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A characterization of cooperative TU-games with large monotonic core

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Abstract: Cooperative TU-games with large core were introduced by Sharkey (1982) and the concept of Population Monotonic Allocation Scheme was defined by Sprumont (1990). Linking these two concepts, Moulin (1990) introduces the notion of large monotonic core giving a characterization for three-player games. In this paper we prove that all games with large monotonic core are convex. We give an effective criterion to determine whether a game has large monotonic core and, as a consequence, we obtain a characterization for the four-player case.
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Resum: El concepte de joc cooperatiu amb large core és introduït per Sharkey (1982) i el de Population Monotonic Allocation Scheme és definit per Sprumont (1990). Inspirat en aquests conceptes, Moulin (1990) introdueix la noció de large monotonic core donant una caracterització per a jocs de tres jugadors. En aquest document provem que tots els jocs amb large monotonic core són convexes. A més, donem un criteri efectiu per determinar si un joc té large monotonic core o no, i daquí obtenim una caracterització pel cas de quatre jugadors.

1 Introduction

A cooperative game with transferable utility assigns to each coalition of players a real number, which represents what the coalition can achieve on its own, its worth. Most studies on cooperative games assume that the grand coalition forms. Then, the analysis reduces to determine the payoff to the members of the grand coalition. When sharing a divisible good among the players of the grand coalition, Sharkey (1982) introduce the concept of acceptable vectors or aspirations.

An aspiration for a given game is a payoff vector that summarizes predictions about what the resulting payoffs of their members will be.

When an efficient allocation is to be determined, Sharkey (1982) defines the notion of largeness of the core: a game has large core if each aspiration has a core allocation as its lower bound. He proves that convexity implies the largeness of the core.

From the viewpoint of variable population, any cooperative game can be seen as a stream: one subgame for each coalition of players. A generalized allocation now determines an allocation for every subgame.

Sprumont (1990) introduces the concept of Population Monotonic Allocation Scheme (PMAS) for cooperative games. A PMAS selects a core allocation for every subgame of a game in such a way that the payoff of any player can not decrease as the coalition to which he belongs enlarges. He shows that all convex games have a PMAS (for example, the extended Shapley value).

To solve fair allocation problems, Moulin (1990) introduces the notion of large core within the context of variable population of players. Thus, he adapts the idea of the largeness of the core by defining the notion of monotonic aspiration, a generalized aspiration with the population monotonic property, and the notion of largeness of the monotonic core (the set of all PMAS): a game has large monotonic core if each monotonic aspiration has a PMAS as its lower bound.

The paper is organized as follows. In Section 2, we introduce basic concepts and notations to be used later on, and we show that convexity is a necessary (but not sufficient) condition to have large monotonic core. This generalizes what was already proved by Moulin in 1990 for three-player games. Section 3, contains the main results of this paper. We give a necessary and sufficient condition for a game to have large monotonic core in terms of some computable parameters associated to its characteristic function. Finally, in Section 4, we characterize those four-player games with large monotonic core in terms of their characteristic function.

2 Preliminaries and notations

A cooperative game with transferable utility (a *game*) is a pair (N, v), where $N = \{1, 2, \cdots, n\}$ is a finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function with $v(\varnothing) = 0$. A subset S of N, $S \in 2^N$, is a coalition of players, $s = |S|$ its cardinality and $v(S)$ is interpreted as the worth of coalition S. We denote by $P(N) := \{S \subseteq N \mid S \neq \varnothing\}$ the set of nonempty coalitions of N. Given $S \in P(N)$, we denote by (S, v_S) the subgame of (N, v) related to coalition S (i.e. $v_S(R) = v(R)$ for all $R \subseteq S$). The class of games with player set N is denoted by G^N. We identify each game of G^N with its characteristic function.

As usual, a game $v \in G^N$ is *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for all $S, T \subseteq N$ with $S \cap T = \varnothing$, and it is *convex* if $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$.

A *payoff vector* is $z = (z_i)_{i \in N} \in \mathbb{R}^N$, where z_i is the payoff to player i, and for $S \in P(N)$ we write $z(S) := \sum\limits_{i \in S} z_i$ and $z(\varnothing) := 0$.

The *core* of a game $v \in G^N$ is the set

$$C(v) := \left\{ z \in \mathbb{R}^N \mid z(N) = v(N) \text{ and } z(S) \geq v(S) \text{ for all } S \in P(N) \right\}.$$

Thus the core $C(v)$ is a compact and convex (possibly empty) polyhedral subset of $\mathbb{R}^N$.

A game $v \in G^N$ is said to be *balanced* if it has a nonempty core, and it is *totally balanced* if the subgame (S, v_S) is balanced for all $S \in P(N)$.

Every game $v \in G^N$ can be written as $v = \sum\limits_{S \in P(N)} \lambda_S u_S$ where u_S is the unanimity game associated to coalition S (i.e. $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise) and λ_S is the unanimity coordinate associated to S and v (i.e. $\lambda_S = \lambda_S(v) = \sum\limits_{T \in P(N): T \subseteq S} (-1)^{s-t} v(T)$ where $t = |T|$ and $s = |S|$).

A vector $\bar{y} = (\bar{y}_i)_{i \in N}$ with components $\bar{y}_i \in \mathbb{R}$ is an *aspiration* (Bennett, 1983) or *acceptable vector* (Sharkey, 1982) of a game $v \in G^N$ when $\bar{y}(S) \geq v(S)$ for all $S \in P(N)$. We denote the set of all aspirations of the game v by $A(v)$.

Sharkey (1982) introduces the concept of *large core* for cooperative games. A game $v \in G^N$ has *large core*, when for every $\bar{y} \in A(v)$ there is an element $\bar{x} \in C(v)$ such that $\bar{x} \leq \bar{y}$ (i.e. $\bar{x}_i \leq \bar{y}_i$ for all $i \in N$). Moreover, he proves that all convex games have large core. Kikuta (1988) and Moulin (1990) show that convex games are exactly those games with totally large core (i.e. with all the subgames having large core).

