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Auction theory, sequential local service privatization, and the effects of geographical scale economies on effective competition.

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Abstract:

A sequential weakly efficient two-auction game with entry costs, interdependence between objects, two potential bidders and IPV assumption is presented here in order to give some theoretical predictions on the effects of geographical scale economies on local service privatization performance. It is shown that the first object seller takes profit of this interdependence. The interdependence externality rises effective competition for the first object, expressed as the probability of having more than one final bidder. Besides, if there is more than one final bidder in the first auction, seller extracts the entire bidder's expected future surplus differential between having won the first auction and having lost. Consequences for second object seller are less clear, reflecting the contradictory nature of the two main effects of object interdependence. On the one hand, first auction winner becomes "stronger", so that expected payments rise in a competitive environment. On the other hand, first auction loser becomes relatively "weaker", hence (probably) reducing effective competition for the second object. Additionally, some contributions to static auction theory with entry cost and asymmetric bidders are presented in the appendix.

Resumen:

Un juego secuencial formado por dos subastas débilmente eficientes, donde las valoraciones de los objetos son interdependientes, con dos posibles compradores bajo el supuesto de Valoración Privada Independiente (IPV), se presenta en este trabajo para dar algunas predicciones teóricas sobre los efectos de las economías de escala de tipo geográfico sobre el nivel de éxito en las privatizaciones de servicios de provisión pública local. Se demuestra que el vendedor del primer objeto se beneficia netamente de la interdependencia entre objetos. Esta interdependencia alienta la competencia efectiva por el primer objeto, expresada en términos de probabilidad de tener más de un comprador ex post. Además, si hay más de un comprador en la primera subasta, el vendedor consigue extraer enteramente el diferencial de excedente neto esperado futuro entre el hecho de haber ganado la primera subasta y el de haberla perdido. Las consecuencias para el vendedor del objeto que se subasta en segundo lugar son menos claras, lo cual refleja la naturaleza contradictoria de los efectos principales de la interdependencia entre objetos. Por un lado, el ganador de la primera subasta se hace "fuerte", al valorar más el segundo objeto, y por lo tanto los pagos esperados aumentan si la competencia es suficiente. Por otro lado, el perdedor de la primera subasta se "debilita" con respecto al ganador, con lo cual probablemente se reduce el nivel de competencia por el segundo objeto, en términos de probabilidad de participación de ambos compradores. En el apéndice a este trabajo, adicionalmente, se presentan avances teóricos relacionados relativos a la teoría de subastas con costes de entrada y compradores asimétricos.

1. Introduction

Since classical papers like Chadwick (1859) and Demsetz (1968), it is said that natural monopolies such as public utilities need not be owned or regulated by public organizations anymore. Demsetz’s idea was that, instead of regulating utility performance, proceeding to periodically auctioning off the right to supply the service would yield good efficiency results while saving information costs necessarily linked to service regulation.

The debate on privatization of local services has evolved from the earlier 70’s through nowadays. During the 80’s, there was a wide consensus on the goodness of service privatization, understood as periodically auctioning it off to private firms. Cost savings were reflected into better service conditions for citizens or better fiscal balances for municipalities.

In the nineties, some economists as Lopez-de-Silanes, Schleifer and Vishny (1996) tried to find out why privatization was not rapidly spreading over all municipalities and services despite its apparently clear advantages. Meanwhile, other economists started reconsidering the possibility that privatizing a public service could not be that beneficial for municipalities (Sclar, 2000). Competition issues were thought to be affecting privatization results. There is a trend towards progressive market concentration in the procurement and contracting-out sectors. Consequences of this process are definitely not good for municipalities and citizens. Here are two tables showing some data collected and summarized about a sample of Spanish municipalities’ privatized local services.

Table 1: Market concentration in the privatized refuse collection and treatment service in Spanish municipalities, 2000.

