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Stores, Prices, and Currency Substitution

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This Version: January 2000

Abstract.

We study endogenous currency substitution in a decentralized trade environment. Sellers maximize profits from sales of imperfectly substitutable goods by posting prices in either one of two currencies. A unique symmetric equilibrium exists where goods are priced only in the local currency. This occurs if foreign trade is sporadic, there is sufficient but not excessive liquidity, and discounting is low. Excess or scarcity of liquidity, however, induces sellers to extract all surplus from buyers. This destroys the monetary equilibrium and shuts down trade. Equilibria with and without currency substitution coexist on some region of the parameter space, and may be multiple. We prove that purchasing power parity may hold even if foreign trade is costly and the currency's value differs across countries. International circulation of money may expand the extent of the market hence enhance welfare.

(*) PRELIMINARY NOTES. Comments welcome. We gratefully acknowledge the financial support of Purdue's CIBER, and the Purdue Research Foundation. Corresponding author Gcamera@mgmt.purdue.edu.

1. Introduction

We develop a theoretical study of the determinants of currency substitution in a two-country setting where trade is decentralized and subject to frictions. By currency substitution we denote a situation in which different currencies are used as media of exchange in an economy.[1] Ours is a study of its theoretical underpinning, and it complements a limited number of recent papers on multi-currency economies where trade is decentralized and currencies are not primitives (see Matsuyama et Al, 1995, and Trejos and Wright, 1996, 2000). We do so by means of a search-theoretic general equilibrium model where two types of inconvertible and intrinsically worthless pieces of paper have value *in equilibrium* in that they can be exchanged for goods whose consumption generates utility.

Because search-theoretic models have been so successful in describing how some fiat money can come to be used in an economy, an obvious next step is to explain what economic parameters affect its acceptability abroad. In particular we ask the following questions. Under what conditions will a foreign currency become a domestic medium of exchange, how will prices be affected, and what are its welfare implications? Are international trade frictions sufficient to cause departures from the law of one price? Can the return from holding a currency differ across regions where its purchasing power is identical?

We show that it is the liquidity services provided by a currency (relative to another) that may induce currency substitution. We also prove that, even in a stylized trade environment as ours, departures from purchasing power parity are not *necessarily* due to the presence of decentralized exchange, inter-regional trade impediments, or differing currency valuations. Rather, they appear to stem from pricing decisions of profit maximizing sellers who sometimes may find it worthwhile to price discriminate against those buyers holding a foreign currency. In fact, we prove that both currencies will complement each other as means of payment within the same region when this improves the extent of the market, in which case they may (or may not) be treated as perfect substitutes. This, in turn, implies that currency substitution may have welfare enhancing effects when there is scarcity of local liquidity.

2. Multiple Currencies in Models of Decentralized Trade.

While the analysis of general equilibrium models with two-country and two-currencies has been the subject of a large number of studies, the greatest majority of them has focused on economic environments where both national and international trade takes place in organized markets.[2] It is well known, however,

[1] Currency substitution, sometimes referred to as "dollarization", has been defined differently in the literature, ranging from a very broad to a very narrow definition. For a survey see Calvo and Vegh (1996) and Giovannini and Turtleboom (1992).

[2] Monetary exchange with several fiat objects has alternatively been motivated by some form of market incompleteness (see the overlapping generations model of Kareken and Wallace, 1981, on indeterminacy of exchange rates), the existence of Clower constraints (see for instance the work of Lucas, 1982, on the determination of prices and exchange rates), or by placing assets into utility functions (see the work of Calvo, 1985, on currency substitution).

that only some commodities (and few manufactured goods) are traded simultaneously and multilaterally on organized exchanges.

A more congenial representation of the trade process may perhaps be described by a search process according to which spatially dispersed sellers are matched to buyers, at different points in time. Molding exchange as a decentralized process, in turn, has implications for international trade flows and policies, relative prices, the purchasing power and the circulation of different currencies in different economic regions (for instance, because of their proximity). These issues have been recently recognized not only by trade theorists (see the work of Rauch, 1996 and 1999), but also by monetary theorists. In an influential paper Matsuyama et Al. (1995) have developed a general equilibrium model of decentralized trade with multiple fiat currencies and multiple countries. In it, they provide conditions for a currency to circulate internationally, under the assumption of fixed terms of trade. Trejos and Wright (1996, 2000) have subsequently extended this early model by incorporating endogenous price determination with the objective of characterizing the currencies' relative purchasing power in the presence of different amounts of liquidity, trade frictions, and monetary policies.[3]

Crucial in Trejos and Wright's analysis is the assumption of a bargaining protocol according to which buyers (who hold indivisible money) make a take-it-or-leave-it quantity offer to sellers (who have no money) who, consequently, receive zero surplus in any trade (and zero lifetime utility). While this simple pricing rule simplifies the model substantially and provides a good benchmark, it also has some consequences for the existence and nature of equilibria. First, currency substitution is not an equilibrium unless the countries are sufficiently identical economies.[4] Additionally, when both currencies circulate in both countries, prices are independent of the currency used and hinge *exclusively* on the seller's nationality. It follows that currencies have always equal purchasing power within a country, but unequal purchasing power across countries (unless they are identical economies). Surprisingly, these results are completely reversed when currency substitution takes place only in one economy. In such a case purchasing power parity may hold but only if the countries are *not* symmetric, and the two currencies will *never* be perfect substitutes in the economy where both have value (the foreign currency having greater purchasing power). Finally, the value of a currency (hence its purchasing power) is independent of the value of the competing money, even in equilibria with currency substitution.

We address these shortcomings by adopting a different (and probably more reasonable) pricing protocol

[3] Random matching environments have recently been used to study nominal exchange determination when currency exchange is also possible and prices are either endogenous (Craig and Waller, 1999, and Head and Shi, 1998), or exogenous (Zhou, 1997). Craig and Waller, in particular, study currency portfolios numerically and characterize the distribution of nominal exchange rates.

[4] On the other hand, as the authors point out, the bargaining protocol adopted implies that sellers would *always* be willing to accept any currency, however small the associated return. Hence currency substitution would be the *only* equilibrium unless barter is possible, or bargaining or other transaction costs (direct or indirect) are borne by sellers.

based on the ordinary observation that in retail transactions it is the seller who chooses both the currency in which to trade and the price(s) to post in his store. The analysis is carried out in a prototypical search model where several types of sellers and buyers participate in random pairwise monetary exchange both locally or across two different regions (or countries). Endogenous price formation is the result of sellers choosing the prices to be posted in any currency they desire. A seller specializes in the production of one type of good, and essentially acts as a specialty "store" attempting to maximize the profit from sales to buyers which are homogenous across countries but rank goods differently. Buyers' type and nationality, however, is unobserved so that stores may only take into account the currency offered, hence post prices to maximize the profit from *expected* sales in each currency denomination. This can potentially generate price discrimination over currencies, whereby different quantities of goods (or none at all) are sold for different currencies.

The main consequence is that the purchasing power of a currency is a function of both its denomination *and* the location of stores (domestic or foreign). We prove existence of equilibria where domestic stores post prices in terms of both the national and the foreign currency (or vice-versa). In doing so, we confirm and extend the work of Trejos and Wright (1996, 2000) by showing that prices are perfectly flexible and respond to trade frictions and changes in the quantity of money in both countries. However, we prove that trade frictions are not sufficient *per se* to generate purchasing power inequality across countries. In fact, we provide conditions for the existence of a class of equilibria with both currency substitution and purchasing power parity even with fundamentally different economies and substantial international trade frictions. In one instance we show that the foreign currency may have identical purchasing power across countries, even if it is valued *more* domestically than abroad, and even in the presence of "home good bias" (with domestic stores giving discounts to buyers making a purchase with domestic currency). Because equilibria with local and international currencies coexist, we also show that currency substitution may lead to a welfare superior outcome.[5]

The contribution of this study, however, goes beyond proving existence of equilibria with currency substitution. By adopting a seller-posting-price protocol we also extend and complement previous work on existence of monetary equilibria in search theoretic models of money with endogenous price formation (see Trejos and Wright, 1995, Shi, 1995), with some novel results.[6] Specifically, we find that, unlike nearly all previous random matching models, symmetric monetary equilibria may exist only if the available amount of currency is neither excessive nor too limited, and if individuals are sufficiently patient. Prices *may* be efficient even in the presence of a positive discount factor, in the sense that there may be a quantity of money

[5] Kocherlakota and Krueger (1998), argue that it may be socially beneficial to have multiple national moneys. This is obtained in a random matching model where preferences are defined over the nationality of goods. It follows that if preferences are sufficiently heterogeneous over the different nationalities of goods, having two moneys leads to a Pareto superior allocation.

[6] See also the recent paper of Soller and Wright (1999).

such that the output sold is equivalent to the solution to a social planner's problem. We also find that prices fall for a marginal increase in the measure of buyers because of the increased extent of the market faced by sellers. A high degree of impatience or an excessively large proportion of buyers, however, would induce a seller to increase prices and cater to a smaller segment of the demand for his good. Most remarkably, there is a liquidity threshold beyond which sellers would charge premium prices with the intent to extract all surplus from those buyers who have the highest valuation for the commodity offered. In a symmetric equilibrium, this behavior would deprive money of value. This would also occur for limited amounts of currency, when too infrequent sales would induce the posting of premium prices, even in the absence of time discounting.

We present the environment in the following section, and discuss the symmetric stationary equilibrium strategies, value functions,and distribution of money in section 4. Section 5 focuses on equilibria with local currencies, and currency substitution. Section 6 concludes.

3. Environment

Time is continuous and goes on forever. There is a continuum of individuals and goods of measure one. Individuals and goods may originate from one of two countries, denoted by $d \in \{D, F\}$: a fraction $P < 1$ of individuals belongs to the domestic economy, D, and the remaining fraction $1 - P$ lives in the foreign economy, F. In each country there are N types of agents and N types (or sets) of goods, each of identical proportion $\frac{1}{N}$. Let $i \in \{1, .., N\}$ denote an individual of type i and let $n \in \mathbf{N} \equiv \{1, .., N\}$ denote a good belonging to set n.

Individuals derive different utility from each different subset (type) of goods available. All agents have the same kind of *cardinal* preferences but their *ordinal* preferences over types of goods differ (see Deneckere and Rothschild, 1992) so that consumption of an identical quantity of different goods provides identical utility to different agents, only if goods are identically ranked. For example, individuals of two different types derive the same utility from consumption of an identical quantity of their most preferred good, but their preferences differ in the rank ordering of every type of commodity. For tractability we specify preferences such that the resulting distribution over the rank ordering of good types is uniform, and symmetric. More specifically, preferences of i are defined on the N sets of goods produced in the economy, with ranking defined as monotonically decreasing in $i + k$ (modulo N), for $k = \{0, .., N - 1\}$. That is, preferences over good types are ordered clockwise so that individual i likes the most any good in the set $n = i$, and the least goods in the set $i - 1$ (modulo N). The temporary utility of i from consumption of q units of good n is

$$u_i(q_n) = (q a_{i,n})^{\gamma}, \tag{1}$$

where $\gamma \in (0, 1)$, $a_{i,n} = a_{i+1,n+1}$ $\forall i, n$ (mod. N) and $1 = a_{i,i} > a_{i,i+1} > ...a_{i,N} > ...a_{i,i-1} = 0$ $\forall i$ (mod. N). Here $a_{i,n}$ characterizes $i's$ marginal valuation for good n, and captures the degree of differentiation of goods.

We will refer to "higher valuation" buyers as those individuals having larger $a_{i,n}$, relative to buyer $j \neq i$. An extreme case of this is the ordering of Kiyotaki and Wright (1989), where $N = 3$, $a_{i,n} = 1$ if $n = i$, and zero otherwise. Note that the resulting distribution over the rank ordering of goods is uniform and symmetric across countries, types of individuals, and goods.

After having consumed, any agent i can produce q units of a perishable good of type $i - 1$ (mod. N) suffering disutility $c(q) = q$. Initially a measure $M \in (0, 1)$ of the population is randomly and identically endowed with one unit of indivisible fiat money, domestic or foreign. We denote the initial supply of domestic (foreign) currency by M_D (M_F), thus

$$M_D + M_F = M. \tag{2}$$

For the sake of tractability we impose a unit storage upper bound on money holdings, assume that production cannot take place when holding money, and rule out free disposal (as in Kiyotaki and Wright, 1993). The remaining $1 - M$ fraction of individuals is initially able to produce. Since own production does not provide utility ($a_{i,i-1} = 0$), exchange is required for consumption to take place, so we call buyers the agents with money, and sellers the others. Let m_d (m'_d) be the proportion of domestic (foreign) traders with currency d. Hence

$$m = m_D + m_F \tag{3}$$

$$m' = m'_D + m'_F \tag{4}$$

represent, respectively, the proportion of *buyers* in the domestic and foreign economy. In equilibrium, all money must be held:

$$Pm_d + (1 - P)m'_d = M_d, \quad d \in \{D, F\}, \tag{5}$$

therefore a proportion $1 - m$ of domestic individuals are *sellers* ($1 - m'$ in the foreign country). Since money is identically distributed across types m (m') also denotes the proportion of domestic (foreign) buyers of type i.

The economy suffers from an extreme restriction on coalition formation in that only bilateral matches are allowed, hence each match experiences absence of double coincidence of wants (because of preferences and technologies). While the partner's inventory and the actions taken in the match are perfectly observable, trading histories are private information (a feature that rules out credit), and so are the type (i.e. preferences) and nationality. These last features will play a crucial role in the determination of prices, while the other assumptions have the main implication that trade needs to be facilitated by money.

It is assumed that sellers are located at fixed points (see Burdett, Trejos and Wright, 1995 for a motivation of this feature), while buyers search for sellers and meet them randomly and bilaterally at each date according to a Poisson process with arrival rates which differ across countries: domestic (foreign) buyers meet domestic sellers at rate α_D (α'_D), and foreign sellers at rate α_F (α'_F). Since the frequency of matches between traders in country D and F must be identical to the frequency of matches between traders in country F and D, then α_F and α'_D must satisfy

$$P\alpha_F = (1-P)\alpha'_D.$$

For this reason we let

$$\alpha_D = Pk, \;\; \alpha_F = (1-P)k', \;\; \alpha'_F = (1-P)k, \;\; \alpha'_D = Pk' \tag{6}$$

which satisfy the restrictions above, where $k' > 0$ denotes the degree of economic integration between countries, while $k > 0$ determines the matching rate within a country.[7]

4. Symmetric Stationary Equilibria

We consider stationary equilibria, where identical individuals adopt identical time-invariant strategies, and where the distribution of objects is stationary. We will denote the foreign country by a prime, strategies by small Greek letters, the strategies of a specific individual by the corresponding capital letter, and equilibrium strategies and best responses by a star. Finally, we consider rational expectations equilibria where the individual's conjecture of distributions is correct in equilibrium, and where the expected payoffs are based on the correct evaluation of the probability and gains from trade.

4.1 Strategies and Prices

An individual i can be either a buyer or a seller depending on her holdings of currency. Contingent on her state, she choose her actions in order to maximize the expected lifetime utility she derives from consumption of goods. In doing so she takes as given the trade opportunities available in the economy, the strategies of others, and the distribution of prices. If at date t she is a seller, her strategy encompasses two choices: (i) which currency to accept and, contingent on that, (ii) the price(s) she intends to post. If she is a buyer, her only relevant choice is whether to buy (at the specified price) the commodity offered by a randomly encountered seller.