A *Population Monotonic Allocation Scheme* (or *PMAS*) of a game $v \in G^N$ (Sprumont, 1990) is a vector $x = \left(x_i^S\right)_{S \in P(N),\ i \in S}$, with components $x_i^S \in \mathbb{R}$, that satisfies the following conditions:

$$\sum_{i\in S} x_i^S = v(S) \text{ for all } S \in P(N) \ . \tag{1}$$

$$x_i^R \leq x_i^S \text{ for all } R, S \in P(N),\ R \subseteq S, \text{ and all } i \in R. \tag{2}$$

The first condition tells us that, for each coalition $S \in P(N)$, the payoff vector $x^S := \left(x_i^S\right)_{i\in S} \in \mathbb{R}^S$ is a distribution of the amount $v(S)$ among the players of S. The second condition guarantees that each one of the players of S does not receive more in any subcoalition R in which he takes part. Thus if x is a PMAS of a game v, then x is an element of the vector space $\prod\limits_{S\in P(N)} \mathbb{R}^S$, which has dimension $2^{n-1}\cdot n$. Moreover, from conditions (1) and (2), it follows that each payoff vector x^S belongs to $C(v_S)$ for all PMAS x. Therefore, if a game v has a PMAS, then it is totally balanced. Sprumont (1990) proves that all convex games have a PMAS (for example, its extended Shapley value). Norde and Reijnierse (2002) give a finite number of conditions to determine whether a game has a PMAS or not. In the case of a four-player game its characteristic function must satisfy sixty inequalities.

Moulin (1990) defines the *monotonic core* of the game $v \in G^N$ as the set of all its PMAS. We denote this set by

$$MC(v) := \left\{ x \in \prod_{S\in P(N)} \mathbb{R}^S \,\middle|\, x \text{ is a PMAS of } v \right\}.$$

Thus, the monotonic core $MC(v)$ is a compact and convex (possibly empty) subset of $\prod\limits_{S\in P(N)} \mathbb{R}^S$.

A *monotonic aspiration* of a game $v \in G^N$ (Moulin, 1990) is a vector $y = \left(y_i^S\right)_{S\in P(N),\ i\in S}$, with components $y_i^S \in \mathbb{R}$, that satisfies the following conditions:

$$\sum_{i\in S} y_i^S \geq v(S) \text{ for all } S \in P(N).$$

$$y_i^R \leq y_i^S \text{ for all } R, S \in P(N),\ R \subseteq S, \text{ and all } i \in R.$$

Then, a monotonic aspiration of a game selects an aspiration for every subgame in such a way that the individual aspiration of a player cannot decrease as the coalition to which he belongs enlarges. The set of *monotonic aspirations* of the game v is denoted by

$$MA(v) := \left\{ y \in \prod_{S\in P(N)} \mathbb{R}^S \,\middle|\, y \text{ is a monotonic aspiration of } v \right\}.$$

Notice that $MA(v)$ is a nonempty closed and convex subset of $\prod_{S\in P(N)} \mathbb{R}^S$, but it is non bounded.

Moulin (1990) introduces the concept of *large monotonic core* for cooperative games. A game (N,v) has *large monotonic core* when for every $y \in MA(v)$ there is an $x \in MC(v)$ such that $x \leq y$ (i.e. $x_i^S \leq y_i^S$ for all $S \in P(N)$, $i \in S$). Thus a game with large monotonic core has a PMAS. In the case of three-player games Moulin characterizes the above concept as follows:

Lemma 2.1 *(Moulin, 1990) A three-player game v has large monotonic core if and only if it is superadditive and it satisfies:*

$$v(12) + v(13) + v(23) \leq v(123) + v(1) + v(2) + v(3) . \square$$

Notice that the above characterization can be rewritten in terms of the unanimity coordinates as

$$\lambda_S \geq 0 \text{ for all } S \subseteq N \text{ with } |S| \geq 2,$$

which implies the convexity of the game. Moulin (1990) says that "whether convexity is still a necessary condition for games with an arbitrary number of players is not clear". However, a convex game does not necessarily have large monotonic core (see, for instance, the three-player convex game $v := u_{12} + u_{13} + u_{23} - u_{123}$).

The following proposition gives an affirmative answer to the above posed question.

Proposition 2.2 *Let $v \in G^N$ be an arbitrary game. If v has large monotonic core, then v is a convex game.*

Proof: We show that the game v has totally large core and, therefore, it is convex by Kikuta (1988) or Moulin (1990). Let $S \in P(N)$ and $y' \in A(v_S)$ fixed. To show that the game v_S has large core, we prove that there is an $x' \in C(v_S)$ such that $x' \leq y'$.

Let $\bar{y}$ be an aspiration of the game v such that $\bar{y} \mid_S = y'$. Then, the vector $y := (\bar{y} \mid_T)_{T\in P(N)} \in MA(v)$. By hypothesis, there is an $x \in MC(v)$ such that $x \leq y$. In particular, the vector $x' := x^S \in C(v_S)$ and $x' \leq y^S = \bar{y} \mid_S = y'$. $\square$

3 The large monotonic core

In this section, we present a characterization of games with large monotonic core, in terms of its characteristic function. This characterization can be put into practice to decide whether a given game has large monotonic core or not.

We start with two remarks. First, a game $v \in G^N$ has large monotonic core if and only if the game v_o has large monotonic core, where v_o is the 0-normalized game associated to v (i.e. $v_o(S) := v(S) - \sum_{i \in S} v(i)$ for $S \in P(N)$).

Second, if a game $v \in G^N$ has large monotonic core, then (S, v_S) has large monotonic core for all $S \in P(N)$.

After that, we consider the set of monotonic aspirations that are efficient for all coalitions of size below a fixed upper bound t.

Definition 3.1 *Let it be $v \in G^N$ and $t = 0, 1, \ldots, n$. The set*

$$MA_t(v) := \left\{ y \in MA(v) \mid y^S(S) = v(S) \text{ for all } S \in P(N) \text{ with } s \le t \right\}.$$

is the set of monotonic aspirations at level t.

