Property Insurance with Conversion Options: Upper Limits and Deductibles

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Abstract

The paper compares models of insurance of real property when owners may convert damaged property to another use instead of restoring it to the pre-damage use. First it describes the optimum insurance. The main result is that the deductible and the upper limit are connected by the equation: upper limit plus deductible equals conversion point. An alternative to the full optimum is a policy having a variable upper limit and a fixed deductible. It is interesting for theoretical reasons and for descriptive reality. The comparative statics of the full optimum and the fixed-deductible alternative are essentially the same. In the fixed-deductible model, the response of the upper limit to changes in parameters is always in the same direction but of lesser magnitude than in the full optimum.

1 Introduction

The topic here is insurance of real property, that is, of structures and other improvements of land. The interesting questions are why does it have upper limits chosen by the consumer? and what is the right way to model it? These are open questions because much of the insurance literature focuses on insurance of wealth, while.the defining feature of property insurance contracts is that they are written in terms of damage. Damage is measured as the cost to restore the property to its undamaged condition. In most cases it is different from loss of wealth. For instance, everyone can envision situations in which the destroyed property is not very valuable – loss of wealth is low – but the cost to reproduce it – the damage – is high. These situations predominate because of opportunities to use the land for something different. When a house is destroyed, the owner converts the site to an apartment, mall, or parking lot, or perhaps just to a slightly different, more modern house, and the value of the conversion limits the loss of wealth to something less than the cost of restoring the original house.

The first step is to examine the optimum contract. When prices are fair, damage is covered without coinsurance up to the level that triggers conversion. With loading, the optimum contract has a deductible and an upper limit on coverage and the two are connected by the relation: upper limit plus deductible equals conversion point. The entire reduction in coverage caused by the loading has the form of a deductible.

The theoretical optimum is at odds with reality. In actual insurance contracts, insurers and consumers focus solely on the upper limit of coverage. Deductibles are usually very small and seem intended only to discourage trivial claims. Even when deductibles are substantial, as in earthquake insurance where they may be as much as 15%, they are not coordinated with the upper limit on coverage using knowledge of the conversion point. Observed practice leads to a model for insurance of real property in which the deductible is not a choice variable. A fixed-deductible model has essentially the same comparative statics as the fully optimum one. Thus on grounds of realism it is better to study the fixed-deductible contract with the understanding that if one did study the fully optimum one, the results would not be much different. Issues involving background risk in contracts with a fixed deductible are addressed in a companion paper.

2 Optimum insurance

It is observed above that conversion options impose an upper bound on loss of wealth. Notation is needed to formalize this point. Look at the situation that exists when an event has caused damage. The realization of the random variable for damage is t.

The property in undamaged condition has a value of v. When the owner of damaged property restores it, he attains wealth $v - t$. On the other hand, there is some best option for converting the property. The conversion chosen by the property owner is the one that maximizes net value. Let the value of the land and improvements in the highest valued use be v^*, and let the cost of building the best improvements be c. Among the conversion options, the greatest attainable net value is $v^* - c$, which is also the value of the land. The decision to restore or convert is the decision to select the greater of $v - t$ and $v^* - c$. At the critical level of damage, $q = v - (v^* - c)$, the options are equally valuable. After some rewriting, the option to convert becomes the option to possess $\max[v - t, v - q]$ or, equivalently, $v - \min[t, q]$. Thus, $v - q$ is a floor beneath which wealth cannot fall, no matter the extent of damage. The second expression is convenient and equally intuitive. It says that the loss of wealth due to damage is no more than q.

Let $h(t)$ be the probability density function of damage and assume that $h(t)$ is never zero on $[0, T]$ but is always zero elsewhere. Denote the cumulative distribution function of t by $H(t)$, and represent integration using $h(t)dt$ by dH. Let λ be the loading factor. Define $x(t)$ to be the premium for loss up to t, and let $I(t)$ denote the indemnity associated with a loss of t. Then

$$x(t) = \int_0^t (1 + \lambda) I(t) dH \tag{1}$$

Letting T be the largest possible damage, the whole premium is $x(T)$. In order to compress notation, it will be denoted x_T.

An expression of $I(t)$ is needed. From the above

$$x'(t) = (1 + \lambda) I(t) h(t) \tag{2}$$

and consequently

$$I(t) = \frac{x'(t)}{(1 + \lambda) h(t)} \tag{3}$$

With insurance as immediately above, wealth in damage-state t is

$$v - x_T - \min[t, q] + \frac{x'(t)}{(1 + \lambda) h(t)} \tag{4}$$

This model of insurance demand is based on the premise of specialization. Insurance firms are specialists that supply wealth contingent upon damage t, but the demander is a generalist who pays the premium in certainty wealth. Thus the indemnity $I(t)$ can be positive or zero, but never negative, leading to the constraint

$$x'(t) \geq 0 \tag{5}$$

A corollary restraint is that the premium must be nonnegative. The constraint of nonnegative indemnity arises not from traditions or legal constraints, but from economic factors that underly them. For perspective notice that specialization does not apply in all insurance markets. Members of assessible mutuals make contingent payments in addition to a fixed premium. Thus the constraint of nonnegative indemnity is not applicable in their case, the notion of a premium paid for insurance is not well defined, and they are outside the scope of this analysis[1].

2.1 The problem:

The problem is to maximize, through choice of $x(.)$, the function

$$\int_0^T u(v - x_T - \min[t, q] + \frac{x'(t)}{(1+\lambda)h(t)})dH \tag{6}$$

subject to the constraints

$$x(0) = 0 \tag{7}$$

and

$$x'(t) \geq 0 \tag{8}$$

The endpoint T is fixed, but the premium x_T is free to be chosen optimally.