Specifically, consider a domestic seller i. At each date she chooses her strategy taking as given the strategies of all others, the value functions, and the distribution money. Recall also that she does not observe type and nationality of he buyers she meets. Since she has to choose whether to sell in exchange for currency, and what quantity to offer, we break her problem in two sub-components, moving backward. Let $Q_{d,i-1} \geq 0$ denote her offer of commodity $i-1$, *contingent on* the fact that she accepts currency d. In this way we can denote by $\frac{1}{Q_{d,i-1}}$ the price she posts in terms of the unit of currency d. Also let $Q = \{Q_{d,i}\}_{\forall d,i}$

[7] These two parameters essentially control the ease of "local" versus "regional" exchange. We can choose P, k, and k' so that, for instance, $\alpha_D > \alpha_F$ but $\alpha'_F < \alpha'_D$, i.e. in country D regional trade is more difficult that local, while the opposite is true for country F.

encompass the choices of quantities for each possible i, d combination (Q' when sellers are foreign). The offers of all other sellers, taken as given, are represented by the vectors $q = \{q_{d,i}\}_{\forall i,d}$, and $q' = \{q'_{d,i}\}_{\forall i,d}$.For any given an quantity offer, let Π_d be the probability that the domestic seller accepts currency d.[8] Let $\Pi = \{\Pi_D, \Pi_F\}$, and $\Pi' = \{\Pi'_D, \Pi'_F\}$ if the seller is foreign. The acceptance strategies of all others, are denoted by $\pi = \{\pi_D, \pi_F\}$ and $\pi' = \{\pi'_D, \pi'_F\}$.

Consider a domestic buyer i holding currency d. Given the quantity offered by a randomly encountered domestic seller h, $q_{d,h}$, buyer i chooses whether to buy or whether to pass on and wait for a better price offer, a strategy we denote by $B_{d,i}(q_{d,h}) \in [0, 1]$. In making her decision she takes as given the distribution of prices in all other possible matches, the strategies of everyone else, the value functions, and the distribution of money. Since buyer i can meet also foreign sellers, we defined the domestic (foreign) buyer's strategy vector $B = \left\{B_{d,i}(q_{d,h}), B_{d,i}(q'_{d,h})\right\}_{\forall i,h,d}$ (B'). The strategies of all other domestic (foreign) buyers are denoted by $\beta = \left\{\beta_{d,i}(q_{d,h}), \beta_{d,i}(q'_{d,h})\right\}_{\forall i,h,d}$ (β'). Finally, let $\boldsymbol{\beta} = \{\beta, \beta'\}$, $\mathbf{B} = \{B, B'\}$, $\boldsymbol{\pi} = \{\pi, \pi'\}$, $\mathbf{\Pi} = \{\Pi, \Pi'\}$, $\mathbf{q} = \{q, q'\}$, and $\mathbf{Q} = \{Q, Q'\}$.

4.2 Distribution of money

Consider the domestic country. The distribution of money holdings of currency d in that country, is stationary when the inflow of domestic sellers, who have received one unit of d (from foreign or domestic buyers) in exchange for their good, is equal to the outflow of domestic buyers who have bought some good (either in the domestic economy or abroad) with their currency d. This implies that in a symmetric strategy equilibrium, the law of motion for $d \in \{D, F\}$ must satisfy

$$\begin{aligned}\dot{m}_d \equiv \sum_{i \in \mathbf{N}} \frac{1}{N} \pi^*_d (1-m) \left[\alpha_D m_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta^*_{d,h}(q^*_{d,i-1}) + \alpha_F m'_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta'^*_{d,h}(q^*_{d,i-1})\right] \\ - \sum_{i \in \mathbf{N}} \frac{1}{N} m_d \left[\alpha_D (1-m) \pi^*_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta^*_{d,i}(q^*_{d,h}) + \alpha_F (1-m') \pi'^*_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta^*_{d,i}(q'^*_{d,h})\right]\end{aligned} \tag{7}$$

for domestic holders of currency d. For a foreign holder of currency d the law of motion is

$$\begin{aligned}\dot{m}'_d \equiv \sum_{i \in \mathbf{N}} \frac{1}{N} \pi'^*_d (1-m') \left[\alpha'_F m'_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta'^*_{d,h}(q'^*_{d,i-1}) + \alpha'_D m_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta^*_{d,h}(q'^*_{d,i-1})\right] \\ - \sum_{i \in \mathbf{N}} \frac{1}{N} m'_d \left[\alpha'_F (1-m') \pi'^*_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta'^*_{d,i}(q'^*_{d,h}) + \alpha'_D (1-m) \pi^*_d \sum_{h \in \mathbf{N}} \frac{1}{N} \beta'^*_{d,i}(q^*_{d,h})\right].\end{aligned} \tag{8}$$

[8] Note that we do not need to index Π_d by the seller's type, i, since we concentrate on symmetric equilibria, were sellers adopt identical strategies, indipendent of their type. We can do so because of the symmetry imposed on preferences, distribution of types i, and sets of goods. Note also that the strategy is non-discriminatory, since if seller i sells in exchange for one unit of currency d to a domestic buyer, he will also sell to a foreign buyer for the same amount of currency, given that buyers' nationality is private information.

In a stationary equilibrium

$$\dot{m}_d = \dot{m}'_d = 0 \ \ \forall d. \tag{9}$$

The explanation of the law of motion is simple. Consider (7), for instance. The proportion of domestic individuals holding currency d increases when a domestic seller finds it worthwhile to exchanges his good for the currency d held by either a domestic or a foreign buyer, in which case the seller sets $\pi_d^* = 1$. Since there is a proportion $\frac{1}{N}$ of each type i of sellers, $\sum\limits_{i\in\mathbf{N}} \frac{1}{N}(1-m)$ is the proportion of domestic sellers. They can meet domestic buyers of type h who hold currency d, at rate $\alpha_D m_d \frac{1}{N}$. These buyers agree to trade if $\beta_{d,h}^*(q_{d,i-1}^*) = 1$. Since there are N types of buyers who might want to purchase the good sold by any seller i at price $\frac{1}{q_{d,i-1}^*}$then $\alpha_D m_d \sum_{h\in\mathbf{N}} \frac{1}{N}\beta_{d,h}^*(q_{d,i-1}^*)$ represents the rate at which seller i meets buyers with currency d and willing to buy. Similar considerations can be made for matches with foreign buyers holding currency d. Hence the first line of (7) yields the total *inflow of domestic sellers* into domestic buyers with currency d. Now consider the second line. The proportion of domestic individuals holding currency d decreases when they buy some good h from either domestic or foreign sellers. There is a proportion $\frac{m_d}{N}$ of domestic individuals of type i holding money. They meet a domestic seller h at rate $\alpha_D(1-m)\frac{1}{N}$. He is willing to sell for currency d if $\pi_d^* = 1$, and buyer i buys if $\beta_{d,i}^*(q_{d,h}^*) = 1$. Since there are N types of sellers, then $\alpha_D(1-m)\pi_d^* \sum_{h\in\mathbf{N}} \frac{1}{N}\beta_{d,i}^*(q_{d,h}^*)$ is the rate at which a domestic buyer with currency d buys from domestic sellers. By adding all transactions with foreign sellers, yields the second line of (7) which thus represents the total *outflow of domestic buyers* with currency d, at each point in time.

Observe that the equations above simplify somewhat in a stationary symmetric equilibrium. Consider once again (7). The rates at which domestic sellers trade with domestic buyers holding currency d (part of the first line), and the rate at which domestic buyers holding currency d trade with domestic sellers (part of the second line) cancel out in equilibrium, since they are identical (domestic buyers and sellers just swap positions on who's holding currency d). Additionally, some multiplicative terms common to both remaining terms can be factored out, hence cancel out in the steady state ($\alpha_F \frac{1}{N}$). The same occurs for the two summations over i,since in a symmetric equilibrium $\sum\limits_{i\in\mathbf{N}} \frac{1}{N} \sum_{h\in\mathbf{N}} \beta_{d,h}^{'*}(q_{d,i-1}^*) = N\frac{1}{N}\sum_{h\in\mathbf{N}} \beta_{d,h}^{'*}(q_{d,i-1}^*)$ and $\sum\limits_{i\in\mathbf{N}} \frac{1}{N}\sum_{h\in\mathbf{N}} \beta_{d,i}^*(q_{d,h}^{\prime *}) = N\frac{1}{N}\sum_{h\in\mathbf{N}} \beta_{d,i}^*(q_{d,h}^{\prime *})$ for all i. Thus in a symmetric equilibrium equations (7) and (8) imply, respectively,

$$\dot{m}_d \propto \pi_d^*(1-m)m'_d \sum_{h\in\mathbf{N}} \beta_{d,h}^{'*}(q_{d,i-1}^*) - m_d(1-m')\pi_d^{'*} \sum_{h\in\mathbf{N}} \beta_{d,i}^*(q_{d,h}^{\prime *}) \tag{10}$$

$$\dot{m}'_d \propto \pi_d^{'*}(1-m')m_d \sum_{h\in\mathbf{N}} \beta_{d,h}^*(q_{d,i-1}^{\prime *}) - m'_d(1-m)\pi_d^* \sum_{h\in\mathbf{N}} \beta_{d,i}^{'*}(q_{d,h}^*). \tag{11}$$

Finally, let $\mathbf{m} = \{m_d, m'_d\}_{d\in\{D,F\}}$ denote the steady state equilibrium vector of the proportion of money traders in the economy.

4.3. Value Functions

In equilibrium, a domestic seller i choose his strategy in order to maximize his expected discounted lifetime utility. Let $V_{s,i}$ and $V_{d,i}$ denote the stationary expected lifetime utility of, respectively, a domestic seller and buyer i with currency d (a prime denotes a foreign individual). Taking as given the equilibrium strategies of all others, $\boldsymbol{\pi}^*$, $\boldsymbol{\beta}^*$, $\mathbf{q}^*$, and the distribution of money, $\mathbf{m}$, seller's i steady state value function must satisfy

$$rV_{s,i} = \sum_{d\in\{D,F\}} \max_{\{\Pi_d, Q_{d,i-1}\}} \Pi_d \left[\alpha_D \frac{m_d}{N} \sum_{h\in\mathbf{N}} \beta^*_{d,h}(Q_{d,i-1}) + \alpha_F \frac{m'_d}{N} \sum_{h\in\mathbf{N}} \beta'^*_{d,h}(Q_{d,i-1}) \right] (V_{d,i} - Q_{d,i-1} - V_{s,i}) \tag{12}$$

for a domestic seller i, and

$$rV'_{s,i} = \sum_{d\in\{D,F\}} \max_{\{\Pi'_d, Q'_{d,i-1}\}} \Pi'_d \left[\alpha'_D \frac{m_d}{N} \sum_{h\in\mathbf{N}} \beta^*_{d,h}(Q'_{d,i-1}) + \alpha'_F \frac{m'_d}{N} \sum_{h\in\mathbf{N}} \beta'^*_{d,h}(Q'_{d,i-1}) \right] (V'_{d,i} - Q'_{d,i-1} - V'_{s,i}) \tag{13}$$

for a foreign seller i.

Similarly, for a domestic buyer i holding currency d, her steady state value function must satisfy

$$\begin{aligned} rV_{d,i} = &\sum_{h\in\mathbf{N}} \max_{B_{d,i}(q^*_{d,h})} \alpha_D \frac{(1-m)}{N} \pi^*_d B_{d,i}(q^*_{d,h}) \left[V_{s,i} + u_i(q^*_{d,h}) - V_{d,i} \right] \\ &+ \sum_{h\in\mathbf{N}} \max_{B_{d,i}(q'^*_{d,h})} \alpha_F \frac{(1-m')}{N} \pi'^*_d B_{d,i}(q'^*_{d,h}) \left[V_{s,i} + u_i(q'^*_{d,h}) - V_{d,i} \right] \end{aligned} \tag{14}$$

while for a foreign buyer holding currency d her steady state value function must satisfy

$$\begin{aligned} rV'_{d,i} = &\sum_{h\in\mathbf{N}} \max_{B'_{d,i}(q^*_{d,h})} \alpha'_D \frac{(1-m)}{N} \pi^*_d B'_{d,i}(q^*_{d,h}) \left[V'_{s,i} + u_i(q^*_{d,h}) - V'_{d,i} \right] \\ &+ \sum_{h\in\mathbf{N}} \max_{B'_{d,i}(q'^*_{d,h})} \alpha'_F \frac{(1-m')}{N} \pi'^*_d B'_{d,i}(q'^*_{d,h}) \left[V'_{s,i} + u_i(q'^*_{d,h}) - V'_{d,i} \right] \end{aligned} \tag{15}$$

For instance, consider (12). A domestic seller of type i maximizes his expected discounted lifetime utility by choosing (i) whether or not to accept currency d, and (ii) given that he accepts it ($\Pi_d = 1$) the price at which he intends to sell his good, that is how much to produce in exchange for one unit of d ($Q_{d,i-1}$). Since the nationality of the buyer is private information the seller cannot discriminate across buyers, even if buyers of different nationalities may be willing to pay different prices. In posting his price the seller considers the expected profit, i.e. he will consider the expected demand from both domestic and foreign buyers. He may decide to charge a high price and capture only part of the world demand for his type of good, or charge a lower price, and sell to a larger subset of buyers. The functional equations shows that the seller meets a domestic buyer h holding currency d at rate $\alpha_D m_d \frac{1}{N}$. The buyer h agrees to buy good $i-1$ at the posted price, $\beta^*_{d,h}(Q_{d,i-1}) = 1$, only if he weakly prefers it to walking away. In that case the seller nets $V_{d,i} - Q_{d,i-1} - V_{s,i}$, since the seller suffers a production costs $Q_{d,i-1}$, and acquires currency d thus changing state, $V_{d,i} - V_{s,i}$. A similar gain is made when the seller meets a foreign buyer h with currency d and who is willing to buy.

Equation (14) describes the lifetime utility of a domestic buyer of type i holding currency d. She maximizes her lifetime utility by deciding whether to buy the goods offered, at the prices posted. She can buy both from domestic and from foreign sellers. meets She meets a domestic seller offering $q^*_{d,h}$ units of good h in exchange for currency d at rate $\alpha_D(1-m)\frac{1}{N}\pi^*_d$. If consumption of good h yields positive utility to buyer i, and if the price $\frac{1}{q^*_{d,h}}$ is not too large, buyer i will obtain a non-negative trade surplus from the transaction, in which case she will accept by setting $B_{d,i}(q^*_{d,h}) = 1$. Her flow payoff is given by the sum of the utility from consumption of the offered quantity of good h, $u_i(q^*_{d,h})$, plus the net continuation payoff $V_{s,i} - V_{d,i}$ since he changes state (from buyer to seller). Finally, let $\mathbf{V} = \left\{V_{s,i}, V'_{s,i}, V_{d,i}, V'_{d,i}\right\}_{i \in \mathbf{N}, d \in \{D,F\}}$ denote the vector of steady state value functions.

4.4 Equilibrium Strategies and Prices

We now discuss the equilibrium strategies of a representative individual (the case of a foreign trader is similar). Note that in choosing her optimal strategy as a buyer or a seller, the representative individual takes as given prices in all other matches, $\mathbf{q}$, strategies of all others, $\boldsymbol{\pi}$ and $\mathbf{B}$, distribution of money, $\mathbf{m}$, and value functions, $\mathbf{V}$. Recall also that we consider rational expectations equilibria where individual actions are based on the correct evaluation of the gains from trade in each possible match.

Buyer. In a random match with a seller h, a domestic buyer i holding currency d accepts to buy commodity h at the posted price $\frac{1}{q_{d,h}}$ only if her gains from trade are positive, and she rejects it otherwise so that $B_{d,i}(q_{d,h}) \in \{0,1\}$.[9] Specifically, her *best response* to the price offer of any randomly encountered domestic seller must satisfy

$$B^*_{d,i}(q_{d,h}) = \begin{cases} 1 & \text{if } V_{s,i} + u_i(q_{d,h}) - V_{d,i} \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Encounters with foreign sellers elicit identical responses, in which case $q'_{d,h}$ is the argument of (16). In a symmetric stationary equilibrium

$$\beta^* = B^*. \tag{17}$$

Seller. At each date domestic seller i chooses which currencies to accept and the quantities he is willing to produce in exchange. He does so to maximize his expected profit from trade. We find it convenient to subdivide the seller's problem in two sub-components, moving backward.

[9] Consideration of only pure strategies is without loss of generality since sellers can always induce "marginal" buyers to buy, by posting a price which leaves "marginal" buyers an $\varepsilon > 0$ small payoff. This is due to perfect divisibility of goods, as it will be clear in the following subsection.