Notice that each $MA_t(v)$ is a closed and convex (possibly empty) subset of $\prod_{S \in P(N)} \mathbb{R}^S$, but, for $t < n$, it is not upper bounded. According to the preceding definition, we have the following decreasing chain of sets

$$MA(v) = MA_0(v) \supseteq MA_1(v) \supseteq \cdots \supseteq MA_{n-1}(v) \supseteq MA_n(v) = MC(v).$$

Moreover, notice that for every $y \in MA(v)$ there is a $z \in MA_1(v)$ with $z \le y$. Indeed, it is sufficient to take $z_i^i := v(i)$ for all $i = 1, \ldots, n$ and $z^R := y^R$ for all $R \in P(N)$ with $|R| \ge 2$. So $MA_1(v) \ne \varnothing$.

Fixed an integer $t \ge 2$, the following proposition shows an effective criterion to decide when a give monotonic aspiration of level $t-1$ has a monotonic aspiration of level t below it.

Proposition 3.2 *Let it be $v \in G^N$, $t = 2, 3, \cdots, n$ and $y \in MA_{t-1}(v)$. Then the following statements are equivalent:*

(a) There exist $z \in MA_t(v)$ such that $z \le y$.

(b) For all $T \in P(N)$ with $|T| = t$ we have $\sum_{i \in T} \left(\max_{R \subsetneq T : i \in R} y_i^R \right) \le v(T)$.

Proof: $(a) \Rightarrow (b)$ We suppose the existence of $z \in MA_t(v)$ such that $z \le y$. Let $T \in P(N)$ be a coalition with size $|T| = t$. Then we have

$$z^R = y^R \text{ for all } \varnothing \ne R \subsetneq T,$$
$$\sum_{i \in T} \left(\max_{R \subsetneq T : i \in R} y_i^R \right) = \sum_{i \in T} \left(\max_{R \subsetneq T : i \in R} z_i^R \right) \le \sum_{i \in T} z_i^T = v(T),$$

which proves (b).

$(b) \Rightarrow (a)$ Let $T \in P(N)$ be a coalition with $|T| = t$. Then, by hypothesis (b), we have

$$\sum_{i \in T} y_i^T \geq v(T), \quad \sum_{i \in T} \left(\max_{R \subsetneq T : i \in R} y_i^R \right) \leq v(T), \quad \max_{R \subsetneq T : i \in R} y_i^R \leq y_i^T \text{ for all } i \in T.$$

Thus, by Lemma A.1 in the Appendix, there exists $x^T \in \mathbb{R}^t$ such that

$$x^T(T) = v(T) \text{ and } \max_{R \subsetneq T : i \in R} y_i^R \leq x_i^T \leq y_i^T \text{ for all } i \in T.$$

Now, we define z as follows:

$$\begin{aligned} z^T &:= x^T \text{ for } T \in P(N) \text{ with } |T| = t, \\ z^R &:= y^R \text{ for } R \in P(N) \text{ with } |R| \neq t. \end{aligned}$$

It is not hard to check that $z \in MA_t(v)$ and $z \leq y$, which proves statement (a). $\square$

Notice that statement (b), in the above proposition, can be checked in a finite number of steps for a given $y \in MA_{t-1}(v)$. Notice also that the sum on statement (b) can be rewritten as

$$\begin{aligned} \sum_{i \in T} \left(\max_{R \subsetneq T : i \in R} y_i^R \right) &= \sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) \\ &= \max_{\{j_i\}_{i \in T} \in \prod_{i \in T} T \setminus \{i\}} \left(\sum_{i \in T} y_i^{T \setminus \{j_i\}} \right). \end{aligned} \tag{3}$$

Corollary 3.3 *Let it be $v \in G^N$ and $y \in MA(v)$ satisfying*

$$\sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) \leq v(T) \textit{ for all } T \in P(N) \textit{ with } |T| \geq 2.$$

Then there exists $x \in MC(v)$ such that $x \leq y$.

Proof: Note first that if $z \in MA(v)$ with $z \leq y$, then z also satisfies

$$\sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} z_i^{T \setminus \{j\}} \right) \leq v(T) \text{ for all } T \in P(N) \text{ with } |T| \geq 2.$$

Applying Proposition 3.2 repeatedly, we obtain the result. $\square$

The following theorem gives a necessary and sufficient condition for a game to have a large monotonic core.

Theorem 3.4 *Let it be $v \in G^N$. The game v has a large monotonic core if and only if the inequality*

$$\sum_{i\in T}\left(\max_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right) \leq v(T)$$

holds for all $T \in P(N)$ with $t := |T| \geq 2$ and all $y \in MA_{t-1}(v)$.

PROOF: First, let us suppose that v has large monotonic core. Take then $T \in P(N)$ with $t := |T| \geq 2$ and $y \in MA_{t-1}(v)$. By hypothesis, there exists $x \in MC(v) \subseteq MA_t(v)$ such that $x \leq y$. Then, by Proposition 3.2, we obtain the desired inequality.

Conversely, let us suppose that v satisfies the above inequality for all $T \in P(N)$ with $t := |T| \geq 2$ and all $y \in MA_{t-1}(v)$. Let y be an arbitrary monotonic aspiration of v. We must show that there is $x \in MC(v)$ such that $x \leq y$. By Proposition 3.2, we can recursively associate a sequence $\{y^t\}_{t=0,1,\cdots,n}$ such that

$$y^0 := y,\ y^t \in MA_t(v) \text{ and } y^t \leq\ y^{t-1} \text{ for } t = 1,\cdots,n.$$

Therefore, taking $x :=\ y^n \in MA_n(v) = MC(v)$, we obtain

$$x =\ y^n \leq\ y^{n-1} \leq \cdots \leq\ y^1 \leq\ y^0 = y. \ \square$$

As a consequence of the above theorem, we can identify a class of games with large monotonic core.

