - w)\ell(w)\}. \tag{31}$$

where the employer's steady state 'labor supply' is

$$\begin{aligned} \ell(w) &= \frac{\eta_0 s_0 + \eta_1 \int_{\underline{w}}^{w} s(z)dG(z)}{r + \delta + \lambda s(w)[1 - F(w)]} && (32) \\ &= \left(\frac{m(\theta)/\theta}{r + \delta + \lambda s(w)[1 - F(w)]}\right)\left[\frac{\delta + m(\theta)\int_{\underline{w}}^{w} s(z)dG(z)}{\delta + m(\theta)\int_{\underline{w}}^{\bar{w}} s(z)dG(z)}\right] && (33) \end{aligned}$$

from equations (21) and (??).[5] The fact that a higher wage increases the probability that an employed applicant will accept and decreases the rate at which existing employees quit to take higher paying jobs, the employer has and exploits dynamic monopsony power.

A labor market equilibrium with wage posting is defined as follows:

Definition 4 A steady state wage posting equilibrium is a function tuple $(s(w); v(p); w(p))$ and the associated labor market steady state that satisfy the optimality conditions, equations (2), (8), and (31).

As demonstrated by Burdett and Mortensen (1998) and by Bontemps et al. (1998), $w(p)$ is unique, strictly increasing, $\underline{w} = R$; and $R < w(p) < p$ when the distribution of p is atomless just as in the linear rent sharing case.[6] Furthermore, the optimal vacancy choice, which solves $c_f'(v(x)) = \max_{w \geq 0}\{(p - w)\ell(w)\}$, is also increasing in p. Consequently, the number of vacancies posted by a firm offering wage w is also increasing. Because the wage posting wage rule and vacancy policy satisfies all the same qualitative properties as under rent sharing, the existence of a non-trivial wage posting equilibrium follows by essentially the same argument used above given a slightly stronger condition on the distribution of productivity.

[5] Of course, $\ell(w)$ is the product of the acceptance probability and the present value of a unit future stream of income discounted at a rate equal to the sum of the interest and separation rates. However, in the case of $r = 0$, $\ell(w)$ is equal to the expected long run size of the labor force of an establishment that pays wage w by equation (15). This fact ties the model to that studied in Burdett and Mortensen (1998).

[6] One difference is that the lowest wage is equal to the reservation wage, R, even if the lower support of the distribution of employer productivity is larger. The reason is that the employer paying the lowest wage hires no employed workers and loses all employees to higher paying firms. Hence, that employers has no incentive to pay more than the wage that all unemployed workers find acceptable.

Theorem 5 *If the measure of employers who with productivity p or less, $\mu(p)$; is differentiable, if the difference between the upper support of the distribution, $\bar{p}$; and the unemployment benefit, $R = b$; is positive and finite, and if the offer arrival rate per unit of search effort, $m(\theta)$; is non-negative, increasing, continuous and concave, then a non-trivial wage posting equilibrium exists under Assumptions 1 and 2.*

Proof. See the Appendix. ■

4 Estimation

An estimation strategy that uses matched employer-employee data which include the size of an employer's labor force and employer specific worker flows is sketched in this section. As an example, consider the Danish Integrated Database for Labor Market Research (IDA) file. It includes annual observations on the employment of all Danish establishments, employee identifiers, the earning of each employee, and employee characteristics for all year 1980-1995. (See Bingley et al. (1999).) These observations over a single year are sufficient to identify the structure of the model constructed above.

Specifically, non-parametric search and recruiting policy functions $s(w)$ and $v(p) = v(p)$ together with the transition parameter vector $(\delta; \lambda; \eta_0; \eta_1)$ can be estimated using a maximum likelihood procedure for given offer and wage distribution functions F and G. For this purpose, it is natural to use non-parametric estimates of the two distribution functions based on the wages received by all workers hired during the year and wages earned by all workers employed at the beginning, both of which are observable in the IDA data. Once these initial estimates are obtained, the associated wage and offer distributions implied by the aggregate labor market and steady state conditions, equations (22)-(25), can be computed. Obviously, one can use these computed distribution functions and the same maximum likelihood procedure to obtain second stage estimates of the policy functions and transition parameters. If the sequence of estimates obtained by continued iteration in this fashion converges, then the limiting estimates incorporate all the structure implied by the model. A comparison of the offer and wage distributions actually observed with the final estimates provide a goodness of fitness test. Furthermore, a cost of search function and a cost of recruiting function can be inferred. The interactive procedure represents a computationally simple way to compute the equilibrium wage and offer distribution while at the same time estimating the structure that underlies the distributions.