(a) Contingent on accepting currency d, $\Pi_d = 1$, seller i chooses the quantity offer $Q_{d,i-1}$. The existence of private information over buyers' nationality and type implies the seller cannot observe buyers' ranking of preferences over the commodity offered, and their overall valuation of the trade proposed. Therefore i cannot extract the entire surplus from *each* buyer encountered, in general, but only from buyers of one specific type and nationality. In particular, seller i will choose $Q_{d,i-1}$ in order to maximize the *expected profit* from sales denominated in currency d

$$\Omega(Q_{d,i-1}) \equiv \left[\alpha_D \frac{m_d}{N} \sum_{h \in \mathbf{N}} \beta_{d,h}(Q_{d,i-1}) + \alpha_F \frac{m'_d}{N} \sum_{h \in \mathbf{N}} \beta'_{d,h}(Q_{d,i-1})\right] (V_{d,i} - Q_{d,i-1} - V_{s,i}). \tag{18}$$

Equation (18) shows that sales to both domestic and foreign buyers (holding currency d) contribute, in general, to $i's$ expected profit $\Omega(Q_{d,i-1})$. Since different types of buyers like good $i-1$ in different degrees (i.e. $a_{h,i-1}$) the seller faces a trade-off in choosing $Q_{d,i-1}$. A higher price (lower $Q_{d,i-1}$) increases his net *payoff* in each single transaction, since $V_{d,i} - Q_{d,i-1} - V_{s,i}$ would grow for any given $\mathbf{V}$, but may also induce *some* buyers to abstain from the purchase (see (16)) which would decrease the expected demand $\sum_{h \in \mathbf{N}} \left[\alpha_D \frac{m_d}{N} \beta_{d,h}(Q_{d,i-1}) + \alpha_F \frac{m'_d}{N} \beta'_{d,h}(Q_{d,i-1})\right]$, and expected profit. In particular, (16) implies that any $Q_{d,i-1}$ leaving buyer h with an arbitrarily small surplus will be accepted. Hence we define the set of all possible offers $Q_{d,i-1}$ that will leave exactly one type of buyer (domestic or foreign) with zero trade surplus, and all others with strictly positive or negative surpluses:

$$\chi_{d,i-1} \equiv \left\{Q_{d,i-1} \geq 0 | V_{s,h} + u_h\left(Q_{d,i-1}\right) - V_{d,h} = 0 \text{ or } V'_{s,h} + u_h\left(Q_{d,i-1}\right) - V'_{d,h} = 0,\ \forall h \neq i\right\}. \tag{19}$$

This is a *discrete* set containing at most $(N-1)^2$ elements (there are $N-1$ types of buyers in each country, but since offers must be non-negative the set may be smaller).

In the appendix we formalize the fact that a *necessary* condition for $Q_{d,i-1}$ to be optimal, is that *at least* one type of buyer h gets zero surplus (in the limit). Offers that leave *all* buyers with a positive surplus are suboptimal, since they do not increase the probability of a sale, while raising production costs in all possible trades. Consequently we refer to $\chi_{d,i-1}$ as the *seller's choice set* for $Q_{d,i-1}$ (foreign sellers choose $Q'_{d,i-1} \in \chi'_{d,i-1}$). It follows that, contingent on $\Pi_d = 1$, the optimal quantity offer of domestic seller i, $Q^*_{d,i-1}$, must maximize expected profits, i.e.

$$Q^*_{d,i-1} \equiv \begin{cases} \left\{Q_{d,i-1} \in \chi_{d,i-1} \,\middle|\, \Omega(Q_{d,i-1}) \geq \Omega(\tilde{Q}_{d,i-1}), \quad \forall \tilde{Q}_{d,i-1} \in \chi_{d,i-1}/Q_{d,i-1}\right\} \\ 0 \quad \text{if} \quad \Omega(Q_{d,i-1}) \leq 0, \quad \forall Q_{d,i-1} \in \chi_{d,i-1} \end{cases}. \tag{20}$$

($\Omega'(Q'^{*}_{d,i-1})$ for a foreign seller). We stress two features of $Q^*_{d,i-1}$. First, while (20) does not imply uniqueness of $Q^*_{d,i-1}$ (since offers are defined on a discrete set), we focus on unique offers resolving all ties in favor of the largest candidate.[10] Second, since (20) specifies offers conditional on $\Pi_d = 1$,

[10] If $\Omega(Q_{d,i-1})$ is convex, in principle there can be two offers $\arg\min\{\chi_{d,i-1}\}$, and $\arg\max\{\chi_{d,i-1}\}$, that can satisfy (20).

once the offer is posted the seller *has to* sell for currency d. In case no offers in $\chi_{d,i-1}$ are *feasible* (i.e. generate positive profits), the only feasible profit-maximizing offer is $Q^*_{d,i-1} = 0$. This is equivalent to set $\Pi^*_d = 0$.[11] Finally, symmetry requires that domestic sellers of the same type charge identical prices,

$$Q^* = q^*. \tag{21}$$

(b) We now discuss domestic seller's i decision of which currency to accept, $\Pi_d \in \{0,1\}$.[12] While the quantity offer depends in part on the arrival rates of matches with different type of buyers, Π_d does not.[13] Recall that seller i correctly evaluate his surplus from a sale, $V_{d,i} - Q_{d,i-1} - V_{s,i}$, but once matched to a buyer with d his uncertainty about the buyer's type and nationality is still unresolved. Thus, i would sell only if his *expected* trade surplus is strictly positive for *some* quantity offer among the ones which *some* buyers would find acceptable.[14] Conditional having encountered a buyer with currency d, the conditional probability that the buyer is domestic (foreign) is Pm_d/M_d $((1-P)m'_d/M_d)$, thus the expected surplus from selling at price $\frac{1}{Q_{d,i-1}}$ is

$$S(Q_{d,i-1}) \equiv \frac{1}{M_d N}\left[Pm_d \sum_{h\in\mathbf{N}} \beta_{d,h}(Q_{d,i-1}) + (1-P)m'_d \sum_{h\in\mathbf{N}} \beta'_{d,h}(Q_{d,i-1})\right](V_{d,i} - Q_{d,i-1} - V_{s,i}).$$

If $\Omega(Q_{d,i-1})$ is concave there may be two adjacent offers in $\chi_{d,i-1}$, that can satisfy (20). Given the discreteness of the choice set, this event has a small probability of occurrence (we never encountered in numerical simulations) as long as every other seller adopts pure strategies. This, however, can be amended by considering equilibria where different fractions of sellers post different prices but have the same expected profit. It is easy to show that equilbria of this type exist (see Soller and Wright, 1999).

[11] In equilibrium, $\Pi^*_d = 0$ and $Q^*_{d,i-1} > 0$ are inconsistent: either the profit to the seller is non-positive, hence $Q^*_{d,i-1} = 0$, or if it is positive, then $\Pi^*_d = 1$.

[12] The mixed strategy $\Pi_d \in (0,1)$ is not a symmetric equilibrium if $Q_{d,i-1}$ leaves positive surplus to some buyer type. The seller could offer a bit less, say $\tilde{Q}_{d,i-1}$, would still be able to sell to some buyers, and do better by obtaining a positive surplus:

$$V_{d,i} - \tilde{Q}_{d,i-1} - V_{s,i} > V_{d,i} - Q_{d,i-1} - V_{s,i} = 0.$$

If $Q_{d,i-1}$ leavse zero surplus to both buyers and sellers a mixed strategy Π_d could be supported only if currency d were to be accepted with probability one in the other country. An uninteresting case since d currency trades in this case would not contribute at all to the lifetime value of holding it, in the country where the mixed strategy is an equilibrium.

[13] If, for instance, seller i expected to meet no buyers with d, he would not to post a price $1/Q_{d,i-1}$. However, strategies must be subgame perfect. Thus should he meet a buyer with d, the seller should decide whether to accept the currency, for some $Q_{d,i-1}$.

[14] If the seller could recognize type and nationality of buyers then he would only consider the surplus in choosing Π_d. This would make a difference in the out-of-equilibrium occurrence where the seller meets someone who (according to the conjectured distribution of money) was not *expected* to be holding that particular currency. For instance, suppose that currency F is not accepted (hence circulating) in the domestic economy, and that the seller meets a domestic buyer with F (out of equilibrium). Suppose also that there is an offer which is mutually advantageous. Since the seller attaches zero probability to the event of having met a domestic buyer with F, then the best he can do is to reject F.

If we let $\min \chi_{d,i-1}$ denote the smallest element of $\chi_{d,i-1}$, then seller's i best response must satisfy

$$\Pi_d^* = \begin{cases} 1 & \text{if } S(Q_{d,i-1}) > 0 \text{ for some } Q_{d,i-1} \geq \min \chi_{d,i-1} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

That is to say, in the subgame where a match with a buyer holding currency d has taken place, a domestic seller will post a price in terms of currency d only if he believes there is a positive probability that the buyer met may want to accept his offer. If the seller believes that there is no price at which trade could be mutually advantageous, then he won't post any. This occurs when there is no feasible price capable of generating a positive payoff from the sale ($V_{d,i} - Q_{d,i-1} - V_{s,i} \leq 0$ for all $Q_{d,i-1} \geq \min \chi_{d,i-1}$). It also occurs when the seller assigns zero-probability to the event that the buyer met is willing to accept any of the prices the store chooses from, *given* the distribution of currencies across countries.[15] In either case $S(Q_{d,i-1}) \leq 0$ and not accepting d is the best option, so that we write $Q_{d,i-1}^* = 0$ when $\Pi_d^* = 0$, with no loss in generality.[16] In a symmetric equilibrium

$$\Pi^* = \pi^*. \tag{23}$$

4.5 Equilibrium

We can now discuss the equilibrium. Since we focus on Nash best responses, and equilibrium outcome in this environment must be such that production and trading decisions are individually optimal given correctly perceived strategies and distribution of money. Additionally, since we focus only on stationary outcomes where strategies are symmetric, equilibrium production and trading decisions must be time-invariant and identical for individuals of identical type. Finally, because of the assumption of rational expectations, the equilibrium pricing rule adopted must be based on the correct evaluation of gains from trade.

That is, a *symmetric stationary monetary equilibrium* is a list of strategy vectors $\{\boldsymbol{\beta}^*, \boldsymbol{\pi}^*\}$, production decisions vectors, $\mathbf{q}^*$, a vector of proportions of money traders, $\mathbf{m}$, and a vector of value functions $\mathbf{V}$ such that:

i) individuals maximize their expected lifetime utilities using symmetric Nash strategies, i.e. given $\{\boldsymbol{\beta}^*, \boldsymbol{\pi}^*\}$ and $\mathbf{m}$, $\mathbf{V}$ must satisfy (12)-(15), $\{\mathbf{B}, \boldsymbol{\Pi}\}$ must satisfy the best response functions (16)-(17) and (22)-(23), and $\mathbf{Q}$ must satisfy the condition for maximization of expected profits (20)-(21),

[15] This is a consequence of the unobservability of buyers' nationality. For instance, suppose a US store would sell to a US customer offering yen. If they assign zero probability to the event that the customer met is a US citizen, however, the store would choose yen-denominated prices only under the conjecture of having encountered a Japanese citizen. If there are no feasible prices at which a yen-denominated trade with a Japanese can occur, then the US store would not post any yen-denominated prices at all.

[16] Mixed strategies π_d can be formally ruled out by assuming the existence of a menu-cost $\varepsilon > 0$ (the cost of posting prices), and then looking at the limiting case when $\varepsilon \to 0$.

ii) the distribution of money is stationary, i.e. given $\mathbf{V}$ and $\{\boldsymbol{\beta}^*, \boldsymbol{\pi}^*\}$, $\mathbf{m}$ satisfies the stationarity conditions (7) through (9).

Some features of the equilibrium are worth noticing at this point. First, all sellers belonging to the same country in equilibrium choose an identical price in terms of one specific currency, independent of their type i. This is because the demand for commodity $i-1$ depends on preferences of buyers, but types are uniformly distributed over $\mathbf{N}$, and the ordering of preferences and production is symmetric, so that *all* sellers located in any given country, face exactly the same expected demand schedule, irrespective of their production.[17] Consequently, in a given country the equilibrium price of goods $i \neq j$ must be identical, so that with no loss in generality we can drop the subscript $i-1$ and let

$$q^*_{d,i-1} = q^*_d, \quad q'^*_{d,i-1} = q'^*_d \tag{24}$$

$\forall i \in \mathbf{N}$, and $d \in \{D, F\}$. Within a country, (24) implies that the equilibrium value functions must be identical across types i, i.e.

$$V_{s,i} = V_s, \quad V_{d,i} = V_d, \quad V'_{s,i} = V'_s, \quad V'_{d,i} = V'_d, \tag{25}$$

$\forall i \in \mathbf{N}$, and $d \in \{D, F\}$. Notice, however, that V'_s (V'_d) and V_s (V_d) need not be equal, and that arrival rates for buyers need not be identical across countries. For this reason, the equilibrium price of good $i-1$, and buyer's strategies are not necessarily equal *across* countries.

5. A Simple Class of Symmetric Equilibria

We now discuss the existence of a simple class of symmetric equilibria, simple in that each store optimally chooses to post prices which will result in a purchase by *any* buyer who has a positive marginal valuation for the commodity offered and has the currency in which the price is posted. We confine our investigation to the three most interesting types of symmetric Nash equilibria: (i) the case in which the two currencies circulate only locally, (ii) outcomes where currency substitution occurs only in one country so that one currency circulates internationally and one locally, and (iii) equilibria where currency substitution occurs in both countries in which case both currencies circulate internationally.[18] To limit the dimensionality of the state space and the complexity of the treatment we set $N = 3$. We define $\rho = rN$, and let $a_{i,i+1} = a \in (0, 1)$, so that we denote by $\bar{q} = a^{\frac{\gamma}{1-\gamma}} < 1$ that q which satisfies $(aq)^\gamma - q = 0$. In what follows we focus the discussion on domestic agents, where no confusion arises.

[17] For a given price, the expected demand for good $i-1$ is a function of the location of the seller (domestic of foreign), because of possible differences in the arrival rates of buyers.

[18] Obviously, there always exists a non-monetary equilibrium where none of the currencies are accepted. There also exist equilibria where one currency is never accepted, while the other circulates locally or internationally.

5.1 Local Currencies

We start with the discussion of the case where the currencies circulate only locally, that is to say that currency d is accepted in exchange for goods only in country d. We conjecture an equilibrium in which domestic (foreign) stores do not accept currency F (D), but only their national currency, that is

$$\pi_F^* = \pi_D'^* = 0, \;\; \pi_D^* = \pi_F'^* = 1. \tag{26}$$

Using (26) and (3)-(4) for $M_D < P < 1 - M_F$, the equilibrium distribution of money holdings is:

$$m_F = m'_D = 0, \;\; m = m_D = \frac{M_D}{P}, \;\; m' = m'_F = \frac{M_F}{1-P}. \tag{27}$$

Since domestic seller i does not post prices in terms of currency F (we will verify it shortly), $Q^*_{D,i-1}$ is the only relevant choice. Recalling that $m'_D = 0$, and $N = 3$, the choice set $\chi_{D,i-1}$ contains only *two* quantities, i.e.

$$\chi_{D,i-1} \equiv \{Q_{D,i-1}|\; V_{s,i-1} + u_{i-1}(Q_{D,i-1}) - V_{D,i-1} = 0,\; V_{s,i+1} + u_{i+1}(Q_{D,i-1}) - V_{D,i+1} = 0\}.$$

This implies the possibility of posting two prices. A *high price* which leaves zero surplus to domestic buyers $i-1$, the ones who have the highest valuation for good $i-1$ ($a_{i-1,i-1} = 1$). At this price no other type would buy. Because of (1), type i would not wish to consume *any* quantity of good $i-1$ (she derives no utility from it, since $a_{i,i-1} = 0$), while type $i+1$ would sustain only a net loss since she has a lower valuation than buyer $i-1$ ($a_{i+1,i-1} = a < 1$). A *low price* at which domestic buyers $i-1$ and $i+1$ are both willing to purchase the good. Since $i+1$ likes the good the least, this lower price takes away her entire surplus.