Corollary 3.5 *Let it be $v \in G^N$ satisfying*

$$\sum_{i\in T} v(T\setminus\{i\}) \leq v(T) + (t-2)\sum_{i\in T} v(i)$$

for all $T \in P(N)$ with size $t \geq 2$. Then v has large monotonic core.

PROOF: Let $T \in P(N)$ be a coalition with $t := |T| \geq 2$ and $y \in MA_{t-1}(v)$. Then we have

$$\begin{aligned}
\sum_{i\in T}\left(\max_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right) &\leq \sum_{i\in T}\left(\sum_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}} - (t-2)\,v(i)\right) \\
&= \sum_{i\in T}\left(\sum_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right) - \sum_{i\in T}(t-2)\,v(i) \\
&= \sum_{j\in T}\left(\sum_{i\in T\setminus\{j\}} y_i^{T\setminus\{j\}}\right) - (t-2)\sum_{i\in T} v(i) \\
&= \sum_{j\in T} v(T\setminus\{j\}) - (t-2)\sum_{i\in T} v(i) \\
&\leq v(T),
\end{aligned}$$

where the first inequality follows from Lemma A.2 in the Appendix, only noting that

$$v(i) = y_i^i \leq y_i^{T\setminus\{j\}} \text{ for all } i \in T \text{ and } j \in T\setminus\{i\},$$

and the second one follows by assumption. Thus, according to Theorem 3.4, we obtain that v has large monotonic core. □

After that, we are going to give a reformulation of Theorem 3.4 to obtain a criterion to determine whether a game has large monotonic core. This criterion will allow obtain in Section 4 a necessary and sufficient condition for the case of four-player games, that can be applied in practice.

We take a game $v \in G^N$ and a coalition $T \in P(N)$ with size $t := |T| \geq 2$. Recall that in the proof of Corollary 3.5 we have seen that

$$\sum_{i\in T} v(i) \leq \sum_{i\in T}\left(\max_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right) \leq \sum_{i\in T} v(T\setminus\{i\}) - (t-2)\sum_{i\in T} v(i), \tag{4}$$

for all $y \in MA_{t-1}(v)$. Thus, the set defined by

$$\Sigma_v^T := \left\{\sum_{i\in T}\left(\max_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right) \mid y \in MA_{t-1}(v)\right\} \subseteq \mathbb{R},$$

is a compact set. Σ_v^T is defined to be empty if $MA_{t-1}(v) = \varnothing$.

Therefore, when $MA_{t-1}(v) \neq \varnothing$ (this occurs, for example, if v has a PMAS) we define the constant

$$K_v^T := \sup \Sigma_v^T = \max_{y\in MA_{t-1}(v)}\left\{\sum_{i\in T}\left(\max_{j\in T\setminus\{i\}} y_i^{T\setminus\{j\}}\right)\right\} \in \mathbb{R}, \tag{5}$$

and, by (4), we have

$$\sum_{i\in T} v(i) \leq K_v^T \leq \sum_{i\in T} v(T\setminus\{i\}) - (t-2)\sum_{i\in T} v(i).$$

By Proposition 3.2, part $(b) \Rightarrow (a)$, for a given integer $t = 2,\ldots,n-1$, if $MA_{t-1}(v) \neq \varnothing$ and $K_v^T \leq v(T)$ for all $T \in P(N)$ with $|T| = t$, then $MA_t(v) \neq \varnothing$ and, therefore, $K_v^{T'}$ is defined for all $T' \in P(N)$ with size $t+1$.

Now, we may reformulate Theorem 3.4 as follows:

Theorem 3.6 *A game $v \in G^N$ has large monotonic core if and only if $K_v^T \leq v(T)$ for all $T \in P(N)$ with $|T| \geq 2$.*□

As a consequence, we are really interested in computing these constants K_v^T for a game v and a coalition T. In fact, we need to find the maximum of a convex function, $\sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right)$, over the following compact set

$$MA_{t-1}^0(v) := \left\{ \left(y^S\right)_{S \in P(N), |S| \leq t-1} \mid y \in MA_{t-1}(v) \right\}.$$

This is a problem of convex programming that can be solved by an appropriate optimization software. Moreover, this problem can be reduced to solving a finite number of linear programming problems since by (5) and (3) we obtain

$$\begin{aligned} K_v^T &= \max_{y \in MA_{t-1}^0(v)} \left\{ \max_{\{j_i\}_{i \in T} \in \prod_{i \in T} T \setminus \{i\}} \left(\sum_{i \in T} y_i^{T \setminus \{j_i\}} \right) \right\} \\ &= \max_{\{j_i\}_{i \in T} \in \prod_{i \in T} T \setminus \{i\}} \left\{ \max_{y \in MA_{t-1}^0(v)} \left(\sum_{i \in T} y_i^{T \setminus \{j_i\}} \right) \right\}. \end{aligned}$$

Therefore, given an arbitrary game, in order to decide whether it has large monotonic core or not firstly, for each $T \in P(N)$ with size $t = 2$ we compute the constant K_v^T and check if the inequality $K_v^T \leq v(T)$ holds. If this is the case, we compute K_v^T for coalitions of size $t = 3$ and check if the same inequalities holds. We go on in this way increasing at each step the size of the coalitions.

4 Four-player games with large monotonic core

For a game $v \in G^N$, by means of computing the constants K_v^T for all coalitions $T \in P(N)$, we have an effective criterion to decide whether the game v has large monotonic core or not. This is what we do for $n = 4$.