4.1 Maximum Likelihood Estimates

For each $i \in \{1, \ldots, N\}$ in a representative sample of employers in a given labor market,

$$n_i = n_i^s + n_i^d \text{ and } n_i^0 = n_i^h + n_i^s \tag{34}$$

where n_i is the number of employees at the beginning of the year, n_i^s is a count of the "stayers," workers present in the firm's labor force both at the beginning and at the end of the year, n_i^d is the number of workers present at the beginning of the year who leave during the year, and n_i^h is the number hired during the year who reported in the employer's labor force at the year. Let w_i represent the average wage paid by the employer and let p_i denote output per worker.. Under the assumption that workers are identical but are paid different wages across employing establishments, the model 'explains' cross employer variation in the triple $(n^0; n^s; n;)$ by cross employer variation in w and p.[7] Let

$$L_i(h_i; d_i) = Pr\{n^0 = n_i^0 | n^s = n_i^s; n = n_i\} Pr\{n^s = n_i^s | n = n_i\} Pr\{n = n_i\} \quad (35)$$

represent the contribution of the ith observation to the overall likelihood function.

As demonstrated above, an employer's establishment size is a stochastic birth-death process defined by the birth frequency h_i and death rate d_i. The invariant distribution of this continuous time first order Markov process on the non-negative integers is the Poison with mean equal to h_i/d. Hence, the size distribution for establishment i is

$$Pr\{n = n_i\} = \frac{e^{-\frac{h_i}{d_i}}}{n_i!}\left(\frac{h_i}{d_i}\right)^{n_i} \quad (36)$$

Given the fact that a job spell with a particular employer is exponentially distributed with parameter equal to the separation rate d, the conditional probability that exactly n^s of the original n workers stayed the entire year and the remainder did not is distributed according to the binomial with parameter e^{-d}

$$Pr\{n^s = n_i^s | n = n_i\} = \binom{n_i}{n_i^s} e^{-d_i n_i^s}(1 - e^{-d_i})^{n_i - n_i^s}. \quad (37)$$

Of course, $n^h = n^0 - n^s$ represent only measured hires, a lower bound on the number actually hired during a year to the extent that some of those hired left before the year ended Still, if the separation rate d were 'small'

$$Pr\{n^0 = n_i^0 | n^s = n_i^s; n = n_i\} \approx \frac{e^{-h_i}}{(n_i^0 - n_i^s)!}(h_i)^{n_i^0 - n_i^s}$$

since new hires arrive at the Poisson rate h over the unit interval. However, one can use the model to improve on this approximation.

Assign the integers 1 to $n^0 - n^s$ to those hired and present at the end of the interval and assign larger numbers to those who were hired and left. Let j

[7] To emperically implement this assumption, one needs to divide the sample into subsamples of workers that are as homogenous as possible and who participate in separate labor markets. In the case of the IDA, workers are identified by occupation categories, managers, office workers, skilled and unskilled workers, that approximate this requirement.

denote this index. Then, any worker with index $j \leq n^0 - n^s$ must have been hired at some unobserved instant $t \in [0,1]$ within the period and stayed for a period of length $[t,1]$ while any one with index j greater did not stay. Since the instant hired in the interval is uniformly distributed, the probability of actually observing someone who was hired during the interval still in the firm at the end is

$$\int_0^1 e^{-(1-t)d}dt = e^{-d}\int_0^1 e^{dt}dt = e^{-d}\left(\frac{e^{dt} - e^{0}}{d}\right) = \frac{1-e^{-d}}{d}.$$