We can rule out the high price, otherwise the trade surplus generated in each transaction would *entirely* go to sellers. This would imply $V_{D,i} = 0$ for *any* buyer i, because under the conjectured strategy buyers would not be able to buy anywhere else. This contradicts the conjectured equilibrium valuation of money, $0 < V_s < V_D$. Therefore, the low price is the unique profit maximizing choice for any seller i, and using (1), (21), (24), and (25), it must satisfy

$$V_s + (aq_D^*)^\gamma - V_D = 0. \tag{28}$$

It follows that only buyer i receives positive surplus from purchases of goods i, and

$$\beta^*_{D,i}(q^*_{D,h}) = 1, \quad h \in \{i, i+1\}, \tag{29}$$

(and zero otherwise) is the equilibrium strategy for any i.

Using (26), (27), (28), and (29) the value function of any domestic seller i in equilibrium must satisfy

$$\rho V_s = 2\alpha_D m_D (V_D - q_D^* - V_s) \tag{30}$$

where we note that in equilibrium a domestic seller expects no matches with domestic buyers holding foreign currency ($m_F = 0$), or foreign buyers with domestic currency ($m'_D = 0$). Similarly, (14) implies

$$\rho V_D = \alpha_D(1 - m_D) \left[V_s + (q_D^*)^\gamma - V_D\right]. \tag{31}$$

Proposition 1. *A unique equilibrium where currencies circulate only locally exists if* $\mathbf{q}^*$ *satisfies*

$$q_D^* = \left\{ \frac{a^\gamma \left[\rho + \alpha_D(1+m)\right] - \alpha_D(1-m)}{2m\alpha_D} \right\}^{\frac{1}{1-\gamma}},$$

$$q_F^{'*} = \left\{ \frac{a^\gamma \left[\rho + \alpha'_F(1+m')\right] - \alpha'_F(1-m')}{2m'\alpha'_F} \right\}^{\frac{1}{1-\gamma}},$$

$$q_F^* = q_D^{'*} = 0,$$

$\mathbf{m}$ *satisfies (27), and the following sufficient conditions hold:*

$$\{m, m' \in (\underline{m}, \bar{m}), \ \ \rho \leq \bar{\rho}, \ \ \alpha_F \leq \bar{\alpha}_F, \ \ \alpha'_D \leq \bar{\alpha}'_D\}$$

where

$$\underline{m} \equiv \frac{1 - a^\gamma}{1 + a^\gamma}, \ \ \bar{m} \equiv \frac{(2-a)(1-a^\gamma)}{2 - a(1+a^\gamma)}, \ \ \bar{\rho} = \min\{\rho_D, \rho_F\},$$

$$\rho_D \equiv \frac{\alpha_D \left[ma^{1+\gamma} + (2-a)(1 - m - a^\gamma)\right]}{(2-a)a^\gamma}, \ \rho_F \equiv \frac{\alpha'_F \left[m'a^{1+\gamma} + (2-a)(1 - m' - a^\gamma)\right]}{(2-a)a^\gamma}$$

$$\bar{\alpha}_F \equiv \frac{2\alpha_D m \left[(aq_D^*)^\gamma - q_D^*\right] + \rho a q_F^{'*}}{(1-m')\left(q_F^{'*}\right)^\gamma \left[1 + a^\gamma - 2a^\gamma \left(q_F^{'*}\right)^{1-\gamma}\right]}, \ \ \bar{\alpha}'_D \equiv \frac{2\alpha'_F m' \left[(aq_F^{'*})^\gamma - q_F^{'*}\right] + \rho a q_D^*}{(1-m)\left(q_D^*\right)^\gamma \left[1 + a^\gamma - 2a^\gamma \left(q_D^*\right)^{1-\gamma}\right]}.$$

Proof. In Appendix.

We provide intuition by considering the actions taken by a representative domestic individual. Recall that in this equilibrium *all* sellers i must post the low price $1/q_D^*$ so that *all* buyers who have a positive valuation for commodity $i-1$ buy (types $i-1$ and i). The problem is that substantial differentiation (small a) requires sellers to incur large production costs, since a substantial quantity must be delivered *in each sale* to both low and high valuation buyers. Since money can be spent only the period following a sale, sellers cannot be too impatient and a low discounting factor is necessary for the low price to be an equilibrium ($\rho \leq \bar{\rho}$). The frequency of transactions (governed by amount of money and arrival rates) has also implications for the price chosen. In particular, the smaller the price posted the larger must be the frequency of trade for the store to keep posting the low price. Since trade matches are proportional to the measure of buyers, enough liquidity is necessary, $m > \underline{m}$ (note that $\underline{m} \to 1$ as $a \to 0$). If liquidity were too limited, or sellers too impatient, the representative store would have an incentive to deviate from the proposed "low price". Expected profits could be increased if all surplus were to be extracted from the high-valuation buyers, by posting premium prices. This would decrease the frequency of sales but would also generate extra return $(1-a)q_D^*$ per sale. If all stores were to post such high prices no buyer would enjoy a positive trade surplus and money would be valueless. A similar argument explains why $m < \bar{m}$ is necessary: an excessive measure of buyers would also induce stores to exploit the trade off between higher prices and lower frequency of sales.[19]

Finally stores won't post prices in terms of the foreign currency whenever that money is perceived to be a bad medium of exchange. Since stores know that the foreign currency is not accepted in domestic transactions, the return from deviating and accepting it is a function only of the ease of international trade. If the latter is sufficiently small, $\alpha_F < \bar{\alpha}_F$, selling for the foreign currency would be a dominated action, even when asking for premium prices.

Corollary 1. *Suppose an equilibrium with two local currencies exists. A marginal increase in the initial level of the domestic (foreign) money supply will lower domestic (foreign) prices. Prices will be efficient if* $m = m^e$ *and* $m' = m'^e$*, where*

$$m^e \equiv \frac{(\alpha_D+\rho)\left[\alpha_D - a^\gamma(\alpha_D+\rho)\right]}{\alpha_D\left[(\alpha_D+\rho)(1+a^\gamma-2\gamma)+\gamma\right]}, \quad m'^e \equiv \frac{(\alpha'_F+\rho)\left[\alpha'_F - a^\gamma(\alpha'_F+\rho)\right]}{\alpha'_F\left[(\alpha'_F+\rho)(1+a^\gamma-2\gamma)+\gamma\right]}.$$

Proof: In appendix.

The fact that prices are decreasing in the initial stock of the money supply, in equilibrium, is not strange as it may appear at first blush. Consider for a moment the case $k' = 0$ where only local exchange is possible. We have seen that the representative store would take advantage of a very "liquid" economy by price discriminating and attempting to sell only to high valuation buyers. For this reason a symmetric monetary equilibrium can exist only when stores do not meet buyers too frequently, that is to say when there isn't excessive liquidity in circulation. Because of this limited frequency of sales, each store maximizes profits by posting low prices thus selling to as many buyers as possible. A marginal increase in the initial stock of money increases the measure of buyers (relative to sellers) and so increases the return from being a seller. It also increases the return from holding money because it boosts the frequency of trade matches. It follows that stores must offer better prices to low-valuation customers to induce a purchase. As m keeps increasing, however, stores can take advantage of the trade-off between prices and frequency of sale, and start charging premium prices. The problem is that by charging higher prices domestic money would lose its value (taking strategies and prices in all other matches as given), since all domestic buyers who make a purchase would be left with no surplus. Because of the discreteness of the support of the distribution of

[19] In the prototypical search models of money and prices (Shi, 1995, and Trejos and Wright, 1995) $a = 0$ in which case $\underline{m}, \bar{m} \to 1$. Thus money would not be valued if sellers were posting prices, which explains the common assumption of buyers making take-it-or-leave-it offers, or Nash bargaining with equal weights.

types, and the existence of only high and low valuation agents, this would lead to a "jump" in prices which would destroy the symmetric monetary equilibrium.[20]

Additionally, the prices posted by stores may be efficient, in that the quantities exchanged solves the problem of a planner who intends to maximizes the ex-ante lifetime utility in each country when he takes as given the pricing protocol. We note that this result does not require the absence of time discounting (as in Trejos and Wright, 1995), rather it hinges on the existence of an optimal quantity of liquidity.

Our findings are illustrated by means of numerical example in which countries are taken to be very different economies.[21] The white rectangle in figure 1 delimits the space M_DxM_F on which the equilibrium in which *both* currencies circulate only locally. As noted before, too little liquidity implies that there are not enough buyers. The limited frequency of trade induces stores to extract all surplus from the high valuation buyers. By doing so, however, the monetary equilibrium ceases to exist. An increase in the supply of local currency makes local trade easier and lowers local prices (but does not influence the trade or prices in the other economy), until the liquidity level does not exceed an upper bound. Note also that since the domestic economy is three times larger than the foreign, the stock of money that can be circulated domestically is larger than the foreign stock (hence the region of existence is shaped as a narrow rectangle).

5.2 One International Currency

In this section we consider equilibria where only domestic stores post prices in both currencies, while foreign stores accept solely their national currency,

$$\pi_D^* = \pi_F^* = \pi_F'^* = 1, \;\; \pi_D'^* = 0. \tag{32}$$

Because only domestic stores sell for currency D, *some* buyer i with currency D must receive a positive surplus from trade, otherwise $V_{D,i} = 0$ for all i and currency D would have no value. Therefore the equilibrium $Q_{D,i}$ must still satisfy (28) $\forall i$. This does not apply to $Q_{F,i}$ because domestic stores sell to both domestic and foreign buyers with F, hence in choosing $Q_{F,i}$ domestic seller $i+1$ considers also the foreign demand. Since in general $V \neq V'$, four possible quantity offers can be posted (two per nationality) which satisfy

$$\begin{aligned}\chi_{F,i} \equiv \{Q_{F,i} | V_{s,i} + u_i(Q_{F,i}) - V_{F,i} = 0,\; V_{s,i-1} + u_{i-1}(Q_{F,i}) - V_{F,i-1} = 0,\\ V'_{s,i} + u_i(Q_{F,i}) - V'_{F,i} = 0,\; V'_{s,i-1} + u_{i-1}(Q_{F,i}) - V'_{F,i-1} = 0\}.\end{aligned} \tag{33}$$

We cannot rule out any of these quantities, a priori, not even the smallest $Q_{F,i}$ at which one buyer type of only one country buys. This because even if all gains from trade go to the domestic seller the

[20] In a more general environment where $N > 3$ an increase in M_D would cause a sequence of jumps in prices until the monetary equilibrium ceased to exist.

[21] Our benchmark is $r = 0.1$, $P = 0.75$, $\gamma = 0.9$, $a = 0.8$, $k = 10$, and $k' = 0.25$. In this case $\alpha_D = 7.5 > \alpha'_F = 2.5 > \alpha'_D = 0.1875 > \alpha_F = 0.0625$.

buyer may still be able to get a positive payoff from foreign purchases. The discussion for the quantity offer of any foreign seller $i+1$ is identical and so is her choice set, $\chi'_{F,i} \equiv \chi_{F,i}$ (where $Q'_{F,i}$ replaces $Q_{F,i}$). Note that $q^*_{F,i}$ need not necessarily equal $q^{'*}_{F,i}$ and there are multiple price vectors which can potentially be equilibria. Furthermore, since a buyer's gains from trade generally depend on the store's nationality, the buyer's strategies will in general be a function of both their nationality (i.e. $B \neq B^{'}$), and the nationality of stores (i.e. $B_{F,h}(q^*_{F,i}) \neq B_{F,h}(q^{'*}_{F,i})$).[22]

The number of possible equilibria is greatly reduced when we consider the simple class of equilibria where every store sells to every buyer who desires the good (and has the right currency). This implies that stores in both countries post an identical lowest price in currency F, and since $\chi'_{F,i} \equiv \chi_{F,i}$ the associated quantity offer must satisfy

$$q^*_F = q'^*_F \quad \text{such that} \quad u_{i-1}(q^*_{F,i}) = \max\left\{V_{F,i-1} - V_{s,i-1},\ V'_{F,i-1} - V'_{s,i-1}\right\}\ \forall i. \tag{34}$$

It follows that in equilibrium agents $h \in \{i-2, i-1\}$ holding F will buy from both domestic and foreign sellers i,

$$\beta^*_{F,h}(q^*_{F,i}) = \beta^*_{F,h}(q^{'*}_{F,i}) = \beta^{'*}_{F,h}(q^{'*}_{F,i}) = \beta^{'*}_{F,h}(q^*_{F,i}) = 1, \quad h \in \{i-2, i-1\}, \tag{35}$$

and since q^*_D satisfies (28)

$$\beta^*_{D,h}(q^*_{D,i}) = \beta^{'*}_{D,h}(q^*_{D,i}) = 1, \quad h \in \{i-2, i-1\}. \tag{36}$$

When $\boldsymbol{\pi}^*$ satisfies (32), q^*_F satisfies (34), $\boldsymbol{\beta}^*$ satisfies (35)-(36), and $M_D < P$, (7)-(9) and (5) with $d = D$ imply $m'_D = 0$, $m'_F = m'$, and $m_D = \frac{M_D}{P}$. Determination of m_F and m' is accomplished by solving the system of two equations (5) and (7) for $d = F$

$$\begin{cases} (1-m)(\alpha_D m_F + \alpha_F m'_F) - m_F[\alpha_D(1-m) + \alpha_F(1-m')] = 0 \\ M_F = (1-P)m'_F + P m_F. \end{cases}$$

Using (4) the stationary distribution of money is

$$m_F = \frac{M_F(P - M_D)}{P(1-M_D)} \in (0,1),\ m'_F = \frac{M_F}{1-M_D} \in (0,1),\ m_D = \frac{M_D}{P} \in (0,1),\ m'_D = 0. \tag{37}$$

Given the conjectured equilibrium strategies (32) and (35)-(36), prices (34), the distribution of money holdings (37), we define the value functions. Equations (1) and (12)-(13) imply

$$\rho V_s = \alpha_D m_D 2(V_D - q^*_D - V_s) + 2\left(\alpha_D m_F + \alpha_F m'_F\right)(V_F - q^*_F - V_s) \tag{38}$$

[22] There are sixteen possible equilibrium vectors $\mathbf{q}^*$, where q^*_D is uniquely determined by (28). Each of these has implications for the strategies of buyers. We will give numerical examples of this multiplicity of outcomes.

One of these proposed equilibria, however, can be ruled out. Namely, the case where sellers in both countries post the highest price in terms of F (in which case $V_{F,i} = V'_{F,i} = 0$, and currency F is not valued).

$$\rho V_s' = 2\left(\alpha_D' m_F + \alpha_F' m_F'\right)\left(V_F' - q_F'^* - V_s'\right), \tag{39}$$

while (14)-(15) in equilibrium imply

$$\rho V_D = \alpha_D(1-m)\left[V_s + (q_D^*)^\gamma - V_D\right]. \tag{40}$$

Since $q_F^* = q_F^{'*}$

$$\rho V_F = \left[\alpha_D(1-m) + \alpha_F(1-m_F')\right]\left[2\left(V_s - V_F\right) + (q_F^*)^\gamma + (aq_F^*)^\gamma\right] \tag{41}$$

$$\rho V_F' = \left[\alpha_D'(1-m) + \alpha_F'(1-m_F')\right]\left[2\left(V_s' - V_F'\right) + (q_F^*)^\gamma + (aq_F^*)^\gamma\right]. \tag{42}$$

To find the equilibrium quantities recall that q_D^* is uniquely determined by (28), and we are considering equilibria where (34) holds. Unfortunately (41) and (42) do not tell us the relative size of $V_F - V_s$ vs. $V_F' - V_s'$ so the pricing rule is not uniquely identified.