Proposition 4.1 *Let $v \in G^N$ and let $T \in P(N)$.*

(a) If $t = 2$ and $T = \{i, j\}$, then

$$K_v^T = v(i) + v(j).$$

(b) If $t = 3$, $T = \{i, j, k\}$ and

$$K_v^S \leq v(S) \;\; \text{for all } S \in P(N) \;\; \text{of size } s = 2,$$

then

$$K_v^T = v(ij) + v(ik) + v(jk) - (v(i) + v(j) + v(k)).$$

(c) If $t=4$, $T=\{i,j,k,l\}$ and

$$K_v^S \leq v(S) \text{ for all } S \in P(N) \text{ of size } s=2 \text{ or } 3,$$

then

$$K_v^T = v(ijk)+v(ijl)+v(ikl)+v(jkl)-\delta_T(v),$$

where $\delta_T(v)$ is

$$\max\left\{\begin{array}{ll} v(jk)+v(jl)+v(kl)+2v(i), & v(ik)+v(il)+v(jk)+v(jl), \\ v(ik)+v(il)+v(kl)+2v(j), & v(ij)+v(il)+v(jk)+v(kl), \\ v(ij)+v(il)+v(jl)+2v(k), & v(ij)+v(ik)+v(jl)+v(kl). \\ v(ij)+v(ik)+v(jk)+2v(l), & \end{array}\right\}$$

The proof of this proposition is in the Appendix.

As a straightforward result of Theorem 3.6 and Proposition 4.1, we can state the following characterization of four-player game with large monotonic core in terms of its characteristic function.

Proposition 4.2 *A four-player game v has large monotonic core if and only if the following three conditions are satisfied:*

(i) $v(i)+v(j) \leq v(ij)$ for all $i,j \in N$ with $i<j$.

(ii) $v(ij)+v(ik)+v(jk) \leq v(ijk)+v(i)+v(j)+v(k)$ for all $i,j,k \in N$ with $i<j<k$.

(iii) $v(123)+v(124)+v(134)+v(234) \leq v(1234)+\delta_{\{1,2,3,4\}}(v)$, where $\delta_{\{1,2,3,4\}}(v)$ is defined in part (c) of Proposition 4.1.

To finish, we show an example where Proposition 4.1 is applied to games with $n \geq 4$ players. It is a game with nonnegative unanimity coordinates, so convex, which does not have large monotonic core.

Example 4.3 *Let it be $\mu_2,\mu_3,\mu_4>0$ and μ such that $0 \leq \mu < \min\{\mu_2,\mu_3,\mu_4\}$. Then the n-player game defined by*

$$v := \mu_2 u_{12} + \mu_3 u_{13} + \mu_4 u_{14} + \mu u_{1234} \in G^N$$

does not have large monotonic core.

Indeed, by (c) of Proposition 4.1 applied to $T=\{1,2,3,4\}$, we have

$$\begin{aligned} K_v^T &= \mu_2+\mu_3+\mu_2+\mu_4+\mu_3+\mu_4+0-\max\{0,\mu_2+\mu_3,\mu_2+\mu_4,\mu_3+\mu_4\} \\ &= \mu_2+\mu_3+\mu_4+\min\{\mu_2,\mu_3,\mu_4\} \\ &> \mu_2+\mu_3+\mu_4+\mu = v(T); \end{aligned}$$

which, by Theorem 3.6, shows that v does not have large monotonic core.

A Appendix

Lemma A.1 *Let $\nu \in \mathbb{R}$, $y_1, \ldots, y_t \in \mathbb{R}$ such that $y_1 + \ldots + y_t \geq \nu$ and let $a_1, \ldots, a_t \in \mathbb{R}$ such that $a_1 + \ldots + a_t \leq \nu$ and $a_i \leq y_i$ for $i = 1, \ldots, t$. Then there exist $x_1, \ldots, x_t \in \mathbb{R}$ such that $x_1 + \ldots + x_t = \nu$ and $a_i \leq x_i \leq y_i$ for $i = 1, \ldots, t$.* □

Lemma A.2 *Let $\nu \in \mathbb{R}$ and $y_1, \ldots, y_r \in \mathbb{R}$.*

(a) If $\nu \leq y_j$ for $j = 1, \ldots, r$, then $\max\{y_1, \ldots, y_r\} \leq y_1 + \ldots + y_r - (r-1)\nu$.

(b) If $\nu \geq y_j$ for $j = 1, \ldots, r$, then $\min\{y_1, \ldots, y_r\} \geq y_1 + \ldots + y_r - (r-1)\nu$. □

Proof of Proposition 4.1: We know that $MA_1(v) \neq \varnothing$.

(a) Assume $t = 2$, so, without lost of generality, we can suppose that $T = \{1, 2\}$. According to Definition 3.1, for any $y \in MA_1(v)$ we have $y_i^i = v(i)$ for all $i \in N$, and

$$\sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) = y_1^1 + y_2^2 = v(1) + v(2);$$

so, by (5), (a) holds.

(b) Assume $t = 3$. By hypothesis, $K_v^{\{i,j\}} \leq v(ij)$ for all $i, j \in N$ with $i < j$ (i.e. $v(i) + v(j) \leq v(ij)$). Then, by a remark before Theorem 3.6, we know that $MA_2(v) \neq \varnothing$; so the constant K_v^T is defined. We can suppose that $T = \{1, 2, 3\}$. Then we must prove:

$$K_v^T = v(12) + v(13) + v(23) - (v(1) + v(2) + v(3)). \tag{6}$$

For any $y \in MA_2(v)$ we have

$$\begin{aligned} \sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) &= \max\left\{y_1^{12}, y_1^{13}\right\} + \max\left\{y_2^{12}, y_2^{23}\right\} + \max\left\{y_3^{13}, y_3^{23}\right\} \\ &\leq \left(y_1^{12} + y_1^{13} - v(1)\right) + \left(y_2^{12} + y_2^{23} - v(2)\right) + \left(y_3^{13} + y_3^{23} - v(3)\right) \\ &= v(12) + v(13) + v(23) - (v(1) + v(2) + v(3)), \end{aligned}$$

where the above inequality is obtained by Lemma A.2 and the last equality by Definition 3.1. This proves the inequality

$$K_v^T \leq v(12) + v(13) + v(23) - (v(1) + v(2) + v(3)).$$

Since, by assumption, $v(ij) - v(j) \geq v(i)$ for all $i, j \in N$, we can consider a monotonic aspiration $y = \left(y^S\right)_{S \in P(N)} \in MA_2(v)$ such that

$$\begin{aligned}
y^{12} &= (v(12) - v(2), v(2)),\\
y^{13} &= (v(1), v(13) - v(1)),\\
y^{23} &= (v(23) - v(3), v(3)),\\
y^{i} &= (v(i)) \text{ for all } i \in N.
\end{aligned}$$

Thus we obtain

$$K_v^T \geq \sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) = v(12) - v(2) + v(13) - v(1) + v(23) - v(3),$$

and equality (6) is proved.