Hence,

$$\Pr\{n^0 = n_i^0 | n^s = n_i^s; n = n_i\} = \sum_{j=n_i^0-n_i^s}^{\infty} \frac{e^{-h_i}}{j!}(h_i)^j \left(\frac{1-e^{-d_i}}{d_i}\right)^{n_i^0-n_i^s}\left(1-\frac{1-e^{-d_i}}{d_i}\right)^{j-(n_i^0-n_i^s)}$$

because we know that at least $j \geq n^0 - n^s$ were hired, those that were observed at the firm stayed, and those hired but not observed left. Finally, after some tedious algebra, one obtains

$$\begin{aligned}\Pr\{n^0 &= n_i^0 | n^s = n_i^s; n = n_i\} = e^{-h_i}\left(\frac{\frac{1-e^{-d_i}}{d_i}}{1-\frac{1-e^{-d_i}}{d_i}}\right)^{n_i^0-n_i^s} \\ &\times \Gamma\left(n_i^0 - n_i^s; h_i\left(1-\frac{1-e^{-d_i}}{d_i}\right)\right) \\ &\times \left(\frac{e^{-h_i}}{(n_i^0-n_i^s)!}(h_i)^{n_i^0-n_i^s}\right)\left(\frac{e^{-\frac{h_i}{d_i}}}{n_i!}\left(\frac{h_i}{d_i}\right)^{n_i}\right)\end{aligned} \tag{38}$$

where $\Gamma(x;y)$ is the incomplete Gamma function, the CDF of the gamma random variate.[8]

Conditional on non-parametric estimates of the wage and offer distributions, F and G, the hiring and separation rate are related to each employer's productivity and wage by

$$\begin{aligned} h_i &= \left[\eta_0 s_0 + \eta_1 \int_R^{w_i} s(z)dG(z)\right] v(p_i; w_i) \\ d_i &= \delta + \lambda s(w_i)[1 - F(w_i)]. \end{aligned} \tag{39}$$

For purpose of identification, the following normalizations are imposed.

$$s_0 = s(R) = 1 = v(\bar{p}; \bar{w}) \tag{40}$$

[8] This final step in the derivation is a contribution from George Neumann.

Essentially, these assumptions determine the units in which search effort and vacancies are measured. Finally, the first stage estimates solve

$$\left(\hat{\delta}, \hat{b}_0, \hat{b}_1, \hat{v}(p;w), \hat{\lambda}, \hat{s}(w)\right) = \arg\max \left\{ \sum_{i=1}^{N} \ln(L_i(h_i; d_i)) \right\} \quad \text{s.t. (39) and (40)}. \tag{41}$$

Call the parameter and policy function estimates obtained in this case by applying (41) the first stage estimates.

4.2 Distribution and Matching Cost Estimates

The equilibrium wage offered and wage earned distributions implied by the first stage estimates of the transition parameters and policy functions can be constructed using equations (23)-(25): Specifically, the equilibrium sample offer distribution implied by the estimates of the recruiting policy function is

$$\hat{F}(w) = \frac{\sum_{w_i \le w} \hat{v}(p_i; w_i)}{\sum_{i \in N} \hat{v}(p_i; w_i)} \tag{42}$$

where w_i is the actual wage paid by employer i and p_i is the same employer's labor productivity. The associated steady state wage distribution is the unique solution to

$$\delta\hat{G}(w) + \lambda[1 - \hat{F}(w)] \int_R^w \hat{s}(z) d\hat{G}(w) = \delta\hat{F}(w): \tag{43}$$

Second stage estimates of the parameters and policy functions can be computed by using these functions in the equations of (39) and then resolving the maximum likelihood estimation problem defined by equation (41). The limit of the sequence of estimates obtained by iterating between these two equations and equations (42) and (43) satisfy all the structure imposed by the model. Finally, these estimates can be compared with the actual distributions observed in the data to determine the extent to which the model explains the wage and offer distributions.

That equations (42) and (43) are the offer and wage distributions implied by the model for both the rent sharing and wage posting hypotheses is worthy of note. The reason for the independence is that both wage determination models have the same qualitative implications about the vacancy function, namely that the number of vacancies posted increases with the wage paid. Given an estimate of this function which is monotone, both the offer distribution and the steady state wage distribution implied by the model is the same under any wage determination mechanism that implies that vacancies increase with the wage.

