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On the monotonic core

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On the monotonic core

Abstract: The monotonic core of a cooperative game with transferable utility (T.U.-game) is the set formed by all its Population Monotonic Allocation Schemes. In this paper we show that this set always coincides with the core of a certain game associated to the initial game.

Keywords: Cooperative games, monotonic core, population monotonic allocation schemes, restricted games.

JEL Classification: C71

Resum: El monotonic core d'un joc cooperatiu amb utilitat transferible (T.U.-game) és el conjunt format per totes les seves Population Monotonic Allocation Schemes. En aquest treball provem que aquest conjunt coincideix sempre amb el core de cert joc associat al joc inicial.

1 Introduction

A cooperative game with transferable utility (a game) assigns to each coalition of players a real number representing the worth of the coalition, that is, what it can achieve on its own.

Sprumont (1990) introduces the concept of Population Monotonic Allocation Scheme (PMAS) in cooperative games. A PMAS selects a core allocation for every subgame of a game in such a way that the payoff of any player cannot decrease as the coalition to which he belongs enlarges. As a consequence, a game with a PMAS has a nonempty core. Sprumont shows that all convex games have a PMAS (for example, the extended Shapley value). Therefore, PMAS can be proposed for many well-known models, for instance bankruptcy problems (see Grahn and Voorneveld, 2002). Moulin (1990) also applies PMAS to the problem of cost-sharing of public goods. Moreover, he gives a theoretical characterization of the class of games with a PMAS.

Norde and Reijnierse (2002) give some conditions (in finite number) to determine whether a game has a PMAS or not. In the case of a four-player game there are sixty inequalities that its characteristic function must satisfy in order to have a PMAS.

Moulin (1990) defines the concept of the monotonic core of a game as the set formed by all its PMAS. The monotonic core of a game has topological and algebraic properties similar to those of the core, since both sets are convex and compact polyhedrons in their respective Euclidean spaces.

The aim of this paper is to show that the monotonic core of a game is the core of another game. To this end, we consider the model of games with restricted cooperation introduced by Faigle (1989). In this model, the characteristic function of the game is defined only on a system of coalitions, called feasible coalitions. Other references on games with restricted cooperation are Bilbao (2000) and Algaba et al. (2001).

The outline of this paper is as follows. After some preliminaries and notations in Section 2, in Section 3 we show that the monotonic core of an arbitrary game is the core of a game with restricted cooperation which is introduced from the initial game. In Section 4 we prove that the monotonic core of a game having a PMAS always coincides with the core of another game related to the previous game but now without restricted cooperation.

2 Preliminaries and notations

A cooperative game with transferable utility (a game) is a pair (N, v), where $N = \{1, 2, \cdots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function with $v(\varnothing) = 0$. A subset S of N, $S \in 2^N$, is a coalition of players, $s = |S|$ its cardinality and $v(S)$ is interpreted as the worth of the coalition S. We denote by $P(N) := \{S \subseteq N \mid S \neq \varnothing\}$ the set of nonempty coalitions of N. Given $S \in P(N)$, we denote by (S, v_S) the subgame of (N, v) related to coalition S, i.e. $v_S(R) = v(R)$ for all $R \subseteq S$. The class of games with player set N is denoted by G^N. We identify each game of G^N with its characteristic function.

As usual, a game $v \in G^N$ is **superadditive** if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \varnothing$.

A payoff vector is $z = (z_i)_{i \in N} \in \mathbb{R}^N$, where z_i represents the payoff to player i, and for $S \in P(N)$ we write $z(S) := \sum\limits_{i \in S} z_i$ and $z(\varnothing) := 0$.

The **core** of a game v is the set

$$C(v) := \left\{ z \in \mathbb{R}^N \mid z(N) = v(N) \text{ and } z(S) \geq v(S) \text{ for all } S \in P(N) \right\}.$$

Thus the core $C(v)$ is a convex and compact (possibly empty) polyhedral subset of $\mathbb{R}^N$.

