Self-Confidence and Timing of Entry*

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Abstract

This paper analyzes the impact of overconfidence on the timing of entry in markets, profits, and welfare. To do that the paper uses an endogenous timing model where (i) players have private information about costs and (ii) one player is overconfident and the other is rational. The paper shows that for moderate levels of self-confidence there is a unique *cost-dependent* equilibrium where the overconfident player has a higher ex-ante probability of entering the market before the rational player. In this equilibrium self-confidence reduces the profits of the rational player but can increase the profits of the overconfident player provided that cost asymmetries are small. Finally, we show that overconfidence reduces welfare, except when cost asymmetries are very small.

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1 Introduction

This paper studies the impact of self-confidence on the timing of entry into a market. Our main research question is whether overconfident players enter markets before rational players or not. We also evaluate the impact of overconfidence on players' profits and on welfare. To perform this analysis we use Branco's (1998) endogenous timing model where two players are privately informed about their cost (which can be either high or low), compete in quantities, and must decide whether to enter the market at date 1 or at date 2.[1] The novelty here is the assumption that one of the players is overconfident while the other one is rational.

The rational player has a correct belief about his cost of production. The overconfident player can be mistaken about his cost with positive probability. More precisely, we assume that if the overconfident player's cost is low, his perception is correct and he thinks that he has a low cost. However, if the overconfident player's cost is high, his perception can be mistaken and he might think that he has low cost.

The main finding of the paper is that there exists a unique *cost-dependent* equilibrium where the overconfident player has a higher ex-ante probability of moving at date 1 than the rational player. In other words, the overconfident player is more likely to be the leader than the rational player. We also show that this equilibrium only exists if the level of self-confidence is moderate.

The intuition behind this result is as follows. In a cost-dependent equilibrium a player with a low cost perception enters the market at date 1 whereas a player with a high cost perception enters the market at date 2. Since an overconfident player has an higher ex-ante probability of having a low cost perception than a rational player, he also has a higher ex-ante probability of entering the market before the rational player. This equilibrium breaks down if self-confidence is high since a rational player with low cost would be better off by deviating and producing at date 2.[2]

Next we study the effects of self-confidence on players' profits and welfare in the cost-dependent equilibrium. We find that moderate self-confidence is good for the overconfident player as long as cost asymmetries are small. The impact of higher self-confidence on the overconfident player's profits depends on a trade-off between a "leadership gain" and an "overproduction loss." It is a well known result that, if players compete in quantities, the Stackelberg leader's profits are higher than those of the follower. The overconfident player's mistaken perception gives him a Stackelberg leadership gain since higher self-confidence increases the probability that he enters the market before the rational player.

[1] Endogenous timing models endogeneize the entry decision in markets. In these models there are usually two players and two productions periods. Players can produce in the first period or they can delay their decision until the second period after observing the action of the other player (if the other player decided to produce in the first period) or observing that the other player also decided to wait.

[2] For all levels of self-confidence there exist two *cost-independent* equilibria where one of the players produces at date 1 independently of his cost perception and the other player produces at date 2.

However, the mistaken perception leads to overproduction which lowers market price and reduces profits. If cost asymmetries are small the leadership gain dominates the overproduction loss and the bias is beneficial for the overconfident player. If cost asymmetries are large the opposite happens and the bias reduces the overconfident player's profits. We also show that the bias of the overconfident player always hurts the rational player.

Finally, we show that overconfidence reduces welfare (the sum of consumer surplus and players' profits) for most values of the parameters of the model. In fact, we find that overconfidence can only increase welfare if cost asymmetries are very small. Overconfidence increases market output which increases consumer surplus. However, the increase in market output also reduces aggregate profits. If cost asymmetries are very small the increase in consumer surplus is of first-order but the reduction in aggregate profits is of second-order and so self-confidence increases welfare. When cost asymmetries are moderate or large the reverse happens and self-confidence reduces welfare. These findings are consistent with the theory of the second-best. It is well known that in a world where at least one distortion is present, introducing a new distortion can increase or reduce welfare. Since the duopoly market structure of the endogenous timing game is a distortion, introducing overconfidence (a new distortion) can increase or reduce welfare.

Our paper is related with two branches of economic literature: endogenous timing and self-confidence. The literature on endogenous timing provides conditions and criteria under which firms play either a sequential-move Stackelberg game or a simultaneous-move Cournot game in oligopolistic markets. A seminal paper is Hamilton and Slutsky (1990). They assume that two players must decide a quantity to be produced in one of the two periods before the market clears. If a player commits to a quantity in the first period, he acts as a leader but he does not know if the other player has chosen to commit also in the first period or not. If a player waits until the second period to do a commitment, then he observes the action of the other player in the first period. This game has three subgame perfect Nash equilibria: one Cournot equilibrium in the first period, and two Stackelberg equilibria where one firm leads and the other follows. Only the Stackelberg equilibria survive elimination of weakly dominated strategies. Branco (1998) extends Hamilton and Slutsky's model by assuming that players are privately informed about their costs. He shows that there exists a cost-dependent Perfect Bayesian equilibrium where the player with a low cost produces in the first period and the player with a high cost produces in the second period. As we mentioned, our paper extends Branco's (1998) by assuming that one of the agents is overconfident.

Our paper is also related to the fast growing literature in economics and management on the implications of overconfidence for individual decision-making and strategic interactions. Heifetz and Spiegel (2000) is the paper that is most closely related to ours. Their paper shows that in a large class of strategic interactions the equilibrium payoffs of overconfident players may be higher than those of rational players. This happens because overconfidence can lead the adversary to change equilibrium behavior, possibly to the benefit of overconfident agent. Our finding that if cost asymmetries are small, the profits of the overconfident player increase while those of the rational player decrease is consistent with Heifetz and Spiegel (2000).

2 The Model

There are two players. One is rational, denoted by r, and the other one is overconfident, denoted by o. The two players produce an homogeneous good whose price is given by $p = a - q_r - q_o$, where is q_r and q_o are the quantities produced by the rational and the overconfident player, respectively, with $a > 0$. To produce the good players incur a cost. We assume that marginal cost of production is constant and that there are no fixed costs. The marginal cost of each player, C_i, $i = r, o$, might take on the values 0 (low) and c (high) with equal probability, where $a > c > 0$.