2.2 Conditions:

The derivations below all concern first order conditions – the first variation. These conditions are sufficient for an optimum here because the objective function is concave in $x(t)$, $x'(t)$, and x_T. To show the existence of a deductible it is enough to show that the deductible satisfies first-order conditions.

In order to supply a parallel with notation used by Kamien and Schwartz (1991), let the notation F denote the integrand

$$F(t, x, x', x_T) = u(v - x_T - \min[t, q] + \frac{x'(t)}{(1+\lambda)h(t)})h(t). \tag{9}$$

Four conditions determine the solution:

[1] See Marshall 1976 for a discussion of how assessible mutual are organized.

Deductible: The deductible is non negative. From the constraints of equations (7) and (8), the accumulation of premium satisfies, on the interval $[0, d]$,

$$x'(t) = x(t) = 0. \tag{10}$$

Euler equation: On the interval $(d, T]$, $x'(t)$ satisfies the Euler equation

$$\frac{d}{dt} F_{x'}(t, x, x', x_T) = F_x(t, x, x', x_T) \tag{11}$$

The right hand side is zero because the objective is independent of x. Therefore the condition requires marginal utility of wealth to be constant which implies that

$$v - x_T - \min[t, q] + \frac{x'(t)}{(1+\lambda)h(t)} = c \tag{12}$$

for an unknown constant c. This equation dictates the form of $x'(t)$. The unknown constants at this point are d, c, x_T, and the constant k_1 that is needed when $x'(t)$ is integrated. Two more conditions are needed.

Joining condition: The joining condition governs the way in which the unconstrained solution on $(d, T]$ must link up with the constrained solution on $[0, d]$. The condition is covered in several sources. Still following Kamien and Schwartz (1991), let R represent the constrained solution on $[0, d]$ and let x represent the solution on $(d, T]$. Then the condition is

$$F(d, R, R', x_T) - F(d, x, x', x_T) + x' F_{x'}(t, x, x', x_T) = 0 \tag{13}$$

The first two terms differ because the $R'(t)$ associated with the constrained extremal and the $x'(T)$ associated with the unconstrained extremal are different. The extremal x is continuous at the joining point, but the derivatives are not. In the problem at hand, the condition is

$$\begin{aligned} 0 &= u(v - x_T - d) \\ &- \left[u(v - x_T - d + \frac{x'(t)}{(1+\lambda)h(t)}) - u'(v - x_T - d + \frac{x'(t)}{(1+\lambda)h(t)}) \frac{x'(t)}{(1+\lambda)h(t)} \right] \end{aligned} \tag{14}$$

Terminal condition:

A free terminal value is a standard feature in the calculus of variations, but the presence of the terminal value as an argument of the integrand is a more unusual wrinkle. The reasoning behind the standard problem is that an increase of δx_T in the terminal value makes the slope of $x(t)$ steeper and therefore raises the value of the objective by

$$u'(v - x_T - d + \frac{x'(T)}{(1+\lambda)h(T)})\frac{1}{(1+\lambda)} \times \delta x_T \tag{15}$$

Here, in addition, the increase changes the value of the integrand at every point and that changes value by the amount

$$\left[-\int_0^T u'(v - x_T - d + \frac{x'(t)}{(1+\lambda)h(t)})dH\right] \times \delta x_T \tag{16}$$

The condition of optimality is that any variation in x_T should not raise the value, and thus the condition is

$$\begin{aligned} &u'(v - x_T - \min[T,q] + \frac{x'(T)}{(1+\lambda)h(T)})\frac{1}{(1+\lambda)} \\ &- \int_0^T u'(v - x_T - \min[t,q] + \frac{x'(t)}{(1+\lambda)h(t)})dH = 0 \end{aligned} \tag{17}$$

The condition can be interpreted by recognizing that the second term is the expected value of marginal utility, briefly denoted Eu', and the first term is $\frac{1}{1+\lambda}$ times the marginal utility of wealth at the point T , briefly denoted $u'(T)$. The terminal condition is then rewritten as

$$\frac{u'(T)}{Eu'} = 1 + \lambda \tag{18}$$

The demand price of wealth in the vicinity of the point T is the marginal rate of substitution between state-T wealth and certainty wealth $\frac{Eu'(T)}{Eu'}h(T)$ and the supply price is $(1+\lambda)h(T)$. Thus the condition says that the supply price equals the demand price at the point T. From the result of the Euler condition in equation (12), $u'(T)$ $= u'(t)$ for all t in the interval $(d, T]$ and $u'(t) < u'(T)$ for all t in the interval $[0, d]$. Therefore below the deductible the demand price of contingent wealth is below the supply price and above the deductible the demand price is equal to the supply price. Diagrams using this idea have appeared in Gollier (1992) and Marshall (1992).

Comparison with other derivations: The solution here extends those of Raviv (1979) and of Arrow (1973). In deriving the existence of a deductible, they treated the premium as a fixed constant. Other arguments, explicit or implicit, were used to narrow the range of admissible premiums to those leading to a deductible. Their procedures are intuitive but roundabout. In derivations like those of Arrow and Raviv and of Gollier (1996) the indemnity is associated with the x variable instead of the x' variable, as it is here. That is important because in calculus of variations the x variable is necessarily continuous while the x' variable can have jumps. Some interesting types of insurance demand are discontinuous (e.g. Marshall (1992) and Garratt and Marshall, (1996), and see the section below). Continuity in the present derivation is a result, not an assumption.