A game (N, v) is said to be **balanced** if it has a nonempty core, and **totally balanced** if the subgame (S, v_S) is balanced for all $S \in P(N)$.

A **Population Monotonic Allocation Scheme** (or **PMAS**) of a game v (Sprumont, 1990) is a vector $x = \left(x_i^S\right)_{S \in P(N),\ i \in S}$, with components $x_i^S \in \mathbb{R}$, that satisfies the following conditions:

$$\sum_{i \in S} x_i^S = v(S) \text{ for all } S \in P(N) \ . \tag{1}$$

$$x_i^R \leq x_i^S \text{ for all } R, S \in P(N),\ R \subseteq S, \text{ and all } i \in R. \tag{2}$$

The first condition tells us that, for each coalition $S \in P(N)$, the payoff vector $x^S := \left(x_i^S\right)_{i \in S} \in \mathbb{R}^S$ is a distribution of the amount $v(S)$ among the players of S. The second condition guarantees that each one of the players of S does not receive more in any subcoalition R in which he takes part. Thus if x is a PMAS of a game v, then x is an element of the vector space $\prod\limits_{S \in P(N)} \mathbb{R}^S$, which has dimension $2^{n-1} \cdot n$. Moreover, from conditions (1) and (2), it follows that each payoff vector x^S belongs to $C(v_S)$ for all PMAS x. Therefore, if a game v has a PMAS, then it is totally balanced.

Moulin (1990) defines the **monotonic core** of the game v as the set of all its PMAS. We denote this set by $MC(v)$:

$$MC(v) := \left\{ x \in \prod_{S \in P(N)} \mathbb{R}^S \,\middle|\, x \text{ is a PMAS of } v \right\}.$$

Thus the monotonic core $MC(v)$ is a compact and convex (possibly empty) subset of $\prod_{S \in P(N)} \mathbb{R}^S$. Therefore, the monotonic core has topological and algebraic properties similar to those of a core, and it seems reasonable to hope that it may coincide with the core of another game with $2^{n-1} \cdot n$ players.

In Section 4 we will show that $MC(v)$ is the core of a game with $2^{n-1} \cdot n$ players. In order to define such a game, the following concepts, from Faigle (1989) and Algaba et al. (2001), are needed.

A **game with restricted cooperation** (Faigle, 1989) is a 4-tuple $\Gamma = (\mathsf{N}, \mathcal{F}, w, w_0)$ satisfying the following conditions:

1. N is a finite set.

2. $\mathcal{F} \subseteq 2^{\mathsf{N}}$ is a selection of coalitions of N, called **feasible coalitions**, such that $\varnothing \in \mathcal{F}$.

3. $w : \mathcal{F} \longrightarrow \mathbb{R}$ is a function with $w(\varnothing) = 0$.

4. $w_0 \in \mathbb{R}$ is the **value** of game Γ.

Faigle (1989) also defined the **core** of game Γ by

$$C(\Gamma) := \left\{ x \in \mathbb{R}^{\mathsf{N}} \mid x(\mathsf{N}) = w_0 \text{ and } x(\mathsf{T}) \geq w(\mathsf{T}) \text{ for all } \mathsf{T} \in \mathcal{F} \right\}.$$

Thus the core $C(\Gamma)$ is a convex and closed (possibly empty) subset of $\mathbb{R}^{\mathsf{N}}$, but, in general, it is not bounded. However, if each individual coalition is a feasible coalition (i.e. $\{\mathsf{i}\} \in \mathcal{F}$ for all $\mathsf{i} \in \mathsf{N}$), then this core is bounded and therefore compact. Moreover, note that if $\mathsf{N} \in \mathcal{F}$ and $w_0 < w(\mathsf{N})$ then $C(\Gamma) = \varnothing$. For more details on this see Derks and Reijnierse (1998).