Players are privately informed about their costs. Each player receives a signal X_i that is correlated with his cost, where $X_i \in \{0, c\}$. For the rational player the relation between the signal and his cost his given by $\Pr(X_r = c | C_r = c) = 1$ and $\Pr(X_r = 0 | C_r = 0) = 1$, that is, the rational player is perfectly informed about his cost. For the overconfident player the relation between the signal and his cost is given by $\Pr(X_o = c | C_o = c) = 1 - s$, $\Pr(X_o = 0 | C_o = c) = s$, and $\Pr(X_o = c | C_o = 0) = 0$, where $s \in [0, 1]$. The parameter s captures the degree of self-confidence since it represents the probability that the overconfident player receives a signal that his cost is 0 when his cost is c. If $s = 0$ there is no overconfidence and the model collapses to Branco (1998). If $s = 1$ we have the maximum level of self-confidence since the overconfident player always thinks that his cost is 0. Higher values of s imply a higher level of self-confidence.

2.1 Timing of Decisions

Players must decide a quantity to be produced at one of two dates. At date 1 they simultaneously decide how much to produce. These decision are then publicly revealed. Any player who does not produce at date 1, may decide his level of production at date 2. Finally, at date 3, given the production decisions, the market clears. The timing of the model is:

1. Nature draws players' costs
2. Each player receives a signal about his cost.
3. Players may decide the quantity to be produced at date 1.
4. A player who has not produced at date 1 decides his quantity at date 2.
5. The market clears at date 3.

For a player there is a clear trade-off between the timing decisions. A player that produces at date 1 gets the possible benefit of acting as a leader, producing first and influencing the other player's decision, if the latter has decided to produce at date 2. However, by producing at date 1, a player chooses his quantity without observing the timing of the opponent's move, risking that the opponent also produces at date 1. To the contrary, a player who decides to wait and produce at date 2, cannot influence the rival's decision, if the rival has decided to wait, but will have more information when deciding since he can observe the quantity chosen by the rival or the rival's decision to wait. The choices of the players regarding the date of production can be described in terms of the leader-follower dichotomy: a player who produces at date 1 acts as the leader, while a player who produces at date 2 acts as a follower. Thus, the structure of the model provides a framework for the study of the choice of moment of production in a quantity setting duopoly, with asymmetric information about costs and overconfidence as the driving forces.

3 Equilibria

In this section we analyze the equilibria of the model. The equilibrium concept used to solve this game is the Perfect Bayesian Equilibrium (PBE) which requires that strategies yield a Bayesian equilibrium in every "continuation game" given the posterior beliefs of the players, and beliefs are required to be updated in accordance with Bayes' law whenever it is applicable.

To incorporate overconfidence into this setting we follow the approach introduced by Squintani (2006) who considers events where the players self-perception may be mistaken but such that player i is playing a given game, that player j thinks that player i thinks that player j's perception is correct, that player i knows that player j believes that her perception is correct and so on. In other words, we assume that the rational knows that if the overconfident player's cost is c the overconfident player can think that his cost is 0 with probability s. In turn, the overconfident player knows that the rational player thinks that if the overconfident player's cost is c the overconfident player can think that his cost is 0 with probability s. However, the overconfident player thinks that the rational player is mistaken about that. So, in this model players might agree to disagree. Therefore, the rational player knows that the ex-ante probability that the overconfident player perceives a signal of high cost is

$$\begin{aligned}\Pr(X_o = c) &= \Pr(C = c)\Pr(X_o = c|C = c) + \Pr(X_o = 0)\Pr(X_o = c|C = c) \\ &= \frac{1}{2}(1-s) + \frac{1}{2}0 = \frac{1}{2}(1-s),\end{aligned}$$

and the ex-ante probability that the overconfident player perceives a signal of low cost is

$$\begin{aligned}\Pr(X_o = 0) &= \Pr(C = 0)\Pr(X_o = 0|C = 0) + \Pr(C = c)\Pr(X_o = 0|C = c) \\ &= \frac{1}{2}1 + \frac{1}{2}s = \frac{1}{2}(1+s).\end{aligned}$$

As in Branco (1998), we characterize the set of equilibria of this game by describing the players' equilibrium strategies and we distinguish between two types of equilibria: cost-dependent and cost-independent. In a cost-dependent equilibria each player chooses a different period to produce according to his cost perception (high or low). In cost-independent equilibria each player chooses to produce in a certain period independently of his cost perception.

Our first result characterizes the cost-dependent equilibrium of the game.

Proposition 1: *If* $\omega(s) \leq x \leq \frac{5+s}{9+3s}$, *and* $s \leq s(x)$, *with* $x = c/a$, $\omega(s) = \frac{41-2s+2s^2-2\sqrt{6}(2-s)(4+s)}{29+28s-10s^2}$, $s(x) = \frac{3\sqrt{3}\sqrt{5+68x+314x^2+588x^3+369x^4}-(10+28x+106x^2)}{7+34x+67x^2}$, *then here is a cost-dependent equilibrium in which the players will have the following strategies:*

Overconfident player

1. If the overconfident player has the perception that his cost is equal to 0:

(a) He produces at date 1;

(b) He produces $q_o = \frac{3a+c+(a+c)s}{8+2s}$;

(c) If he had not produced at date 1 and the rational player had produced q_r *at date 1, he would produce according to* $q_o = \frac{a-q_r}{2}$, *at date 2;*

(d) If neither player had produced at date 1, he would produce $q_o = \frac{2a+c}{6}$ *at date 2;*

2. If the overconfident player has the perception that his cost is equal to c:

(a) He produces at date 2;

(b) If he were to produce at date 1, he would produce $q_o = \frac{9a-13c+(3a-c)s}{24+6s}$;

(c) If the rational player has produced q_r *at date 1, he will produce according to* $q_o = \frac{a-c-q_r}{2}$, *at date 2;*

(d) If the rational player has not produced at date 1, he will produce $q_o = \frac{a-c}{3}$, *at date 2;*