Firm's contracts	Name	Total number of contracts	Contracts market share (%)	Total anual treated waste (Tn)	Treated waste per contract (Tn)	Treated waste market share (%)
46	FCC	46	29.68	1,069,516.37	23,250.36	46.86
18	CESPA	18	11.61	371,923.36	20,662.41	16.30
8	Vicens Orts	8	5.16	52,025.43	6,503.18	2.28
7	BF-Iacsa	7	4.52	65,779.01	9,397.00	2.88
4	Urbaser	4	2.58	201,879.53	50,469.88	8.85
4	Ferran Vila	4	2.58	13,538.30	3,384.58	0.60
3	3 firms	9	5.81	42,729.86	4,747.76	1.87
2	10 firms	20	12.90	153,160.03	7,658.00	6.71
1	39 firms	39	25.16	311,766.46	7,994.01	13.66
	Total sample	155	100.00	2,282,318.34	14,724.63	100.00

Source: Bel and Miralles (2004).

Table 2: Market concentration in the privatized water supply service in Spanish municipalities, 2000.

Firms	Contracts market share (%)	Population served market share (%)	Domestic consump. Market share (%)	Industrial consump. Market share (%)	Households served market share (%)	Industrial consumers market share (%)
AGBAR	75.00	95.49	94.84	93.57	95.28	96.86
Prodaisa	6.25	0.13	0.18	0.10	0.16	0.07
Seragua-FCC	3.13	1.02	1.32	2.04	1.22	1.32
CASSA	3.13	0.79	0.72	0.92	0.70	0.48
ABSA	3.13	0.58	0.58	0.34	0.55	0.62
SOGESUR	1.56	0.75	0.88	0.11	0.74	0.00
Aigües Vilanova	1.56	0.36	0.28	0.27	0.33	0.33
ATCA	1.56	0.06	0.12	0.46	0.04	0.02
Aigües de Vilassar	1.56	0.52	0.61	0.12	0.63	0.13
Aigües de Catalunya	1.56	0.05	0.06	0.02	0.04	0.02
AICSA	1.56	0.24	0.42	2.06	0.30	0.14
Total sample	64 (contracts)	3,241,563 (citizens)	166,132,498 (m^3)	82,675,301 (m^3)	1,347,434 (users)	167,517 (users)
HH index	0.5708	0.9122	0.8999	0.8766	0.9083	0.9384

Source: Bel and Miralles (2004).

Market concentration in these sectors becomes higher as time goes by. Municipalities that had not yet privatized some public service could then be afraid that this loss of effective competition would harm potential gains arising from privatization. On the other hand, once a firm is effectively established in some little region, it is true that municipalities there that had not yet privatized could take profit of fixed costs savings through contracting this firm. But these fixed cost savings represent a clear advantage for the firm already established against firms that could compete for entering the region, thus explaining the trend towards a lesser degree of competition.

Auction theory is one of the Economics fields that could address the issue of the trade-off between geographic economies of scale and successive competition for contracts among firms. In this paper, I construct a simple auction game that tries to get some light to the issue, being applicable to cases like the following one.

Imagine that there is a little region with two municipalities that have some public service owned and managed by their own. There are two ex ante identical firms that would like to manage the service, because both are more efficient than the public providers are. Municipality 1 decides to privatize the service, putting it into contest. The two firms are potential contestants, but they have to choose carefully whether to enter the contest process or not, because doing so is costly. Firm 1 wins the contest and therefore the contract. Years later, municipality 2 privatizes the service, hence auctioning it off. Firm 1 could provide the service cheaply, since it is already installed in the region. These are good news for municipality 2. The bad news are that firm 2 may decide not to enter the auctioning process given its cost disadvantage, so that firm 1 could both win the contract and enjoy almost the whole efficiency gain generated by the privatization. What will the aggregated effect be for municipality 2? Will it be better off if municipality 1 had not previously privatized the service? Does municipality 1 obtain good contract conditions thanks to the fact that there is a second municipality in the region that has not privatized the service?

Some answers derived in this paper and applied to the questions above are summarized as follows. First, municipality 1 takes great profit of the fact that municipality 2 is going to privatize later. In fact, if both firm 1 and firm 2 participate in the first contest, municipality 1 fully extracts the expected future surplus differential between being already installed in the region and having to enter the region lately. Second, the aggregated effect for municipality 2 is not so clear, hence reflecting the contradictory nature of the consequences of municipality 1's privatization process. In some cases it could have been better for municipality 2 if municipality 1 had not previously privatized the service, while in other cases the opposite happens.