Assuming that $\{\mathsf{i}\} \in \mathcal{F}$ for all $\mathsf{i} \in \mathsf{N}$, we can associate a game defined on all the coalitions to any game with restricted cooperation Γ, since each coalition $\mathsf{S} \subseteq \mathsf{N}$ can be partitioned in feasible coalitions, at least considering its one-player subcoalitions. It is standard to define the game $(\mathsf{N}, w^{\mathcal{F}})$ by

$$w^{\mathcal{F}}(\mathsf{S}) := \max_{P \in \mathcal{P}^{\mathsf{F}}(\mathsf{S})} \left(\sum_{\mathsf{T} \in P} w(\mathsf{T}) \right) \tag{3}$$

for all nonempty coalition $\mathsf{S} \subseteq \mathsf{N}$, where $\mathcal{P}^{\mathcal{F}}(\mathsf{S})$ denotes the set of all partitions of S in feasible coalitions of N, and $w^{\mathcal{F}}(\varnothing) := 0$.

It follows from the previous definition that the game $w^{\mathcal{F}} \in G^{\mathsf{N}}$ is superadditive and its restriction to $\mathcal{F}$ is greater than or equal to w: $w^{\mathcal{F}}(\mathsf{T}) \geq w(\mathsf{T})$ for all $\mathsf{T} \in \mathcal{F}$. Moreover, $w^{\mathcal{F}}$ is the smallest game of G^{N} with these properties;

i.e. if $\mathsf{v} \in G^{\mathsf{N}}$ is a superadditive game such that $\mathsf{v}(\mathsf{T}) \geq w(\mathsf{T})$ for all $\mathsf{T} \in \mathcal{F}$, then $\mathsf{v}(\mathsf{S}) \geq w^{\mathcal{F}}(\mathsf{S})$ for all $\mathsf{S} \subseteq \mathsf{N}$. We also have $w^{\mathcal{F}}(\mathsf{i}) = w(\mathsf{i})$ for all $\mathsf{i} \in \mathsf{N}$, and $w^{\mathcal{F}}(\mathsf{S}) \geq \sum_{\mathsf{i} \in \mathsf{S}} w(\mathsf{i})$ for all $\mathsf{S} \subseteq \mathsf{N}$.

Note that the game $(\mathsf{N}, w^{\mathcal{F}})$ does not depend on the value w_0 of the game Γ, so the cores $C(\Gamma)$ and $C\left(w^{\mathcal{F}}\right)$ are not equal in general. However these cores coincide if both are in the same hyperplane of efficiency as the following lemma establishes.

Lemma 2.1 (Faigle, 1989) If $w_0 = w^{\mathcal{F}}(\mathsf{N})$, then $C(\Gamma) = C\left(w^{\mathcal{F}}\right)$.

3 Restricted cooperation and PMAS

Let (N, v) be a game. We begin this section building from (N, v) a game with restricted cooperation $\widehat{\Gamma} = \left(\widehat{N}, \widehat{\mathcal{F}}, \widehat{v}, \widehat{v}_0\right)$ in such a way that its core, $C\left(\widehat{\Gamma}\right)$, coincides with the monotonic core $MC(v)$.

We index the components x_i^S of every PMAS x of the game v by elements of the set

$$\widehat{N} := \{(S, i) \mid \; S \in P(N), \; i \in S\};$$

so the vectorial space $\mathbb{R}^{\widehat{N}} = \prod_{S \in P(N)} \mathbb{R}^S$ and the set $\widehat{N}$ has $\widehat{n} := 2^{n-1} \cdot n$ players. Note that each coalition $S \subseteq N$ can be identified with the coalition $[S]$ of $\widehat{N}$ defined by:

$$[S] := \{(S, i) \mid \; i \in S\} \subset \widehat{N} \qquad ([\varnothing] := \varnothing).$$

via the one-to-one mapping

$$\begin{array}{ccc} 2^N & \longrightarrow & 2^{\widehat{N}} \\ S & \longmapsto & [S] \; . \end{array}$$