The fundamental assumption of the search equilibrium `flow' approach to labor market analysis is that frictions in the form of search and recruiting costs are important. Accounting data simply cannot provide a test of this assumption because accounting practice does not separate out these costs from general overhead. One advantage of the structural framework outlined here is that it permits inference about the magnitude of these costs. Specifically, the marginal search cost function implied by the estimates,

$$c_w^0(\hat{s}(w_i)) = \hat{\lambda} \int_{w_i}^{\overline{w}} \frac{[1 - \hat{F}(\tilde{w})]d\tilde{w}}{r + \hat{\delta} + \hat{\lambda}\hat{s}(\tilde{w})[1 - \hat{F}(\tilde{w})]} \tag{44}$$

by equation (26), which is determined up to a specification of the discount rate r. Because empirically the separation rate is much larger than any estimate of the discount rate, even the upper bound on the marginal cost obtained by setting $r = 0$ is a good approximation. Similarly, equation (28) implies that the marginal cost of recruiting function estimate in the rent sharing case is

$$c_f^0(\hat{v}(p_i; w_i)) = \frac{[\hat{b}_0 + \hat{b}_1 \int_{\underline{w}}^{w_i} \hat{s}(z)d\hat{G}(z)](p_i - w_i)}{r + \hat{\delta} + \hat{\lambda}\hat{s}(w)[1 - \hat{F}(w)]}: \tag{45}$$

Of course, in the rent sharing case, this estimate of recruiting cost is only identified up to a choices of the share parameter β and reservation wage $R = b$:

5 Future Research

The model explored in this paper is a prototype for a more general framework that might be useful for the structural interpretation and analysis of matched employer-employee data. A more complex version is needed to confront the richness of the data actually available and the set of questions that are of interest.

Extending the empirical approach to the full panel of observations available in most matched employer-employee data sets is an obvious next step. Time variation in employer productivity will be a feature of these data that one cannot ignore. How does one model this feature? Should the observed variations be regarded as noise or as the realizations of a stochastic process with a law of evolution known to the decision makers in the model? Of course, births and deaths of employing establishments will also be observed. How does one acknowledge these events in the model and in the estimation strategy? Although the formal model is much more complicated when the employer productivity parameter is treated as a process, it has the potential for providing insight into the interpretation of job and worker flow data, particularly those components that are associated with the births and deaths of establishments.

Another generalization would explicitly account for within establishment worker heterogeneity. In the case of the IDA, the usual demographic and education of each worker is known as well as occupation. The availability of the data permit a joint theoretical and empirical study of labor force composition effects. For example, the model can be modified to include an establishment level production function that could be estimated using observed time series variation in a given employers labor force composition as well as cross section differences. The extent to which employee composition explains cross employer productivity differences is also a question that one should be able to address from a structural perspective.

6 Appendix: Existence Proofs

6.1 Rent Sharing Case

Proof. Under the hypothesis, the lowest wage offered and the least productive employer's productivity both equal the reservation wage, i.e., $\underline{w} = \underline{p} = R$. Hence, any fixed point of the map $T : Z \to Z$ defined in the text is a rent sharing equilibrium when $w(p)$ is the wage rule (9) where $f = (s(w), v(p))$ is an element of Z, a subset of continuous bounded real valued function pairs defined on the compact interval $[R, w(\bar{p})] \times [R, \bar{p}]$. Hence, Schauder's fixed point theorem, as stated by Lucas and Stokey (1989, p. 520), implies existence if Z is non-empty, closed, bounded and convex, the operator T is continuous and the family of functions $T(Z)$ is equicontinuous.

The vacancy function $v(p)$ maps $[R, \bar{p}]$ into $[0, \bar{v}]$ and the search function $s(w)$ maps $[R, w(\bar{p})]$ into $[\underline{s}, \bar{s}]$ where $\underline{s}$ is specified in Assumption 1, $\bar{v}$ is specified in Assumption 2, and $\bar{s}$ is uniquely determined by equations (27) and (29). As $\mu(R)$ is increasing in R, $\bar{s}$ is monotone decreasing in R. Hence, for any finite $\bar{w} > R$, $1 > \bar{s} > \underline{s}$ for all $\underline{s} > 0$ chosen sufficiently small from equation (26). Consequently, Z is nonempty, closed, bounded and convex.