The model is an independent private-values second-price sealed-bid sequential auction game with separate entry costs. More details are further explained. Other possible setups and assumptions could have been applied here, as common-values, first-price, … The model chosen here has the advantage of being mathematically handy. The private-values assumption manages to give importance to the firm's cost advantage fact, which is the central issue in this paper. The second-price sealed-bid auction is guaranteed to be efficient among the potential bidders that finally take part in the auction[1], while this efficiency is not guaranteed in a first-price setup when bidders are not ex ante identical. Entry is assumed to be costly, as it is usual in this kind of contracting processes.

The paper is organized as follows. Next section presents a short overview of auction theory with entry costs and of sequential auctions. Third section presents the model and derives several results. Section four analyses some implications of these results. Final section concludes. An appendix is included in order to make some points clearer.

2. Auction theory with entry costs and sequential auctions

Typically, the number of bidders that take part in an auction has been taken as exogenously fixed by a major part of the literature on auction theory. General results given this assumption are found and summarized in Milgrom (1989) and McAfee and McMillan (1987a).

Nevertheless, there is a recent but growing literature that analyses bidders' entry cost and its consequences on the entry decision (McAfee and McMillan, 1987b; Engelbrecht-Wiggans, 1993; Levin and Smith, 1994; Menezes and Monteiro, 1994; Stegeman, 1996; Campbell, 1998; Menezes and Monteiro, 2000; Lixin, 2002; Kaplan and Sela, 2002; Gal, Landsberger and Nemirovski, 2002; Tan and Yilankaya, 2003; Pevnitskaya, 2003; Landsberger and Tsirelon, 2003). Key contributions are Levin and

[1] This is the concept of weak efficiency, as explained in Armstrong (2000). Any other auction mechanism that is weakly efficient in this sense could have been used here instead of the second-price sealed-bid one, due to the Revenue Equivalence Theorem.

Smith (1996) and Menezes and Monteiro (2000), since they develop the basis of auctions with entry that is currently in use.

Levin and Smith (1994) paper constitutes in part a critique against Engelbrecht-Wiggans setup. In the latter, participation decision is a pure strategy. This leads in many cases to asymmetric equilibria even if potential participants are ex ante identical, which sounds unintuitive. Levin and Smith (1994) conceive participation decision as a mixed strategy. In their setup, one single indivisible object is to be sold by an auctioneer to one out of N potential bidders. Every potential buyer i has the same distribution function F with support on $[0,v^*]$, which reflects other bidder's beliefs about i^{th}-bidder's possible valuation (or signal). Each final participant has to pay a constant positive cost c, which is common knowledge. Participation decision is taken before having knowledge of the own valuation (or signal). The auction mechanism, the number of potential entrants, and the number of final bidders become common knowledge as well. There is no reservation price. In this setup, they find interesting results, as an extension of the revenue equivalence to endogenous entry cases. They also find that market thickness (a high number of potential participants) may not be profitable for the seller.

Menezes and Monteiro (2000) present a very similar model, but in this case participation decision is to be taken after having had knowledge of the own valuation (or signal). In this case, it is shown that bidders play so-called *cut-off strategies* or *threshold strategies* in the equilibrium. Each one bids obviously only if he participates, and he participates if his known valuation is higher than some own threshold value. Menezes and Monteiro find similar results to the ones of Levin and Smith, that is, revenue equivalencies and the fact that market thickness may not be good for the seller.

In this paper, I use the Menezes and Monteiro's setup, because it seems more appealing for the cases I analyze. I focus on cost features that are firm specific, so that they are private and previously known by the firm before engaging into the contest process. It is precisely contest participation what entails participation costs. Hence, valuations are typically known before taking the participation decision.

Stegeman (1996) and Lixin (2002) study optimal auctions under participation costs. Campbell (1998) insights, among other things, into the possibility of having asymmetric equilibria even with symmetric players, in the Menezes-Monteiro setup. This possibility is deeply analyzed in Tan and Yilankaya (2003), who also analyze asymmetric bidders in the Menezes-Monteiro setup with second-price sealed-bid auctions. Kaplan and Sela

(2002) introduce an interesting variation in which valuations are common knowledge but entry costs are private uncorrelated information. Pevnitskaya (2003) focuses on first-price sealed-bid auctions with entry costs and risk-aversion variability among potential bidders. Gal, Landsberger and Nemirovski (2002) investigate to which extent partial participation cost rebating could be beneficial for the seller. Landsberger and Tsileron (2003), finally, contribute to the study of auctions with entry when valuations (signals) are correlated among potential bidders. They arrive to the conclusion that threshold equilibrium concept should be revisited when signals are correlated. In this paper, I get rid of the correlation assumption, hence avoiding tedious and difficult calculation. As noted above, the central issue of this paper is the effect of firm's costs advantages on competition and contract conditions. Common features of each contract that is to be auctioned off are assumed to be non-random common knowledge.