Now we can identify the feasible coalitions of $\widehat{N}$. Given $R, S \in P(N)$ with $R \subseteq S$, and given $i \in R$, we define the following coalition of $\widehat{N}$:

$$[S, R, i] := \{(S, i)\} \cup \{(R, j) \mid j \in R \setminus \{i\}\} \subset \widehat{N}.$$

In other words, coalition $[S, R, i]$ is defined from $[R]$ replacing player (R, i) by player (S, i) of $[S]$. Observe that if $R = \{i\}$, then the coalition $[S, R, i]$ is reduced to the single player (S, i); i.e. we have $[S, \{i\}, i] = \{(S, i)\}$ for all $S \in P(N), i \in S$. Moreover, if $R = S$, then $[S, R, i]$ is the coalition $[S]$; i.e. we have $[S, S, i] = [S]$ for all $S \in P(N), i \in S$.

The set of feasible coalitions of $\widehat{N}$ is defined by:

$$\widehat{\mathcal{F}} := \{[S,R,i] \mid R,S \in P(N) \text{ with } R \subseteq S,\ i \in R\} \cup \{\varnothing\} \subset 2^{\widehat{N}}.$$

By the above observation, notice that the individual coalitions of $\widehat{N}$ are always feasible coalitions and that $[S] \in \widehat{\mathcal{F}}$ for all $S \in P(N)$. However, the coalition $\widehat{N}$ is not a feasible coalition (i.e. $\widehat{N} \notin \widehat{\mathcal{F}}$). An example of the partition of $\widehat{N}$ in feasible coalitions is the set $\{[S] \mid S \in P(N)\}$. Moreover, there are

$$\begin{aligned}\left|\widehat{\mathcal{F}}\right| &= \sum_{S\in P(N)} \left(\sum_{R\in P(N),\ R\subset S} |R| + 1 \right) + 1 = \sum_{s=1}^{n} \binom{n}{s} \cdot \left(\sum_{r=1}^{s-1} \binom{s}{r} \cdot r + 1 \right) + 1 \\ &= \sum_{s=1}^{n} \binom{n}{s} \cdot \left(2^{s-1} \cdot s - s + 1\right) + 1 = 3^{n-1} \cdot n \ - \ 2^{n-1} \cdot (n-2)\end{aligned}$$

feasible coalitions, where the term $3^{n-1} \cdot n$ is obtained by differentiating the equality $1 + \sum_{s=1}^{n} \binom{n}{s} t^s = (t+1)^n$ with respect to t and replacing t by 2.

Next, we illustrate the preceding construction of the pair $\left(\widehat{N}, \widehat{\mathcal{F}}\right)$ from N with an example. Let $N = \{1,2,3\}$. Then $\widehat{N} = \bigcup_{S\in P(N)} [S]$ has $\widehat{3} = 12$ players and there are $\left|\widehat{\mathcal{F}}\right| = 23$ feasible coalitions. The nonempty feasible coalitions are the following:
(A) $\{(S,i)\}$ for $(S,i) \in \widehat{N}$ (12 coalitions),
(B) $[S]$ for $S \in P(N)$ with $s \geq 2$ (4 coalitions),
(C) $[N, \{i,j\}, i] = \{(N,i), (\{i,j\}, j)\}$ for $i \in N$ and $j \in N \backslash \{i\}$ (6 coalitions).