Since currency F is the international currency, we focus on a subset of the simple class of equilibria here considered. Namely, we conjecture (and then verify) the existence of an equilibrium where currency F generates the largest net return to domestic sellers, i.e.

$$V_F - V_s \geq V_F' - V_s' > 0, \tag{43}$$

so that (34) implies

$$(aq_F^*)^\gamma = V_F - V_s. \tag{44}$$

Using (44) and defining the following positive constants

$$b_1 \equiv \rho + \alpha_D\left(1 + m_D - m_F\right), \quad b_2 \equiv \alpha_D(1-m), \quad b_3 \equiv 2\alpha_D m_D, \quad b_4 \equiv 2\left(\alpha_D m_F + \alpha_F m_F'\right)$$

$$c_1 \equiv \rho + \alpha_D\left(1 - m_D + m_F\right) + \alpha_F(1 + m_F'), \quad c_2 \equiv \alpha_D(1-m) + \alpha_F(1-m_F'),$$

$$d_1 \equiv \rho + 2\alpha_D'\left(1 - m_D\right) + 2\alpha_F', \quad d_2 \equiv \left[\alpha_D'(1-m) + \alpha_F'(1-m_F')\right]\left(1 + a^\gamma\right), \quad d_3 \equiv 2\left(\alpha_D' m_F + \alpha_F' m_F'\right),$$

then (38), and (40)-(42) imply

$$\left(V_D - V_s\right) b_1 = b_2(q_D^*)^\gamma + b_3 q_D^* - b_4\left(V_F - q_F^* - V_s\right), \tag{45}$$

$$\left(V_F - V_s\right) c_1 = c_2\left(q_F^*\right)^\gamma + b_4 q_F^* - b_3\left(V_D - q_D^* - V_s\right), \tag{46}$$

$$\left(V_F' - V_s'\right) d_1 = d_2\left(q_F^*\right)^\gamma + d_3 q_F^*. \tag{47}$$

The right hand side of (47) is positive for all $q_F^* > 0$, but from (46)-(47) it is not clear whether $V_F - V_s \geq V_F' - V_s'$, hence to find the equilibrium we proceed as follows in subsequent steps. First, we provide conditions under which there is a unique pair $\{q_D^*, q_F^*\}$ which solves the system (45)-(46). Then, using (47), we provide sufficient conditions for (43) to hold. Finally we provide conditions guaranteeing sellers' maximization of profits.

Using (28) and (44) the system (45)-(46) reduces to

$$\begin{cases} (q_D)^\gamma \left[\frac{a^\gamma b_1 - b_2}{b_3}\right] + \frac{b_4}{b_3}\left[(aq_F)^\gamma - q_F\right] = q_D \\ (q_F)^\gamma \left[\frac{a^\gamma c_1 - c_2}{b_4}\right] + \frac{b_3}{b_4}\left[(aq_D)^\gamma - q_D\right] = q_F. \end{cases} \tag{48}$$

The system of equations (48) defines a map which has a fixed point $q_D = q_F = 0$ corresponding to the non-monetary equilibrium. Under certain conditions, it also has a unique positive fixed point $\{q_D^*, q_F^*\}$, as specified in the following lemma.

Lemma 1. *The map defined by (48) has a unique fixed point $\{q_D^*, q_F^*\} \in (0, \bar{q})^2$, if*

$$a \in (a_L, a_H) \text{ and } \rho < \rho_H$$

where:

$$q_F^* = \left(\frac{a^\gamma c_1 - c_2 + a^\gamma b_3 \delta}{b_4 + b_3 \delta^{\frac{1}{\gamma}}}\right)^{\frac{1}{1-\gamma}}, \quad q_D^* = q_F^* \delta^{\frac{1}{\gamma}}, \tag{49}$$

$$\delta \equiv \frac{a^\gamma(\rho + c_2) - c_2}{a^\gamma(\rho + b_2) - b_2} > 1, \quad \rho_H \equiv \frac{b_2(b_3 + b_4)}{c_2 - b_2} > 0, \quad a_L \equiv \left(\frac{c_2}{c_1 + b_3}\right)^{\frac{1}{\gamma}} < a_H < \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}},$$

and a_H is the unique value of a that satisfies $[a^\gamma(\rho + c_2) - c_2] + \left(\delta^{\frac{1-\gamma}{\gamma}} - 1\right)(a^\gamma c_1 - c_2) = 0$.

Proof. In appendix.

Recall that since stores intend to induce a purchase by every buyer who has positive valuation for the good, they choose prices by considering the preferences of low valuation buyers. The wider the gap between high and low valuation (the smaller the a) the lower the price to be posted and the higher the quantity to be produced *per sale*. Because sellers must obtain a positive surplus from each transaction, however, they will not find it worthwhile selling to all prospective buyers if this entails production of too large quantities of output ($a > a_L$). Buyers, on the other hand, expect to obtain positive surplus only from a $1/N$ fraction of the stores visited. The value attached to money holdings, therefore, depends only on the surplus obtained when the preferred commodity is acquired. Since the size of this surplus is inversely related to a, it follows that a cannot be too small either ($a < a_H$). Furthermore, sellers won't produce at all if they are too impatient because money receipts can only be used only in future purchases. This explains why the rate of time discounting cannot be too high ($\rho < \rho_H$).

Clearly to be an equilibrium $\{q_D^*, q_F^*\}$ must support the proposed π^* strategies listed in (32). We do so in the next lemma where we also provide a condition sufficient to support (43), in which case currency F is valued more by domestic than foreign sellers.

Lemma 2. *When the conditions listed in lemma 1 hold and* $\{q_D^*, q_F^*\}$ *is the equilibrium production vector, then* $\Pi_d^* = 1\ \forall d$, $\Pi_F'^* = 1$, *and there exists some* $k \geq k' > 0$ *and* $P \in [0.5, 1)$ *such that if* $M_D \leq 2P - 1$ *then* $V_F - V_s \geq V_F' - V_s'$.

Proof. In appendix.

Interestingly, we note that the sufficient condition for foreign money to be valued relatively more by domestic sellers *does not* require the existence of a substantial degree of economic integration between the two countries. Rather, it requires a lack of domestic liquidity. When $P \geq 0.5$ and $k \geq k'$ the domestic economy is larger (relative to the foreign) and trade matches with domestic buyers are more likely (than with foreign buyers). However, if the domestic money supply is low ($M_D \leq 2P - 1$) the domestic sellers will prize the foreign currency more than foreign sellers. Finally, we provide sufficient conditions such that $\{q_D^*, q_F^*\}$ is consistent with sellers' maximization of profits, and foreign sellers do not accept currency D.

Lemma 3. *When the conditions listed in lemmas 1-2 hold and* $\{q_D^*, q_F^*\}$ *is the equilibrium production vector, then if* a *is in a neighborhood of* a_L *and* k' *is sufficiently small then (i)* q_D^* *satisfies (20) for domestic sellers (ii) if* $V_F - V_s = V_F' - V_s'$ *then* q_F^* *satisfies (20) for both domestic and foreign sellers, and (iii)* $\Pi_D'^* = 0$.

Proof. In appendix.

An existence proposition follows.

Proposition 2. *The conditions set forth in lemmas 1-3 are sufficient for the existence of a unique equilibrium where only one currency circulates internationally (currency* F*), and where prices of identical commodities are identical across countries even if the equilibrium valuation of the two currencies may differ across countries.*

Proof. See Appendix.

We illustrate the proposition with the help of figure 1. The equilibrium where currency substitution takes place in the domestic economy corresponds to the dotted area. It arises only when the domestic stock of liquidity is sufficiently low, and does not coexist with the one with only local currencies. Because of this lack of liquidity domestic stores would rather sell less frequently but charge premium prices by extracting all surplus of the high valuation buyers (destroying the monetary equilibrium). The lack of liquidity, however, can be lessened if domestic traders would adopt also the foreign currency in their transactions. Even if foreign trade is subject to substantial frictions ($\alpha_D = 7.5 > \alpha_F = 0.0625$) currency substitution enlarges the extent of the market and keeps the stores from posting premium prices in an illiquid economy. Note from the picture that the foreign economy has "sufficient" liquidity ($P = 0.75$ and $M_F > M_D$ in the area where the equilibrium exists) so that, given the extent of international trade frictions, foreign sellers do not find it worthwhile to use both currencies as media of exchange.

Interestingly, the international currency has identical purchasing power in both countries, even if they are very different economies. Because of the existence of substantial international trade frictions, however, the two currencies are not perfect substitutes in that there is a "home goods bias" ($q_D^* > q_F^*$). This is because domestic stores value the local currency more since the frequency of trade it allows is greater than the competing money.

5.3 Two international currencies.

We now prove the existence of equilibria where currency substitution takes place in both economies

$$\pi_D^* = \pi_F^* = \pi_F'^* = \pi_D'^* = 1, \tag{50}$$

in which case any store posts prices in terms of both currencies. Because buyers can buy locally or abroad, seller $i+1$ considers both the foreign and domestic demand in choosing $Q_{d,i}$. However, since in general $V \neq V'$, the seller can choose to post one of four possible quantity offers for each denomination:

$$\chi_{d,i} \equiv \{Q_{d,i} | V_{s,i} + u_i(Q_{d,i}) - V_{d,i} = 0,\ V_{s,i-1} + u_{i-1}(Q_{d,i}) - V_{d,i-1} = 0,$$
$$V'_{s,i} + u_i(Q_{d,i}) - V'_{d,i} = 0,\ V'_{s,i-1} + u_{i-1}(Q_{d,i}) - V'_{d,i-1} = 0\}.$$

As explained in the previous section, none of these quantities can be ruled out a priori, not even the smallest. Even if, say, domestic sellers obtained the entire surplus generated by d-denominated trades, foreign buyers could still be able to get a positive surplus, sometimes, if foreign sellers posted lower prices. Clearly, monetary equilibria in which all stores post the highest price could not be supported (in which case $V_{d,i} = V'_{d,i} = 0$), but several other possible combinations of prices can be conjectured $(4^4 - 1)$. The set of possible outcomes, however, is once again greatly reduced when we consider the simple class of equilibria where every store sells to every buyer who desires the good. This implies that stores in both countries post the lowest d-denominated price (highest possible quantity).

The gains from trade for buyer h, however, still depend on his nationality because the returns from holding d may differ in different locations (due to dissimilar market frictions, for instance). For this reason (and since $\chi'_{d,i} \equiv \chi_{d,i}$) the optimal quantity offer must generally satisfy

$$q_d^* = q_d'^* \quad \text{such that} \quad u_{i-1}(q_{d,i}^*) = \max\{V_{d,i-1} - V_{s,i-1},\ V'_{d,i-1} - V'_{s,i-1}\}\ \forall d, i, \tag{51}$$

so that any buyer $h \in \{i-2, i-1\}$ holding d buys from any stores i :

$$\beta_{d,h}^*(q_{d,i}^*) = \beta_{d,h}^*(q_{d,i}'^*) = \beta_{d,h}'^*(q_{d,i}'^*) = \beta_{d,h}'^*(q_{d,i}^*) = 1. \tag{52}$$

When the equilibrium strategies $\boldsymbol{\pi}^*$ and $\boldsymbol{\beta}^*$satisfy (50) and (52), and q_d^* satisfies (51), then (7)-(9) and (5) imply the distribution of money

$$m_d' = m_d = M_d \quad \forall d, \tag{53}$$

hence $m = m' = M$. This and (1), (12)-(13), in turn imply the value functions

$$\rho V_s = 2\left(\alpha_D + \alpha_F\right)\left[m_D(V_D - q_D^* - V_s) + m_F(V_F - q_F^* - V_s)\right] \tag{54}$$

$$\rho V_s' = 2\left(\alpha_D' + \alpha_F'\right)\left[m_D(V_D' - q_D^* - V_s') + m_F(V_F' - q_F^* - V_s')\right], \tag{55}$$

and since in general $V_d - V_s \neq V_d' - V_s'$, then (14)-(15) imply

$$\rho V_d = (1-m)(\alpha_D + \alpha_F)\left[2\left(V_s - V_d\right) + (q_d^*)^\gamma + (aq_d^*)^\gamma\right] \tag{56}$$

$$\rho V_d' = (1-m)(\alpha_D' + \alpha_F')\left[2\left(V_s' - V_d'\right) + (q_d^*)^\gamma + (aq_d^*)^\gamma\right], \tag{57}$$

for all d.

Unfortunately (54)-(57) do not tell us the relative size of $V_d - V_s$ vs. $V_d' - V_s'$, so that we cannot pin down a unique equilibrium quantity using (51). Once again we conjecture (and then verify) the existence of an equilibrium where it is the domestic stores that have the largest net return from holding each currency, $V_d - V_s \geq V_d' - V_s' > 0\ \forall d$, so that (51) implies

$$(aq_d^*)^\gamma = V_d - V_s. \tag{58}$$

Using this "pricing rule", we prove the following.

Proposition 3. *There exists a unique equilibrium with currency substitution where* $q_d^* = q_d^{'*} = q^*\ \forall d$, *if* $\mathbf{m}$ *satisfies (53), and the following sufficient conditions hold:*

$$\{m = m' \in (\underline{m}, \bar{m}),\ \ \rho \leq \hat{\rho}\},$$

with

$$q^* \equiv \left\{\frac{a^\gamma\left[\rho + (\alpha_D + \alpha_F)(1+m)\right] - (\alpha_D + \alpha_F)(1-m)}{2m(\alpha_D + \alpha_F)}\right\}^{\frac{1}{1-\gamma}},$$

$$\hat{\rho} = \min\{\rho_{D1}, \rho_{F1}\},\ \ \rho_{D1} \equiv \frac{(\alpha_D + \alpha_F)\left[ma^{1+\gamma} + (2-a)\left(1 - m - a^\gamma\right)\right]}{(2-a)a^\gamma}$$

and ρ_{F1} *is the unique positive value of* ρ *that solves*

$$(q^*)^{1-\gamma} = \frac{(\alpha_D' + \alpha_F')(1-m)(1+a^\gamma)}{\rho(2-a) + 2(\alpha_D' + \alpha_F')(2-a-m)}.$$

Corollary. *Equilibria where both currencies are international may coexist with equilibria without currency substitution, but the former also exist on regions of the parameter space which do not support the latter. Thus the possibility of currency substitution may raise welfare.*

Proof. In Appendix.

The union of the black, white and dotted areas in figure 1 represents the region which supports the equilibrium discussed in this section. It clearly coexists with the two types of equilibria with partial and no currency substitution. Note, for instance, that even if the equilibrium with two local currencies does not exists for too small domestic or foreign money stocks, an equilibrium where both currencies circulate internationally does. This because by using both denominations for both local and international trade the extent of the market increases for both local and foreign sellers. This generates a greater frequency of trade and allows individual stores to post lower prices. Similarly, an equilibrium with two international currencies exists for currency stocks larger than the one delimiting the area of existence of the two local currency equilibrium. Larger foreign currency stocks make it optimal for domestic sellers to accept the foreign currency (and vice-versa) as long as every stores posts prices in both denominations. Because the two currencies are substitutable, however, as the aggregate stock of moneys rises prices in *both* countries will drop.

While we focus on a simple class of equilibria where currency substitution takes place and the international currency has identical purchasing power in both countries, multiple equilibria with currency substitution may exist outside of this class. Whereas attempting to characterize all of them analytically is a daunting task, we provide some examples of what we think are interesting outcomes. Figure 2 in particular, shows that on some regions of the parameter space there are equilibria with international price dispersion in that each national currency has a larger purchasing power abroad. These equilibria are interesting because they are symmetric, and because they coexist with equilibria without price dispersion.

6. Concluding remarks.

We have considered existence of monetary equilibria in a two-country, two-currencies world with endogenous price formation by means of a seller-posting-price protocol. We have proved the existence of a unique equilibrium where domestic trade is entirely facilitated by domestic currency. In this case our preliminary findings show that, in contrast to earlier random matching models of money, symmetric monetary equilibria may exist only on a subset of the support of the distribution of money, and only if individuals are sufficiently patient. Interestingly we find that, under certain conditions, this pricing mechanism allows for the possibility that stores post efficient prices, in the sense that the equilibrium prices are identical to the one a social planner would choose. Additionally, we have shown that if the money supply is too large or too small sellers would find it optimal to price discriminate against the low valuation buyers, and extract all trade surplus from the high valuation buyers. This, in turn, would deprive money of its value and shut down trade. We have shown that in this case currency substitution could be prove to be socially beneficial in the sense that it would allow the achievement of Pareto superior allocations where trade occurs.