Rational player

1. If the rational player has cost equal to 0:

(a) He produces at date 1;

(b) He produces $q_r = \frac{3a+c-2sc}{8+2s}$;

(c) If he had not produced at date 1 and the overconfident player had produced q_o *at date 1, he would produce according to* $q_r = \frac{a-q_o}{2}$, *at date 2;*

(d) If neither player had produced at date 1, he would produce $q_r = \frac{2a+c}{6}$ *at date 2;*

2. *If the rational player has cost equal to* c:

(a) He produces at date 2;

(b) If he were to produce at date 1, he would produce $q_r = \frac{9a-13c+(3a-9c-2cs)s}{(3+s)(8+2s)}$;

(c) If the overconfident player has produced q_o *at date 1, he will produce according to* $q_r = \frac{a-c-q_o}{2}$, *at date 2;*

(d) If the overconfident player has not produced at date 1, he will produce $q_r = \frac{a-c}{3}$ *at date 2*;

Proposition 1 says that if cost differences are sufficiently high and the level of overconfidence is moderate, then there exists a cost-dependent equilibrium where a player with a low cost perception produces at date 1 and a player with a high cost perception produces at date 2.[3] Thus, the production moments reveal the players' cost perceptions.[4]

More importantly, in the cost-dependent equilibrium the overconfident player has a higher ex-ante probability of producing at date 1 than the rational player. When the overconfident player's cost is low the timing decision is not affected by overconfidence since it keeps playing in the first period. However, if the overconfident player's cost is high, then the player can have a mistaken perception which leads him to enter the market at date 1. In contrast, the rational player's ex-ante probability of moving at date 1 is not affect by the overconfidence of the rival. Thus, for overconfident player the ex-ante probability of moving at date 1 is equal to his ex-ante probability of getting a low cost signal, $(1+s)/2$, whereas for the rational player it is equal to the ex-ante probability of having low cost, $1/2$.

The strategies described in Proposition 1 are an equilibrium if and only if cost differences are sufficiently high, that is, $c/a > \omega(s)$, and overconfidence is moderate, that is, $s < s(x)$. When overconfidence is moderate but cost differences are not sufficiently high, the previous strategies will not be an equilibrium. A player with a high cost perception would gain by deviating and producing at date 1. However, both players producing at date 1 is not an equilibrium. One can show that if overconfidence is sufficiently low but cost differences are not sufficiently high, then the player with a low cost perception will still produce at date 1, but the player with a high cost perception will randomize between producing at date 1 or waiting to produce at date 2.[5]

When cost differences are sufficiently high but the level of overconfidence is greater than s^*, the strategies described in Proposition 1 will not be an equilibrium. The existence of a upper bound for overconfidence is quite intuitive since a high level of overconfidence implies that an overconfident player who follows his cost-dependent equilibrium strategy produces at date 1 with a very high probability. However, if the rational player knows that there is a very high probability that the overconfident player produces at date 1, he does not have incentives to play according his cost-dependent equilibrium strategy. Particularly, if the rational player has low cost he would gain by deviating and producing at date 2.

Are there are any circumstances in which a rational player can enter the market with a higher ex-ante probability than an overconfident player? The answer to this question is yes. The endogenous timing game has two pure strategy cost-independent equilibria: (i) the overconfident player produces at

[3] For the equilibrium to be well defined it must also be the case that cost differences are not so high that entering the market is not attractive for a rational follower with cost equal to c. This is guaranteed by $c \leq \frac{5+s}{9+3s}a$.

[4] Proposition 1 also tells us that four outcomes are possible: a Cournot outcome will result, if both players wait to produce at date 2; a Stackelberg outcome will emerge, if one player produces at date 1 and the other does it at date 2 (there are two of these outcomes); and a double leadership outcome appears if both players produce at date 1.

[5] The implications of overconfidence are the same for both types of cost-dependent equilibria (high and low cost differences). Therefore we do not characterize the cost-dependent equilibrium with low cost differences.

date 1 and the rational player at date 2 and (ii) the rational player produces at date 1 and the overconfident player at date 2. Proposition 2 shows that there are no other equilibria in our model.

Proposition 2: *For $\omega(s) \leq x \leq \frac{5+s}{9+3s}$, and $s \leq s(x)$ there does not exist a Perfect Bayesian equilibria whose strategy profiles differ from the cost-independent equilibrium of Proposition 1 or from the two cost-independent equilibria.*

4 Self-Confidence, Profits and Welfare

In this section we characterize effects of self-confidence on profits and welfare. Let $\Pi_o(s)$ and $\Pi_r(s)$ denote the ex-ante profits of the overconfident and the rational players, respectively, as a function of s. We take as benchmark scenario the endogenous timing game played between two rational players. Let $\Pi(0)$ denote the ex-ante profits of a player in an endogenous timing game played between two rational players. We have the following result.

Proposition 3: *In the cost-dependent equilibrium:*
(i) $\Pi_o(s) > \Pi(0)$ *for all* $0 < s \leq s(x)$, *if* $\omega(s) \leq x \leq \tau(s)$;
(ii) $\Pi_o(s) < \Pi(0)$ *for all* $0 < s \leq s(x)$, *if* $\tau(s) < x \leq \frac{5+s}{9+3s}$;
(iii) $\Pi_r(s) < \Pi(0)$ *for all* $0 < s \leq s(x)$.

This result shows that, in the cost-dependent equilibrium, moderate self-confidence can increase the profits of the overconfident player provided that cost asymmetries are small. The intuition is as follows. By making a mistake the overconfident player has a "leadership gain" because it will produce at date 1 instead of date 2. However, the mistaken perception of the overconfident player will lead him to choose a quantity that is higher than the optimal one given his true cost. This leads to a loss which increases with the value of c since the difference between the optimal quantity and the quantity chosen increases with c. Therefore, for low values of c the "leadership gain" more than compensates the loss incurred by not playing the optimal quantity. For high values of c the reverse happen.