2.3 Derivations

Consider the joining condition in equation (14). Because utility is concave, the only possible solution is

$$x'(d) = 0 \tag{19}$$

where it is remembered that this $x'(t)$ belongs to the unconstrained part of the solution, that pertaining to $t \geq d$. This implies that the indemnity function is continuous at the point d. Using $x'(d) = 0$ and the Euler condition result at equation (12) evaluated at the point d, it follows that

$$c = v - x_T - d \tag{20}$$

The same Euler equation holds for all $t \in (d, T]$, with the consequence that

$$\frac{x'(t)}{(1+\lambda)h(t)} = \min[t, q] - d \tag{21}$$

From the initial conditions it is already known that $x'(t) = x(t) = 0$ on the interval $[0, d]$. Thus is proved:

Theorem 1 *In the solution of the problem in equations (6,7, and 8)* $I(t) = 0$ *on the interval* $[0, d]$ *and elsewhere is*

$$I(t) = \min[t, q] - d.$$

An immediate corollary to the theorem is

Corollary 1 *The upper limit on insurance in the problem in equations (6,7, and 8) is* b *satisfying* $b + d = q$.

Fairly priced insurance leads to a deductible of zero, as shown in

Corollary 2 *If* $\lambda = 0$ *in the problem in equations (6,7, and 8),* $d = 0$ *and* $b = q$.

Proof. Suppose for purposes of later contradiction that $d > 0$. From equation (17) and the assumption that $\lambda = 0$, derive the condition

$$u'(v - x_T - \min[T, q] + \frac{x'(T)}{(1+\lambda)h(T)}) = \int_0^d u'(v - x_T - \min[t, q])\frac{h(t)}{H(d)}dt \qquad (22)$$

From the theorem (and assuming $t \leq q$), that reduces to

$$u'(v - x_T - d) = \int_0^d u'(v - x_T - t)\frac{h(t)}{H(d)}dt \qquad (23)$$

This condition says that the expected marginal utility conditional on $t \in [0, d]$ is the same as the terminal marginal utility. On $t \in [0, d]$,

$$u'(v - x_T - d) \geq u'(v - x_T - t) \qquad (24)$$

with strict inequality at every point except d. Thus the condition in equation (17) cannot be satisfied. Since the existence of a solution is not in doubt, this contradicts the premise that $d > 0$. ■

The effects of variations in the utility function and in the parameters λ, v, and q are important in themselves and for comparison to the fixed-deductible model considered farther below.

Corollary 3 *If the utility function* u *in the problem in equations (6,7, and 8) is replaced by a utility function* $\hat{u}$ *that possesses decreasing absolute risk aversion and is for all* t *strictly more risk averse than* u*, the result is a decrease in* d *and an increase in* x_T.

Proof. From equation (17) define the expression

$$\Gamma(d) \equiv u'(v - x_T - d)(\frac{1}{1+\lambda} - 1 + H(d)) - \int_0^d u'(v - x_T - t)dH \qquad (25)$$

Let d^* be the solution to $\Gamma(d) = 0$. As usual, all results are invariant to affine transformation of the utility function. Therefore without loss of generality assume that $u'(v - x_T - d^*) = \hat{u}'(v - x_T - d^*)$. It follows that for all $w > v - x_T - d^*$, $\hat{u}'(w) < u'(w)$. Thus when the hat utility is substituted in equation (25), the result is

$$\hat{\Gamma}(d^*) \equiv \hat{u}'(v - x_T - d^*)(\frac{1}{1+\lambda} - 1 + H(d^*)) - \int_0^{d^*} \hat{u}'(v - x_T - t)dH > 0 \qquad (26)$$

To show that a decrease in d is needed to restore optimality, it is necessary to show that $\Gamma'(d^*) > 0$. Differentiate and substitute using $A(w) = \frac{-u''(w)}{u'(w)}$, with the result that

$$\hat{\Gamma}'(d^*) = A(v - x_T - d^*) \cdot$$
$$\left[(\hat{u}'(v - x_T - d^*)(\frac{1}{1+\lambda} - 1 + H(d^*)) - \int_0^{d^*} \frac{A(v - x_T - t)}{A(v - x_T - d^*)}\hat{u}'(v - x_T - t)dH\right] \qquad (27)$$

Compare the quantity is square brackets to the quantity in equation (26), which is already positive. The only difference is the factor under the integral, and that factor is less than unity by DARA. It follows that $\hat{\Gamma}'(d^*) > 0$. ■

When the optimum varies with the parameters λ, v, and q, price effects and income effects are sometimes in contrary directions – a common situation. Some clarity can be achieved by looking at the case of CARA, effectively neutralizing the wealth effects. Thus the comparative statics are shown in two versions. Proofs are in the Appendix.

Corollary 4 *Assuming that the utility function obeys constant absolute risk aversion in the problem in equations (6,7, and 8), the comparative statics are*

$$\begin{array}{lll} \frac{dd}{d\lambda} > 0 & \frac{dd}{dv} = 0 & \frac{dd}{dq} = 0 \\ \frac{db}{d\lambda} < 0 & \frac{db}{dv} = 0 & \frac{db}{dq} = 1 \\ \frac{dx_T}{d\lambda} > 0 \textit{ near } \lambda = 0 & \frac{dx_T}{dv} = 0 & \frac{dx_T}{dq} > 0 \end{array} \qquad (28)$$

Under decreasing absolute risk aversion, the comparative statics are

$$\begin{array}{lll} \frac{dd}{d\lambda} > \textit{or} < 0 & \frac{dd}{dv} > 0 & \frac{dd}{dq} < 0 \\ \frac{db}{d\lambda} > \textit{or} > 0 & \frac{db}{dv} < 0 & \frac{db}{dq} > 1 \\ \frac{dx_T}{d\lambda} > 0 \textit{ near } \lambda = 0 & \frac{dx_T}{dv} < 0 & \frac{dx_T}{dq} > 0 \end{array} \qquad (29)$$

3 Further implications

Realistic modifications of the model lead to further implications. Two of them are discussed in this section. Derivations are only sketched because the results are transparent and because the implications – unlike the modifications – are not realistic. For various reason these properties are rarely observed in contracts of insurance of real property.