Another point of the model I present is that auctions with entry costs are sequentially undertaken. Menezes and Monteiro (1994) have presented a similar model to the one of this paper. The difference with my assumptions is that the former assumes that once a bidder wins the first object, he is not interested in the second one, while my model imposes no limit on the number of objects a bidder could obtain. Bremzen (2003) also studies sequential auctions and associated entry deterrence strategies, in two sequential auctions for identical objects where there is a potential new entrant in the second auction.

Finally, I shall quote the work of Gandal (1997), which is an empirical paper that deals with the issue I try to explain. The paper studies sequential cable television license sales taken at Israel. The winner of the auction for a region had a cost advantage in further auctions to be done in bordering regions. The paper assesses this cost reduction and concludes that in the case under study, the cost-reduction effect has outweighed the loss-of-competition effect. Gandal observes that there is no theory trying to explain this issue. Here it is.

3. The model

Consider the following sequential auction game, which will be called the ORIGINAL GAME in what follows. There are two risk-neutral bidders competing for obtaining either one or both of two objects. Objects are sold sequentially and separately in second-price sealed-bid auctions with no reservation prices. For the first one, ex ante valuations are believed by bidders to follow a distribution F over the support $[0,v^*]$. Valuations are always independent between bidders. Bidders' beliefs in the second stage depend on the outcome of the first stage:

a) If bidder 1 wins the first auction, his valuation will follow a distribution function G over the previous support, while bidder 2 will maintain the same distribution function F. G first order stochastically dominates F.

b) If bidder 2 wins the first auction, the converse will happen.

c) If none of the bidders enter the first auction, then both bidders will have the same distribution function F again.

Taking part in each auction has a known cost c ($0 < c < v^*$) for each participant, so that participation decisions in any, none or both auctions become endogenous. Before each bidder $i \in \{1,2\}$ decides whether to participate in the second auction or not, he gets knowledge of his own real valuation v_i^2. Before each bidder i decides on participation in the first auction, he gets knowledge of his valuation for the first object v_i^1, but not for the second (v_i^2). There is a common unity discount factor between auctions.

The game is readily solved by backward induction. Hence, I shall start by considering stage 2, the auction of the second object, in any of the cases a, b and c.

Case a. Bidder 1 has won the first auction, and we proceed to the second one:

In this kind of auctions with entry, with no consequences on the future, it is well known that bidder's strategy involves two actions: a bidding function (given participation) and a participation function, which depend on bidder's valuation. Given participation, a weakly dominating bidding function consists of revealing the true valuation. This is standard in Vickrey auctions and is not explained here. From this optimal bidding, an expected profit function given participation is derived. This helps us find the participation function. Expected profits are increasing in the own valuation. Valuations below or equal to some threshold point θ_i ($c \le \theta_i \le v^*$) suggest bidder i deter from participating, as expected profits of doing so are non-positive. Above this threshold, participating is the right decision, as expected profits are positive. Hence, this sequence of actions is usually called threshold strategy, which typically depends on the other bidder's threshold strategy (take into account that the other bidder's threshold value positively affects expected profits given participation). Optimal responses are represented by the optimal threshold function $\theta_i^{\#}(\theta_j)$, as the bidding function is already known. Threshold equilibrium or cut-off equilibrium (θ_i^*,θ_j^*) is reached where Nash equilibrium conditions hold, that is, at some point where each threshold strategy is optimal for any bidder.