Proposition 3 also shows that self-confidence always reduces the profits of the rational player. This happens because the mistaken perceptions of the overconfident player lead to a reduction of market share for the rational player. If the rational player has low cost, then he produces at date 1. However, since the rational player knows that the overconfidence player is likely to overproduce, the rational player must produce a smaller Stackelberg leader's quantity than if he faced a rational opponent. If the rational player has high cost, then he produces at date 1. In this case, no matter if the overconfident player produces at date 1 or at date 2, the rational player will have a smaller market share than if he faced a rational rival.

We now discuss the impact of self-confidence on players' profits in the cost-independent equilibria. In the cost cost-independent equilibrium where the rational player leads and the overconfident player follows, the impact of self-confidence on players' ex-ante profits is similar to that in the cost-dependent equilibrium, that is, self-confidence always hurts the rational player but can be advantageous for the overconfident player. In this equilibrium, the rational player produces at date 1 no matter if his cost is high or low and the overconfident player produces at date 2. Since the rational player knows that the overconfident player can overproduce with positive probability at date 2, he must lower his leadership output by comparison with a situation where he would be faced with a rational opponent. The mistaken beliefs of the overconfident follower allow him to increase is market share at the expense of the rational player's market share. However, if cost asymmetries are large the overconfident player will be worse off since the optimization mistake loss will be less than the gain from the increase in market share.

In the cost-independent equilibrium where the overconfident player is the leader and the rational player the follower, overconfidence hurts both players. In this equilibrium, the overconfident player always produces at date 1 so there is no leadership gain from holding mistaken beliefs. However, the mistaken beliefs will lead the overconfident leader to overproduce which originates a loss. The rational player is also worse off because he will have a lower market share than if he would be faced with a rational opponent.

Our last result characterizes the impact of self-confidence on welfare (the sum of profits and consumer surplus) in the cost-dependent equilibrium. Let $W(0)$ denote ex-ante welfare in the endogenous timing game played between two rational players and $W(s)$ denote the ex-ante welfare in the endogenous timing game played between a rational player and an overconfident player with self-confidence of $s > 0$.

Proposition 4: *In the cost-dependent equilibrium:*
(i) $W(s) > W(0)$ *for all* $0 < s \leq s(x)$, *if* $\omega(s) \leq x \leq \psi(s)$;
(ii) $W(s) < W(0)$ *for all* $0 < s \leq s(x)$, *if* $\psi(s) < x \leq \frac{5+s}{9+3s}$.

Proposition 4 tells us that moderate levels of self-confidence reduce welfare in the cost-dependent equilibrium, except when cost asymmetries are very small. The intuition behind this result is as follows. First, self-confidence increases market output. As we have seen, self-confidence increases the output of the overconfident player but reduces the output of the rational player. The net effect is an overall increase in market output since the reduction in the output of the rational player is less than the increase in that of the overconfident player. The increase in market output increases consumer surplus which improves welfare.

Second, self-confidence reduces aggregate profits. For low levels of self-confidence the increase in the ex-ante profits of the overconfident player is less than the decrease in the ex-ante profits of the rational player. For high levels of overconfidence the ex-ante profits of both players decrease. This effect of self-confidence reduces welfare.

When cost asymmetries are very small the increase in consumer surplus is of first-order and the reduction in profits is of second-order and so self-confidence increases welfare. However, when cost asymmetries are not very small the reverse happens and self-confidence reduces welfare.

5 Extensions

The endogenous timing model in this paper makes several simplifying assumptions. For example, cost are assumed to take only two values, high or low. Low cost is equal to 0 and high cost is equal to $c > 0$. The prior probability that true cost is high or low is 1/2. Demand and costs are assumed to be linear. It would be straightforward to extend the model by assuming that costs can be $\underline{c}$ or $\overline{c}$ where $0 \leq \underline{c} < \overline{c}$. The assumption that the prior probability is $1/2$ could also be relaxed. This would complicate the algebra but it would not change the main findings of the paper.

Another possible extension would be to allow for a continuum of types instead of two discrete types. In this case, as is mentioned in Branco (1998), the separating equilibrium would require that players use cutoff strategies, with a player with a cost perception below the cutoff value committing to a quantity in first period, while a player with a cost perception above it would prefer to wait and produce in the second period. We are convinced that allowing for more general demand or cost functions would not change qualitatively the main findings of the paper.

6 Conclusion

In this paper we characterize the impact of self-confidence on the timing of entry in a market, profits, and welfare. To do that we extend the endogenous timing model of Branco (1998) by assuming that one of the players is overconfident. We find that, in a cost-dependent equilibrium, the overconfident player has a higher ex-ante probability of being the leader than the rational player. We show that this result is valid only if the level of self-confidence is moderate. We also show that self-confidence always hurts the rational player. However, the self-confident player can be better off by being overconfident (although he does not know it) provided that cost asymmetries are small. Finally, we show that self-confidence increases consumer surplus but reduces aggregate profits. The second effect dominates for most parameter values of the model.

7 References

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8 Appendix

Proof of Proposition 1: Suppose player r plays according to the strategy defined. Player o has to produce according to a best response whenever possible. The proof proceeds by showing that the eight steps that describe the strategy of player o form a best response. First, we determine the optimal production levels for o in each contingency.
1. Player o has the perception that his cost is equal to 0:
(i) Player o produces at date 1: it may be that r will also produce at date 1, if he has cost equal to 0, or he will produce at date 2, if he has cost equal to c; hence, the quantity produced by o must solve the following problem:

$$\max_{q_o} \frac{1}{2}\left(a - q_o - \frac{3a + c - 2cs}{8 + 2s}\right) q_o + \frac{1}{2}\left(a - q_o - \frac{a - c - q_o}{2}\right) q_o.$$

The solution to this problem is:

$$q_o = \frac{3a + c + (a + c)s}{8 + 2s}.$$

(ii) Player o produces at date 2, knowing that r has produced the quantity q_r at date 1: then o must produce the quantity that solves the following problem:

$$\max_{q_o} (a - q_o - q_r)\, q_o,$$

which leads to production of:

$$q_o = \frac{a - q_r}{2}.$$

(iii) Player o produces at date 2, knowing that r has not produced at date 1: then o infers that r has cost equal to c and that he will produce $(a - c)/3$ at date 2; thus o must produce a quantity that solves the following problem:

$$\max_{q_o} \left(a - q_o - \frac{a - c}{3}\right) q_o,$$

which leads to production of:

$$q_o = \frac{2a + c}{6}.$$

2. Player o has the perception that his cost is equal to c:
(i) Player o produces at date 1: it may be that r will also produce at date 1, if he has cost equal to 0, or he will produce at date 2, if he has cost equal to c; hence, the quantity produced by o must solve the following problem:

$$\max_{q_o} \frac{1}{2}\left(a - q_o - \frac{3a + c - 2cs}{8 + 2s} - c\right) q_o + \frac{1}{2}\left(a - q_o - \frac{a - c - q_o}{2} - c\right) q_o.$$