Discontinuous deductible Consider the cost function. It assumes that the cost of making an indemnity $I(t)$ is a constant fraction $\lambda I(t)$. A step toward realism is to consider that the cost has a fixed component and is of the form $\gamma + \lambda I(t)$. The fixed cost has the interesting implication that

$$\begin{aligned} x'(t) &= 0 \text{ if } I(t) = 0 \\ &= \gamma + \lambda I(t) \text{ if } I(t) \geq 0 \end{aligned} \tag{30}$$

Thus indemnity in its non-zero range is

$$I(t) = \frac{x'(t) - \gamma}{1 + \lambda} \tag{31}$$

At the joining point, where $t = d$, the condition $x'(d) = 0$ no longer applies. Instead, $x'(d) > 0$, meaning that the indemnity jumps from zero to some positive value. That makes perfect sense because once the fixed cost is overcome, the indemnity should be pushed to the point at which marginal cost is equal to marginal value.

Round shoulder The model has q known with certainty. Suppose instead that q is random with distribution $G(q|t)$. Suppose that levels of q are non contractible, and suppose further that the distribution of q is common knowledge to insurer and client. These are realistic assumptions. Then the problem is as before except that a new utility function must be used, namely

$$\breve{u}(v - x_T + I(t); t) = \int_{q_0}^{q_1} u(v - x_T - \min[t, q] + I(t)) dG(q|t) \tag{32}$$

This utility function is the same as u for $t < q_0$, but it displays progressively lower marginal utility as t increasingly exceeds q_0 and grows towards q_1.

$$\begin{aligned}\breve{u}'(v - x_T + I(t); t) &= u'(v - x_T - t + I(t)) \text{ if } t < q_0 \\ &< u'(v - x_T - t + I(t)) \text{ if } q_0 < t < q_1 \\ &= 0 \text{ if } t \geq q_1 \end{aligned} \quad (33)$$

At equation (12) the derivation is changed. It is still true that the marginal utility is constant, but the implication of constant consumption above the deductible is no longer valid. With the new utility function, consumption is constant on $[d, q_0]$, but for $t > q_0$ it must decline in order to satisfy the Euler equation. This happens by a reduction in the marginal indemnity – that is, by a coinsurance – starting at $t = q_0$ and becoming progressively more marked until at $t = q_1$ and beyond, the marginal indemnity is zero. Instead of a sharply defined upper limit, the optimum contract has a round shoulder.

Actual policies have neither of these properties, which is okay because optimality for the consumer is not a perfect guide to real insurance. Indeed, other implications of the model require a similar scrutiny. Specifically, more thought should be given to the implication that the deductible is linked to the upper limit. Descriptive realism suggests that the deductible is seldom important and in any event not linked to the upper limit in the way required by theory. The following sections therefore derive a model of insurance of real property that has a fixed deductible. It turns out that the comparative statics of this type of insurance are not much different from those of the fully optimum type.

4 Fixed deductible

Optimum insurance contracts are not always the best ones to study. In many practical instances the deductibles are absent or very small. Moreover, the optimizing relation between the upper bound the deductible is not enforced: when the client says he wants to insure his house for \$300K, the theory requires the selling agent to say "Okay. I know your conversion value is \$350K, so that means your deductible is \$50K." The agent does not say any such thing because the conversion value is unknown to him and he is, if anything, looking to the consumer's choice of upper limit as an indicator of it. The deductible is already set in the contract or is computed by arbitrary formula, for instance, as a fraction of the upper limit. For that reason it is interesting also to consider the demand for insurance when the deductible is not a variable. To facilitate comparison, adopt a differential calculus framework. Consider contracts with upper limits and a fixed deductible.

The insurance indemnity is therefore $I(t) = \min[t-d, b]$ on $(d, T]$ and zero elsewhere. Wealth in damage state $t > d$ is

$$v - x_T - \min[t, q] + \min[t-d, b] \tag{34}$$

The objective can be written

$$F(d, b; \lambda, v, q) = \int_0^d u(v - x_T - t)dH$$
$$+ \int_d^{d+b} u(v - x_T - d)dH + \int_{d+b}^T u(v - x_T - \min[t, q] + b)dH \tag{35}$$

The problem is to choose b to maximize expected utility subject to the constraint of a loaded premium

$$x_T = (1+\lambda)\left[\int_d^{d+b}(t-d)dH + b\int_{d+b}^T dH\right] \tag{36}$$

When both d and b are chosen optimally, the solution is the same as in the previous section. In present notation it satisfies

$$\begin{aligned} F_d(d, b; \lambda, v, q) &= 0 \\ F_b(d, b; \lambda, v, q) &= 0 \end{aligned} \tag{37}$$

Denote the solutions at the maximum by $d(\lambda, v, q)$ and $b(\lambda, v, q)$. The objective has critical points other than the global maximum – for instance at $d = 0$, $b = 0$, when $\lambda = 0$ – but they are not a concern here. At the maximum, the matrix of second partials must be negative semi-definite. In order to avoid non-generic complications, this derivation assumes that is it negative definite[2].

4.1 Comparison of the models

Increased risk aversion. The final topic is the effect of increased risk aversion. Suppose that u and $\hat{u}$ are utility functions and that $\hat{u}$ is more risk averse: that is

$$\text{for } w \in (0, \infty), \quad \hat{A}(w) \geq A(w). \tag{38}$$

[2] Meyer and Ormiston study second-order conditions in the case that only the deductible is variable. Their result is that the objective is quasi-concave. It cannot be extended to the present situation because the objective here possesses a local minimum at zero under fair prices, and at small positive values of b and d under positive loading.