In case a, bidder 1's expected profits when participating, given some bidder 2's threshold strategy θ_2 and bidder 1's known valuation v_1^a [2], is

$$F(\theta_2)v_1^a + [1-F(\theta_2)]E\left(max\{0, v_1^a - v_2^a\} \middle| v_2^a > \theta_2\right) - c =$$
$$= F(\theta_2)v_1^a + I\{v_1^a > \theta_2\}\int_{\theta_2}^{v_1^a}(v_1^a - x)f(x)dx - c =$$
$$= \int_0^{v_1^a} F(max\{\theta_2, x\})dx - c$$

Here, v_2^a is bidder 2's valuation, unknown by bidder 1, and $I\{\}$ is an index function defined as usual. I have skipped some steps but arriving to these results is not extremely difficult. Last equality is reached by means of integration by parts. Bidder 1's optimal threshold strategy is the valuation that makes expected profits given participation equal zero

$$\int_0^{\theta_1^{\#a}(\theta_2)} F(max\{\theta_2, x\})dx = c$$

whenever this optimal strategy is lower than v^*. The optimal threshold would be equal to v^* if and only if[3]

$$\int_0^{v^*} F(max\{\theta_2, x\})dx \le c$$

Notice that it always happens that $\theta_i^{\#}(v^*) = c$, as bidder i knows for sure that the other bidder is almost never going to take part in the auction.

Bidder 2's expected profits under similar conditions become

$$\int_0^{v_2^a} G(max\{\theta_1, x\})dx - c$$

and optimal threshold strategy is derived analogously. A cut-off equilibrium for case a is a pair $(\theta_1^a*, \theta_2^a*) \in [c, v*]^2$ such that $\theta_1^{\#a}(\theta_2^a*) = \theta_1^a*; \theta_2^{\#a}(\theta_1^a*) = \theta_2^a*$. There could be

[2] The superscript refers to case a, and both valuation and threshold value refer to the second auction.
[3] Notice that I restrict attention to threshold strategies defined on the compact interval [0,*v**] (in practice, they become restricted to [*c*,*v**]). While this is not necessary, it makes further calculations clearer.

multiple equilibria. Existence has already been proved (see Miralles, 2002). We can also state (see Lemma 1) in this case that there will exist an equilibrium $(\theta_1^a*, \theta_2^a*)$ such that $\theta_1^a* \le \theta_2^a*$. As G dominates F, $G(x) \le F(x) \forall x \in [0, v*]$, and, supposing that both bidders play the same threshold strategy, bidder 2's expected profits given participation are lower or equal than bidder 1's ones. Hence, playing the same strategies may seldom be threshold equilibrium, and a possible "natural" readjustment suggests that the weakest bidder should rise threshold strategy in the equilibrium, while the strongest one should lower it.

Lemma 1:[4]

Consider a two-neutral-bidder, independent private-values, second-price sealed-bid auction with common and known entry cost c ($0 < c < v*$ *), where each bidder knows his own valuation before taking a decision about participating in the auction. Bidder 1's valuation is an observation from a random variable with distribution function G with support on* [0,*v**]. *Bidder 2's valuation is extracted from a distribution function F with identical support. G first order stochastically dominates F, and both are common knowledge.*

Then, this auction game has at least one cut-off equilibrium $(\theta_1*, \theta_2*) \in [c, \theta_F] \times [\theta_G, v*]$*, where* $c < \theta_F \le \theta_G < v*$ *and thetas are defined as the unique solutions for*

$$F(\theta_F)\theta_F = c$$
$$G(\theta_G)\theta_G = c$$

Proof:

By the definition of expected profits with participation given above, it can be seen that $\theta_1^{\#}(\theta_F) = \theta_F$, $\theta_2^{\#}(\theta_G) = \theta_G$. First order domination implies $G(x) \le F(x) \forall x \in [0, v*]$, which consequentially implies $\theta_F \le \theta_G$. These fixed points exist as $\theta_1^{\#}(x)$ and $\theta_2^{\#}(x)$ exist, are continuos and their dominia and codominia are both [0,*v**].

[4] This is a refinement on a special case of Tan and Yilankaya (2003) proposition 4. I also generalize their proposition in the appendix.

For any bidder $i, \theta_i^{\#}(x)$ is strictly decreasing on x whenever $\theta_i^{\#}(x) < v*$. This can be readily understood: increasing the other bidder's threshold value increases own expected profits given participation for any possible valuation, making the own optimal threshold strategy decrease.