The solution to this problem is:

$$q_o = \frac{9a - 13c + (3a - c)s}{24 + 6s}.$$

(ii) Player o produces at date 2, knowing that r has produced the quantity q_r at date 1: then o must produce the quantity that solves the following problem:

$$\max_{q_o} (a - q_o - q_r - c)\, q_o,$$

which leads to production of:

$$q_o = \frac{a - c - q_r}{2}.$$

(iii) Player o produces at date 2, knowing that r has not produced at date 1: then o infers that r has cost equal to c and that he will produce $(a - c)/3$ at date 2; thus o must produce a quantity that solves the following problem:

$$\max_{q_o} \left(a - q_o - \frac{a - c}{3} - c \right) q_o,$$

which leads to production of:

$$q_o = \frac{a - c}{3}.$$

Now, the optimal moment of production of the overconfident player is determined by looking at the associated expected profits at dates 1 and 2:
1. Player o has the perception that his cost is equal to 0:
(i) If o produces at date 1, his perceived expected profit will be:

$$\begin{aligned}
\pi_o^1 = &\frac{1}{2} \left(a - \frac{3a + c + (a + c)s}{8 + 2s} - \frac{3a + c - 2cs}{8 + 2s} \right) \frac{3a + c + (a + c)s}{8 + 2s} \\
&+ \frac{1}{2} \left(a - \frac{3a + c + (a + c)s}{8 + 2s} - \frac{a - c - \frac{3a + c + (a + c)s}{8 + 2s}}{2} \right) \frac{3a + c + (a + c)s}{8 + 2s} \\
&= \frac{3(3a + c)^2 + 3\left(4a + (a + c)(2 + s)\right)(a + c)s}{(16 + 4s)^2}.
\end{aligned}$$

(ii) If o produces at date 2, his perceived expected profit will be:

$$\begin{aligned}
\pi_o^2 = &\frac{1}{2} \left(a - \frac{a - \frac{3a + c - 2cs}{8 + 2s}}{2} - \frac{3a + c - 2cs}{8 + 2s} \right) \frac{a - \frac{3a + c - 2cs}{8 + 2s}}{2} \\
&+ \frac{1}{2} \left(a - \frac{2a + c}{6} - \frac{a - c}{3} \right) \frac{2a + c}{3} \\
&= \frac{1}{2} \left(\frac{5a - c + 2s(a + c)}{16 + 4s} \right)^2 + \frac{1}{2} \left(\frac{2a + c}{6} \right)^2
\end{aligned}$$

Comparing the two possible profits of o, one obtains:

$$\frac{3(3a+c)^2+3\left(4a+(a+c)(2+s)\right)(a+c)s}{(16+4s)^2} - \frac{1}{2}\left(\frac{5a-c+2(a+c)s}{16+4s}\right)^2 - \frac{1}{2}\left(\frac{2a+c}{6}\right)^2 = \frac{(5a^2+158ac-19c^2)+2(8+s)(a^2+10ac+7c^2)s}{18(16+4s)^2},$$

which, given the restrictions on the parameters, is positive. So, when player o perceives that his cost is equal to 0, his perceived expected profit from producing at date 1 is greater than that of producing at date 2.

2. Player o has the perception that his cost is equal to c:

(i) If o produces at date 1, his expected profit will be:

$$\pi_o^1 = \frac{1}{2}\left(a - \frac{9a-13c+(3a-c)s}{6(4+s)} - \frac{3a+c-2cs}{8+2s} - c\right)\frac{9a-13c+(3a-c)s}{6(4+s)} + \frac{1}{2}\left(a - \frac{3a-15c+3as+3cs}{12(4+s)} - c\right)\frac{9a-13c+(3a-c)s}{6(4+s)}$$

Doing some algebra we find that:

$$\pi_o^1 = \frac{(81a^2-234ac+169c^2)+(54a^2-96ac+26c^2+(9a^2-6ac+c^2)s)s}{3(16+4s)^2}.$$

(ii) If o produces at date 2, his expected profit will be:

$$\pi_o^2 = \frac{1}{2}\left(a - \frac{a-c-\frac{3a+c-2cs}{8+2s}}{2} - \frac{3a+c-2cs}{8+2s} - c\right)\frac{a-c-\frac{3a+c-2cs}{8+2s}}{2} + \frac{1}{2}\left(a - \frac{a-c}{3} - \frac{a-c}{3} - c\right)\frac{a-c}{3} = \frac{1}{2}\left(\frac{5a-9c+2as}{16+4s}\right)^2 + \frac{1}{2}\left(\frac{a-c}{3}\right)^2.$$

Comparing the two expected profits of o, he will prefer to produce at date 2 if:

$$\frac{(81a^2-234ac+169c^2)+(54a^2-96ac+26c^2+(9a^2-6ac+c^2)s)s}{3(16+4s)^2} \leq \frac{1}{2}\left(\frac{5a-9c+2as}{16+4s}\right)^2 + \frac{1}{2}\left(\frac{a-c}{3}\right)^2$$

Solving this expression with respect to c, one concludes that o will prefer to produce at date 2 if:

$$c > \frac{41-2s+2s^2-2\sqrt{6}\sqrt{64-32s-12s^2+4s^3+s^4}}{28s-10s^2+29}a = \frac{41-2s+2s^2-2\sqrt{6}\,(s+4)\,(2-s)}{29+28s-10s^2}a = \omega(s)a$$

We now determine the optimal production levels of the rational player in each contingency.