Once the set of feasible coalitions is defined, in order to complete the game with restricted cooperation $\left(\widehat{N}, \widehat{\mathcal{F}}, \widehat{v}, \widehat{v}_0\right)$, we now define, from our original game v, the mapping $\widehat{v} : \widehat{\mathcal{F}} \longrightarrow \mathbb{R}$ by

$$\widehat{v}\left([S,R,i]\right) := v\left(R\right) \text{ for } [S,R,i] \in \widehat{\mathcal{F}}, \text{ and } \widehat{v}\left(\varnothing\right) := 0.$$

Notice that by definition we have $\widehat{v}\left(\{(S,i)\}\right) = v\left(i\right)$ for all $(S,i) \in \widehat{N}$, and that $\widehat{v}\left([S]\right) = v\left(S\right)$ for all $S \in P(N)$.

Finally, we take as value $\widehat{v}_0 := \sum_{S\in P(N)} v\left(S\right).$

Summarizing, the extended game $\widehat{\Gamma} = \left(\widehat{N}, \widehat{\mathcal{F}}, \widehat{v}, \widehat{v}_0\right)$ associated to a cooperative game (N, v) is the game with restricted cooperation defined as

1. $\widehat{N} = \{(S,i) \mid \ i \in S\}$.

2. $\widehat{\mathcal{F}} = \{[S,R,i] \mid R, S \in P(N) \text{ with } R \subseteq S, \ i \in R\} \cup \{\varnothing\}$,
where each $[S,R,i] = \{(S,i)\} \cup \{(R,j) \mid j \in R \setminus \{i\}\}$.

3. $\widehat{v}([S,R,i]) = v(R)$ for all $[S,R,i] \in \widehat{\mathcal{F}}$ and $\widehat{v}(\varnothing) = 0$.

4. $\widehat{v}_0 = \sum_{S \in P(N)} v(S)$.

In the next theorem we will show that the monotonic core of a cooperative game (N,v) can always be viewed as the core of its extended cooperative game with restricted cooperation $\widehat{\Gamma}$.

Theorem 3.1 Let (N,v) be a game. Then we have

$$MC(v) = C\left(\widehat{\Gamma}\right),$$

where $\widehat{\Gamma} = \left(\widehat{N}, \widehat{\mathcal{F}}, \widehat{v}, \widehat{v}_0\right)$.

Proof: First we show the inclusion $MC(v) \subseteq C\left(\widehat{\Gamma}\right)$. Let $x \in MC(v)$, we will prove that $x \in C\left(\widehat{\Gamma}\right)$. To begin with, we prove efficiency (i.e. $x\left(\widehat{N}\right) = \widehat{v}_0$). From condition (1), we obtain

$$x\left(\widehat{N}\right) = \sum_{(S,i)\in\widehat{N}} x_i^S = \sum_{S\in P(N)} \left(\sum_{i\in S} x_i^S\right) = \sum_{S\in P(N)} v(S) = \widehat{v}_0.$$

We prove now the coalitional rationality for the feasible coalitions; i.e. $x(\mathsf{T}) \geq \widehat{v}(\mathsf{T})$ for all $\mathsf{T} \in \widehat{\mathcal{F}}$. Since x is a PMAS of v, from conditions (2) and (1) and the definition of $\widehat{v}$ we deduce

$$x(\mathsf{T}) = x_i^S + \sum_{j\in R\setminus\{i\}} x_j^R \geq \sum_{j\in R} x_j^R = v(R) = \widehat{v}(\mathsf{T}) \text{ for all } \mathsf{T} = [S,R,i] \in \widehat{\mathcal{F}}.$$

Now we show the other inclusion, $C\left(\widehat{\Gamma}\right) \subseteq MC(v)$. Let $x \in C\left(\widehat{\Gamma}\right)$, in order to prove that x is a PMAS of (N,v), we first prove that x satisfies condition (1). As a consequence of the coalitional rationality for the feasible coalitions, we obtain

$$\sum_{i\in S} x_i^S = x([S]) \geq \widehat{v}([S]) = v(S) \text{ for all } S \in P(N).$$

Furthemore, from the efficiency of x, we have

$$\sum_{S\in P(N)}\left(\sum_{i\in S}x_i^S\right)=x\left(\widehat{N}\right)=\widehat{v}_0=\sum_{S\in P(N)}v\left(S\right).$$