Form a weighted sum for $a \in [0,1]$

$$u(a;w) = a\hat{u}(w) + (1-a)u(w) \tag{39}$$

It can be checked that the absolute risk aversion satisfies

$$\begin{aligned} \text{for } a &\in (0,1], \;\; A(a,w) > A(w) \\ \text{for } a &\in (0,1), \;\; A_a(a,w) > 0 \end{aligned} \tag{40}$$

Corollary 5 *Suppose that $u(a,w)$ is the utility function in the problem in equations (35 and 36) and $b(a)$ is the solution. Then $b'(0) > 0$.*

Proof. Let

$$F_b(a,b) = \int_{d+b}^{T} u_w(a,w)dH - (1+\lambda)\int_{d+b}^{T} dH \int_{t=0}^{T} u_w(a,w)dH \tag{41}$$

so that

$$F_b(a,b(a)) = 0 \tag{42}$$

defines the solution function $b(a)$. Then

$$b'(0) = \frac{-F_{ba}(0,b(0))}{F_{bb}(0,b(0))} \tag{43}$$

Because the $b(0)$ is an optimum for the utility u, the objective function must be concave and therefore the denominator is negative. Differentiate to find that the numerator is

$$F_{ba}(0,b(0)) = \int_{d+b}^{T} u_{wa}dH - (1+\lambda)\int_{d+b}^{T} dH \int_{t=0}^{T} u_{wa}dH \tag{44}$$

From equation (39)

$$u_{wa}(a,w) = \hat{u}'(w) - u'(w) \tag{45}$$

Expand and substitute

$$\begin{aligned} F_{ba}(0,b(0)) &= \int_{d+b}^{T} [\hat{u}' - u']dH(1 - (1+\lambda)\int_{d+b}^{T} dH) \\ &\quad -(1+\lambda)\int_{d+b}^{T} dH \left[(\hat{u}'(v-x_T) - u'(v-x_T)) \int_{0}^{d+b} dH \right] \end{aligned} \tag{46}$$

Because von Neumann-Morgenstern utility is unique only up to affine transformation, it can be assumed without loss of generality that

$$\hat{u}'(v - x_T) = u'(v - x_T) \tag{47}$$

The final term in equation (46) vanishes. All the arguments in the domain $(d + b, T]$ are for wealths less than $v - x_T$. The more risk averse utility function has higher marginal utility in that range. The term $1 - (1 + \lambda) \int_{d+b}^{T} dH$ is positive because, were it negative, the reduction of one dollar on $(d + b, T]$ would save more than a dollar on the entire domain $[0, T]$. Therefore

$$F_{b\alpha}(0, b(0)) > 0 \tag{48}$$

It follows that $b'(0) > 0$. ■

Parameters:

In the problem studied here, the deductible is fixed at $\bar{d} = d(\lambda, v, q)$, and at that point the condition for optimum b is,

$$F_b(\bar{d}, b; \lambda, v, q) = 0 \tag{49}$$

The solution at a maximum of this problem is denoted $\hat{b}(\lambda, v, q; \bar{d})$. There are other solutions to the first-order conditions, representing local minima with low levels of the upper limit, but the focus here is solely on the maximum. The local behavior of $\hat{b}$ at the maximum is given in the usual way by

$$\hat{b}_\lambda = -\frac{F_{b\lambda}}{F_{bb}} \tag{50}$$

Through the assumption on the matrix of second partials, $F_{bb} < 0$. It is shown in the appendix that the pure price effect of λ on F_b is negative so that for CARA utility and moderate degrees of DARA utility, $F_{b\lambda} < 0$. It is conceivable, as usual, that the income effect of a high degree of DARA would overwhelm the price effect and change the sign of $F_{b\lambda}$, but that case is not considered here. Thus $\hat{b}_\lambda$ is negative.

The main finding is that $\hat{b}(\lambda, v, q; \bar{d})$ and $b(\lambda, v, q)$ have similar behavior in the neighborhood of the point $(\lambda, v, q; \bar{d})$. Start with coinsurance rate λ. The derivatives of solutions to the fully optimized system are connected by the equation (among others)

$$F_{bd} d_\lambda + F_{bb} b_\lambda = -F_{b\lambda} \tag{51}$$

Dividing through by F_{bb} and substituting from equation (50) gives

$$\hat{b}_\lambda = b_\lambda + d_\lambda \frac{F_{db}}{F_{bb}} \tag{52}$$

To this point the derivation uses no facts specific to the situation under study. The salient fact from previous sections is that the optimum satisfies $b + d = q$, which implies

$$b_\lambda = -d_\lambda \tag{53}$$

The consequence is that

$$\hat{b}_\lambda = b_\lambda (1 - \frac{F_{db}}{F_{bb}}) \tag{54}$$

From corollary 4, it is known that the unconstrained solution satisfies $b_\lambda < 0$, and from equation (50) and the ensuing discussion the constrained solution is also negative. Therefore, the multiplier connecting $\hat{b}_\lambda$and b_λ in equation (54) must be positive.

The remaining task is to show that the multiplier is less than unity. The fact that $F_{db} < 0$, when the utility function possesses constant absolute risk aversion (CARA) or decreasing absolute risk aversion (DARA) is shown in the appendix. Thus $\frac{F_{db}}{F_{bb}}$ is positive and, from the positivity of the multiplier, it is less than unity. Thus the multiplier in equation (54) must be a positive number and less than unity. Summarizing the results so far, the effect of increased loading is to reduce the upper limit on insurance. In the case of fixed deductible, the reduction is less.