(c) Assume $t = 4$. By hypothesis, $K_v^{\{i,j\}} \leq v(ij)$ for all $i, j \in N$ with $i < j$ (then, $MA_2(v) \neq \varnothing$) and $K_v^{\{i,j,k\}} \leq v(ijk)$ for all $i, j, k \in N$ with $i < j < k$ (as a consequence, $MA_3(v) \neq \varnothing$). Therefore, by (a) and (b), we have that the unanimity coordinates are

$$\lambda_S \geq 0 \text{ for all } S \in P(N) \text{ with } |S| = 2 \text{ or } 3,$$

and so, the subgames v_S are convex for all $S \in P(N)$ with $|S| = 3$.

Without loss of generality, we can suppose that $T = \{1, 2, 3, 4\}$. Then we must prove the following equality:

$$K_v^T = v(123) + v(124) + v(134) + v(234) - \delta_T(v).$$

Let $y \in MA_3(v)$. Then we have

$$\begin{aligned}
\sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) &= \max\{y_1^{123}, y_1^{124}, y_1^{134}\} + \max\{y_2^{123}, y_2^{124}, y_2^{234}\} + \\
&\quad + \max\{y_3^{123}, y_3^{134}, y_3^{234}\} + \max\{y_4^{124}, y_4^{134}, y_4^{234}\}.
\end{aligned}$$

We want to find an optimal upper bound for the above expression. From Lemma A.2, for $i = 1$ we have

$\max\{y_1^{123}, y_1^{124}, y_1^{134}\} \leq$

$$\cdots \leq \max\{y_1^{123} + y_1^{124} - y_1^{12}, y_1^{134}\} \leq \begin{cases} y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{12} + y_1^{13}), \\ y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{12} + y_1^{14}), \end{cases}$$

or

$$\cdots \leq \max\{y_1^{123} + y_1^{134} - y_1^{13}, y_1^{124}\} \leq \begin{cases} y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{12} + y_1^{13}), \\ y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{13} + y_1^{14}), \end{cases}$$

or

$$\cdots \leq \max\{y_1^{123}, y_1^{124} + y_1^{134} - y_1^{14}\} \leq \begin{cases} y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{12} + y_1^{14}), \\ y_1^{123} + y_1^{124} + y_1^{134} - (y_1^{13} + y_1^{14}). \end{cases}$$

Summarizing, for $i = 1$ we have obtained

$$\max\left\{y_1^{123},y_1^{124},y_1^{134}\right\}\leq\begin{cases} y_1^{123}+y_1^{124}+y_1^{134}-\left(y_1^{12}+y_1^{13}\right), \\ y_1^{123}+y_1^{124}+y_1^{134}-\left(y_1^{12}+y_1^{14}\right), \\ y_1^{123}+y_1^{124}+y_1^{134}-\left(y_1^{13}+y_1^{14}\right). \end{cases}$$

In the same way as above, for $i=2$ we have

$$\max\left\{y_2^{123},y_2^{124},y_2^{234}\right\}\leq\begin{cases} y_2^{123}+y_2^{124}+y_2^{234}-\left(y_2^{12}+y_2^{23}\right), \\ y_2^{123}+y_2^{124}+y_2^{234}-\left(y_2^{12}+y_2^{24}\right), \\ y_2^{123}+y_2^{124}+y_2^{234}-\left(y_2^{23}+y_2^{24}\right). \end{cases}$$

For $i=3$ we obtain

$$\max\left\{y_3^{123},y_3^{134},y_3^{234}\right\}\leq\begin{cases} y_3^{123}+y_3^{134}+y_3^{234}-\left(y_3^{13}+y_3^{23}\right), \\ y_3^{123}+y_3^{134}+y_3^{234}-\left(y_3^{13}+y_3^{34}\right), \\ y_3^{123}+y_3^{134}+y_3^{234}-\left(y_3^{23}+y_3^{34}\right). \end{cases}$$

And, for $i=4$ we have

$$\max\left\{y_4^{124},y_4^{134},y_4^{234}\right\}\leq\begin{cases} y_4^{124}+y_4^{134}+y_4^{234}-\left(y_4^{14}+y_4^{24}\right), \\ y_4^{124}+y_4^{134}+y_4^{234}-\left(y_4^{14}+y_4^{34}\right), \\ y_4^{124}+y_4^{134}+y_4^{234}-\left(y_4^{24}+y_4^{34}\right). \end{cases}$$

Now, we add the last terms in all possible ways, picking one for each i, according to Definition 3.1. Comparing the sums obtained we see that

$$K_v^T\leq v(123)+v(124)+v(134)+v(234)-\delta_T(v).$$

On the other hand, to prove the converse inequality, by symmetry, we can reduce to the following two cases:

$$\begin{array}{l} \text{I: } \delta_T(v)=v(23)+v(24)+v(34)+2v(1). \\ \text{II: } \delta_T(v)=v(12)+v(13)+v(24)+v(34). \end{array}$$

Case I: $\delta_T(v)=v(23)+v(24)+v(34)+2v(1)$. Thus, then we have

$$\lambda_{13}+\lambda_{14}\leq\lambda_{34},\quad \lambda_{12}+\lambda_{14}\leq\lambda_{24},\quad \lambda_{12}+\lambda_{13}\leq\lambda_{23}; \tag{7}$$

and, in particular, we obtain

$$\lambda_{13}\leq\lambda_{34},\quad \lambda_{12}\leq\lambda_{24},\quad \lambda_{13}\leq\lambda_{23}\leq\lambda_{23}+\lambda_{34}. \tag{8}$$