Hence, x satisfies condition (1). We prove now that x also satisfies condition (2). Let $R,S\in P(N)$ with $R\subseteq S$ and let $i\in R$. As a consequence of condition (1) and the coalitional rationality for the feasible coalitions, we obtain

$$x_i^R+\sum_{j\in R\setminus\{i\}}x_j^R=v\left(R\right)=\widehat{v}\left([S,R,i]\right)\le x\left([S,R,i]\right)=x_i^S+\sum_{j\in R\setminus\{i\}}x_j^R,$$

which implies $x_i^R\le x_i^S$. So x satisfies condition (2). This finishes the proof of the theorem. ¥

4 The monotonic core as a core of a T.U.-game

Let (N,v) be a game. Let $\widehat{\Gamma}=\left(\widehat{N},\widehat{\mathcal{F}},\widehat{v},\widehat{v}_0\right)$ be the extended game with restricted cooperation associated to (N,v) in Section 3. As we have noted above, all individual coalitions $\{(S,i)\}=[S,\{i\},i]$ are feasible coalitions. Therefore, we can define the cooperative game $\left(\widehat{N},\ \widehat{v}^{\widehat{\mathcal{F}}}\right)$ associated to $\widehat{\Gamma}$ following definition (3). Thus, for every non empty coalition $\mathsf{S}\subseteq\widehat{N}$ we have, by definition, that

$$\widehat{v}^{\widehat{\mathcal{F}}}\left(\mathsf{S}\right)=\max_{P\in\mathcal{P}^{\widehat{\mathsf{F}}}\left(\mathsf{S}\right)}\left(\sum_{[S,R,i]\in P}v(R)\right)\tag{4}$$

where $\mathcal{P}^{\widehat{\mathcal{F}}}\left(\mathsf{S}\right)$ denotes the set of all partitions of S in feasible coalitions of $\widehat{N}$.

In this section we see that the monotonic core $MC\left(v\right)$ coincides with the core $C\left(\widehat{v}^{\widehat{\mathcal{F}}}\right)$ when our game v has a PMAS.

We know, from Section 2, that $\widehat{v}^{\widehat{\mathcal{F}}}$ is the smallest superadditive game of $G^{\widehat{N}}$ whose restriction to $\widehat{\mathcal{F}}$ is greater than or equal to $\widehat{v}$. In fact, in the case that v is totally essential (i.e. $\sum_{i\in S}v(i)\le v(S)$ for all coalition $S\in P(N)$) we have that $\widehat{v}^{\widehat{\mathcal{F}}}([S,R,i])=v(R)$ for all feasible coalitions $[S,R,i]$. In particular, in this case we have $\widehat{v}^{\widehat{\mathcal{F}}}\left([S]\right)=v(S)$ for all $S\subseteq N$.

One particular partition of $\widehat{N}$ in feasible coalitions, $\widehat{N} = \bigcup_{S \in P(N)} [S]$, shows that

$$\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) \geq \widehat{v}_0 = \sum_{S \in P(N)} v(S) = \sum_{S \in P(N)} \left(\sum_{i \in S} x_i^S\right) = x\left(\widehat{N}\right), \tag{5}$$

for all PMAS x of our game v (if there are any). Unfortunately, the equality $\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) = \widehat{v}_0$ is not true in general, not even in the case that v is totally balanced, as the following example shows.

