

On the Consequences of State Dependent Preferences for the Pricing of Financial Assets

by

Jean-Pierre Danthine
Université de Lausanne, FAME and CEPR

John B. Donaldson
Columbia University

Christos Giannikos
Baruch College, City University of New York

Hany Guirguis
Manhattan College

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Abstract

This paper introduces state dependent utility into the standard Mehra and Prescott (1985) economy by allowing the representative agent's coefficient of relative risk aversion to vary with the underlying economy's growth rate. Existence of equilibrium is proved and its asymptotic properties analyzed. This generalization leads to level dependent marginal rates of substitution, a property that sharply distinguishes this model from the standard construct. For very low coefficients of relative risk aversion, the equilibrium risk free and risky security returns are demonstrated to have volatilities and an associated equity premium that substantially exceed what is found in the data. This provides a contrasting perspective on the classic "equity premium puzzle."

Keywords: state dependent utility, equity premium, equity premium puzzle

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1. Introduction

This paper explores the implications for asset pricing of allowing the representative agent's coefficient of relative risk aversion to vary with the economy's growth rate of consumption. That such preference representations are consistent with the choice theoretic foundations of utility theory was first argued in Savage (1972). Myerson (1991) provides an axiomatic base for the precise form considered here.

In light of the increasing popularity of the state dependent utility framework, no doubt motivated by observations suggesting the pervasiveness of changes in the markets' tolerance for risk, it seems appropriate to explore the full implications of this modeling hypothesis for asset pricing in particular. Our main result is that, at least for the very natural formulation that we adopt, the consequences of allowing risk aversion to be state dependent are extreme. They arise from the fact that agents' demand for securities will depend not only upon the growth rate of consumption (as in the canonical case, cf. Mehra and Prescott (1985)) but upon its level as well. The end-effect is to introduce another source of variation to the pricing kernel, one that is unrelated to the volatility of the underlying fundamentals and whose importance grows disproportionately as the economy itself is growing.

The implications of this latter feature are such as to alter dramatically the form of the classic asset pricing puzzles: for standard parameterizations, the equity premium is easily matched or exceeded, the risk free rate is asymptotically too low, the Hansen-Jagannathan bounds are easily satisfied, and the standard deviations of equity and risk free returns are too large relative to the data. At the minimum, our uncovering of this disturbing property should serve as a warning signal for those relying on state dependent preferences to help them resolve some outstanding puzzles.

General support for the hypothesis that risk aversion is state dependent may be found in recent results from the experimental psychology and economics literatures. Isen and Patrick (1983), Isen and Geva (1978) and Nygren et al. (1996) present evidence suggesting that happy decision makers – those who have received a consumption increase – are much less willing to gamble than control groups. Isen (1996) interprets these results as suggesting that persons in a “good mood” are more reluctant to gamble because losing might undermine their good mood. Bosh-Domènech and Silvestre (1999) report the results of an experiment in which the subjects were given title to a random payout of money and were asked if they wished to insure against a 20% chance of having their personal monetary realization taken from them. Half of the subjects chose to insure, but only if their income realization fell within the high level category, a response that associates greater risk aversion with higher income levels. Strictly speaking, these experiments do not focus on pure growth rate effects, and we are unaware of any experiment that does. Nevertheless, they are consistent with the postulate of associating higher risk aversion with greater consumption growth and higher consumption levels. Broadly speaking, this is the perspective that risk aversion is procyclical.

Strong empirical evidence to the contrary is provided by Gordon and St-Amour (2002) who postulate a model with time varying risk aversion not unlike the one to be considered here, and estimate the implied process on risk aversion arising from per capita consumption and financial return data. Their basic finding is that risk aversion is strongly countercyclical, rising during recessions and falling during expansions. In addition, the Gordon and St.-Amour (2002) CRRA estimate moves opposite to the University of Michigan index of consumer confidence, a fact that is also broadly consistent with their finding of countercyclical risk aversion.

These results are also in harmony with the basic postulates underlying the habit formation literature where, *ceteris paribus*, higher current consumption levels (and thus growth rates *ex post*) are associated with locally lower risk aversion on the part of the representative agent. We adopt the countercyclical perspective as our benchmark. The results we obtain are independent, however, of the specific pattern of this variation: the same implications are associated with procyclical or countercyclical risk aversion.

Other relevant literature includes Campbell and Cochrane (1999), Constantinides (1986), Mehra and Sah (2001), and Kraus and Sagi (2002). Constantinides (1986) initiated the by-now-substantial habit formation literature. Using this device Campbell and Cochrane (1999), more recently, present a precisely calibrated model that is broadly consistent with the stylized facts of the financial markets. The level of *effective* risk aversion they employ is, however, quite high. The present paper replicates many of the results in their model with much lower effective risk aversion; this is possible because the underlying risk variation mechanism is fundamentally different from that in Campbell and Cochrane (1999). Also of direct relevance is Mehra and Sah (2002), who derive the partial equilibrium effects of small fluctuations in agents' subjective preferences (discount rates and risk aversion) on the volatility of asset prices. Our model may be viewed as the return-focused, general equilibrium counterpart of the Mehra and Sah (2002) analysis. A related (the underlying mechanism is similar) but much more elaborate model can be found in Kraus and Sagi (2002). We discuss this paper and Melino and Yang (2001) in more detail in a later section.

An outline of the paper is as follows: Section 2 presents the model in the context of the classic setting of Mehra and Prescott (1985), while the solution methodology is detailed in the associated appendix. Section 3 provides a full overview and interpretation of a numerical

analysis of the model, and explores the consequences of admitting a more general stochastic structure. A comparison with other models in the literature is provided in Section 4 while Section 5 concludes the paper.

2. The Model

2.1 The Model

In this section we present an adaptation of the Mehra and Prescott (1985) model and detail the methodology for computing its financial equilibrium. Since the model is so familiar the description will be parsimonious.

There is a financial market in which an equity claim to an exogenous output stream $\{y_t\}$ is traded. Output grows at the stationary stochastic growth rate x_t according to

$$(2.1) \quad y_{t+1} = x_{t+1}y_t.$$

An infinitely lived representative agent with state dependent utility organizes his preference for random consumption paths according to

$$(2.2) \quad E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, \alpha_t) \right\},$$

where $u(c_t, \alpha_t) = \frac{c_t^{1-\alpha_t}}{1-\alpha_t}$ is the agent's period utility function and E the expectations operator.

The agent's CRRA, α_t , is presumed to vary stochastically through time. Subsequently we will relate α_t closely to the output growth rate x_t but for the moment it will be useful to preserve the added generality. In principle, the state of this economy is therefore the triple (y_t, x_t, α_t) .