Turning to the effect of increased equity v, the derivation is the same as far as the results

$$\hat{b}_v = -\frac{F_{bv}}{F_{bb}} \tag{55}$$

and

$$\hat{b}_v = b_v (1 - \frac{F_{db}}{F_{bb}}) \tag{56}$$

The specific information needed here is, as shown in the appendix, in the case of DARA,

$$F_{bv} < 0 \tag{57}$$

In case of CARA, the value is zero. That means $\hat{b}_v < 0$ in DARA and $\hat{b}_v = 0$ in CARA. Because the term in parentheses in equations (54) and (56) is known to be positive and less than unity, b_v and $\hat{b}_v$ are zero under CARA and negative under DARA. Moreover, under DARA the response of the constrained $\hat{b}_v$ is less than that of the unconstrained b_v.

Variation in the conversion point is slightly different because the condition that $d + b = q$ translates to

$$d_q + b_q = 1 \tag{58}$$

Consequently the ending formulas are

$$\hat{b}_q = -\frac{F_{bq}}{F_{bb}} \tag{59}$$

$$\hat{b}_q = \frac{F_{db}}{F_{bb}} + b_q(1 - \frac{F_{db}}{F_{bb}}) \tag{60}$$

From corollary 4, the sign of b_q is positive. Then the sign of $\hat{b}_q$ is also positive because each of the additive components is. Under CARA, $b_q = 1$ and consequently, from the preceding equation $\hat{b}_q = 1$. Under DARA, $b_q > 1$ and $b_q > \hat{b}_q > 1$. Thus, again, the response of the constrained upper limit is in the same direction, but of lesser magnitude, than the response of the unconstrained upper limit.

5 Concluding remarks

The question of why insurance contracts have upper limits was once a puzzling one. In some applications, an adverse selection argument applies, that is, insurers impose upper limits to screen out bad risks. The argument doesn't work for insurance of real property because there typically it is the client who determines the upper limit. Instead, the reason for upper limits is that loss of wealth is bounded and may be much less than the cost to repair damage. Scrupulous consumers do not want insurance for damage beyond the point at which loss of wealth ceases, and prudent insurers do not want to supply it.

Why are contracts written for damage and not for loss of wealth? The reason is that damage is measurable, tangible, and therefore contractible. Damage – the cost to restore property to its pre-damage condition – is relatively easy to estimate, and it can be assessed by methods that do not vary much from one instance to another. In contrast, loss of wealth is a nebulous concept because of uncertainty about the conversion point. Resolving the uncertainty is costly. A more puzzling question is why insurance contracts are not written contingent on conversion. Why not make an explicit link between indemnity and the realized conversion value instead of relying on an upper limit? The reason must be that conversion values cannot be specified in a way that makes the contracts enforceable. The conversion value is never known in a way that could be enforced by an impartial court of law.

These points are illuminated by making a contrast. In the closely associated case of automobile collision insurance, conversion values are in effect contractible. Since the car does not occupy a piece of land, the associated land value is zero. Moreover, the value of the car is known from the so-called "blue-book" values, which are based upon numerous transactions in cars of every make, model, year, and condition. Within the collision contract the client does not choose an upper limit. The upper limit is understood by contract to be the future blue-book value, a random variable whose realized value can be enforced by an future court. Further damage can occur after the maximum loss of wealth is reached, as happens when a car worth $2000 in the blue book receives damage that would cost $4000 to repair. The automobile collision contract covers this situation without difficulty. In contrast, there are no blue book values for homes and vacant land, and insurance contracts for real property are not contingent on conversion values.

A question raised in the introduction is what model of insurance of real property is most appropriate for further research on the effects of background risks? Coinsurance models are not attractive because there is little coinsurance in practical insurance of real property and none in the optimum contract. The fully optimum contract itself has unrealistic features such as discontinuous deductibles, round shoulders, and a tight link between the upper limit and the deductible. That tight link does not exist in practice because neither the insurer nor the client knows with any accuracy what the conversion point will be at that later time, if any, when the insurer must pay the indemnity. Compared to the fully optimum model, the fixed-deductible model is more realistic in appearance and has essentially the same comparative statics. Study of insurance of real property using the fixed-deductible model is undertaken in Garratt and Marshall (1999A).

6 Appendix

Proof. [Proof of Corollary 4] Two equations link d, x_T, and λ, namely, from equation (17) and other parts of the derivation

$$u'(v - x_T - d)(\frac{1}{1+\lambda}) - \int_0^T u'(v - x_T - \min[t, d])dH = 0 \tag{61}$$

The other condition is, from the definition of the terminal value x_T, the premium,

$$(1+\lambda)\int_d^T (\min[t, q] - d)dH - x_T = 0 \tag{62}$$

The problem has the form

$$\begin{aligned} J_1(d, x_T; \lambda, v, q) &= 0 \\ J_2(d, x_T; \lambda, v, q) &= 0 \end{aligned} \tag{63}$$

and comparative statics come from the system:

$$\begin{bmatrix} dd \\ dx_T \end{bmatrix} = \frac{1}{J_{11}J_{22} - J_{12}J_{21}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \cdot \begin{bmatrix} -J_{13} & -J_{14} & -J_{15} \\ -J_{23} & -J_{24} & -J_{25} \end{bmatrix} \cdot \begin{bmatrix} d\lambda \\ dv \\ dq \end{bmatrix} \tag{64}$$

To begin, then

$$\begin{aligned} J_{11} &= -u''(v - x_T - d)\left[\frac{1}{1+\lambda} - \int_d^T dH\right] \\ J_{12} &= -u''(v - x_T - d)\frac{1}{1+\lambda} + \int_0^T u''(v - x_T - \min[t, d])dH \\ J_{21} &= -(1+\lambda)\int_d^T dH \\ J_{22} &= -1 \end{aligned}$$