As optimal threshold functions are decreasing, continuos and defined over the compact set $[0,v*]^2$, $\theta_1^{\#}(\theta_F) = \theta_F$ and $\theta_2^{\#}(\theta_G) = \theta_G$ imply that $\theta_1^{\#}(x)$ is closed under $[c,\theta_F] \times [\theta_F, v*]$ (recall $\theta_1^{\#}(v*) = c$) and that $\theta_2^{\#}(x)$ is closed under $[c,\theta_G] \times [\theta_G, v*]$. Hence, both functions are closed under the compact intersection set $[c,\theta_F] \times [\theta_G, v*]$. By Brower's fixed-point theorem, this implies that an intersection between both optimal threshold functions, i.e. a cut-off equilibrium, must be in this set. **QED**

An important generalization of this lemma can be found in the appendix. Lemma 1 has proven that, in our case a, there will exist at least one equilibrium $(\theta_1^a *, \theta_2^a *)$ such that $\theta_1^a * \leq \theta_2^a *$. Of course, there could be many such equilibria. There could be, though not necessarily, "not natural" equilibrium $(\theta_1^a ', \theta_2^a ')$ such that $\theta_1^a ' > \theta_2^a '$ (more concretely in the set $\{(\theta_1^a ', \theta_1^a ') \in [\theta_G, v*] \times [c, \theta_F]\})$ [5]. For practical purposes, a more formal definition of what is considered "natural" equilibrium follows below:

Definition 1: A *"natural" cut-off equilibrium* of a second-price sealed-bid entry-cost Menezes-Monteiro auction with two bidders where:

- bidder 1's valuation follows a distribution function *G*() over [0,*v**]
- bidder 2's valuation follow a distribution function *F*() over [0,*v**]
- *G* first order stochastically dominates *F*

is the one and unique that comes as a result of iterated best responses when original threshold coordinates are either (θ_F, θ_F) or (θ_G, θ_G), where

$$F(\theta_F)\theta_F = c$$
$$G(\theta_G)\theta_G = c$$

, regardless the identity of the bidder who first reacts to these original coordinates. We denote this "natural" equilibrium by the pair (θ_L, θ_H).

The logic of this definition is straightforward and appealing. Imagine that there is a "first primitive" auction game where both players have the same distribution *F*. This game has one symmetric threshold equilibrium (θ_F, θ_F), so we start from this point. Suppose that bidder 1 changes its original distribution function by *G*, which dominates the previous one. (θ_F, θ_F) is then no longer an equilibrium (only bidder 1 would be in an optimal position), so departing from it bidder 2 reacts optimally by choosing $\theta_2^{\#}(\theta_F) \geq \theta_F$. Bidder 1 counter-reacts by choosing $\theta_1^{\#}(\theta_2^{\#}(\theta_F)) \leq \theta_F$. Bidder 2 reacts again by means of $\theta_2^{\#}(\theta_1^{\#}(\theta_2^{\#}(\theta_F))) \geq \theta_2^{\#}(\theta_F)$, and so on. The process ends in the limit when the equilibrium (θ_L, θ_H) is reached. Notice that

$$(\theta_L, \theta_H) = (max\{\theta \in [c, \theta_F] : (\theta, \theta_2^{\#}(\theta)) \in E\}, \theta_2^{\#}(max\{\theta \in [c, \theta_F] : (\theta, \theta_2^{\#}(\theta)) \in E\}))$$

, where *E* is the set of cut-off equilibria of this game. Now, imagine instead that we start from a "second primitive" game were both bidders' valuations are extractions from an identical distribution function *G*. (θ_G, θ_G) is the induced symmetric cut-off equilibrium in this case. By analogous reasoning, if we now change bidder 2's valuation distribution function by *F*, we would arrive again to (θ_L, θ_H). As a corollary, observe that the "natural" cut-off equilibrium when both players have identical distribution functions (they weakly dominate each other) is the symmetric equilibrium. I assume from now on that the "natural" equilibrium (θ_L, θ_H) is going to be played in case a.