Taking security prices as given the representative agent solves

$$(2.3) \quad \begin{aligned} & \max E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t, \alpha_t) \right\} \\ \text{s.t.} \quad & c_t + z_{t+1} p_t \leq z_t (p_t + y_t) \\ & 0 \leq z_t \leq 1, \end{aligned}$$

where $p_t = p(y_t, x_t, \alpha_t)$ denotes the period t price of the equity claim, and $z_t = z(y_t, x_t, a_t)$ the fraction of the perfectly divisible share held by the agent. By well known arguments, equilibrium for this economy is characterized by the asset price function $p(y_t, x_t, a_t)$ that solves:

$$(2.4) \quad u_1(y_t, a_t) p(y_t, x_t, a_t) = \beta \int u_1(y_{t+1}, \alpha_{t+1}) [p(y_{t+1}, x_{t+1}, \alpha_{t+1}) + y_{t+1}] \cdot dF(y_{t+1}, x_{t+1}, \alpha_{t+1}; y_t, x_t, \alpha_t)$$

In a like fashion, the period t price of a risk free asset, paying one unit of the output good in every state next period and assumed to be in zero net supply, is given by

$$(2.5) \quad q(y_t, x_t, a_t) = \beta E_t \left\{ \frac{u_1(y_{t+1}, x_{t+1}, \alpha_{t+1})}{u_1(y_t, x_t, \alpha_t)} \right\}.$$

The corresponding conditional period rates of return for these security, denoted respectively by $r^e(y_{t+1}, x_{t+1}, a_{t+1}; y_t, x_t, a_t)$ and $r^f(y_t, x_t, a_t)$ are thus:

$$r^e(y_{t+1}, x_{t+1}, a_{t+1}; y_t, x_t, a_t) = \frac{p(y_{t+1}, x_{t+1}, a_{t+1}) + y_{t+1}}{p(y_t, x_t, a_t)} - 1, \text{ and}$$

$$r^f(y_t, x_t, a_t) = \frac{1}{q(y_t, x_t, a_t)} - 1.$$

Equation (2.4), in particular, is obtained from substituting the market clearing conditions $c_t = y_t$ and $z_t = 1$ into the representative agent's necessary and sufficient first order condition. For the specialized functional forms specified later in the paper, we prove that an equilibrium exists constructively. See Appendix 1.

Of crucial significance for asset pricing is the behavior of the representative agent's equilibrium intertemporal marginal rate of substitution (IMRS). Under the utility specification considered here, this quantity assumes the form

$$(2.6) \quad \begin{aligned} \text{IMRS}_{t,t+1} &= \beta \frac{u_1(y_{t+1}, x_{t+1}, \alpha_{t+1})}{u_1(y_t, x_t, \alpha_t)} \\ &= \beta \frac{(y_t)^{\alpha_t - \alpha_{t+1}}}{(x_{t+1})^{\alpha_{t+1}}} \end{aligned}$$

Two features stand out relative to the $\text{IMRS}_{t,t+1}$ in the standard Mehra and Prescott

(1985) economy where $a_t = a$, and $\text{IMRS}_{t,t+1} = \beta \left(\frac{1}{x_{t+1}^\alpha} \right)$:

(i) the $\text{IMRS}_{t,t+1}$ depends on the output level y_t , and

(ii) there is an added source of volatility, a_t , which is fundamentally different from the customary pure consumption uncertainty. With variation in a_t the agent's period utility varies even if there is no uncertainty in his consumption growth rate. To distinguish this latter source of risk from the variation in utility arising from consumption uncertainty alone, it will be referred to as "mood" or "outlook" uncertainty in recognition of the observation that an individual's mood can substantially affect his assessment of his objective circumstances. Adding this feature to an otherwise parsimonious-in-the-extreme model makes explicit, in one particularly simple way, the assertion that such mood swings can potentially influence an individual's economic behavior. Note also that these mood swings are fully anticipated by the agent and thus may be fully hedged. The results to follow are thus unrelated to any aspect of market incompleteness.

It is the associated level effect, however, that gives these mood swings potency for asset pricing. From (2.6) we see that as y_t increases the standard deviation of the $\text{IMRS}_{t,t+1}$ will similarly increase as $a_t - a_{t+1}$ is stochastically negative and then positive. As a result, the agent

will increasingly desire to smooth his consumption without in fact being able to do so.

Anticipating somewhat the results to follow this effect gives rise asymptotically to very high equilibrium risk free asset prices (and thus low risk free rates) and to very low equity prices (and, simultaneously, high equity returns). A high equity premium follows accordingly.

2.2 Sources of the Level Effect

The period utility function $u(c) = \frac{c^{1-\tilde{\alpha}}}{1-\tilde{\alpha}}$ is a special case of the specification $u(c) = \tilde{\eta} c^{1-\tilde{\alpha}}$, with $\tilde{\eta}$ a bounded random variable with $\tilde{\eta} < 0$ for $\tilde{\alpha} > 1$. The question to be explored is whether the level effect is due in any way to variation in the multiplicative factor $\tilde{\eta}$, rather than variation in $\tilde{\alpha}$ itself, or to some complex interaction of $\tilde{\eta} \left(\frac{1}{1-\tilde{\alpha}} \right)$ with $\tilde{\alpha}$. There are a number of ways to tackle this issue, two of which are considered below.

2.2.1 If the results are solely attributable to the multiplicative factor $\tilde{\eta}$, then the specification $u(c) = -c^{1-\tilde{a}}$ should rule them out. But it does not. As before, let $\tilde{a} \in \{a_1, a_2\}$ with $a_1 = a(x_1) < a(x_2) = a_2$. With this simplified utility specification, the $IMRS_{t, t+1}$ now has the form

$$(2.7) \quad IMRS_{t, t+1} = \frac{(y_t)^{\alpha_t - \alpha_{t+1}} (1 - \alpha_{t+1})}{(x_{t+1})^{\alpha_{t+1}} (1 - \alpha_t)} .$$

Assuming $a_t > 1$ for all t , the standard deviation of this quantity will also grow without bound as y_t increases. At least at this superficial level there is no indication that the multiplicative factor is the source of the results.

2.2.2 Let us next consider the other extreme where it is assumed that $u(c) = \frac{\tilde{\eta} c^{1-\alpha}}{1-\alpha}$: no variation in the CRRA, but variation in the multiplicative factor.

The associated maximization problem of the representative agent is

$$(2.8) \quad \max_{\{z_t\}} E \left(\sum_{t=0}^{\infty} \beta^t \left[\tilde{\gamma}_t \frac{c_t^{1-a}}{1-a} \right] \right)$$

$$\text{s.t.} \quad c_t + p_t z_{t+1} \leq (p_t + y_t) z_t$$

where z_t denotes the fraction of the equity security demanded. In equilibrium

($c_t = y_t$, $y_{t+1} = \tilde{x}_{t+1} y_t$, and $z_t = 1$), the equity pricing function must satisfy

$$p(x_t, \theta_t) = \beta \int \frac{\tilde{\theta}_{t+1}}{\theta_t} (\tilde{x}_{t+1})^{-\alpha} [p(x_{t+1}, \theta_{t+1}) + y_{t+1}] dG(x_{t+1}, \theta_{t+1}; x_t, \theta_t).$$

The same solution procedure as in Mehra and Prescott (1985) will work for the above equation because the level of output is not present in the pricing kernel; that is, the IMRS is level independent. Its standard deviation does not grow with output. Variations in the multiplicative factor $\tilde{\gamma}$ appear not to drive the results. We note that problem (2.8) is formally equivalent to the standard Mehra and Prescott (1985) formulation where the discount factor is stochastic, a model with no unusual asset pricing features.¹

Whatever unexpected asset pricing features this model may possess, these comments suggest that CRRA variation can be their only source. It remains only to calibrate the model and to explore its remaining implications numerically.

2.3 Probability Structures.

We follow Mehra and Prescott (1985) exactly and specify a two state Markov chain governing the growth rate of output:

¹ We need to qualify this comment somewhat. In a model with Epstein-Zinn/Weil preferences where the CRRA and the elasticity of intertemporal substitution (EIS) can be specified independently, Melino and Yang (2001) find that state dependent time preferences can have substantial effects, when employed in conjunction with a variable CRRA. In particular, in conjunction with a variable CRRA, it can serve as a substitute for variation in the EIS in achieving a high premium.

$$(2.9) \quad x_1 = \begin{cases} x_1 = 1 + \mu + d \\ x_2 = 1 + \mu - d \end{cases} \begin{matrix} x_1 & x_2 \\ \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{211} \end{bmatrix} \end{matrix}$$

In the spirit of the habit formation literature, we first assume $a_t \in \{a_1, a_2\}$ where $a_1 = a(x_1) < a(x_2) = a_2$: the agent is relatively less risk averse in the high-growth-in-output state.