Example 4.1 Let v be the glove market game with four players, $N = \{1,2,3,4\}$, partitioned in two coalitions $L = \{1,2\}$ and $R = \{3,4\}$. Each player of L (resp. R) possesses a left (resp. right) hand glove. A left-right pair of gloves can be sold at a market price of 1 monetary unit while a single glove has no value. Thus $v(S) = \min\{|S \cap L|, |S \cap R|\}$ for all $S \subseteq N$. For this game we have the inequality

$$\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) > \widehat{v}_0.$$

Indeed, if in the partition $\widehat{N} = \bigcup_{S \in P(N)} [S]$ we replace the union of the coalitions

$$[\{1,3\}],\ [\{2,3\}],\ [\{2,4\}],\ [\{1,2,3\}] \text{ and } [\{2,3,4\}]$$

by the union of the feasible coalitions

$$[\{1,2,3\},\{1,3\},1],\ [\{1,2,3\},\{1,3\},3],\ [\{1,2,3\},\{2,3\},2],$$
$$[\{2,3,4\},\{2,3\},3],\ [\{2,3,4\},\{2,4\},2] \text{ and } [\{2,3,4\},\{2,4\},4],$$

then we obtain a new partition of $\widehat{N}$ in feasible coalitions that shows the above inequality, since

$$2\left(v(13) + v(23) + v(24)\right) = 6 > 5 = v(13) + v(23) + v(24) + v(123) + v(234).$$

Moreover, this game is totally balanced but it has no PMAS (cf. Sprumont, 1990).

Theorem 4.2 Let (N, v) be a game having a PMAS. Then we have

$$MC(v) = C\left(\widehat{v}^{\widehat{\mathcal{F}}}\right),$$

where $\left(\widehat{N}, \widehat{v}^{\widehat{\mathcal{F}}}\right)$ is the game defined by (4).

Proof: By Theorem 3.1 we have that the monotonic core $MC(v) = C\left(\widehat{\Gamma}\right)$, and by Lemma 2.1 we know that this last core is $C\left(\widehat{\Gamma}\right) = C\left(\widehat{v}^{\widehat{\mathcal{F}}}\right)$ when $\widehat{v}_0 = \widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right)$. Therefore to prove the theorem it is sufficient to see the inequality

$$\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) \leq \widehat{v}_0 \tag{6}$$

since $\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) \geq \widehat{v}_0$ holds by (5).

Let $\{\mathsf{T}_j\}_{j\in J}$ be an arbitrary partition of $\widehat{N}$ in feasible coalitions. Thus each $\mathsf{T}_j = [S_j, R_j, i_j]$ with $i_j \in R_j \subseteq S_j \subseteq N$, and by definition $\widehat{v}(\mathsf{T}_j) = v(R_j)$. Taking into account (4), we must prove the inequality

$$\sum_{j\in J} v(R_j) \leq \widehat{v}_0 \tag{7}$$

to see (6). We observed above that, given $i \in S \subseteq N$, we have $(S,i) \in [S] \subseteq \hat{N} = \bigcup_{j\in J} \mathsf{T}_j$. Thus, there is a unique $j \in J$ such that $(S,i) \in \mathsf{T}_j$. Hence one of the following cases occurs:

$$(S,i) = (S_j, i_j) \quad \text{or} \quad (S,i) = (R_j, k) \;\text{ with } k \in R_j \backslash \{i_j\}.$$

Or, in other words,

$$S = S_j \supseteq R_j \text{ and } i = i_j \in R_j, \quad \text{or} \quad S = R_j \text{ and } i \neq i_j.$$

Therefore, in both cases we have $i \in R_j \subseteq S$.

Let $x \in MC(v)$. Now note that every part of (7) can be written as follows

$$\begin{aligned}\sum_{j\in J} v(R_j) &= \sum_{j\in J}\left(\sum_{i\in R_j} x_i^{R_j}\right) = \sum_{i\in N}\left(\sum_{j\in J: R_j \ni i} x_i^{R_j}\right),\\ \widehat{v}_0 &= \sum_{S\in P(N)} v(S) = \sum_{S\in P(N)}\left(\sum_{i\in S} x_i^S\right) = \sum_{i\in N}\left(\sum_{S\in P(N): S\ni i} x_i^S\right),\end{aligned}$$

since $N = \bigcup_{S\in P(N)} S = \bigcup_{j\in J} R_j$ as observed before. In consequence, it is sufficient to prove the inequalities

$$\sum_{j\in J: R_j \ni i} x_i^{R_j} \leq \sum_{S\in P(N): S\ni i} x_i^S \quad \text{for } i \in N,$$

to show the inequality (7). But the above inequalities are directly obtained from (2) and the following lemma.