To show that T is continuous on Z it is necessary and sufficient to prove that both of its component operators, T_1 and T_2, are continuous. First, consider $T_1 : Z \to \Re_+ \times \Delta$ defined by equations (23)-(25). Without belaboring the details, continuity of the transformation is implied by that fact that the integration operator on any continuous bounded real valued function is continuous in the space of continuous bounded functions with the sup norm. The transformation $T_2 : \Re_+ \times \Delta \to Z$ defined by the integral equation (26) is continuous and equation (28) defines a pointwise continuous functional relationship between $v(w)$ and $(\theta, s(w), F(w))$ for all $w \in [R, w(\bar{p})]$.

The family $T(F)$ is equicontinuous if every $f \in T(Z)$ is uniformly continuous on $[R, w(\bar{p})] \times [R, \bar{p}]$ and if the continuity is uniform for all functions $T(Z)$, i.e., if for every $x \in [R, w(\bar{p})] \times [R, \bar{p}]$ and $\varepsilon_0 > 0$ there exists a $\delta_0 > 0$ such that

$$|x - y| < \delta_0 \text{ implies } \|f(x) - f(y)\| < \varepsilon_0, \text{ all } f \in T(Z) \tag{46}$$

where

$$\|f(x) - f(y)\| \equiv \max \langle |v(x) - v(y)|, |s(x) - s(y)| \rangle.$$

The following demonstration that this condition holds amounts to showing that the gradient of $f(x)$ is uniformly bounded for all functions $T(Z)$.

By equation (26),

$$c''_w(s(w))s'(w) = \frac{-m(\theta)[1 - F(w)]}{r + \delta + m(\theta)s(w)[1 - F(w)]}.$$

Hence, $m(\theta)$ increasing and $\theta \le \bar{\theta}$ imply

$$|s'(w)| \le \frac{m(\bar{\theta})}{c''_w(s(w))(r + \delta)}$$

where $\bar{\theta}$ is defined in equation (29). As a consequence,

$$|x-y|<\delta_0 \Rightarrow |s(x)-s(y)| \le \max_{w\in[b,\bar{w}]}\{|s'(w)|\}|y-x| < \delta_0 \max_{w\in[R,\bar{w}]}\left\{\frac{m(\bar{\theta})}{c_w''(s(w))(r+\delta)}\right\} = \varepsilon_0 \tag{47}$$

where $\bar{\theta}$ is the uniform upper bound on market tightness defined in (29) given that the search cost function is strictly convex.

Either $v(p)$ equals the upper bound $\bar{v}$ or $v(w)$ is an interior solution. In the latter case,

$$c_f'(v(p)) = (1-\beta)(p-R)\ell(w(p))$$

by equations (28) where $w(p) = \beta p + (1-\beta)R$ and

$$\ell(w) = \left(\frac{m(\theta)/\theta}{r+\delta+\lambda s(w)[1-F(w)]}\right)\left[\frac{\delta+m(\theta)\int_{\underline{w}}^{w} s(z)dG(z)}{\delta+m(\theta)\int_{\underline{w}}^{\bar{w}} s(z)dG(z)}\right]. \tag{48}$$

Because $s(w)$ and $F(w)$ are both differentiable given any $f = (s(w), v(p)) \in Z$ by equations (2) and (23), so is $\ell(w)$. Now in general, $m(\theta)/\theta$ is monotone increasing in θ and $m(\theta)/\theta \to \infty$ as $\theta \to 0$. If so, then a $\infty > \underline{\theta}(p) > 0$ exists such that $v(p) = \bar{v}$ for all $\theta \le \underline{\theta}(p)$. Hence,

$$|x-y|<\delta_0 \Rightarrow |v(x)-v(y)| \le \max_{p\in[R,\bar{p}]}\{v'(p)\}|x-y| \tag{49}$$

$$< \delta_0 \max_{p\in[b,\bar{p}]}\left\{\left(\frac{m(\underline{\theta}(p))}{c_f''(v(p))\underline{\theta}(p)}\right)(1-\beta)[\ell(w(p))+(p-w)\beta\ell'(w(p))]\right\} = \varepsilon_0$$