1. Player r has cost equal to 0:

(i) Player r produces at date 1: it may be that o will also produce at date 1, if he perceives that he has cost equal to 0, or he will produce at date 2, if he perceives that he has cost equal to c; hence, the quantity produced by r must solve the following problem:

$$\max_{q_r} \frac{1}{2}(1+s)\left(a - q_r - \frac{3a+c+(a+c)s}{8+2s}\right) q_r + \frac{1}{2}(1-s)\left(a - q_r - \frac{a-c-q_r}{2}\right) q_r.$$

The solution to this problem is

$$q_r = \frac{3a+c-2cs}{8+2s}.$$

(ii) Player r produces at date 2, knowing that o has produced the quantity q_o at date 1: then r must produce the quantity that solves the following problem:

$$\max_{q_r} (a - q_o - q_r)\, q_r,$$

which leads to production of:

$$q_r = \frac{a-q_o}{2}.$$

(iii) Player r produces at date 2, knowing that o has not produced at date 1: then r infers that o has perceived that his cost is equal to c and that he will produce $(a-c)/3$ at date 2; thus r must produce a quantity that solves the following problem:

$$\max_{q_r} \left(a - q_r - \frac{a-c}{3}\right) q_r,$$

which leads to production of:

$$q_r = \frac{2a+c}{6}.$$

2. Player r has cost equal to c:

(i) Player r produces at date 1: it may be that o will also produce at date 1, if he perceives that he has cost equal to 0, or he will produce at date 2, if he perceives that he has cost equal to c; hence, the quantity produced by r must solve the following problem:

$$\max_{q_r} \frac{1}{2}(1+s)\left(a - q_r - \frac{3a+c+(a+c)s}{8+2s} - c\right) q_r + \frac{1}{2}(1-s)\left(a - q_r - \frac{a-c-q_r}{2} - c\right) q_r.$$

The solution to this problem is

$$q_r = \frac{9a - 13c + (3a - 9c - 2cs)s}{(3+s)(8+2s)}$$

(ii) Player r produces at date 2, knowing that o has produced the quantity q_o at date 1: then r must produce the quantity that solves the following problem:

$$\max_{q_r} (a - q_o - q_r - c)\, q_r,$$

which leads to production of:

$$q_r = \frac{a - c - q_o}{2}.$$

(iii) Player r produces at date 2, knowing that o has not produced at date 1: then r infers that o has perceived that his cost is equal to c and that he will produce $(a-c)/3$ at date 2; thus r must produce a quantity that solves the following problem:

$$\max_{q_r} \left(a - q_r - \frac{a-c}{3} - c\right) q_r,$$

which leads to production of:

$$q_r = \frac{a-c}{3}.$$

Now, the optimal moment of production of the rational player is determined by looking at the associated expected profits at dates 1 and 2:
1. Player r has cost equal to 0:
(i) If r produces at date 1, his expected profit will be:

$$\begin{aligned}
\pi_r^1 &= \frac{1+s}{2}\left(a - \frac{3a+c-2cs}{8+2s} - \frac{3a+c+(a+c)s}{8+2s}\right)\frac{3a+c-2cs}{8+2s} \\
&\quad + \frac{1-s}{2}\left(a - \frac{3a+c-2cs}{8+2s} - \frac{a-c-\frac{3a+c-2cs}{8+2s}}{2}\right)\frac{3a+c-2cs}{8+2s} \\
&= \frac{3(3a+c)^2 + (9a^2 - 30ac - 11c^2 + 4(2c - 3a + cs)cs)s}{(16+4s)^2}.
\end{aligned}$$

(ii) If r produces at date 2, his expected profit will be:

$$\begin{aligned}
\pi_r^2 &= \frac{1+s}{2}\left(a - \frac{a - \frac{3a+c+(a+c)s}{8+2s}}{2} - \frac{3a+c+(a+c)s}{8+2s}\right)\frac{a - \frac{3a+c+(a+c)s}{8+2s}}{2} \\
&\quad + \frac{1-s}{2}\left(a - \frac{2a+c}{6} - \frac{a-c}{3}\right)\frac{2a+c}{3} \\
&= \frac{1+s}{2}\left(\frac{5a - c + s(a-c)}{16+4s}\right)^2 + \frac{1-s}{2}\left(\frac{2a+c}{6}\right)^2
\end{aligned}$$

Comparing the two expected profits of r, he will prefer to produce at date 1 if:

$$\frac{3(3a+c)^2+(9a^2-30ac-11c^2+4(2c-3a+cs)cs)s}{(16+4s)^2} - \frac{1+s}{2}\left(\frac{5a-c+s(a-c)}{16+4s}\right)^2 - \frac{1-s}{2}\left(\frac{2a+c}{6}\right)^2 \geq 0$$

or

$$\frac{1}{18(16+4s)^2}\left\{(5a^2+158ac-19c^2)-(25a^2+214ac+193c^2)s +(13a^2+22ac+145c^2)s^2+(7a^2+34ac+67c^2)s^3\right\} \geq 0$$

Solving this expression with respect to s, one concludes that r will prefer to produce at date 1 if:

$$s \leq \frac{3\sqrt{3}\sqrt{5a^4+369c^4+588ac^3+68a^3c+314a^2c^2}-(10a^2+28ac+106c^2)}{7a^2+34ac+67c^2}.$$

2. Player r has cost equal to c:
(i) If r produces at date 1, his expected profit will be:

$$\pi_r^1 = \frac{1+s}{2}\left(\frac{6a-14c+5as-9cs+as^2-cs^2}{14s+2s^2+24}\right)\frac{9a-13c+(3a-9c-2cs)s}{(3+s)(8+2s)} + \frac{1-s}{2}\left(\frac{a-c}{2}+\frac{13c-9a-3as+9cs+2cs^2}{28s+4s^2+48}\right)\frac{9a-13c+(3a-9c-2cs)s}{(3+s)(8+2s)}$$

After doing some algebra this expression simplifies to

$$\pi_r^1 = \frac{81a^2-234ac+169c^2+\left(54a^2-240ac+234c^2\right)s}{16\left(48+40s+11s^2+s^3\right)} + \frac{\left(9a^2-90ac+133c^2\right)s^2-12acs^3+36c^2s^3+4c^2s^4}{16\left(48+40s+11s^2+s^3\right)}.$$