Notice under this specification that output growth and the CRRA are perfectly negatively correlated. This latter feature can be relaxed in a straightforward and simple way by specifying a joint process on a and x as governed by a Markov chain of the form

$$(2.10) \quad \begin{matrix} & (x_1, \alpha_1) & (x_1, \alpha_2) & (x_2, \alpha_1) & (x_2, \alpha_2) \\ \begin{matrix} (x_1, \alpha_1) \\ (x_1, \alpha_2) \\ (x_2, \alpha_1) \\ (x_2, \alpha_2) \end{matrix} & \left| \begin{array}{cccc} \Phi & \pi & \sigma & H \\ \pi - \Delta & \Phi - \Delta & H & \sigma \\ \sigma & H & \Phi - \Delta & \pi + \Delta \\ H & \sigma & \pi & \Phi \end{array} \right| \end{matrix}$$

In the above formulation, the entries F , p , s , $?$, and H may be selected to admit any specification of $\text{corr}(a_t, a_{t-1})$, $\text{corr}(y_t, y_{t-1})$, and $\text{corr}(a_t, y_t)$. This added generality will be seen to leave the fundamental results of the paper largely unaltered, however. For this reason we take (2.9) as the benchmark output process for most of our discussion.

Let us next specialize equations (2.4) and (2.5) to accommodate our specific output process; with this substitution they become, respectively:

$$(2.11) \quad p(y, x_i) = \beta \sum_{j=1}^2 \frac{(y x_j)^{-\alpha_j}}{(y)^{-\alpha_i}} y x_j \pi_{ij} + \beta \sum_{j=1}^2 \frac{(y x_j)^{-\alpha_j}}{(y)^{-\alpha_i}} p(y x_j, x_j) \pi_{ij}, \text{ and ,}$$

$$(2.12) \quad q(y, x_i) = \beta \sum_{j=1}^2 \frac{(y x_j)^{-\alpha_j}}{(y)^{-\alpha_i}} \pi_{ij}$$

Equation (2.11), although the direct analogue of the pricing equation (7) in Mehra and Prescott (1985), is fundamentally more complex in structure. In particular the pricing function, which solves (2.11), cannot be homogeneous of degree one in output (as in Mehra and Prescott (1985)) so that (2.11) represents a countably infinite set of (level dependent) linear equations. Furthermore, the equilibrium pricing function cannot be expressed as the fixed point of a contraction. Nevertheless, there is a certain recursivity present in the structure of equation (2.11), which can be readily exploited to construct its solution. See also Appendix 1.

2.4 Asymptotic Properties of the Risk Free Security Price

In the calibration to follow, we will need to choose reasonable values of a_1 and a_2 and the risk free security price will be seen to yield clues as to what they should be. For this reason it will be useful to explore the asymptotic behavior of the risk free security price as output grows without bound through time. In order to simplify the discussion further, let us also restrict the probability transition structure to the two state Mehra and Prescott (1985) case; with $p_{11} = p_{22} = p$. Since the ultimate goal is to explore relative model performance, such a restriction is appropriate at this point anyway.

Under these restrictions, for any output level y_t , the prices of the risk free asset in the high and low growth states are, respectively

$$(2.13) \quad q(y_t, x_1) = \beta \left\{ p \frac{(x_1 y_t)^{-a_1}}{(y_t)^{-a_1}} + (1-p) \frac{(x_2 y_t)^{-a_2}}{(y_t)^{-a_1}} \right\}$$

$$= \beta \left\{ p(x_1)^{-a_1} + (1-p)x_2^{-a_2} (y_t)^{a_1 - a_2} \right\}, \text{ and}$$

$$(2.14) \quad q(y_t, x_2) = \beta \left\{ p(x_2)^{-a_2} + (1-p)x_1^{-a_1} (y_t)^{a_2 - a_1} \right\}.$$

Since $a_1 - a_2 < 0$, $q(y_t, x_1) \mapsto \beta \pi(x_1)^{-\alpha_1}$ as $y_t \mapsto \infty$, and

$r^f(y_t, x_1) \mapsto \frac{1}{\beta p(x_1)^{-a_1}} - 1$, finite. For the low growth state, however,

$q(y_t, x_2) \mapsto \infty$ as $y_t \mapsto \infty$ because $a_2 - a_1 > 0$. As a result, $r^f(y_t, x_2) \mapsto -1$. With each state having equal asymptotic likelihood, the unconditional average risk free rate Er^f satisfies,

$$(2.15) \quad Er^f \mapsto \frac{1}{2} \left[\frac{1}{\beta p(x_1)^{-a_1}} \right] - 1, \text{ as } y_t \mapsto \infty.$$

Furthermore, it is monotonically decreasing with time.

In a like fashion the unconditional asymptotic standard deviation satisfies

$$(2.16) \quad SDr^f \mapsto \frac{1}{8} \left[\frac{(x_1)^{2a_1}}{\beta^2 p^2} \right]; \text{ it is monotonically increasing.}$$

Notice that in expression (2.15) (also in (2.16)) only the high-growth-low-risk-aversion values are represented. Otherwise, the intuition encapsulated therein is pretty much standard: agents with higher subjective discount factors β bid up security prices thereby lowering the risk free rate. In a like fashion, a higher p suggests greater risk: consumption growth rates are either highly persistently favorable or unfavorable; risk averse agents value risk free assets more highly in these circumstances similarly bidding up prices. Somewhat counterintuitively, as the agent becomes more risk averse in the high growth state (larger a_1), asymptotic risk free security prices in that same state decline, a fact that leads to a higher asymptotic standard deviation.

It is important also to realize that these results do not depend critically on the assumption of greater risk aversion specifically in the low growth state. If the converse were true; that is, if states one and two change roles so that $a_1 > a_2$, a scenario perhaps more in harmony with the experimental psychology literature, then $q(y_t, x_2) \mapsto \beta \pi(x_2)^{-a_2}$ and $q(y_t, x_1) \mapsto \infty$. The asymptotic formulae for Er^f and SDr^f are altered accordingly (in particular,

$E r^f \mapsto \frac{1}{2} \left[\frac{1}{\beta p \alpha_1 (x_1)^{-a_1}} \right] - 1$ as $y_t \mapsto \infty$), but their essential form remains the same. Which state

displays the greater risk aversion is thus not significant for the majority of the results to follow.

Recognizing that the price of the risk free asset effectively identifies the state contingent MRS, the remarks above have the implication that the expected MRS effectively becomes asymptotically unbounded, and it is this fundamental feature that drives many of the results.

2.5 Calibration: Benchmark Formulation

To the extent possible, we initially rely on the original Mehra and Prescott (1985) calibration. In particular, Mehra and Prescott (1985) choose $\beta = .96$, $\mu = .018$, $d = .036$, and $p = .43$ in order to match the mean, standard deviation, and first order autocorrelation in the growth rate of U.S. per capita consumption. It remains first to specify a_1 and a_2 . As mentioned earlier, we look to the risk free security price to provide clues.