Now, let us see that we can always find a monotonic aspiration $y=\left(y^S\right)_{S\in P(N)}\in MA_3(v)$ satisfying

$$\begin{aligned} y^{123}&=\left(v(1),v(23)-v(13)+v(1),v(123)-v(23)+v(13)-2v(1)\right), \\ y^{124}&=\left(v(124)-v(24),e,e'\right), \\ y^{134}&=\left(v(1),v(13)-v(1),v(134)-v(13)\right), \\ y^{234}&=\left(v(234)-v(34),v(13)-v(1),v(34)-v(13)+v(1)\right), \end{aligned}$$

$$
\begin{aligned}
y^{12} &= \left(v\left(1\right), v\left(12\right) - v\left(1\right)\right), \\
y^{13} &= \left(v\left(1\right), v\left(13\right) - v\left(1\right)\right), \\
y^{14} &= \left(v\left(1\right), v\left(14\right) - v\left(1\right)\right), \\
y^{23} &= \left(v\left(23\right) - v\left(13\right) + v\left(1\right), v\left(13\right) - v\left(1\right)\right), \\
y^{24} &= \left(e, e'\right), \\
y^{34} &= \left(v\left(13\right) - v\left(1\right), v\left(34\right) - v\left(13\right) + v\left(1\right)\right), \\
y^{i} &= \left(v\left(i\right)\right) \text{ for all } i \in N,
\end{aligned}
$$

where $e, e' \in \mathbb{R}$ satisfy the following inequalities

$$
\begin{array}{rll}
v\left(12\right) - v\left(1\right) \leq & e & \leq v\left(234\right) - v\left(34\right), \\
v\left(14\right) - v\left(1\right) \leq & e' & \leq v\left(34\right) - v\left(13\right) + v\left(1\right), \\
 & e + e' & = v\left(24\right).
\end{array}
$$

Notice first that, by Lemma A.1, there always exist e, e' satisfying the above inequalities since, applying (7) and (8), we obtain

$$
\begin{aligned}
& v\left(12\right) - v\left(1\right) \underset{(8)}{\leq} v\left(24\right) - v\left(4\right) \leq v\left(234\right) - v\left(34\right), \\
& v\left(14\right) - v\left(1\right) \underset{(7)}{\leq} v\left(34\right) - v\left(13\right) + v\left(1\right), \\
& v\left(12\right) - v\left(1\right) + v\left(14\right) - v\left(1\right) \underset{(7)}{\leq} v\left(24\right) \underset{(8)}{\leq} v\left(234\right) - v\left(34\right) + v\left(34\right) - v\left(13\right) + v\left(1\right).
\end{aligned}
$$

The other monotonic inequalities can be obtained from (7) and (8), since we have

$$
\begin{aligned}
& v\left(23\right) - v\left(13\right) + v\left(1\right) \underset{(7)}{\geq} v\left(12\right) - v\left(1\right), \\
& v\left(123\right) - v\left(23\right) + v\left(13\right) - 2v\left(1\right) \geq v\left(13\right) - v\left(1\right), \\
& v\left(124\right) - v\left(24\right) \geq v\left(1\right), \\
& v\left(134\right) - v\left(13\right) \geq v\left(14\right) - v\left(1\right), \\
& v\left(134\right) - v\left(13\right) \geq v\left(34\right) - v\left(13\right) + v\left(1\right), \\
& v\left(234\right) - v\left(34\right) \geq v\left(23\right) - v\left(13\right) + v\left(1\right) \\
& v\left(23\right) - v\left(13\right) + v\left(1\right) \underset{(8)}{\geq} v\left(2\right), \\
& v\left(34\right) - v\left(13\right) + v\left(1\right) \underset{(8)}{\geq} v\left(4\right),
\end{aligned}
$$

Therefore, the claimed $y \in MA_3\left(v\right)$ does exist. Moreover, we have

$$
\begin{aligned}
K_v^T &\geq \sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) \\
&= y_1^{124} + y_2^{234} + y_3^{123} + y_4^{134} \\
&= v\left(124\right) - v\left(24\right) + v\left(234\right) - v\left(34\right) + \\
&\quad\; v\left(123\right) - v\left(23\right) + v\left(13\right) - 2v\left(1\right) + v\left(134\right) - v\left(13\right) \\
&= v\left(123\right) + v\left(124\right) + v\left(134\right) + v\left(234\right) - \delta_T\left(v\right).
\end{aligned}
$$

Case II: $\delta_T(v) = v(12) + v(13) + v(24) + v(34)$. So, we have

$$\lambda_{23} \leq \lambda_{12} + \lambda_{13},\ \lambda_{14} \leq \lambda_{12} + \lambda_{24},\ \lambda_{14} \leq \lambda_{13} + \lambda_{34},\ \lambda_{23} \leq \lambda_{24} + \lambda_{34}, \tag{9}$$
$$\lambda_{14} + \lambda_{23} \leq \lambda_{12} + \lambda_{34}, \quad \lambda_{14} + \lambda_{23} \leq \lambda_{13} + \lambda_{24}. \tag{10}$$

Now, let us see that we can always find a monotonic aspiration $y = \left(y^S\right)_{S \in P(N)} \in MA_3(v)$ satisfying

$$\begin{aligned}
y^{123} &= (a, a', v(123) - v(12)), \\
y^{124} &= (v(124) - v(24), e, e'), \\
y^{134} &= (b, b', v(134) - v(13)), \\
y^{234} &= (v(234) - v(34), f, f'), \\
y^{12} &= (a, a'), \quad y^{23} = (d, d'), \\
y^{13} &= (b, b'), \quad y^{24} = (e, e'), \\
y^{14} &= (c, c'), \quad y^{34} = (f, f'), \\
y^{i} &= (v(i)) \text{ for all } i \in N.
\end{aligned}$$

where $a, a', b, b', c, c', d, d', e, e', f, f' \in \mathbb{R}$ satisfy the following inequalities

$$\begin{array}{l}
v(1) \leq c \leq b \leq a, \quad v(2) \leq d \leq a' \leq e, \\
v(4) \leq c' \leq e' \leq f', \quad v(3) \leq d' \leq f \leq b', \\
a + a' = v(12), \quad b + b' = v(13), \quad c + c' = v(14), \\
d + d' = v(23), \quad e + e' = v(24), \quad f + f' = v(34).
\end{array} \tag{11}$$