Expected profits for bidder 1 at the equilibrium becomes

$$I\{v_1^a > \theta_L\}\left(\int_0^{v_1^a} F(max\{\theta_H, x\})dx - c\right)$$

It is useful, concerning stage 1, to calculate bidder 1's expected profits before he knows his own valuation,

$$\pi_1^a = \pi_{win} = \int_0^{v^*} g(x) I\{x > \theta_L\}\left(\int_0^x F(max\{\theta_H, z\})dz - c\right)dx$$

For bidder 2, equilibrium expected profits are

$$I\{v_2^a > \theta_H\}\left(\int_0^{v_2^a} G(max\{\theta_L, x\})dx - c\right)$$

[5] Tan and Yilankaya's (2003) show some conditions for the impossibility of having these cases.

Notice that

$$I\{v > \theta_H\}\left(\int_0^v G(max\{\theta_L, x\})dx - c\right) \le I\{v > \theta_L\}\left(\int_0^v F(max\{\theta_H, x\})dx - c\right) \forall v \in [0, v*]$$

Bidder 2's expected profits before learning his valuation is

$$\pi_2^a = \pi_{los} = \int_0^{v*} f(x) I\{x > \theta_H\}\left(\int_0^x G(max\{\theta_L, z\})dz - c\right)dx$$

It is easy to check that $\pi_{win} \ge \pi_{los}$.

<u>Case b. Bidder 2 has won the first auction, and we proceed to the second one:</u>

This case is just the opposite of case a. By inverse argumentation, it is seen that there is a natural equilibrium (θ_H, θ_L) which is assumed to be played. Hence:

$\pi_1^b = \pi_{los}$ and $\pi_2^b = \pi_{win}$.

<u>Case c. None of bidders participate in the first auction, and we proceed to the second one:</u>

Is this case, it is readily seen that symmetric cut-off equilibrium (θ_F, θ_F) exists, where $F(\theta_F)\theta_F = c$. Notice that $\theta_L \le \theta_F \le \theta_G \le \theta_H$. This symmetric equilibrium is also a "natural" equilibrium, so it is assumed to be played. Expected profits before knowing own valuations are

$$\pi_1^c = \pi_2^c = \pi_{dra} = \int_0^{v*} f(x) I\{x > \theta_F\}\left(\int_0^x F(max\{\theta_F, z\})dz - c\right)dx$$

It can be readily checked that $\pi_{win} \ge \pi_{dra} \ge \pi_{los}$.

<u>Stage 1. The first object is auctioned:</u>

Now that we have seen all the possibilities in the second stage, we turn to the first one. As the result of the latter will have consequences on the former, these consequences must be taken into account by bidders when maximizing their payoff functions.

Players are symmetric in this phase of the game, and therefore payoff functions are identical. Bidder *i*'s expected profit after knowing his own valuation v_i for object 1 and deciding to participate in the first auction, where *j* is the other bidder and $b_i(\ )$, $b_j(\ )$ are the bidding functions, is

$$\begin{aligned}
\Pi_i &= F(\theta_j)(v_i + \pi_{win}) + \\
&+ \left[1 - F(\theta_j)\right] \cdot \{P(b_i(v_i) > b_j(v_j) | v_j > \theta_j)\big(v_i + \pi_{win} - E(b_j(v_j) | v_j > \theta_j, b_i(v_i) > b_j(v_j))\big) + \\
&+ \left[1 - P(b_i(v_i) \geq b_j(v_j) | v_j > \theta_j)\right] \cdot \pi_{los}\} - c = \\
&= F(\theta_j)(v_i + \pi_{win}) + \left[1 - F(\theta_j)\right] \cdot \pi_{los} - c + \\
&+ \left[1 - F(\theta_j)\right] \cdot P(b_i(v_i) > b_j(v_j) | v_j > \theta_j)\left[v_i + \pi_{win} - \pi_{los} - E(b_j(v_j) | v_j > \theta_j, b_i(v_i) > b_j(v_j))\right]
\end{aligned}$$

Symmetry implies that we can assume symmetric bidding functions. By an argument that is typical of Vickrey auctions, it can be shown that optimal bidding is as follows:

$b_i(v) = b_j(v) = v + \pi_{win} - \pi_{los}$ [6]

This is due by the fact that real valuation for object 1 takes into account the different consequences that winning or loosing will have in the next auction. The effect is the same as the one of shifting the distribution function *F* and bidders' valuations to the right. Hence, a typical Vickrey auction argument follows, and our bid is equal to this *real* valuation.