Since $x_1 > 1$, notice that (2.15) implies that the risk free rate will be made lower if a_1 is close to one.² It is also reasonable to hypothesize that, at a minimum, the representative agent would only be willing to pay a smaller and smaller fraction of his income to purchase one unit of the risk free security as his income grows arbitrarily large. Such a security, after all,

² It is possible, of course, to choose an a_1 so as to match exactly the empirically observed average real risk free rate of .8% (cf. Mehra and Prescott (1985)). For $\beta=.96$, $x_1=1.056$, $p=.43$, working backwards through (2.15) gives an $a_1=15.96$; the agent must be dramatically risk loving in his high growth state. While this result is perhaps not objectionable qualitatively, the magnitude appears too large to be reasonable, a fact that is itself directly attributable to the assumed low persistence in the output growth rate (.43). High persistence ($p=.95$) yields a more conventional $a_1=-1.4$ under the same calculation. As with most asset pricing models, the one considered here is thus likely to yield a counterfactually high $E r_f$. These observations remind us how parsimonious the preference construct of Mehra and Prescott (1985) actually is. In particular the CRRA and the elasticity of the intertemporal marginal rate of substitution are one and the same (a). If we retain the natural assumption of risk aversion then the calculations above suggest that a matching of the empirical risk free rate will require not only a model where the CRRA and the EIMRS may be specified independently but one in which the latter displays considerable variations. This is accomplished in Melino and Yang (2001).

represents a claim on only one unit of additional consumption. To summarize, it is reasonable to require that

$$\frac{q(y_t, x_i)}{y_t} \mapsto 0 \text{ as } y_t \mapsto \infty \text{ for any } x_i \in \{x_1, x_2\}.$$

In the case of $q(y_t, x_1)$, such a requirement is automatically satisfied. In the case of $q(y_t, x_2)$,

$$\begin{aligned} \frac{q(y_t, x_2)}{y_t} &= \beta \frac{\{p(x_2)^{-a_2} + (1-p)(x_1)^{-a_1} (y_t)^{a_2-a_1}\}}{y_t} \\ &= \beta \left\{ \frac{p(x_2)^{-a_2}}{y_t} + (1-p)(x_1)^{-a_1} (y_t)^{a_2-a_1-1} \right\} \end{aligned}$$

Provided $a_2 - a_1 < 1$, $\frac{q(y_t, x_2)}{y_t} \mapsto 0$, as $y_t \mapsto \infty$ as well. Our restrictions on a_1 and a_2 are thus that $a_1 \sim 1$ and $a_2 - a_1 < 1$.³ Both restrictions are convenient, especially the latter one, because it suggests that our subsequent results do not require dramatic changes in the CRRA, a property that would challenge our sense of what is reasonable.

These calculations also remind us that the financial return statistics for this model will be sensitive to the magnitude of output, and thus to the length of the time series we use as the basis of our statistical computations. Denoting this time series length by the variable “S”, we will choose for $S = 120$, in the case of $\beta = .96$ where the length of the period is interpreted to be one year. In the case of $\beta = .99$ (period is one quarter), we choose $S = 400$. Both of these parameterizations are roughly in conformity with the maximum length of available data sets. All

³ This choice is also broadly consistent with the estimate in Gordon and St-Amour (2002). Their estimates for risk aversion are centered at .25.

reported statistics are averages of the indicated quantity computed for one thousand individual time series.

It remains only to fix the initial output level y_0 , a choice which is in some sense “dual” to the choice of the series length with a choice of $S=100$. For all reported cases we choose $y_0=1$

because this choice yields a $E\left(\frac{p}{y}\right)$ which is very similar to its constant CRRA counterpart

$\alpha_t \equiv \alpha = \frac{\alpha_1 + \alpha_2}{2}$.⁴ While the $E\left(\frac{p}{y}\right)$ ratio has limited empirical significance in this model, it

does constitute a natural statistic around which to standardize.⁵

3. Numerical Results

3.1 Replicating the Mehra and Prescott (1985) Experiments

Table (1) below presents a benchmark set of cases. Not only do we report the first and second moments of all the series, but their basic correlation structure as well. In reference to the Hansen-Jagannathan bound, the ratio $SD(IMRS)/E(IMRS)$ is also reported; if the model is to explain the most basic statistical characteristics of the equity premium this ratio must exceed the

value $\frac{E r^p}{\sigma_{r^p}} = .37$ for the U.S. economy.

⁴ There is another sense in which $y_0=1$ is a natural choice of starting point. Note from (2.6) that the SD of the $IMRS_{t,t+1}$ will increase without bound both as $y_t \rightarrow \infty$ and as $y_t \rightarrow 0$. Our results thus all apply to an endlessly shrinking economy as well. These dual asymptotic results suggest $y_0=1$ since departures (in either direction) give rise to the phenomena reported here.

⁵ By “limited empirical significance” we mean that this ratio corresponds to the (value of equity)/GDP ratio only for an economy where all the income is capital income. If we modify the model to accommodate a situation in which only a fraction of the income stream is priced (output less wage income), the resulting $E\left(\frac{p}{y}\right)$ and $SD\left(\frac{p}{y}\right)$ are much lower for the same series length. Mehra (1998) demonstrates that a properly calibrated model can account neither for the range nor the variation in the $\left(\frac{p}{y}\right)$ ratio. His CRRA is constant, however.

Table 1
 Summary Return Statistics: Representative Cases
 (Unless Otherwise Indicated, $\beta = .96$, $p = .43$, $\mu = .018$, $d = .036$, $S = 120$, $y_0=1$)
 All Returns Expressed in Percent^{(i), (ii)}

	A U.S. Data	B Mehra- Prescott (1985) $a_1=a_2=3$	C $a_1 = 1.0$ $a_2 = 1.5$	D $a_1 = .5$ $a_2 = 1.0$	E $a_1 = 4.0$ $a_2 = 4.5$	F $a_1 = 2.5$ $a_2 = 2.0$	G $a_1 = 2.0$ $a_2 = 2.5$ $d=0$
$E r^e$	6.98	9.58	16.95	16.14	21.5	21.57	20.63
S.D. r^e	16.54	4.99	53.04	53.17	51.94	60.95	58.69
$E r^f$.80	9.10	8.47	7.19	15.74	7.69	9.72
S.D. r^f	5.67	1.61	34.21	33.90	35.78	34.95	36.24
$E r^p$	6.18	.48	8.48	8.94	5.76	13.9	10.91
S.D. r^p	16.67	4.70	34.92	35.26	32.75	44.0	40.27
$\text{Corr}(r_t^e, r_{t-1}^e)$	-.03	-.33	-.47	-.47	-.47	-.48	-.48
$\text{Corr}(r_t^f, r_{t-1}^f)$.87	-.15	-.15	-.15	-.14	-.15	-.15
$\text{Corr}(r_t^e, r_t^f)$	-.09	.34	.77	.77	.79	.64	.74
$\frac{SD(IMRS)}{E(IMRS)}$.11	.20	.32	.39	.54	.49
$E\left(\frac{P}{y}\right) / E\left(\frac{P}{y}; \alpha = \frac{\alpha_1 + \alpha_2}{2}\right)$		1	1.05	1.16	1.08	1.14	1.13

- (i) Data Sources: Mehra and Prescott (1985), Mehra (1998), and Jones (2001); the correlations are based on data for the period 1897-1998, where r^f is the real commercial paper rate and r^e is obtained from Cowles Foundation and CRISP data.
- (ii) Panel B reports results from the original Mehra and Prescott (1985) model when the CRRA is fixed at $a = 3$

Lastly, the ratio

$$E\left(\frac{P}{y}; \alpha_1 \neq \alpha_2\right) / E\left(\frac{P}{y}; \alpha_1 = \frac{\alpha_1 + \alpha_2}{2}\right)$$

is reported in order to give some sense of the relative magnitude of the price-output ratio

vis-à-vis its value in a pure Mehra and Prescott (1985) economy. For comparison purposes, Panel A provides the by-now-familiar summary financial statistics for the U.S. economy while Panel B

provides the corresponding Mehra and Prescott (1985) results for a representative choice of α . The classic equity premium puzzle is clearly evident, as are the other dimensions of model failure not emphasized in Mehra and Prescott (1985): the return standard deviations are much too low, the mean security returns are uniformly too high, and the correlation structure generally does not conform to the data. The low value of the $SD(IMRS)/E(IMRS)$ ratio is fully consistent with these results.