We leave the existence of these parameters for the next lemma. Therefore assuming their existence and recalling that, by assumption, $v(ijk) - v(ij) \geq v(ik) - v(i)$ for all $i, j, k \in N$, we obtain the claimed existence of $y \in MA_3(v)$. Moreover, we have

$$\begin{aligned}
K_v^T &\geq \sum_{i \in T} \left(\max_{j \in T \setminus \{i\}} y_i^{T \setminus \{j\}} \right) \\
&= y_1^{124} + y_2^{234} + y_3^{123} + y_4^{134} \\
&= v(124) - v(24) + v(234) - v(34) + \\
&\quad\ v(123) - v(12) + v(134) - v(13) \\
&= v(123) + v(124) + v(134) + v(234) - \delta_T(v). \ \square
\end{aligned}$$

Lemma A.3 *With the above assumptions and notations, there are real numbers $a, a', b, b', c, c', d, d', e, e', f, f'$ satisfying (11).*

PROOF: It is sufficient to show the existence of two real numbers a, e satisfying the following eight inequalities:

$$\begin{array}{l} v\,(12)-v\,(23)+v\,(3)\leq a\leq v\,(12)-v\,(2)\,, \\ v\,(24)-v\,(14)+v\,(1)\leq e\leq v\,(24)-v\,(4)\,, \\ v\,(12)\leq a+e\leq v\,(12)+v\,(13)+v\,(24)-v\,(14)-v\,(23)\,, \\ v\,(14)-v\,(24)\leq a-e\leq v\,(12)+v\,(34)-v\,(23)-v\,(24)\,. \end{array} \tag{12}$$

Indeed, taking

$$\begin{array}{lll} a'=v\,(12)-a, & b=v\,(14)-v\,(24)+e, & b'=v\,(13)-b, \\ e'=v\,(24)-e, & f=v\,(23)-v\,(12)+a, & f'=v\,(34)-f, \end{array}$$

we obtain

$$\begin{array}{ll} v\,(1)\leq b\leq a, & v\,(2)\leq a'\leq e, \\ v\,(4)\leq e'\leq f', & v\,(3)\leq f\leq b', \\ v\,(14)=b+e', & v\,(23)=a'+f, \end{array}$$

and, taking

$$c=b,\quad c'=e',\quad d=a',\quad d'=f,$$

the lemma would be proved.

Now we prove the existence of numbers a and e satisfying (12). Firstly, notice that, by (10), we have

$$\begin{array}{l} v\,(12)\leq v\,(12)+v\,(13)+v\,(24)-v\,(14)-v\,(23)\,, \\ v\,(14)-v\,(24)\leq v\,(12)+v\,(34)-v\,(23)-v\,(24)\,. \end{array}$$

Secondly, the set of the points (a,e) in the Euclidean plane satisfying the first four inequalities (resp. the last four inequalities) in (12) is a rectangle D (resp. D') with the sides l_i, $1\leq i\leq 4$, parallel to the axes (resp. with the sides l_i, $5\leq i\leq 8$, parallel to the quadrant's bisectrix) as the Figure 1 shows.

Therefore, we only have to prove that the intersection of both rectangles is nonempty. For this, we denote by $P_{ij}=(a_{ij},e_{ij})$ the intersection point of the line l_i with the line l_j when l_i and l_j are non parallel lines. By (9), we have the following inequalities

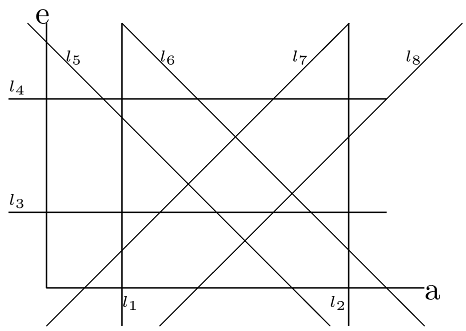

Figure 1: Rectangles D and $D\prime$

$$
\begin{array}{ll}
(1) & a_{57} = \dfrac{v(12) + v(14) - v(24)}{2} \leq v(12) - v(2)\,, \\
(2) & a_{68} = \dfrac{2v(12) + v(13) + v(34) - 2v(23) - v(14)}{2} \geq v(12) - v(23) + v(3)\,, \\
(3) & e_{58} = \dfrac{v(23) + v(24) - v(34)}{2} \leq v(24) - v(4)\,, \\
(4) & e_{67} = \dfrac{v(12) + v(13) + 2v(24) - 2v(14) - v(23)}{2} \geq v(24) - v(14) + v(1)\,, \\
(5) & e_{16} = v(13) + v(24) - v(14) - v(3) \geq v(24) - v(14) + v(1)\,, \\
(6) & e_{25} = v(2) \leq v(24) - v(4)\,, \\
(7) & a_{37} = v(1) \leq v(12) - v(2)\,, \\
(8) & a_{48} = v(12) + v(34) - v(23) - v(4) \geq v(12) - v(23) + v(3)\,.
\end{array}
$$

The inequalities (1) and (2) (resp. (3) and (4)) show that D' has at least a point between the lines l_1 and l_2 (resp. l_3 and l_4), and the inequalities (5) and (6) (resp. (7) and (8)) show that D has at least a point between the lines l_5 and l_6 (resp. l_7 and l_8). From this follows that $D \cap D' \neq \varnothing$ and, therefore, the lemma is proved. $\square$

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