given that the cost of recruiting is strictly convex. ■

6.2 Wage Posting Case

Proof. In this case the wage rule $w(p)$ is endogenous and satisfies (31). Hence, it is appropriate to think of Z as a subset of the space of vector functions $f(x) = (w(p), v(p), s(w(p)))$ defined on the productivity interval $[R, \bar{p}]$ together with the sup norm where

$$w(p) = \arg\max_{w\ge R}\{(p-w)\ell(w)\} \tag{50}$$

$$v(p) = \arg\max_{v\ge 0}\{\max_{w\ge R}\{(p-w)\ell(w)\}v - c_f(v)\} \tag{51}$$

and

$$s(w) = \max_{s \geq 0} \left\{ m(\theta) s \int_w^{\overline{w}} \frac{[1 - F(x)]dx}{r + \delta + \lambda s(x)[1 - F(x)]} - c_w(s) \right\} : \qquad (52)$$

These definitions together with the market equations and steady state conditions (23)-(25) define a map $T : z \to z$ where z is the subspace of continuous bounded real vector function with the sup norm and any fixed point of the map is a non-trivial equilibrium under Assumptions 1 and 2. As z is non-empty, compact and convex and T is continuous on z for the reasons given in the rent sharing case, Schauder's theorem applies in this case as well if (46) holds for this extended definition of z:

In the rent sharing case, we used the fact that $T(z)$ is in the set of differentiable functions given that F is a subset of the continuous bounded functions to prove equicontinuity in the rent sharing case. To apply the same argument in the wage posting case where $w(p)$ is a component of any $f \in F$ we need to show that $w(p)$ is differentiable as well as $v(p)$ and $s(p)$: The added assumption that the measure of employers with productivity less than or equal to p; call it $\mu(p)$; is differentiable is needed for this purpose. Under this assumption, z can be regarded as a subset of the differentiable functions defined on $[R, \overline{p}]$. Namely, given any differentiable $f(p) = (w(p); v(p); s(w(p)))$; it follows that the implied $F(w)$ is twice differentiable by equation (23). As a consequence, $s(w)$ by equation (52) is also twice differentiable and so is $\ell(w)$ by equation (48) given any differentiable $f(p)$. Hence, $w(p)$ is differentiable by equation (50) so that

$$|x - y| < \delta_0 \Rightarrow |w(x) - w(y)| \leq \max_{z \in [R, \overline{p}]} \{w'(z)\} |x - y| < \delta_0 \max_{z \in [R, \overline{p}]} \{w'(z)\} = \varepsilon_0 ; \qquad (53)$$

$$\begin{aligned} |x - y| < \delta_0 \Rightarrow |s(w(x)) - s(w(y))| &\leq \max_{z \in [R, \overline{p}]} |s'(w(z)) w'(z)| |y - x| \qquad (54) \\ &< \delta_0 \max_{z \in [R, \overline{p}]} \left\{ \frac{m(\theta) w'(z)}{c_w''(s(z))(r + \delta)} \right\} = \varepsilon_0 ; \end{aligned}$$

and

$$\begin{aligned} |x - y| < \delta_0 \Rightarrow |v(x) - v(y)| &\leq \max_{z \in [b, \overline{p}]} \{v'(z)\} |y - x| \qquad (55) \\ &< \delta_0 \max_{z \in [R, \overline{p}]} \left\{ \frac{m(\theta) \ell(w(z))}{c_f''(v(z)) \theta} \right\} = \varepsilon_0 : \end{aligned}$$

■

6.3 Non-trivial Equilibrium

Proof. In sum, we have shown that Tf satisfied (46) and consequently $T : z \to z$ has a fixed point by Schauder's theorem in both the rent sharing and wage posting cases. The claim that any ¯xed point is non-trivial follows: Because as constructed every f 2 z is such that s(w) ¸ $\underline{s} > 0$ for all w ¸ b equation (22) implies that µ = V=S < 1: But given that fact, equation (28) requires v(w) > 0 for all w > b and consequently µ = V=S > 0 given $\overline{w} > b$: In short, the trivial equilibrium that always exists is not a ¯xed point of the constructed map.■

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