Panel C reports results under the presumption of both consumption and “outlook” uncertainty and we will take it as the benchmark case. While the coefficient of relative risk aversion in each state is very low, the premium rises to 8.48%.⁶ The standard deviation of all the series have increased enormously and now substantially exceed their corresponding values in the data. Mean security returns are also excessive although less so than for standard deviations, particular in the case of the risk free rate and the premium. The pattern of correlations departs little from the Mehra and Prescott (1985) case and the $SD(IMRS)/E(IMRS)$ more than exceeds the level necessary to resolve the equity premium puzzle. All of this is accomplished in a context in which $E\left(\frac{P}{y}\right)$ does not depart much from its corresponding value in the standard Mehra and Prescott (1985) economy. From the perspective of this particular adaptation of the Mehra and Prescott (1985) model, the “puzzle” is not that the actual premium and security return standard deviations are so high but that they are so low.

Panels D and E provide comparative statistics when the agent is made, respectively, less and more risk averse (relative to Panel C) all the while maintaining the same CRRA risk differential of one half. Comparing Panels C and D we observe that as the agent becomes less

⁶ This results is broadly consistent with the estimates of Gordon and St-Amour who find that consumption risk accounts for less than 1% of the fitted risk premium. In this model, the premium observed when $a_1=a_2=2.25$ (only consumption growth uncertainty) is only 4% of the 7.51% figure.

risk averse, average returns to both equity and debt decline, the more so for the risk free security. As a result, the premium rises. We attribute this latter observation to the fact that as a_1 and a_2 decline, all the while maintaining $a_2 - a_1 = .5$, they become more disparate on a relative basis (as measured, for example, by the ratio of $\frac{a_2}{a_1}$; for Panel C this ratio is .50 while for Panel D it is

2.0). In effect, although the average level of risk aversion declines, with the familiar consequences, relative outlook variation increases. The latter effects dominate with the result that the riskiness of the agent's overall environment increases. On a relative basis the risky asset becomes less valuable and the premium rises. The relative outlook affect diminishes, however, if a_1 and a_2 are increased (compare Panels C and E). In the case of $\frac{a_2}{a_1} = \frac{4.5}{4} = 1.25$. As a result

the premium declines to 5.76%, although mean returns rise and standard deviations are little affected. But this cannot be the full story. Table (2) presents a set of cases in which a_1 and a_2 are progressively increased, all the while maintaining the same differential of $a_2 - a_1 = 1/2$. Panel B contains the analogous results for the corresponding Mehra and Prescott (1985) economies.

What is of interest here is the observation that the consequences of "outlook" uncertainty for the premium increase when the average CRRA decreases. This is the exact opposite of the Mehra and Prescott (1985) result. In fact, the premium eventually turns negative for sufficiently high CRRA. Our strong, and somewhat striking, results are a manifestation of low rather than high average risk aversion. Furthermore, the effects are quite large. For this reason we will typically limit a_1 and a_2 not only to differ by at most .5 but also to be in the neighborhood of one.⁷

⁷ We also limit the difference between a_1 and a_2 because otherwise the return standard deviations become extremely large. In the case of $a_1 = 1$, $a_2 = 1.75$, for example, $Er^e = 35, S.D.r^e = 97$, $Er^f = 13.3, S.D.r^f = 51$,

Table 2
 Comparative Dynamics: Changes in a_1 , a_2 , and
 Comparison with Mehra and Prescott (1985) Results
 ($\beta = 96$, $\mu = .018$, $d = .036$, $p = 43$, $s = 120$, $y_0 = 1$)

	Panel A				Panel B			
	$a_1 = 1.0$ $a_2 = 1.5$	$a_1 = 3$ $a_2 = 3.5$	$a_1 = 11$ $a_2 = 11.5$	$a_1 = 21$ $a_2 = 21.5$	$a = 1.25$	$a = 3.25$	$a = 11.25$	$a = 21.25$
Er^e	16.95	20.10	29.92	33.51	6.49	10.00	20.61	24.76
S.D. r^e	53.04	52.46	49.60	45.06	3.89	5.15	10.87	17.19
Er^f	8.47	13.43	29.46	38.07	6.33	9.47	17.38	16.82
S.D. r^f	34.21	35.38	38.83	40.12	.66	1.75	6.20	10.37
Er^p	8.48	6.68	.46	-4.56	.15	.53	3.23	7.94
S.D. r^p	34.92	33.54	28.78	25.37	3.82	4.82	8.64	12.55
$\text{Corr}(r_t^e, r_{t-1}^e)$	-.47	-.47	-.47	-.45	-.18	-.34	-.53	-.56
$\text{Corr}(r_t^f, r_{t-1}^f)$	-.15	-.15	-.11	-.02	-.15	-.15	-.15	-.15
$\text{Corr}(r_t^e, r_{t-1}^f)$.77	.78	.82	.83	.18	.35	.61	.69
$\frac{SD(IMRS)}{E(IMRS)}$.20	.18	.30	.56	.04	.11	.37	.63
$\frac{E\left(\frac{p}{y}\right)}{E\left(\frac{p}{y}; \alpha = \frac{\alpha_1 + \alpha_2}{2}\right)}$	1.05	1.02	.98	.94	1	1	1	1

We have postulated that agents display less risk aversion when they are confronted by the experience of greater consumption growth. As noted in the introduction, Bosch-Domenech and Silvestre (1999) present indirect evidence, however, that the contrary is the case. For this reason it is appropriate to consider a sample of cases under the orthogonal assumption that agents display more risk aversion in the high growth state, $a_2 > a_1$, an assertion that suggests a strong

$Er^p = 21$ and $S.D.r^p = 70$ (all return measures in percent).

desire “to preserve one’s gains when they are high.” The results of this exercise are summarized in Panel F of Table 1. Comparing Panels C and F we observe that the mean and S.D. of equity returns both increase while the opposite is true for the risk free security. As a result the Er^P rises in Panel F to 13.9%: if anything the results are strengthened when low growth is associated with low risk aversion. Notice that the correlation structure is largely unaffected. Bearing in mind these observations and preferring the hypothesis most consistent with the data, the case of Panel C is retained as the benchmark.

Our latter results strongly suggest that “outlook” variation, rather than consumption variation, is the dominant effect. To confirm this assertion we examined the return characteristics of an economy in which there is no consumption variation at all, i.e., we fix $a_2 > a_1$, and $d = 0$. The results of this exercise are presented in Table 1, Panel G. Relative to Panel C, the premium is even larger (standard deviations are largely unchanged, however). This suggests that within the class of Lucas (1978) asset pricing models, uncertainty in the period utility function of the type considered here will matter overwhelmingly more importantly for the characteristics of asset prices than will any postulated variation in the consumption (dividend) process.⁸

3.2 Asset Prices and the Level of Wealth

Table 2 reminds us that level effects can reverse standard intuition (in particular, higher average risk aversion lowers the premium), a fact that encourages the exploration of the consequences of wealth effects more broadly in the model. Useful information is provided in Table 3, where the length of the output series, S , is allowed to vary for two distinct correlation

⁸ More precisely, while the dividend growth autocorrelations determines the autocorrelation of risk free and risky returns, their mean values and standard deviation are governed by changes in the CRRA.

structures.⁹ Panel A is the standard Mehra and Prescott (1985) output process while Panel B admits a mild degree of positive output growth correlation.