(ii) If r produces at date 2, his expected profit will be:

$$\begin{aligned}\pi_r^2 &= \frac{1+s}{2}\left(\frac{a-c}{2}-\frac{(3a+c+as+cs)}{4s+16}\right)\frac{a-c-\frac{3a+c+(a+c)s}{8+2s}}{2} \\ &\quad +\frac{1-s}{2}\left(a-\frac{a-c}{3}-\frac{a-c}{3}-c\right)\frac{a-c}{3} \\ &= \frac{1+s}{2}\left(\frac{5a-9c+(a-3c)s}{16+4s}\right)^2+\frac{1-s}{2}\left(\frac{a-c}{3}\right)^2.\end{aligned}$$

Comparing the two expected profits of r, he will prefer to produce at date 2 if:

$$\begin{aligned}&\frac{81a^2-234ac+169c^2+\left(54a^2-240ac+234c^2\right)s}{16\left(48+40s+11s^2+s^3\right)}\\&\qquad+\frac{\left(9a^2-90ac+133c^2\right)s^2-12acs^3+36c^2s^3+4c^2s^4}{16\left(48+40s+11s^2+s^3\right)}\\&\qquad\qquad\leq\frac{1+s}{2}\left(\frac{5a-9c+(a-3c)\,s}{16+4s}\right)^2+\frac{1-s}{2}\left(\frac{a-c}{3}\right)^2\end{aligned}$$

Solving this expression with respect to c, one concludes that r will prefer to produce at date 2 if:

$$c>\frac{123+143s+67s^2+11s^3-6\sqrt{2}\sqrt{(1+s)\,(3+s)}\,(2+s)\,(4+s)}{87+53s-5s^2-7s^3}a=\lambda(s)a.$$

Since $\omega(s)\geq\lambda(s)$ for any $s\in[0,1]$, the condition $c>\omega(s)a$ implies $c>\lambda(s)a$. *Q.E.D.*

Proof of Proposition 2: Using the same approach as Branco (1998) we enumerate each type of strategy profile that could be considered and explain why there cannot exist equilibria with such profiles.

Both players produce at date 1, regardless of their cost perceptions:

This cannot be an equilibrium because if both players choose to produce at date 1 regardless of their perceptions they don't have information about the other player and they cannot guarantee a leadership gain. Thus, a player would gain by waiting to see the production of the other player and picking his best response quantity at date 2.

Both players produce at date 2, regardless of their cost perceptions:

This cannot be an equilibrium because if both players wait regardless of their cost perceptions, then have no information gain by waiting. If they deviate by committing to a quantity at date 1 they have a first-mover advantage gain.

A player with a high cost perception produces at date 1 and a player with a low cost perception produces at date 2:

Suppose that there is a cost-dependent equilibrium in which the player with a high cost perception produces at date 1 whereas the player with a low cost perception produces at date 2. In this case, the strategy of the overconfident player in the hypothetical equilibrium would be:

1. If $X_o=c$, then produce a quantity equal to $q_o=\frac{1}{2s-8}\left(4c-3a+as-2cs\right)$ at date 1.

2. If $X_o=0$, then do not produce at date 1. Produce at date 2 according to $q_o=\frac{a}{2}-\frac{1}{2}q_r$ if r has produced q_r at date 1, otherwise produce at date 2 $q_o=\frac{a}{3}$ if neither player has produced at date 1.

The strategy of the rational player in the hypothetical equilibrium would be:

1. If $X_r=c$, then produce a quantity equal to $q_r=\frac{1}{2s-8}\left(4c-3a+2cs\right)$ at date 1.

2. If $X_r = 0$, then do not produce at date 1. Produce at date 2 according to $q_r = \frac{a}{2} - \frac{1}{2}q_o$ if o has produced q_o at date 1, otherwise produce at date 2 $q_r = \frac{a}{3}$ if neither player has produced at date 1.

In this hypothetical cost-dependent equilibrium, the overconfident player with a low cost perception has expected profits equal to $\frac{3}{16(s-4)^2}\left(3a - 4c - as + 2cs\right)^2$. However, if the overconfident player deviates and produces at date 1 a quantity equal to $\frac{1}{24-6s}\left(9a + 4c - 3as + 2cs\right)$, he will obtain expected profits equal to

$$\frac{\left(9a + 4c - 3as + 2cs\right)^2}{48\left(s-4\right)^2} > \frac{3\left(3a - 4c - as + 2cs\right)^2}{16\left(s-4\right)^2}.$$

Therefore, the strategy profiles cannot be a cost-dependent equilibrium. *Q.E.D.*

Proof of Proposition 3: The ex-ante profits of player o are equal to

$$\Pi_o(s) = \frac{1+s}{2}\pi_o^1 - \frac{sc}{2}q_o^1 + \frac{1-s}{2}\pi_o^2.$$

Making use of the expressions obtained for π_o^1, q_o^1, and π_o^2 in Proposition 2 we have that

$$\begin{aligned}\Pi_o(s) = {} & \frac{1+s}{2}\frac{3(3a+c)^2 + 3\left(4a + (a+c)(2+s)\right)(a+c)s}{(16+4s)^2} \\ & - \frac{sc}{2}\frac{3a + c + (a+c)s}{8+2s} + \frac{1-s}{2}\left(\frac{1}{2}\left(\frac{5a - 9c + 2as}{16+4s}\right)^2 + \frac{1}{2}\left(\frac{a-c}{3}\right)^2\right),\end{aligned}$$

which can be simplified to

$$\begin{aligned}\Pi_o(s) = {} & \frac{a^2}{36(16+4s)^2}\left[967 - 998x + 1039x^2 + (637 - 230x - 1271x^2)s\right. \\ & \left. + 2(61 + 40x - 335x^2)s^2 + 2(1 - 2x - 53x^2)s^3\right]. \qquad (1)\end{aligned}$$