By considering a simple class of equilibria where every store posts identical prices, we have proved the existence of an outcome where the sale of domestic output is facilitated by both the domestic and the foreign currency, but not vice-versa. In this context, we have shown the existence of "home good bias", in that the domestic currency may have a larger purchasing power because of international trade frictions. This may occur even if the net return from holding the foreign currency is larger in the domestic economy, then abroad. We have also shown that there are equilibria in which currency substitution takes place in both economies, they are perfect substitutes and their purchasing power is identical across economies. Interestingly, this may occur despite the presence of both international trade frictions and substantial asymmetries in the economic fundamentals of the two countries. Numerically, we have shown that equilibria with currency substitution may be multiple, and may entail price dispersion across countries on certain regions of the parameter space.

We consider this study as a first step in the direction of a better understanding of the phenomenon of currency substitution. In future research we intend to develop this model further to investigate the effect that currency substitution has on both prices and volume of trade, and ultimately welfare.

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Appendix

Suboptimality of $Q_{d,i-1}$ leaving all potential buyers with some positive surplus.

Note first that offers larger than the one leaving zero surplus to buyers with the **lowest** valuation for good $i-1$, are not an equilibrium: a larger quantity offered would not increase the probability of a sale, while raising production costs in all other possible trades. That is $Q_{d,i-1} \leq \bar{Q}_{d,i-1}$where $\bar{Q}_{d,i-1}$ satisfies

$$\min\left\{V_{s,i+1} - V_{d,i+1}, V'_{s,i+1} - V'_{d,i+1}\right\} + u_{i+1}\left(\bar{Q}_{d,i-1}\right),$$

due to our description of preferences and since both domestic and foreign buyers *may* hold currency d.[23] To consider the strictest possible case, suppose that $\bar{Q}_{d,i-1}$ satisfies $V_{d,i} - \bar{Q}_{d,i-1} - V_{s,i} > 0$, so sellers would sell for any $Q_{d,i-1} < \bar{Q}_{d,i-1}$ (since their payoff is positive). Then (omitting the indices d and $i-1$ where they are understood), pick any two offers $Q_L < Q_H < \bar{Q}$, such that they take away the surplus from two *adjacent types of buyers*: a low valuation buyer in the case of Q_H and the next higher valuation buyer in the case of Q_L.

We now show that the profit function $\Omega(Q)$ is strictly decreasing on $Q \in [Q_L, Q_H]$. Let also

$$D_{d,i-1} \equiv \frac{\alpha_D m_d + \alpha_F m'_d}{N} \sum_{h \in \mathbf{N}} \left[\beta_{d,h}(Q_{d,i-1}) + \beta'_{d,h}(Q_{d,i-1})\right]$$

denote the expected demand for the commodity $i-1$ when sold at price $\frac{1}{Q_{d,i-1}}$. For ease of notation omit the subscripts d and $i-1$ from D, hence we write the profit function as

$$\Omega(Q) \equiv D(V_d - Q - V_s)$$

for any Q. Define also the convex combination $Q_\lambda \equiv \lambda Q_L + (1-\lambda)Q_H$ for some $\lambda \in (0,1)$. We now show that offering Q_λ is worse than at least offering Q_L or Q_H. Because of their definition, Q_L and Q_H are "adjacent offers", that is

$$D_L = D_\lambda < D_H,$$

[23] Recall that in a monetary equilibrium, for money to valued there must at least be one of the differences $V_{s,i} - V_{d,i}$ or $V'_{s,i} - V'_{d,i}$ which is **strictly negative**. Hence the $\min\{\cdot\}$ function.

Furthermore, buyer $h = i+1$ is the one who has the lowest *positive* valuation for good $i-1$, since $a_{h,i-1}$ is an increasing sequence for $h \geq i$ (h mod N) with $0 = a_{i,i-1} < a_{i+1,i-1}$. Thus $u_i\left(Q_{d,i-1}\right) = 0$. But then, a monetary equilibrium must satisfy $\min\{V_{s,i} - V_{d,i}, V'_{s,i} - V'_{d,i}\} + u_i\left(Q_{d,i-1}\right) < 0$.

i.e. only if the quantity is increased to Q_H more buyers will buy. Thus

$$\Omega(Q_\lambda) \equiv D_L(V_d - Q_\lambda - V_s) < D_L(V_d - Q_L - V_s) \equiv \Omega(Q_L)$$

But then, if $\max_{Q_L, Q_H} \{\Omega(Q_L), \Omega(Q_H)\} = \Omega(Q_L)$, then Q_λ is a choice dominated at least by Q_L, while if $\max_{Q_L, Q_H} \{\Omega(Q_L), \Omega(Q_H)\} = \Omega(Q_H)$ then Q_λ is dominated by both Q_L and Q_H.■

Proof of Proposition 1.

We only discuss the case for domestic traders (the case of foreign traders is identical). In equilibrium, using (27), (30), (31), (28), and (26),

$$V_D - V_s = \frac{q_D^* \alpha_D \left[(q_D^*)^{\gamma-1} (1-m) + 2m \right]}{\rho + \alpha_D(1+m)}. \tag{59}$$

Equilibrium quantities. Using (59) and (28) in equilibrium

$$q_D^* = \left\{ \frac{a^\gamma \left[\rho + \alpha_D(1+m)\right] - \alpha_D(1-m)}{2m\alpha_D} \right\}^{\frac{1}{1-\gamma}}. \tag{60}$$

Similarly, a foreign seller posts the quantity

$$q_F^{'*} = \left\{ \frac{a^\gamma \left[\rho + \alpha_F'(1+m')\right] - \alpha_F'(1-m')}{2m'\alpha_F'} \right\}^{\frac{1}{1-\gamma}}. \tag{61}$$

A sufficient (but not necessary) condition for $q_D^*, q_F^{'*} > 0$ is

$$m', m > \underline{m} \equiv \frac{1-a^\gamma}{1+a^\gamma} \in (0,1).$$

Buyer's strategies. It is easily verified that $\beta_{D,i}^*(q_{D,h}^*) = 1$ $\forall i$ and $h = i, i+1$, since the equilibrium q^* satisfies (28).

Seller's strategies.

(i) $\pi_D^* = 1$. Using (22), given that q_D^* satisfies (28), and given (29), $\Pi_D^* = 1$ whenever $q_D^* < \bar{q}$. This because $V_D - q_D^* - V_s > 0$, in equilibrium is just $(aq_D^*)^\gamma - q_D^* > 0$, i.e. $(q_D^*)^{1-\gamma} < a^\gamma$.

(ii) Q_D^* maximizes $\Omega(Q_D^*)$. Taking as given the value functions, the expected profit from charging the low price must be the highest. Using (20) this is equivalent to

$$V_D - (V_D - V_s)^{\frac{1}{\gamma}} - V_s \le 2 \left[V_D - \frac{(V_D - V_s)^{\frac{1}{\gamma}}}{a} - V_s \right],$$

which, using (28), is satisfied whenever $(q_D^*)^{1-\gamma} \le \frac{a^\gamma}{2-a}$. Note that if the latter inequality is satisfied this also guarantees $q_D^* < \bar{q}$ (seller's surplus is positive), since $(q_D^*)^{1-\gamma} \le \frac{a^\gamma}{2-a} < a^\gamma$. Substituting (60) in $(q_D^*)^{1-\gamma} \le \frac{a^\gamma}{2-a}$ and rearranging we obtain

$$\alpha_D \left[m a^{1+\gamma} + (2-a)(1-m-a^\gamma) \right] \ge (2-a) a^\gamma \rho.$$

Since $ma^{1+\gamma} + (2-a)(1-m-a^{\gamma}) > 0$ only if

$$m < \bar{m} \equiv \frac{(2-a)(1-a^{\gamma})}{2-a(1+a^{\gamma})} \in (0,1),$$

then q_D^* maximizes $\Omega(q_D^*)$ only if

$$\rho \leq \rho_{_D} \equiv \frac{\alpha_D \left[ma^{1+\gamma} + (2-a)(1-m-a^{\gamma})\right]}{(2-a)a^{\gamma}} \text{ and } m < \bar{m}.$$

These conditions are:

- Necessary for Q_D^* to maximize profits, when it satisfies (28)
- Sufficient to guarantee $q_D^* < \bar{q}$ so that $\Pi_D^* = 1$.

It is easy to show that as $a \to 1$ then $\bar{m} \to 1/2$ (by L'Hospital rule), that both $\underline{m}$ and $\bar{m}$ are decreasing in a, and that $\underline{m} < \bar{m}$ $\forall a$. Therefore there are M_D that satisfy $m \in (\underline{m}, \bar{m})$ $\forall a, P$. Additionally, because $\rho_D > 0$ there are $0 < \rho \leq \rho_{_D}$. In a similar manner $\Pi_F^{'*} = 1$ and $q_F^{'*}$ maximizes $\Omega'(q_F^{'*})$ only when $m' < \bar{m}$ *and* $\rho \leq \rho_{_F} \equiv \frac{\alpha'_F\left[m'a^{1+\gamma}+(2-a)\left(1-m'-a^{\gamma}\right)\right]}{(2-a)a^{\gamma}}$. Finally, if

$$\bar{\rho} \leq \min\{\rho_D, \rho_F\} \quad \text{and} \quad m, m' < \bar{m}$$

these conditions are *sufficient* for $\Pi_D^* = \Pi_F^{'*} = 1$, and *necessary* for q_D^* and $q_F^{'*}$ to satisfy (20) thus be the profit maximizing quantities.

(iii) $\pi_F^* = 0$. Using (22) there must be no quantity in $\chi_{F,i-1}$ the seller would like to produce in exchange for currency F, given the conjectured distribution, value functions, prices, and strategies. In principle a domestic seller could sell for currency F to either a domestic *or* a foreign buyer. However, since $m_F = 0$ domestic sellers expect to meet none of the former, only the latter. Thus, in case the seller deviates to $\Pi_F = 1$, the only (potentially) worthwhile production choices are $Q_F = \frac{1}{a}(V'_F - V'_s)^{1/\gamma} \equiv q_F'^*$ or $Q_F = (V'_F - V'_s)^{1/\gamma} \equiv aq_F'^*$. Since some foreign buyers (the high valuation ones) would buy when offered the smallest quantity $aq_F'^*$ (the highest possible price), then the deviation would offer the possibility to sell to some, at the highest possible price. Clearly eve if by doing so the surplus from trade is non-positive, then the seller would not deviate and choose $\Pi_F = 1$. Therefore a sufficient condition for $\Pi_F^* = 0$ is

$$V_F - aq_F'^* - V_s \leq 0. \tag{62}$$

Out-of-equilibrium, a domestic buyer i holding foreign currency has a lifetime utility which satisfies

$$\rho V_{F,i} = \alpha_F(1-m'_F) \sum_{h\in\mathbf{N}} \max_{B_{F,i}(q_{F,h}^{'*})} B_{F,i}(q_{F,h}'^*) \left[V_{s,i} + u_i(q_{F,h}^{'*}) - V_{F,i}\right]. \tag{63}$$

One problem with the functional equation above is that it contains strategies $B_{F,i}(q_{F,h}'^*)$ which, as complete contingent plans, define actions never observed along the equilibrium path: we don't know if the domestic buyer with F buys from a foreign seller and at what price. However we know that if $V_F - V_s \leq aq_F'^*$ (as required by (62)) then $V_F - V_s < (aq_F'^*)^\gamma$ because $(aq_F'^*)^\gamma > aq_F'^*$ (since $aq_F'^* < 1$). This has implications for the out-of-equilibrium moves of domestic buyers with F : buyer i will buy all commodities i and $i+1$ offered, i.e. $B_{F,i}^*(q_{F,h}'^*) = 1$ $h = 1, i+1$.

Now there are two cases: (i) $aq_F'^* \geq V_F - V_s > 0$, or (ii) $aq_F'^* > 0 \geq V_F - V_s$. In the second instance domestic buyers with F would always buy ($B_{F,i}^*(q_{F,h}'^*) = 1$ $\forall h$) even if consumption of the commodity offered did not provide any utility (recall that there is no free disposal of money so by buying individuals throw away their unwanted money). Suppose that $V_F - V_s > 0$. Then using $B_{F,i}^*(q_{F,h}'^*) = 1$ $h = 1, i+1$ and (63) implies:

$$\rho V_F = \alpha_F(1 - m_F') \left[2V_s - 2V_F + \left(q_F^{'*}\right)^\gamma (1 + a^\gamma)\right] \tag{64}$$

which, together with (28) and (30), implies

$$(V_F - V_s)\left[\rho + 2\alpha_F(1 - m_F')\right] = \alpha_F(1 - m_F')\left(q_F^{'*}\right)^\gamma (1 + a^\gamma) - 2\alpha_D m_D\left[(aq_D^*)^\gamma - q_D^*\right]. \tag{65}$$

It is easy to show that the inequality $aq_F'^* \geq V_F - V_s$ can be rearranged as

$$\alpha_F \leq \bar{\alpha}_F \equiv \frac{2\alpha_D m\left[(aq_D^*)^\gamma - q_D^*\right] + \rho a q_F^{'*}}{(1 - m')\left(q_F'^*\right)^\gamma \left[1 + a^\gamma - 2a^\gamma \left(q_F'^*\right)^{1-\gamma}\right]}$$

where $\bar{\alpha} > 0$ because $1 + a^\gamma - 2a^\gamma \left(q_F'^*\right)^{1-\gamma} > 0$ (its smallest value, achieved at $q_F'^* = 1$, is positive). Note that $\bar{\alpha}_F$ is a number since it is a function only of parameters (q_D^* and $q_F^{'*}$ are determined by (60) and (61)).

Furthermore $V_F - V_s > 0$ whenever the right hand side of (65) is non-positive:

$$\alpha_F \leq \alpha_{F2} \equiv \frac{2\alpha_D m\left[(aq_D^*)^\gamma - q_D^*\right]}{(1 - m')\left(q_F'^*\right)^\gamma (1 + a^\gamma)}. \tag{66}$$

Clearly $\alpha_{F2} < \bar{\alpha}_F$. Therefore:

- $aq_F'^* \geq V_F - V_s > 0$ if $\alpha_F \in (\alpha_{F2}, \bar{\alpha}_F)$. This implies $\Pi_F^* = 0$ and $q_F^* = 0$.
- $aq_F'^* > 0 \geq V_F - V_s$ if $\alpha_F < \alpha_{F2} < \bar{\alpha}_F$. This also implies $\Pi_F^* = 0$ and $q_F^* = 0$.

We conclude that $\alpha_F < \bar{\alpha}_F$ is sufficient for $\Pi_F^* = 0$ and $q_F^* = 0$. Similarly, for a foreign seller a sufficient condition for $\Pi_D'^* = q_D^{'*} = 0$ is $\alpha_D' < \bar{\alpha}_D'$, where

$$\bar{\alpha}_D' \equiv \frac{2\alpha_F' m'\left[(aq_F'^*)^\gamma - q_F'^*\right] + \rho a q_D^*}{(1 - m)\left(q_D^*\right)^\gamma \left[1 + a^\gamma - 2a^\gamma \left(q_D^*\right)^{1-\gamma}\right]}. \tag{67}$$

Clearly, because of (6) there are k' which satisfy both (66) and (67).■

Proof of Corollary to Proposition 1.

Prices decreasing in the money supply. Using the definition for q_D^*, $m_D = m$, $m'_F = m'$, it is easy to see that $\frac{\partial q_D^*}{\partial m} > 0$ if $\rho \leq \frac{\alpha_D(1-a^\gamma)}{a^\gamma}$. Additionally

$$\rho_D < \frac{\alpha_D(1-a^\gamma)}{a^\gamma},$$

so that since $\rho \leq \rho_D$ is necessary for existence of an equilibrium with two local currencies, then $\frac{\partial q_D^*}{\partial m} > 0$. A similar procedure is used to prove that $\frac{\partial q_F^*}{\partial m'} > 0$. The statement in the corollary follows from our definition of "price" in terms of currency d, i.e. $\frac{1}{q_d^*}$.