Table 3
Security Return Statistics
Various Time Series Lengths
(For all cases, $\beta = .96$, $\mu = .018$, $d = 0.36$)

	Panel A $a_1 = .7, a_2 = .9, p = .43$			Panel B $a_1 = .7, a_2 = .9, p = .70$		
	S=120	S=200	S=400	S=120	S=200	S=400
Er^e	7.01	9.89	24.87	6.31	7.79	15.36
S.D. r^e	17.36	31.50	78.46	12.13	22.10	56.23
Er^f	6.01	6.59	8.60	5.51	5.13	2.91
S.D. r^f	13.95	23.25	44.30	8.40	13.37	25.07
Er^{ps}	1.01	3.29	16.27	.80	2.65	12.46
S.D. r^p	9.69	19.27	55.15	8.49	17.06	47.95
$\text{Corr}(r_t^e, r_{t-1}^e)$	-.52	-.52	-.43	-.16	-.23	-.21
$\text{Corr}(r_t^f, r_{t-1}^f)$	-.15	-.15	-.14	.38	.39	.41
$\text{Corr}(r_t^e, r_{t-1}^f)$.83	.80	.73	.73	.65	.54

There are several aspects of Table 3 that merit attention. First, as S increases and the agent's consumption grows, he increasingly resists the purchase of the equity security. This is overwhelmingly a pure "outlook effect": as agents become wealthier they become increasingly concerned for a reversal of fortune, and the equity security provides no hedge against this possibility.

Of greater interest is the differing behavior of the average risk free rate across the two cases. While the mean risk free rate increases with consumption in Panel A, the reverse is true in

⁹ For these discussions we temporarily abandon the benchmark case (Table 1, Panel C), as the base case ($S=120$) with $a_1 - a_2 = .50$ exhibits enormous mean returns and return standard deviations as S grows. For Table 3 we thus

Panel B. It is the difference in the basic risk structure of the two cases that accounts for the difference: Panel B describes the higher risk environment as consumption growth is non trivially persistent. In Panel A, the stochastic growth structure is close to independence.

In contrast to the case of Panel A, the parameterization of Panel B thus implies that periods of negative consumption growth may persist for many periods. As agents become wealthier, this possibility becomes of greater concern, so much so as to reverse the pattern of risk free security prices: these prices are bid up in Panel B as the agents become wealthier. The other manifestation of this same effect is the fact that all security prices are higher under Panel B on a comparative case by case basis. Another way to accentuate the centrality of the level effect is to compute the analogous results for Panels A and B where growth is absent ($\mu=0$). In the case of Panel A, the mean premium is .19% while in Panel B it is .26% (in both cases for all values of S).

At this point where do we stand? There are a few tentative conclusions that may be reached. First, when level effects are admitted into the model, the nature of the asset pricing puzzles in a standard Mehra and Prescott (1985) economy is radically transformed. The standard deviations of equity and risk free returns – as well as of the premium – are typically excessive, the equity premium itself is frequently too large, and the Hansen-Jagannathan bound is easily satisfied. All this is accomplished in an environment where the price to output ratio is within reasonable bounds. On the other hand, the results of Table 3 report nonstationarity in the r^e and r_f series, something that is generally understood to be absent in the data.

3.3 Replicating the Basic Stylized Facts: How Close Can We Come?

In this section we attempt to identify the specific parameters that appear key to presenting a better replication of U.S. data. There is little flexibility along most dimensions if we are to be

reduce a_1 - a_2 to .4.

faithful to Mehra and Prescott's (1985) formulation. In fact, all that is possible is to modify the stochastic structure in order to break the perfect negative correlation between a and x .

Accordingly, we admit independently specified (i.e., $\rho(\tilde{x}, \tilde{\alpha}) \neq 1$) stochastic processes on \tilde{x} , and $\tilde{\alpha}$ as per (2.10). Table 4 presents an initial comparative analysis for a wide class of probability structures. No a priori attempt is made to replicate the specific U.S. output process. The tentative message of these cases is pretty much as before: premia can easily be very large, though usually at the cost of excessive return volatility. As noted earlier, these standard deviations are very sensitive to the CRRA differential. As the differential increases, furthermore, expected equity returns increase while the expected risk free rate decreases, both effects being consistent with the increase in effective risk.

The chief beneficiary of this generalization appears largely to be a potential improvement in the correlation structure of returns vis-à-vis the cases of Table 1. In the case of Panel 2B, for example, $\rho(r_t^e, r_{t-1}^e)$ and $\rho(r_t^f, r_{t-1}^f)$ now almost exactly match the data. While $\rho(r_t^e, r_t^f)$ is still too high, it is much less than in standard formulations. For this particular choice of probability structure, $\rho(x_t, x_{t-1})$ is still low (as in the data) while $\rho(x_t, \alpha_t)$ is slightly positive, a fact that suggests that the version of the model which best reflects the experimental literature (i.e., which asserts that the agent is generally more risk averse in the high payoff state) is also the version that best replicates the data.

Table 4
Statistical Return Summaries
Various Correlation Structures⁽ⁱ⁾
For all Cases: $\mu = .018$, $d = .036$, $\beta = .96$, $S = 120$
All Return Quantities in Percent
(a) mean return (b) standard deviation of return

Panel 1

a ₁ = 1.1, a ₂ = 1.4								
	A		B		C		D	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
r ^e	9.64	26.94	7.18	12.67	9.27	25.20	7.24	13.39
r ^f	6.00	16.86	5.70	5.70	6.56	17.04	5.96	6.70
r ^p	3.64	20.07	1.49	11.17	2.71	17.52	1.29	11.31
$\rho(r_t^e, r_{t-1}^e)$	-.4		.11		-.39		.09	
$\rho(r_t^f, r_{t-1}^f)$.09		.79		.09		.80	
$\rho(r_t^e, r_t^f)$.67		.52		.73		.60	

Panel 2

a ₁ = 1.1, a ₂ = 1.7								
	A		B		C		D	
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
r ^e	20.47	64.78	10.09	32.21	19.60	62.43	10.45	35.37
r ^f	5.37	31.82	3.74	10.11	6.48	32.01	4.36	11.12
r ^p	15.10	51.33	6.35	30.58	13.11	48.33	6.09	33.11
$\rho(r_t^e, r_{t-1}^e)$	-.34		.01		-.33		-.01	
$\rho(r_t^f, r_{t-1}^f)$.09		.79		.09		.78	
$\rho(r_t^e, r_t^f)$.63		.40		.40		.41	

(i)	Correlation Structure			Resultant Matrix (2.10)				
	Corr(x _t , x _{t-1})	Corr(a _t , a _{t-1})	Corr(x _t , a _t)	f	p	s	H	?
Panel A	.1	.1	.1	.5298	.0202	.0247	.4253	.01
Panel B	.1	.1	.8	.8393	.0607	.0742	.0258	.03
Panel C	.8	.8	.1	.5496	.0004	.0034	.4466	.03
Panel D	.8	.8	.8	.8996	.0004	.0034	.0966	.03

Note also that for both panels, cases B and D are closely similar. Both agree only on the (x_t, a_t) dimension, a fact that gives added support to the latter assertion.