The ex-ante profits of player r are given by

$$\begin{aligned}\Pi_r(s) &= \frac{1}{2}\pi_r^1 + \frac{1}{2}\pi_r^2 \\ &= \frac{1}{2}\frac{3(3a+c)^2 + (9a^2 - 30ac - 11c^2 + 4(2c - 3a + cs)cs)s}{(16+4s)^2} \\ & \quad + \frac{1}{2}\left(\frac{1+s}{2}\left(\frac{5a - 9c + (a-3c)s}{16+4s}\right)^2 + \frac{1-s}{2}\left(\frac{a-c}{3}\right)^2\right),\end{aligned}$$

which can be simplified to

$$\begin{aligned}\Pi_r(s) &= \frac{a^2}{36(16+4s)^2}\left[967 - 998x + 1039x^2 + (349 - 1526x + 889x^2)s\right. \\ & \quad \left. - (13 + 478x - 599x^2)s^2 - (7 + 22x - 137x^2)s^3\right]. \qquad (2)\end{aligned}$$

The ex-ante profits of a player in an endogenous timing game with two rational players are:

$$\Pi(0) = \Pi_o(0) = \Pi_r(0) = \frac{a^2}{36}\frac{967 - 998x + 1039x^2}{16^2}. \tag{3}$$

The difference between (1) and (3) is given by:

$$\begin{aligned}\Pi_o(s) - \Pi(0) \quad &= \quad \frac{sa^2}{36 \times 16^2(16+4s)^2}\left[-(458368 + 188144s + 27136s^2)x^2\right. \\ &\quad \left. +(68864 + 36448s - 1024s^2)x + (39296 + 15760s + 512s^2)\right].\end{aligned} \tag{4}$$

From (4) we have that $\Pi_o(s) > \Pi(0)$ as long as

$$x < \frac{2152 + 1139s - 32s^2 + 24(s+4)\sqrt{8137 + 2659s + 96s^2}}{28648 + 11759s + 1696s^2} = \tau(s).$$

The difference between (3) and (2) is:

$$\begin{aligned}\Pi(0) - \Pi_r(s) = &\frac{sa^2}{36 \times 16^2(16+4s)^2}\left[-(94592 + 136720s + 35072s^2)x^2\right. \\ &\left. +(262912 + 106400s + 5632s^2)x + (34432 + 18800s + 1792s^2)\right].\end{aligned} \tag{5}$$

From (5) we see have that $\Pi(0) > \Pi_r(s)$ for all $0 < s < s(x)$ since the restrictions on the parameters imply that $x > x^2$ and $262912 + 106400s + 5632s^2 > 94592 + 136720s + 35072s^2$. *Q.E.D.*

Proof of Proposition 4: From (4) and (5), the change in aggregate profits, $\triangle\Pi$, is equal to

$$\begin{aligned}\triangle\Pi &= \Pi_o(s) - \Pi(0) - (\Pi(0) - \Pi_r(s)) \\ &= \frac{sa^2}{36\,(2048 + 1024s + 128s^2)}\left[152 - 95s - 40s^2\right. \\ &\quad \left. -(6064 + 2186s + 208s^2)x - (11368 + 1607s - 248s^2)x^2\right].\end{aligned}$$

The ex-ante consumer surplus in the model with the overconfident player is equal to

$$\begin{aligned}CS(s) \quad &= \quad \frac{1+s}{4}CS(l,l) + \frac{1+s}{4}CS(l,f) + \frac{1-s}{4}CS(f,l) + \frac{1-s}{4}CS(f,f) \\ &= \quad \frac{1+s}{4}\frac{1}{2}\left(\frac{6a + 2c + as - cs}{2s+8}\right)^2 + \frac{1+s}{4}\frac{1}{2}\left(\frac{11a - 7c + 3as - cs}{4s+16}\right)^2 \\ &\quad + \frac{1-s}{4}\frac{1}{2}\left(\frac{11a - 7c + 2as - 4cs}{4s+16}\right)^2 + \frac{1-s}{4}\frac{1}{2}\left(2\frac{a-c}{3}\right)^2.\end{aligned}$$

After some algebra the above expression simplifies to

$$CS(s) = \frac{a^2}{96\left(192+96s+12s^2\right)}\left[4498+2206s+335s^2+17s^3\right.$$
$$\left.-(3956+20s-806s^2-146s^3)x+(2050+118s-781s^2-163s^3)x^2\right].$$

The ex-ante consumer surplus in the model with two rational players is equal to

$$\begin{aligned} CS(0) &= \frac{1}{4}CS(l,l)+\frac{1}{2}CS(l,f)+\frac{1}{4}CS(f,f) \\ &= \frac{1}{4}\frac{1}{2}\left(\frac{3a+c}{4}\right)^2+2\frac{1}{4}\frac{1}{2}\left(\frac{3a+c}{8}+\frac{5a-9c}{16}\right)^2+\frac{1}{4}\frac{1}{2}\left(2\frac{a-c}{3}\right)^2 \\ &= \frac{a^2}{96^2}\left(2249-1978x+1025x^2\right). \end{aligned}$$

The change in consumer surplus, $\triangle CS$, is equal to

$$\begin{aligned} \triangle CS &= CS(s)-CS(0) \\ &= \frac{sa^2}{96\left(1536+768s+96s^2\right)}\left[-344+431s+136s^2\right. \\ &\quad \left.+(15664+8426s+1168s^2)x-(7256+7273s+1304s^2)x^2\right] \end{aligned}$$

The change in welfare, $\triangle W$, is the change in aggregate profits plus the change in consumer surplus:

$$\begin{aligned} \triangle W &= \triangle\Pi+\triangle CS \\ &= \frac{sa^2}{36\times 96}\left[\frac{-(1079712+377532s+29088s^2)x^2}{\left(1536+768s+96s^2\right)\left(2048+1024s+128s^2\right)}\right. \\ &\quad \left.+\frac{(127296+145944s+27072s^2)x+(-1440+8676s+2016s^2)}{\left(1536+768s+96s^2\right)\left(2048+1024s+128s^2\right)}\right] \end{aligned}$$

From the expression above we have that $\triangle W > 0$ as long as

$$x < \frac{1768+2027s+376s^2+24(s+4)\sqrt{209+1412s+324s^2}}{29992+10487s+808s^2} = \psi(s).$$

Q.E.D.