The coefficient $J_{11} = 0$ if $\lambda = 0$ because the optimum deductible there is zero. Otherwise, the sign of J_{11} is positive for all $\lambda > 0$, which is a consequence of the fact that

$$\frac{1}{1+\lambda} - \int_d^T dH > 0 \tag{65}$$

Suppose for purposes of contradiction that the quantity is negative. Then it could be rewritten as

$$1 < (1+\lambda)\int_d^T dH \tag{66}$$

With loading λ, the right hand term is the saving in certainty dollars of a one dollar reduction in insurance coverage on the interval $[d, T]$. Since the saving is more than a certainty dollar, the consumer would choose to reduce coverage. Therefore this can never happen in an optimum.

The sign of J_{12} requires study. It turns out that in CARA, $J_{12} = 0$, and in DARA, $J_{12} > 0$. See that by denoting absolute risk aversion by

$$A(w) = \frac{-u''(w)}{u'(w)}$$

and using it to rewrite as

$$J_{12} = -u'(v - x_T - d)A(v - x_T - d)\frac{1}{1+\lambda} \tag{67}$$

$$+\int_0^T A(v - x_T - \min[t,d])u'(v - x_T - \min[t,d])dH \tag{68}$$

Under CARA, the A's are the same and the quantity is

$$J_{12} = A(v - x_T - d) \cdot \left[u'(v - x_T - d)\frac{1}{1+\lambda} - \int_0^T u'(v - x_T - \min[t,d])dH\right] \tag{69}$$

From the condition of equation (61) it follows that $J_{12} = 0$. On the other hand, under DARA the coefficient is

$$J_{12} = A(v - x_T - d) \cdot \left[u'(v - x_T - d)\frac{1}{1+\lambda} - \int_0^T \frac{A(v - x_T - \min[t,d])}{A(v - x_T - d)}u'(v - x_T - t)dH\right] \tag{70}$$

Compare the term in square brackets to the condition of equation (61). They differ only in the presence of the factor under the integral sign, and that factor is never greater than unity and sometimes is less. Thus J_{12} is a positive number and is more positive as absolute risk aversion decreases more rapidly.

The other terms are

$$\begin{aligned}
J_{13} &= \frac{-u'(v - x_T - d)}{(1+\lambda)^2} \\
J_{14} &= u''(v - x_T - d)\frac{1}{1+\lambda} - \int_0^T u''(v - x_T - \min[t,d])dH \\
J_{15} &= 0 \\
J_{23} &= \int_d^T (\min[t,q] - d)dH \qquad (= P(d), \text{ the fair premium}) \\
J_{24} &= 0 \\
J_{25} &= (1+\lambda)\int_q^T dH
\end{aligned}$$

Note that J_{14} is the negative of J_{12} and consequently it is zero in CARA, negative in DARA, and more negative as risk aversion decreases more rapidly. Other signs are transparent.

The determinant for the matrix of first partials, denoted $\Delta = J_{11}J_{22} - J_{12}J_{21}$, reduces to

$$\begin{aligned}\Delta &= u''(v - x_T - d)(1+\lambda)\left[\frac{1}{(1+\lambda)} - \int_d^T h(t)dt\right]^2 + \\ &\quad (1+\lambda)\int_d^T h(t)dt \int_0^d u''(v - x_T - t)h(t)dt \qquad (71)\end{aligned}$$

Each additive term is non positive and therefore Δ is non positive. It is negative except in the special case that $\lambda = 0$ and CARA. In that case, Δ is zero because the deductible is zero and the condition in equation (61) is vacuous. The other condition – equation (62) – governs. Since the singular case is rather trivial, it is noted infrequently below. All statements exclude the case in which both CARA and $\lambda = 0$ are present.

The effect of increased loading is

$$\frac{dd}{d\lambda} = \frac{1}{\Delta}(-J_{22}J_{13} + J_{12}J_{23}) \qquad (72)$$

The first additive term is negative, and under CARA the second term is zero. Since Δ is negative, the pure price effect is, as expected positive. DARA makes the second additive term in parentheses into a negative quantity, tending to offset the pure price effect. If risk decreases too fast, the loss of wealth from increased loading stimulates risk aversion so much that more insurance is demanded, not less. Although the sign could become negative, the practical presumption is that increased loading raises the deductible.

The effect of more loading on the premium is also ambiguous, and this time assumptions about risk aversion are not clarifying

$$\frac{dx_T}{d\lambda} = \frac{1}{\Delta}(J_{21}J_{13} - J_{11}J_{23}) \qquad (73)$$

The first term in parentheses is positive. The second term is negative, but at $\lambda = 0$ the J_{11} term vanishes as discussed above and so does the second additive term. Thus at $\lambda = 0$, the net sign is negative because $\Delta < 0$. By continuity, the result holds in some neighborhood of zero.

Changing equity value v has only wealth effects. In particular

$$\frac{dd}{dv} = \frac{J_{14}}{\Delta} \qquad (74)$$

because J_{24} is zero. The term J_{14} is zero under CARA and therefore, absent a wealth effect, the deductible is unchanged. Under DARA, the expression in parentheses is negative. Therefore the net effect is positive. Increased equity raises wealth, lowers risk aversion, and hence raises the deductible, a pure wealth effect.

The effect of equity on premium is

$$\frac{dx_T}{dv} = \frac{J_{21}J_{14}}{\Delta} \tag{75}$$

From the notes on J_{14}, the effect is null under CARA and negative under DARA, in accord with the analysis of the deductible.