With this probabilistic generality all of the conclusions to data remain unaltered: for very low levels of average risk aversion, the mean and standard deviation of returns as well as the premium and its standard deviation are all easily made too large relative to U.S. data.

4. Further Comparison with the Literature

Of interest are other models with changing risk aversion. It is well known that the effective CRRA of a representative agent under habit formation changes with his past consumption history. A recent and particularly comprehensive application of this fact is Campbell and Cochrane (1999), who postulate an external habit. These authors define the consumption “surplus above the habit,” s_t , by $\tilde{s}_t = (c_t - x_t)/c_t$ where x_t is the habit, and demonstrate that

$$\frac{-c_t u_{11}(c_t - x_t)}{u_1(c_t - x_t)} = \frac{a}{\tilde{s}_t}$$

where a is the representative agent’s fixed CRRA. Under the formulation considered in this paper, the CRRA is also stochastic and stationary. For Campbell and Cochrane (1998), however,

the pricing kernel, $\beta \left(\frac{s_{t+1}}{s_t} \times \frac{c_{t+1}}{c_t} \right)^a$, is a stationary random variable: there are no level effects.

Viewing the same issue from a slightly different perspective, the consumption surplus variable in Campbell and Cochrane (1999) is a deterministic function of current consumption. This implies that all the IMRS variation must be consumption driven. Without any additional source of risk, it is not surprising that they require a relatively high CRRA to achieve their results. In both of these respects their model is fundamentally different from the one considered here, which

accepts nonstationarities as the price of relying on an almost trivially low CRRA (unlike in Campbell and Cochrane (1999)).

Another related paper is Danthine and Donaldson (2002), which postulates a variable risk sharing mechanism as a route to constructing a model which simultaneously reflects the stylized facts of the business cycle and the financial markets. This mechanism also leads to an economy with a variable economy-wide CRRA, although their pricing kernel is also stationary. As in the model presented here, the favorable financial properties they report are almost entirely attributable to the variation in risk sharing alone. The one drawback to their model is the excessive dividend volatility required to match the data well, a feature unnecessary in the formulation considered here.

Although they analyze a much more complex multiagent model in continuous time, the construct of Kraus and Sagi (2000) shares many of the features of this model as well as many of its principal conclusions. As in this model, they incorporate randomly changing risk aversion. They also incorporate elements of market incompleteness, however, a feature absent here. Similar to the principal conclusion of this paper, these authors also find that the equity premium “depends not so much on the level of relative risk aversion as on the volatility of the relative risk aversion.” Our much simpler model argues that their incorporation of incompleteness may be an unnecessary complication.

Lastly we mention Melino and Yang (2001) who undertake a similar exercise using Epstein-Zinn/Weil preferences where the CRRA and the elasticity of intertemporal substitution (EIS) can be specified independently. With state dependence in both the CRRA and EIS, they are able to match perfectly the first two moments of equity and risk free returns. This result follows from two features of their model: (1) the certainty equivalent is linear homogenous

which eliminates the level effect, while (2) the variation in two parameters provides additional “degrees of freedom.” In effect, the underlying mechanism is thus not the same as the one considered here.

All of these papers have one thing in common, and it is this: for replicating financial data what is critical is the preference characteristics of the representative agents. The characteristics of the dividend process are only marginally relevant.

5. Concluding Comments

State dependent preferences are increasingly popular. In finance the accent is applied to the plausible hypothesis that risk aversion is not constant. In this paper we have shown that the most natural representation of this hypothesis has profound implications for asset pricing. It implies in particular that the IMRS is level dependent and that this dependence introduces another source of volatility potentially more powerful than the volatility of the fundamentals being priced. This in turn leads to a reversal of the paradoxes first identified in the seminal work of Mehra and Prescott (1985). Asymptotically, the risk free rate is too low, the equity premium too high, and the standard deviations of all security returns are much too high. The Hansen-Jagannathan bounds are easily satisfied.

Is the model reasonable? The non-stationarity of the resulting economy is disturbing although it can be argued that it would possibly not be detectable with usual tests and available data sets. At a minimum our exploration should serve as a warning, for those researchers naturally inclined to postulate state dependent preferences, of the powerful mechanism and the unintended non stationarity this hypothesis may thus generate for their model environment.

Appendix 1

A. Constructive Existence of Equilibrium for the Economy of Section 2

We construct the equilibrium price function for probability structure (2.9). Without loss of generality, let us assume $y_0 = 1$, $x_0 = x_i \in \{x_1, x_2\}$, and that $a(x_i) = a_i$. By recursive substitution equation (2.11) becomes:

$$(A.2.1) \quad p(y_0, x_0) = p(1, x_i) = \beta p_{i1} x_1^{1-a_1} + \beta p_{i2} x_2^{1-a_2} + \sum_k \beta^2 p_{ik} p_{k1} (x_k x_1)^{1-a_1} \\ + \sum_k \beta^2 p_{ik} p_{k2} (x_k x_2)^{1-a_2} + \sum_k \sum_\ell \beta^3 p_{ik} p_{k\ell} p_{\ell 1} (x_k x_\ell x_1)^{1-a_1} \\ + \sum_k \sum_\ell \beta^3 p_{ik} p_{k\ell} p_{\ell 2} (x_k x_\ell x_2)^{1-a_2} + \dots,$$

where we have done nothing more than segregate the terms in the sum according to whether the state in each future time period is x_1 or x_2 . We note that the exponent in each term depends only upon the last growth realization. Further grouping of like terms yields

$$(A.2.2) \quad p(1, x_i) = \beta p_{i1} x_1^{1-a_1} + \sum_k \beta^2 p_{ik} p_{k1} (x_k x_1)^{1-a_1} + \sum_k \sum_\ell \beta^3 p_{ik} p_{k\ell} p_{\ell 1} (x_k x_\ell x_1)^{1-a_1} + \dots \\ + \beta p_{i2} x_2^{1-a_2} + \sum_k \beta^2 p_{ik} p_{k2} (x_k x_2)^{1-a_2} + \sum_k \sum_\ell \beta^3 p_{ik} p_{k\ell} p_{\ell 2} (x_k x_\ell x_2)^{1-a_2} \\ \equiv \hat{p}(x_1, x_1) + \hat{p}(x_1, x_2)$$

These semi-prices $\hat{p}(x_i, x_j)$ can, in turn, be computed as the solution to a system of linear equations. To see this, note that

$$(A.2.3) \quad \hat{p}(x_i, x_j) = \beta \pi_{ij} x_j^{1-\alpha_i} + \sum_k \beta^2 \pi_{ik} \pi_{kj} (x_k x_j)^{1-\alpha_i} + \sum_k \sum_\ell \beta^3 p_{ik} p_{k\ell} p_{\ell j} (x_k x_\ell x_j)^{1-\alpha_i} + \dots \\ = \beta p_{ij} x_j^{1-a_j} + \beta^2 p_{i1} p_{1j} (x_1 x_j)^{1-a_j} + \beta^2 p_{i2} p_{2j} (x_2 x_j)^{1-a_j} + \beta^3 p_{i1} \sum_\ell p_{1\ell} p_{\ell j} (x_1 x_\ell x_j)^{1-a_j}$$

$$\begin{aligned}
& +\beta^3 p_{i2} \sum_{\ell} p_{2\ell} p_{\ell 2} (x_2 x_{\ell} x_j)^{1-a_j} + \dots \\
& = \beta p_{ij} x_j^{1-a_j} + \beta p_{i1} x_1^{1-a_j} \left[\beta p_{1j} x_j^{1-a_j} + \beta^2 \sum_{\ell} p_{1\ell} p_{\ell 1} (x_{\ell} x_j)^{1-a_j} + \dots \right] \\
& \quad + \beta p_{i2} x_2^{1-a_j} \left[\beta p_{2j} x_j^{1-a_j} + \beta^2 \sum_{\ell} p_{2\ell} p_{\ell 2} (x_{\ell} x_j)^{1-a_j} + \dots \right] \\
& = \beta \pi_{ij} x_j^{1-\alpha_j} + \beta \pi_{i1} x_1^{1-\alpha_j} [\hat{p}(x_1, x_j)] + \beta \pi_{i2} x_2^{1-\alpha_j} [\hat{p}(x_2, x_j)]
\end{aligned}$$

We observe that (A.2.3) constitutes a system of four linear equations in the four unknowns $\hat{p}(x_i, x_j)$, $i \in \{1, 2\}$, $j \in \{1, 2\}$. They constitute the basic valuation factors for equity securities in this economy in a manner that will now be made apparent.