Efficient prices. Consider a domestic seller and for convenience let $k' = 0$ so that non-local trade is ruled out. An offer is efficient if it solves the social planner's problem of maximizing the ex-ante lifetime utility of individuals in each country, denoted by

$$\rho W = m\rho V_D + (1-m)\rho V_s,$$

by choice of q_D, while taking as given the trading arrangements. By substituting the equilibrium value functions (30) and (31), we obtain

$$\rho W = \frac{\alpha_D(1-m)m}{\rho + \alpha_D(1+m)} \left[2\left(q_D^\gamma - q_D\right)(\alpha_D + \rho) - \rho q_D^\gamma\right],$$

which is concave in q_D since

$$\frac{\partial W}{\partial q_D} \propto 2\left(\gamma q_D^{\gamma-1} - 1\right)(\alpha_D + \rho) - \gamma q_D^{\gamma-1}$$

and

$$\frac{\partial W}{\partial q_D} \propto 2\gamma(\gamma-1)q_D^{\gamma-2}(\alpha_D+\rho) - \gamma(\gamma-1)q_D^{\gamma-2} < 0.$$

It is easy to see that the first order condition implies that if $\alpha_D + \rho > 1/2$ then the efficient quantity must satisfy

$$q_D = q_D^e \equiv \left\{\frac{\gamma\left[2(\alpha_D+\rho)-1\right]}{2(\alpha_D+\rho)}\right\}^{\frac{1}{1-\gamma}}.$$

Using (60), $q_D^* = q_D^e$ if

$$m = m^e \equiv \frac{(\alpha_D+\rho)\left[\alpha_D - a^\gamma(\alpha_D+\rho)\right]}{\alpha_D\left[(\alpha_D+\rho)(1+a^\gamma-2\gamma)+\gamma\right]}.$$

A similar discussion leads to the definition of

$$m'^e \equiv \frac{(\alpha'_F+\rho)\left[\alpha'_F - a^\gamma(\alpha'_F+\rho)\right]}{\alpha'_F\left[(\alpha'_F+\rho)(1+a^\gamma-2\gamma)+\gamma\right]},$$

for the foreign country. It is easy to see that $m'^e, m^e \in (\underline{m}, \bar{m})$ for $\rho > 0$ small and α_D, α'_F in a neighborhood of one. If k' is sufficiently small, then all conditions set in proposition 1 hold.■

Proof of Lemma 1

In what follows we provide conditions sufficient to guarantee the existence and uniqueness of a positive fixed point (q_D^*, q_F^*) on a subset of $(0, \bar{q})^2$.

1. *Finding the fixed point.* Letting $x_D = (aq_D)^\gamma - q_D$, and $x_F = (aq_F)^\gamma - q_F$, rewrite (48) by adding and subtracting $(aq_D)^\gamma$ to the right hand side of the second equation, and $(aq_F)^\gamma$ to the second:

$$\begin{cases} (q_D)^\gamma \left[\frac{a^\gamma b_1 - b_2}{b_3}\right] + \frac{b_4}{b_3} x_F = -x_D + (aq_D)^\gamma \\ (q_F)^\gamma \left[\frac{a^\gamma c_1 - c_2}{b_4}\right] + \frac{b_3}{b_4} x_D = -x_F + (aq_F)^\gamma . \end{cases}$$

Rearranging

$$\begin{cases} (q_D)^\gamma \left[\frac{a^\gamma (b_1 - b_3) - b_2}{b_3}\right] + \frac{b_4}{b_3} x_F + x_D = 0 \\ (q_F)^\gamma \left[\frac{a^\gamma (c_1 - b_4) - c_2}{b_4}\right] + \frac{b_3}{b_4} x_D + x_F = 0. \end{cases}$$

Solve for x_D from the second equation

$$x_D = -\frac{b_4}{b_3} \left[x_F + \frac{a^\gamma (c_1 - b_4) - c_2}{b_4} (q_F)^\gamma \right]$$

and plug it into the first to obtain

$$q_D^* = q_F \left[\frac{a^\gamma (c_1 - b_4) - c_2}{a^\gamma (b_1 - b_3) - b_2} \right]^{\frac{1}{\gamma}} .$$

Note that $c_1 - b_4 > b_1 - b_3 > 0$, and that

$$c_1 - b_4 = \rho + \alpha_D (1 - m) + \alpha_F (1 - m_F') \equiv \rho + c_2$$
$$b_1 - b_3 = \rho + \alpha_D (1 - m) \equiv \rho + b_2 .$$

Therefore let

$$\delta \equiv \frac{a^\gamma (\rho + c_2) - c_2}{a^\gamma (\rho + b_2) - b_2}$$

and rewrite

$$q_D = q_F \delta^{\frac{1}{\gamma}} . \tag{68}$$

Note that since $\frac{c_2}{\rho + c_2} > \frac{b_2}{\rho + b_2}$, then $\delta > 0$ either if

$$a > \left(\frac{c_2}{\rho + c_2} \right)^{\frac{1}{\gamma}}, \quad \text{or} \quad a < \left(\frac{b_2}{\rho + b_2} \right)^{\frac{1}{\gamma}} .$$

Additionally, $\delta < 1$ if $a > \left(\frac{c_2}{\rho + c_2} \right)^{\frac{1}{\gamma}}$ (because $c_2 > b_2$) and $\delta > 1$ if $a < \left(\frac{b_2}{\rho + b_2} \right)^{\frac{1}{\gamma}}$ otherwise (because the negative numerator would be larger, in absolute value, than the negative denominator). Using q_D from (68) into the second equation of the system yields

$$q_F^* = \left(\frac{a^\gamma c_1 - c_2 + a^\gamma b_3 \delta}{b_4 + b_3 \delta^{\frac{1}{\gamma}}} \right)^{\frac{1}{1-\gamma}} . \tag{69}$$

2. *Conditions for* $q_d^* > 0$ *and* $q_d^* < \bar{q}$.

- $q_F^* < \bar{q}$: This inequality is satisfied if

$$a^\gamma b_3 \left(\delta - \delta^{\frac{1}{\gamma}}\right) < -\left[a^\gamma \rho - c_2(1 - a^\gamma)\right].$$

If $a > \left(\frac{c_2}{\rho + c_2}\right)^{\frac{1}{\gamma}}$ then $\delta < 1$, the left hand side is positive, but the right hand is negative. Therefore we need $a < \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}}$, in which case $\delta > 1$, the left had side is negative while the right hand side is positive, so that the inequality is always satisfied.

- $q_d^* > 0 \; \forall d$: Note that since $c_1 > c_2$, and $\delta > 1$, a sufficient condition for $q_F^* > 0$ is

$$a \geq a_L \equiv \left(\frac{c_2}{c_1 + b_3}\right)^{\frac{1}{\gamma}}.$$

It is easy to show that $a_L < \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}}$ if

$$\rho < \rho_H \equiv \frac{b_2(b_4 + b_3)}{c_2 - b_2}.$$

- $q_D^* < \bar{q}$: Using (68) and (69) this inequality can be rearranged as

$$a^\gamma \left(c_1 \delta^{\frac{1-\gamma}{\gamma}} - b_4\right) - c_2 \delta^{\frac{1-\gamma}{\gamma}} < 0$$

i.e.

$$a^\gamma \left(c_2 - c_2 + c_1 \delta^{\frac{1-\gamma}{\gamma}} - b_4\right) - c_2 \delta^{\frac{1-\gamma}{\gamma}} + c_2 - c_2 < 0$$

yielding

$$\left[a^\gamma (\rho + c_2) - c_2\right] + \left(\delta^{\frac{1-\gamma}{\gamma}} - 1\right)(a^\gamma c_1 - c_2) < 0. \tag{70}$$

Note that when $\rho < \rho_H$, $a_L < \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}} < \left(\frac{c_2}{\rho + c_2}\right)^{\frac{1}{\gamma}}$. Therefore, if $a = a_L$ then the left hand side of (70) is strictly negative since $(a^\gamma c_1 - c_2) < 0$ and $a^\gamma (\rho + c_2) - c_2 < 0$. When $a_L < a < \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}}$ then $a^\gamma (\rho + c_2) - c_2 < 0$ still holds. As $a \to \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}}$ then $\delta \to \infty$ so that the left hand side of (70) is strictly positive. By the intermediate value theorem it follows that there exists an $a_H \in \left(a_L, \left(\frac{b_2}{\rho + b_2}\right)^{\frac{1}{\gamma}}\right)$ such that $\left[a^\gamma (\rho + c_2) - c_2\right] + \left(\delta^{\frac{1-\gamma}{\gamma}} - 1\right)(a^\gamma c_1 - c_2) \leq 0$ for all $a \leq a_H$. Therefore (70) is satisfied for all $a \in (a_L, a_H)$. ■

Proof of Lemma 2.

1. *Conditions for* $\Pi_d^* = 1 \;\; \forall d$. In this equilibrium the proposed quantity offers satisfy (28) and (44), that is $(aq_d)^\gamma = V_d - V_s \; \forall d$. From (22), $\Pi_d^* = 1$ only if sellers have a positive trade surplus after producing $q_{d,}^*$ i.e. $q_d^* < \bar{q}$. We have shown that this inequality is satisfied when the conditions listed in lemma 1 hold.

2. *Condition for* $\Pi_F^{'*} = 1$. Note that $q_F^* < \bar{q}$ is not sufficient to guarantee that foreign sellers sell for currency F, $\Pi_F^{'*} = 1$. This because we have conjectured that $V_F - V_s \geq V_F' - V_s'$ (we will provide conditions for this to hold in the following subsection). Therefore, we must find a condition supporting $V_F' - V_s' > q_F^*$ (in which case foreign sellers have a positive surplus from trade, which satisfies (22)). Using (47) this inequality is rearranged as

$$(q_F^*)^{1-\gamma} < \frac{d_2}{d_1 - d_3} \equiv \frac{(1+a^\gamma)\left[\alpha_D'(1-m) + \alpha_F'(1-m_F')\right]}{\rho + 2\left[\alpha_D'(1-m) + \alpha_F'(1-m_F')\right]},$$

whose right hand side is maximized at $\rho = 0$, becoming $\frac{(1+a^\gamma)}{2}$. Since the upper bound of the left hand side is $\bar{q}$, then the inequality is always satisfied because $a^\gamma < \frac{(1+a^\gamma)}{2}$.

3. *Condition for* $V_F - V_s \geq V_F' - V_s'$. We must find a condition supporting $(aq_F^*)^\gamma \geq V_F' - V_s'$. Using (47) this inequality is

$$(q_F^*)^{1-\gamma} \leq \frac{a^\gamma d_1 - d_2}{d_3}. \tag{71}$$

The RHS of this inequality must be positive, that is $\frac{a^\gamma d_1 - d_2}{d_3} > 0$, which can be rearranged as

$$a > a_{LF} \equiv \left[\frac{\alpha_D'(1-m) + \alpha_F'(1-m_F')}{\rho + \alpha_D'(1 - m_D + m_F) + \alpha_F'(1+m_F')}\right]^{\frac{1}{\gamma}}.$$

Because of the equilibrium requirement listed earlier a sufficient condition for the RHS of (71) to be positive is $a_{LF} \leq a_L$, an inequality that, after some algebra, amounts to

$$\rho(k - k')\left[(1-P)(1-m_F') - (1-m)P\right] \leq 0. \tag{72}$$

Using (3), (4), the equilibrium distribution of (37), (and recalling that $M_D < P$), then

$$1 - m_F' = \frac{(1 - M_D - M_F)}{(1 - M_D)}$$
$$1 - m = \frac{(P - M_D)}{P}(1 - m_F').$$

It follows that if $k > k'$, the LHS of (72) is negative whenever $M_D < 2P - 1$ (which obviously also requires $P > 0.5$). If $k < k'$, the LHS of (72) is negative whenever $M_D > 2P - 1$. Therefore (72) (and hence the positivity of the RHS of (71)) is satisfied whenever,

$$\begin{cases} M_D > 2P - 1, & \text{if } k < k' \\ M_D \leq 2P - 1 \text{ and } P \geq \frac{1}{2}, & \text{if } k \geq k'. \end{cases} \tag{73}$$

Finally, we note that when $a \to a_L$ then the LHS of (71) converges to

$$\frac{a^\gamma b_3(\delta - 1)}{b_4 + b_3 \delta^{\frac{1}{\gamma}}},$$

and (using the definition of δ) note that as $k'(1-P) \to 0$, then $c_2 \to b_2$ so that $\delta \to 1$. By continuity there exists large P and small k' such that (71) is satisfied for some $a \in (a_L, a_H)$.■

Proof of Lemma 3.

To show that profits are maximized, by choosing q_d^* satisfying Lemma 1 and (28) and (44), we consider the profit functions $\Omega(Q_d)$ and $\Omega'(Q_d')$ under all possible one-time deviations in offers, taking as given value functions, distributions and prices in all other matches.[24]

1. *Currency D, Domestic Seller.* $\Omega(Q_D)$ must be maximized when q_D^* satisfies (49). Offering a higher quantity is not an equilibrium since the probability of a sale would not increase. The only possible deviation is offering $Q_D = aq_D^*$ which, given (33) and (28), takes away the surplus form *all* domestic buyers with D. No deviation occurs if $\Omega(q_D^*) > \Omega(aq_D^*)$,which under the conjectured strategies amounts to
$$(q_D^*)^{1-\gamma} < \frac{a^\gamma}{2-a} \tag{74}$$
(see proof of proposition 1). Substitution of q_D^* from (49) implies
$$\left(q_F^* \delta^{\frac{1}{\gamma}}\right)^{1-\gamma} < \frac{a^\gamma}{2-a} \implies q_F^* < \frac{a^{\frac{\gamma}{1-\gamma}}}{(2-a)^{\frac{1}{1-\gamma}}\delta^{\frac{1}{\gamma}}},$$
and since a minimum for q_F^* is obtained when $a = a_L$, in which case rearrange the inequality above as
$$\frac{a_L^\gamma b_3(\delta-1)}{b_4 + b_3\delta^{\frac{1}{\gamma}}}(2-a_L)\,\delta^{\frac{1-\gamma}{\gamma}} < a_L^\gamma \implies b_3\delta^{\frac{1}{\gamma}}\left[(\delta-1)(2-a_L)\delta^{-1} - 1\right] < b_4.$$
The left hand side of this latter inequality is non-positive whenever $(\delta-1)(2-a_L)\delta^{-1} - 1$, or
$$\delta \le \frac{2-a_L}{1-a_L}.$$
Since a minimum for $(2-a_L)/(1-a_L)$ is given by 2, and since $\delta \to 1$ as $k'(1-P) \to 0$ then there exists a sufficiently small k' that satisfies (74) when a is in a right neighborhood of a_L.

2. $\Pi_D^{'*} = 0$. From (22) we need to show that $S'(Q_D') \le 0$, $Q_D' \in \chi_D'$. Since $\pi_D^{'*} = 0$, $m_D' = 0$, and nationality is private information, a foreign seller meeting a buyer with D expects her to be a domestic trader. Thus we must show that the foreign seller's surplus form deviating (accepting D) is non-positive even when he offers the smallest quantity $Q_D' \equiv aq_D^*$ at which some domestic buyer with D would buy. A sufficient condition for $\Pi_D^{'*} = 0$ is then
$$V_D' - aq_D^* - V_s' \le 0. \tag{75}$$

[24] Since a domestic seller chooses both Q_D and Q_F, she could simultaneously deviate from both q_D^* and q_F^* , for one period. We do not consider this type of deviations, but only one quantity at a time.