The effects of varying the conversion value are similar.

$$\frac{dd}{dq} = \frac{J_{12}J_{25}}{\Delta} \tag{76}$$

The effect is zero under CARA and negative under DARA. The interpretation is that raising the conversion value raises the risk and reduces expected wealth. The DARA consumer is poorer and therefore more risk averse, leading to decreased deductible. Finally, the effect on the premium is unambiguous

$$\frac{dx_T}{dq} = \frac{-J_{11}J_{25}}{\Delta} \tag{77}$$

The effect is always positive. Intuitively, the increase in exposure always induces an increased expenditure on premium. That completes the proof.

The results for the upper limit are found from through the equation $b = q - d$. The mainly interesting ones are, for the case of DARA,

$$\begin{aligned} \frac{db}{d\lambda} &< 0 \\ \frac{db}{dv} &< 0 \\ \frac{db}{dq} &> 1. \end{aligned} \tag{78}$$

■

Signs of some cross partials: The objective is

$$\begin{aligned} F(d,b;\lambda,v,q) = \int_0^d u(v - x_T - t)dH + \int_d^{d+b} u(v - x_T - d)dH \\ + \int_{d+b}^T u(v - x_T - \min[t,q] + b)dH \end{aligned} \tag{79}$$

where

$$x_T = (1+\lambda)\left[\int_d^{d+b}(t-d)dH + b\int_{d+b}^T dH\right] \tag{80}$$

$$F_b(d,b;\lambda,v,q) = u'(v-x_T-q+b)\int_{d+b}^T dH - (1+\lambda)\int_{d+b}^T dH \cdot \int_0^T u'()dH \tag{81}$$

$$F_d(d,b;\lambda,v,q) = -u'(v-x_T-d)\int_d^{d+b} dH + (1+\lambda)\int_d^{d+b} dH \cdot \left[\int_0^{d+b} u'()dH + \int_{d+b}^T u'()dH\right] \tag{82}$$

$F_{db} < 0$

From the previous equation,

$$\begin{aligned} F_{db}(d,b;\lambda,v,q) &= -u'(v-x_T-d)h(d+b) + (1+\lambda)h(d+b)\cdot\left[\int_0^{d+b} u'()dH + \int_{d+b}^T u'()dH\right] \\ &\quad +(1+\lambda)\int_d^{d+b} dH \cdot \int_{d+b}^T u''()dH \\ &\quad + \left[\int_d^{d+b} u''(v-x_T-d)dH - (1+\lambda)\int_d^{d+b} dH \cdot \int_0^T u''()dH\right]\left[(1+\lambda)\int_d^{d+b} dH\right] \end{aligned} \tag{83}$$

From equation (82) the first line is zero. The middle line is negative. Using $A()$ for absolute risk aversion, the last line is

$$-\left[\begin{array}{c} A(v-x_T-d)u'(v-x_T-d)\int_d^{d+b} dH \\ -(1+\lambda)\int_d^{d+b} dH \cdot \int_0^T A()u'()dH \end{array}\right]\left[(1+\lambda)\int_d^{d+b} dH\right] = 0 \tag{84}$$

Under DARA, A is a maximum at $v - x_T - d$, and by comparison to equation (82), the first expression in square brackets is positive. Thus the whole last line is negative and $F_{db} < 0$ under CARA and DARA.

$F_{b\lambda} < 0$ **except when DARA is too large** From equation (81),

$$\begin{aligned} &F_{b\lambda}(d,b;\lambda,v,q) = \\ &-\left[u''(v-x_T-q+b)\int_{d+b}^T dH - (1+\lambda)\int_{d+b}^T dH \cdot \int_0^T u''()dH\right] \\ &\cdot\left[\int_d^{d+b}(t-d)dH + b\int_{d+b}^T dH\right] - \int_{d+b}^T dH \cdot \int_0^T u'()dH \end{aligned} \tag{85}$$

The last term is negative, the price effect. Under CARA, the first term is square brackets is

$$A\left[u'(v-x_T-q+b)\int_{d+b}^{T}dH-(1+\lambda)\int_{d+b}^{T}dH\cdot\int_{0}^{T\prime}u'()dH\right] \tag{86}$$

because its is the same as the first order condition of equation (81). More generally, the term is

$$\left[A(v-x_T-q+b)u'(v-x_T-q+b)\int_{d+b}^{T}dH-(1+\lambda)\int_{d+b}^{T}dH\cdot\int_{0}^{T\prime}A()u'()dH\right] \tag{87}$$

Under DARA, A is a maximum at $v-x_T-q+b=v-x_T-d$. Therefore the term is positive, and it is more positive as the decrease in risk aversion is more rapid. In extreme cases, the income effect might overcome the price effect. Otherwise, $F_{b\lambda}<0$.

$F_{bv}<0$ **under DARA and** $=0$ **under CARA** From equation (81),

$$F_{bv}(d,b;\lambda,v,q)=u''(v-x_T-q+b)\int_{d+b}^{T}dH-(1+\lambda)\int_{d+b}^{T}dH\cdot\int_{0}^{T}u''()dH \tag{88}$$

Substituting for absolute risk aversion A,

$$F_{bv}(d,b;\lambda,v,q)= \\ -\left[\begin{array}{c} A(v-x_T-q+b)u'(v-x_T-q+b)\int_{d+b}^{T}dH \\ -(1+\lambda)\int_{d+b}^{T}dH\cdot\int_{0}^{T}A()u'()dH\end{array}\right] \tag{89}$$

Under DARA, A is a maximum at $v-x_T-q+b=v-x_T-d$. By comparison to the first-order condition of equation (81), the expression in brackets is positive. Under CARA it is zero. Thus the result.

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