Our objective is to obtain a recursive style representation for the equity security when the current output level is y and the current growth state is $x_i(a_i)$.

$$\begin{aligned}
(A.2.4) \quad p(y, x_i) &= \beta \pi_{i1} \frac{(y x_1)^{1-\alpha_1}}{y^{-\alpha_1}} + \sum_k \beta^2 \pi_{ik} \pi_{k1} \frac{(y x_k x_1)^{1-\alpha_1}}{y^{-\alpha_1}} + \sum_k \sum_{\ell} \beta^3 \pi_{k\ell} \pi_{\ell 1} \frac{(y x_k x_{\ell} x_1)^{1-a_1}}{y^{-a_1}} + \dots \\
& + \beta p_{i2} \frac{(y x_2)^{1-a_2}}{y^{-a_2}} + \sum_k \beta^2 \pi_{ik} \pi_{k2} \frac{(y x_k x_2)^{1-a_2}}{y^{-a_2}} + \sum_k \sum_{\ell} \beta^3 \pi_{k\ell} \pi_{\ell 2} \frac{(y x_k x_{\ell} x_2)^{1-a_2}}{y^{-a_2}} + \dots \\
& = y^{1-\alpha_1+\alpha_i} \hat{p}(x_1, x_1) + y^{1-\alpha_2+\alpha_i} \hat{p}(x_1, x_2).
\end{aligned}$$

Given a sequence of growth rates $\{x_t\}$ generated to respect the chosen transition matrix (2.9) and the resulting output sequence as per (2.1), the corresponding equilibrium price sequence p_t can thus be generated as per

$$(A.2.5) \quad p_t = p(y_t, x_t) = y_t^{1-\alpha_1+\alpha_i} \hat{p}(x_t, x_1) + y_t^{1-\alpha_2+\alpha_i} \hat{p}(x_t, x_2)$$

with the associated equity return sequence is given by

$$r_{t,t+1}^e = \frac{p_{t+1} + y_{t+1}}{p_t} - 1$$

Expression (A.2.5) makes sense if the semi prices $\hat{p}(x_i, x_j)$ are well defined. A strong sufficient condition (but one satisfied by the benchmark case) is that

$$(A.2.6) \quad \beta(x_i)^{1-\alpha_j} < 1, \quad i \in \{1,2\}, \quad j \in \{1,2\}.$$

This development leads to the following theorem:

Theorem A.1.1: Suppose $u(\cdot)$ is strictly increasing, strictly concave, and continuously differentiable on $(0, \infty)$, and that condition (A.2.6) is satisfied. Then an equilibrium price function exists for the economy of Section 2.

Proof: Equation (2.11) represents the representative agents necessary and sufficient first order condition on which the market clearing conditions have been imposed. By construction a price function satisfying (A.2.5) will satisfy (2.11). By (A.2.6) such a price function exists and is well defined.

B. Constructing Sequences of Equilibrium Risk Free Security Prices and Rates of Return

As noted in the text (equation (2.12)), the risk free security is priced according to:

$$(A.2.7) \quad q(y_t, x_t = x_i) = \beta \sum_{j=1}^2 \frac{(y_t x_j)^{-a_j}}{(y_t)^{-a_i}} p_{ij} .$$

Once again, $q(y_t, x_i)$ is level dependent so that equation (A.2.7) also represents a countably infinite system necessitating numerical solution methods.

The procedure is much simpler than in the case of the equity security, however. For the same $\{x_t\}$ and associated $\{y_t\}$ constructed above, the corresponding sequence of risk free asset prices was constructed according to (A.2.6) with the risk free rate series defined according to

$$r^f(y_t, x_t) = \frac{1}{q(y_t, x_t)} - 1.$$

For all cases reported in this paper the lengths of the constructed time series of returns were no less than 10,000. All financial statistics represent averages of those obtained for 10,000 independently constructed time series. As a final check on our procedures we precisely replicated the results in Mehra and Prescott (1985) for the cases of their parameterization. These remarks bring with them the appreciation not only of the elegance and simplicity of the Mehra and Prescott (1985) formulation but also of its highly specialized nature.

C: Pure Monte-Carlo Simulation of Equation (2.11)

Here we provide an alternative to the procedure of sections A and B and one that was used as a verification for computational accuracy.

Our alternative statistical summaries of equity and risk free return distributions are based on return sequences of length 10,000. To generate these latter sequences we began by first generating a time series of random output growth rates $\{x_t\}$ of 10,001 unit length, which respected (2.9). The corresponding output sequence $\{y_t\}_{t=0}^{t=10,000}$ then satisfied.

$$(A.2.8) \quad \begin{aligned} y_0 &\equiv 1, \\ y_t &= x_t y_{t-1}, t = 1, \dots, 10,000. \end{aligned}$$

At each time period, the corresponding equity price $p(y_t, x_t)$ was estimated as follows: suppose for some time period \tilde{t} , $(y_{\tilde{t}}, x_{\tilde{t}}) = (\tilde{y}, \tilde{x}_i)$. Starting from \tilde{x}_i , we next generated 1000

independent growth rate sequences $\{x_s^k\}$, $k = 1, 2, \dots, 1000$, each of length 1,000 ($s = 1, \dots, 1000$) (for every sequence k , $\{x_1^k\} \equiv x_i$), all of which respected (2.9).

To each of these sequences $\{x_s^k\}$, there corresponds an associated output sequence generated as per

$$y_0^k = \tilde{y} \quad s = 0$$

$$y_s^k = y_{s-1}^k x_s^k \quad s = 1, 2, \dots, 1000 .$$

Using these sequences the price of the security, $P^e(\tilde{y}, x_i)$ was then estimated as

$$(A.2.9) \quad p(\tilde{y}, x_i) = \frac{1}{1000} \sum_{k=1}^{1000} \sum_{s=1}^{1000} \beta^s \frac{(y_s^k)^{-a(x_s^k)}}{(\tilde{y})^{-a(x_i)}} y_s^k$$

In a similar fashion, the corresponding price of the risk free security, $q(\tilde{y}, x_i)$ is directly computed via (2.12). The choice of $k = 1000$ and $s = 1000$ is somewhat arbitrary. It is justified by the fact that if $\alpha(x) \equiv \alpha$, and the model is otherwise parameterized as in Mehra and Prescott (1985) (same $\beta, \mu, \delta, \pi_{11}, \pi_{12}, \pi_{21}, \pi_{22}$), then the Monte-Carlo solution methodology replicates the price and return results those authors obtain to three decimal places when $\alpha \leq 10$. This latter upper bound vastly exceeds any values of $\alpha(x_i)$ needed for our principal results.

Both methods gave identical results to two decimal places.

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