Out of equilibrium, a foreign buyer i holding currency D has a lifetime utility which satisfies

$$\rho V'_{D,i} = \alpha'_D(1-m)\sum_{h\in\mathbf{N}} \max_{B'_{D,i}(q^*_{D,h})} B'_{D,i}(q^*_{D,h})\left[V'_{s,i} + u_i(q^*_{D,h}) - V'_{D,i}\right]. \tag{76}$$

We know that if (75) holds, then $V'_D - V'_s < (aq^*_D)^\gamma$ because $aq^*_D < 1$. As previously done (see proof of Porposition1) there are two cases to consider: (i) $aq^*_D \geq V'_D - V'_s > 0$ or (ii) $aq^*_D > 0 \geq V'_D - V'_s$. Suppose $V'_D - V'_s > 0$. Then $B'_{D,i}(q^*_{D,h}) = 1$ only for $h = i, i+1$ and (76) implies

$$\rho V'_D = \alpha'_D(1-m)\left[2V'_s - 2V'_D + (q^*_D)^\gamma(1+a^\gamma)\right]. \tag{77}$$

Using (28) and (39)

$$(V'_D - V'_s)\left[\rho + 2\alpha'_D(1-m)\right] = \alpha'_D(1-m)(q^*_D)^\gamma(1+a^\gamma) - 2(\alpha'_D m_F + \alpha'_F m'_F)(V'_F - V'_s - q^*_F) \tag{78}$$

where $V'_F - V'_s$ in equilibrium is a function of q^*_F, defined by (47). Clearly, the most restrictive case for (75) to hold is when $V'_D - V'_s$ is the largest, which occurs when $V'_F - V'_s - q^*_F \approx 0$ in which case (75) is rearranged as

$$(q^*_D)^{1-\gamma} \geq \frac{\alpha'_D(1-m)}{\rho + 2\alpha'_D(1-m)} \cdot \frac{1+a^\gamma}{a}.$$

Rearranging

$$\alpha'_D \leq \frac{a\rho\,(q^*_D)^{1-\gamma}}{(1-m)\left[1 + a^\gamma - 2a\rho\,(q^*_D)^{1-\gamma}\right]}. \tag{79}$$

It is easy to prove that $1 + a^\gamma - 2a\rho\,(q^*_D)^{1-\gamma} > 0$ always, since $q^*_D < \bar{q}$, therefore there exists a sufficiently small k' that satisfies both (79) and (74), and hence (75).

3. *Currency F, domestic and foreign sellers.* Restrict attention to the case $V'_F - V'_s = V_F - V_s$, which we know exists (see lemma 2) for some choice of the parameters. From (20), $\Omega(Q_F)$ and $\Omega'(Q_F)$ must be maximized when q^*_F satisfies (44). Equilibrium profits are:

$$\Omega(q^*_F) = \frac{2}{N}(\alpha_D m_F + \alpha_F m'_F)(V_F - q^*_F - V_s), \quad \Omega'(q^*_F) = \frac{2}{N}(\alpha'_D m_F + \alpha'_F m'_F)(V'_F - q^*_F - V'_s).$$

Since here we are assuming $V'_F - V'_s = V_F - V_s$, only one other possible quantity can be offered by a sellers (domestic or foreign) $Q_F = aq^*_F \equiv (V_F - V_s)^{\frac{1}{\gamma}}$. The deviation profits are

$$\Omega(Q_F) = \frac{\alpha_D m_F + \alpha_F m'_F}{N}(V_F - aq^*_F - V_s)$$

$$\Omega'(Q'_F) = \frac{\alpha'_D m_F + \alpha'_F m'_F}{N}(V'_F - aq^*_F - V'_s).$$

In the expression above $V'_F - V'_s = (aq^*_F)^\gamma$, and it is easy to verify that q^*_F satisfies $\Omega'(q^*_F) > \Omega'(Q'_F)$, and $\Omega(q^*_F) > \Omega(Q_F)$ if $q^*_F < \frac{a^\gamma}{2-a}$. This latter requirement is always satisfied because $q^*_F < q^*_D$ by (49), and we have shown above that in equilibrium $(q^*_D)^{1-\gamma} < \frac{a^\gamma}{2-a}$.■

Proof of Proposition 2.

This follows from lemmas 1 through 3 and implies $V_F' - V_s' = V_F - V_s$. By continuity, similar equilibria exists in a neighborhood of the parameterization chosen.■

Proof of Proposition 3.

The format of this proof is quite similar to the proof of proposition one, and we will focus on domestic sellers and buyers. In equilibrium, using (50)-(53), the "pricing rule" (58), then (56) implies

$$\rho V_D = (1-m)(\alpha_D + \alpha_F)\left[V_s - V_D + (q_D^*)^\gamma\right]$$

$$\rho V_F = (1-m)(\alpha_D + \alpha_F)\left[V_s - V_F + (q_F^*)^\gamma\right].$$

Since $m = m' = M$ in equilibrium, by defining the constants

$$h_0 = \rho + (1 + m_D - m_F)(\alpha_D + \alpha_F), \quad h_1 = (1-m)(\alpha_D + \alpha_F),$$
$$h_2 = 2(\alpha_D + \alpha_F), \quad h_3 = \rho + (1 + m_F - m_D)(\alpha_D + \alpha_F),$$

then

$$(V_D - V_s)\,h_0 = h_1\,(q_D^*)^\gamma + h_2\,(m_D q_D^* + m_F q_F^*) - h_2 m_F\,(V_F - V_s)$$
$$(V_F - V_s)\,h_3 = h_1\,(q_F^*)^\gamma + h_2\,(m_D q_D^* + m_F q_F^*) - h_2 m_D\,(V_D - V_s)\,.$$

Because of the way prices are formed we can rewrite the system of equations above as

$$\begin{cases} (aq_D^*)^\gamma\,h_0 = h_1\,(q_D^*)^\gamma + h_2\,(m_D q_D^* + m_F q_F^*) - h_2 m_F\,(aq_F^*)^\gamma \\ (aq_F^*)^\gamma\,h_3 = h_1\,(q_F^*)^\gamma + h_2\,(m_D q_D^* + m_F q_F^*) - h_2 m_D\,(aq_D^*)^\gamma\,. \end{cases} \tag{80}$$

Equilibrium quantities. Using (80) there is a non monetary equilibrium $q_d^* = 0$ for all d. Subtracting the second from the first equation in (80) we obtain

$$a^\gamma\left[\rho + (\alpha_D + \alpha_F)(1-m)\right]\left[(q_D^*)^\gamma - (q_F^*)^\gamma\right] = (\alpha_D + \alpha_F)(1-m)\left[(q_D^*)^\gamma - (q_F^*)^\gamma\right].$$

which implies that as long as

$$a \neq \left[\frac{(\alpha_D + \alpha_F)(1-m)}{\rho + (\alpha_D + \alpha_F)(1-m)}\right]^{\frac{1}{\gamma}},$$

there is a unique monetary equilibrium where $q_D^* = q_F^* = q^*$, with

$$q^* = \left\{\frac{a^\gamma\left[\rho + (\alpha_D + \alpha_F)(1+m)\right] - (\alpha_D + \alpha_F)(1-m)}{2m(\alpha_D + \alpha_F)}\right\}^{\frac{1}{1-\gamma}}. \tag{81}$$

A sufficient (but not necessary) condition for $q^* > 0$ is

$$m > \underline{m}.$$

Buyer's strategies. It is easily verified that $\beta^*_{d,i}(q^*_{d,h}) = 1\ \forall d, i$ and $h = i, i+1$, since the equilibrium q^* satisfies (58).

Sellers' strategies.

(i) $\pi^*_d = 1$. Since in this equilibrium $V_D - V_s = V_F - V_s$, then using (22), given that $q^*_d = q^*$ satisfies (81), and given (52), it follows that $\Pi^*_d = 1$ whenever $q^* < \bar{q}$.

(ii) q^* *maximizes* $\Omega(q^*)$. Taking as given the value functions, the expected profit from charging the low price must be the highest. Using (20) this is equivalent to

$$V_d - (V_d - V_s)^{\frac{1}{\gamma}} - V_s \leq 2\left[V_d - \frac{(V_d - V_s)^{\frac{1}{\gamma}}}{a} - V_s\right],$$

which, using (28), is satisfied whenever $(q^*)^{1-\gamma} \leq \frac{a^\gamma}{2-a}$. Note that if the latter inequality is satisfied this also guarantees $q^* < \bar{q}$. Substituting (81) in $(q^*)^{1-\gamma} \leq \frac{a^\gamma}{2-a}$ and rearranging we obtain

$$(\alpha_D + \alpha_F)\left[ma^{1+\gamma} + (2-a)(1-m-a^\gamma)\right] \geq (2-a)a^\gamma \rho.$$

It follows that q^* maximizes $\Omega(q^*)$ only if

$$\rho \leq \rho_{D1} \equiv \frac{(\alpha_D + \alpha_F)\left[ma^{1+\gamma} + (2-a)(1-m-a^\gamma)\right]}{(2-a)a^\gamma} \text{ and } m < \bar{m}.$$

These conditions are necessary for Q^*_d to maximize profits for all d (when Q^*_d satisfies (58)), and are sufficient to guarantee $q^*_d < \bar{q}$ so that $\Pi^*_d = 1\ \forall d$. Recall (from the proof of proposition 1) that as $a \to 1$ then $\bar{m} \to 1/2$, that both $\underline{m}$ and $\bar{m}$ are decreasing in a, and that $\underline{m} < \bar{m}\ \forall a$. Therefore there are M_D and M_F that satisfy $m \in (\underline{m}, \bar{m})\ \forall a$.

(iii) $\pi'_d = 1\ \forall d$, *and* q^* *maximizes* $\Omega'(q^*)$. Since $q'^*_D = q'^*_F = q^*$ and $m = m'$,then in equilibrium

$$V'_d - V'_s = \frac{(\alpha'_D + \alpha'_F)\left[(1-m)(1+a^\gamma)(q^*)^\gamma + 2mq^*\right]}{\rho + 2(\alpha'_D + \alpha'_F)}, \quad \forall d. \tag{82}$$

It follows that $\Pi'^*_d = 1\ \forall d$ if $V'_d - V'_s > q^*$, rearranged as

$$(q^*)^{1-\gamma} < \frac{(1+a^\gamma)(1-m)(\alpha'_D + \alpha'_F)}{\rho + 2(1-m)(\alpha'_D + \alpha'_F)},$$

whose right hand side is maximized at $\rho = 0$, in which case it becomes $\frac{(1+a^\gamma)}{2}$. However, since the upper bound of the left hand side is $\bar{q}$, then the inequality is always satisfied because $a^\gamma < \frac{(1+a^\gamma)}{2}$. Finally q^* maximizes $\Omega'(q^*)$ whenever deviating to charge aq^* does not increase the expected profit, i.e. $V'_d - aq^* - V'_s \leq 2\left[V'_d - q^* - V'_s\right]$. Using (82) this inequality can be rearranged as

$$(q^*)^{1-\gamma} \leq \frac{(\alpha'_D + \alpha'_F)(1-m)(1+a^\gamma)}{\rho(2-a) + 2(\alpha'_D + \alpha'_F)(2-a-m)},$$

which is seen to be satisfied with equality by a unique $\rho_{F1} > 0$ small (q^* increases in ρ). It follows that if

$$\rho \leq \hat{\rho} \equiv \min\{\rho_{D1}, \rho_{F1}\} \quad \text{and} \quad m < \bar{m}$$

these conditions are *sufficient* for $\Pi_d^* = \Pi_d^{'*} = 1\ \forall d$, and *necessary* for q^* to be the profit maximizing quantity for both domestic and foreign sellers.

(iv) $V_d - V_s \geq V_d' - V_s'$. Using (58), (81), and (82) this inequality can be rearranged as

$$\rho\left[\alpha_D' + \alpha_F' - (\alpha_D + \alpha_F)\right] \leq 2(\alpha_D' + \alpha_F')(\alpha_D + \alpha_F)(1 - m),$$

an inequality satisfied by the sufficient condition

$$P \geq 0.5 \ \text{ and } \ k \geq k',$$

for all $\rho > 0$ (by some ρ sufficiently small, otherwise).■

Proof of Corollary to Proposition 3.

Coexistence follows from comparison of the conditions required for the existence of equilibria with two local currencies. Note that for $\rho > 0$ small, the region of the parameter space supporting the equilibria with local currencies is a subset of the parameter space supporting equilibria with two international currencies.

Note that in the case of two local currencies $m = M_D/P$, whereas in the case of two international currencies $m = M_D + M_F$. Recall that the equilibrium with two international currencies does not exist if $M_D < \underline{m}P$. If $M_D + M_F > \underline{m}$ the two international currencies equilibrium exists. It follows that if $M_D \in (\underline{m} - M_F, \underline{m}P)$ an equilibrium with two international currencies exists, whereas the equilibrium with two local currencies does not. It follows that on this region of the parameter space the former equilibrium is clearly a Pareto superior outcome.■

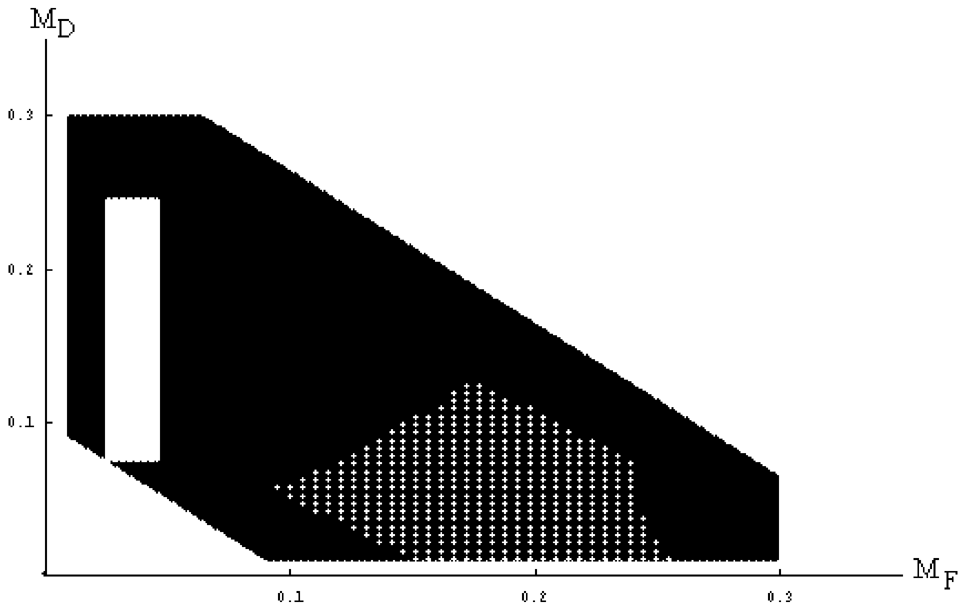

Figure 1: Existence of Equilibria

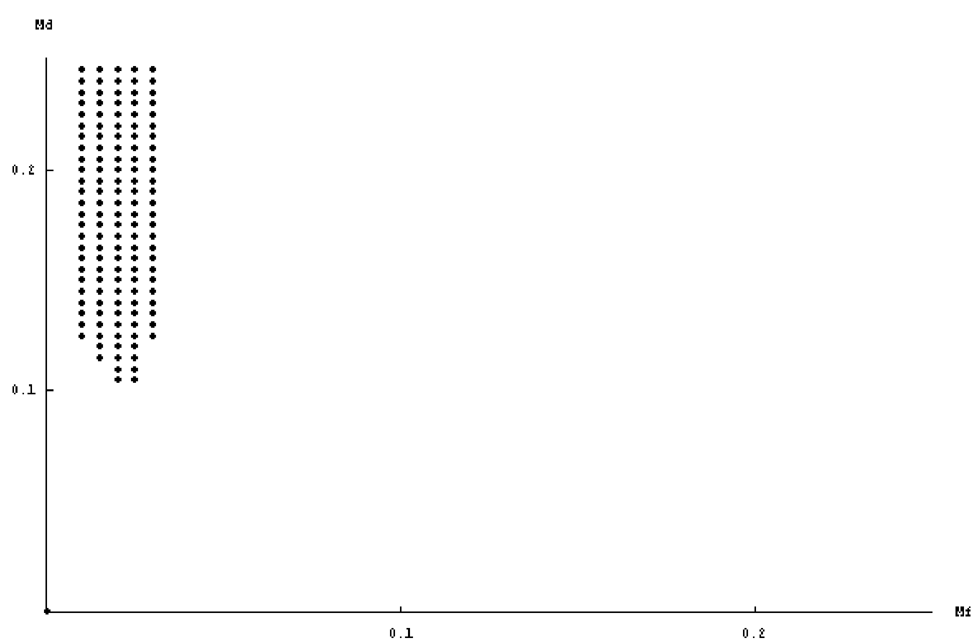

Figure 2: Equilibria with currency substitution and international price dispersion