Lemma 4.3 With the above notations, let $i \in N$. Then there exists a bijective function

$$f : \{S \in P(N) \mid i \in S\} \longrightarrow \{j \in J \mid i \in R_j\},$$

that satisfies the following property:

$$R_{f(S)} \subseteq S \text{ for all } S \in P(N) \text{ with } i \in S.$$

Proof: First we define the function f. Let $S \in P(N)$ with $i \in S$. We know that there is a unique $j \in J$ such $(S,i) \in \mathsf{T}_j = [S_j, R_j, i_j]$ with $i_j \in R_j \subseteq S_j \subseteq N$, and we know that $i \in R_j \subseteq S$. We define $f(S) := j$. Then the function f is well defined and it satisfies the required property.

In order to show that f is a bijective function, we define a mapping

$$g : \{j \in J \mid R_j \ni i\} \longrightarrow \{S \in P(N) \mid i \in S\}$$

such that the compositions $f \circ g$ and $g \circ f$ are the respective identity functions. Let $j \in J$ with $i \in R_j$. We define

$$g(j) := \begin{cases} S_j & \text{if } i_j = i, \\ R_j & \text{if } i_j \neq i. \end{cases}$$

It can be checked directly that the above compositions are the respective identity functions. ¥

Now we can prove that the monotonic core of a game can always be identified as the core of a complete game defined on $\widehat{N}$. To do this we only have to change the efficiency worth of the game $\left(\widehat{N}, \widehat{v}^{\widehat{\mathcal{F}}}\right)$ by $\widehat{v}_0 = \sum_{S \in P(N)} v(S)$.

Corollary 4.4 Let (N, v) be a game. Then we have

$$MC(v) = C(\overline{v}),$$

where $\left(\widehat{N}, \overline{v}\right)$ is defined by

$$\begin{aligned} \overline{v}(\mathsf{S}) &:= \widehat{v}^{\widehat{\mathcal{F}}}(\mathsf{S}) \quad \text{for } \mathsf{S} \subset \widehat{N}, \\ \overline{v}\left(\widehat{N}\right) &:= \widehat{v}_0. \end{aligned} \tag{8}$$

Proof: Observe first that $C(\overline{v}) \subseteq C\left(\widehat{\Gamma}\right)$, since we know that

$$\overline{v}(\mathsf{T}) = \widehat{v}^{\widehat{\mathcal{F}}}(\mathsf{T}) \geq \widehat{v}(\mathsf{T}) \text{ for all } \mathsf{T} \in \widehat{\mathcal{F}}.$$

If the game v does not have a PMAS then, by Theorem 3.1, $C\left(\widehat{\Gamma}\right) = \varnothing$ and, by what we have just observed, $C(\overline{v}) = \varnothing$.

On the other hand, if the game v has a PMAS, then $\widehat{v}^{\widehat{\mathcal{F}}}\left(\widehat{N}\right) = \widehat{v}_0$ (i.e. $\widehat{v}^{\widehat{\mathcal{F}}} = \overline{v}$) as we have seen in the proof of Theorem 4.2, and the corollary is obtained using this same theorem. ¥

From the above corollary, we now deduce the following characterization of the games having a PMAS. It is rather theoretical in nature.

Corollary 4.5 Let (N, v) be a game and let $\left(\widehat{N}, \overline{v}\right)$ be the game defined in (8). Then v has a PMAS if and only if $\left(\widehat{N}, \overline{v}\right)$ is balanced. ¥

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