Given that the optimal bidding strategy $b_i(v_i)$ meets $\frac{\partial \Pi_i}{\partial b_i(v_i)} = 0$, by the Envelope Theorem

$$\frac{d\Pi_i}{dv_i} = \frac{\partial \Pi_i}{\partial v_i} = F(\theta_j) + \left[1 - F(\theta_j)\right] \cdot P(b_i(v_i) > b_j(v_j) | v_j > \theta_j)$$

As $P(b_i(v_i) > b_j(v_j) | v_j > \theta_j) = I\{b_j^{-1}(b_i(v_i)) > \theta_j\} \frac{F(b_j^{-1}(b_i(v_i))) - F(\theta_j)}{1 - F(\theta_j)}$ and $b_j^{-1}(b_i(v_i)) = v_i$, the latter due to symmetry,

$$\frac{d\Pi_i}{dv_i} = F(\theta_j) + I\{v_i > \theta_j\}(F(v_i) - F(\theta_j)) = F(max\{\theta_j, v_i\})$$

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[6] We could also take into account a more general case where there is a common discount factor $0 \leq \delta \leq 1$ and some uncertainty about whether there is going to be a second auction or not, expressed by an ex ante probability $0 \leq p \leq 1$ of having the second auction. It can be seen that in this general case $b_i(v) = b_j(v) = v + \delta \cdot p \cdot (\pi_{win} - \pi_{los})$.

When bidder *i* has the minimum possible valuation for object 1, that is, zero, given participation he can expect to lose the first auction with probability one if the other bidder also takes part in it, so that

$$\Pi_i(0) = -c + F(\theta_j)\cdot\pi_{win} + \left(1 - F(\theta_j)\right)\cdot\pi_{los}$$

From this starting point, we can integrate previous differential equation and obtain

$$\Pi_i = -c + F(\theta_j)\cdot\pi_{win} + \left(1 - F(\theta_j)\right)\cdot\pi_{los} + \int_0^{v_i} F(max\{\theta_j, x\})dx$$

If bidder *i* does not take part in the first auction, given any own known valuation v_i we can express his expected profits as

$$\Pi_i = F(\theta_j)\cdot\pi_{dra} + \left(1 - F(\theta_j)\right)\cdot\pi_{los}$$

Thus, an optimal threshold strategy is given by

$$\begin{aligned}
&-c + F(\theta_j)\cdot\pi_{win} + \left(1 - F(\theta_j)\right)\cdot\pi_{los} + \int_0^{\theta_i^{\#}(\theta_j)} F(max\{\theta_j, x\})dx = \\
&= F(\theta_j)\cdot\pi_{dra} + \left(1 - F(\theta_j)\right)\cdot\pi_{los} \Leftrightarrow \\
&\Leftrightarrow \int_0^{\theta_i^{\#}(\theta_j)} F(max\{\theta_j, x\})dx + F(\theta_j)(\pi_{win} - \pi_{dra}) = c
\end{aligned}$$

whenever this threshold strategy is lower than v^*. Otherwise, the equation becomes a lower-or-equal inequality.

Strategies are symmetric for both players, so we can find a symmetric, “natural” cut-off equilibrium (θ^*, θ^*), which meets

$$F(\theta^*)(\theta^* + \pi_{win} - \pi_{dra}) = c$$

Notice that $\theta^* \le \theta_F \le \theta_H$ [7]. The former equation has close relation to the symmetric cut-off equilibrium of an isolated auction in which both bidders' valuations follow the distribution function $H(x) = F(x - (\pi_{win} - \pi_{dra})), x \in \left[\pi_{win} - \pi_{dra}, v^* + \pi_{win} - \pi_{dra}\right]$. The effect of the interdependence between objects on first auction consists of a shift of bidders' valuations to the right. It is as if valuations increased by the difference between ex ante expected pay-off in the second auction for the buyer of the first object and the same ex ante expected pay-off if no interdependence existed.

[7] This result still holds when taking into consideration the more general case of footnote 5. In the general case, the symmetric equilibrium will accomplish with the condition $F(\theta^*)(\theta^* + \delta \cdot p \cdot (\pi_{win} - \pi_{dra})) = c$. Symmetric equilibrium will not exist if discount factor differs between bidders, but in this paper I am assuming that bidders are symmetric ex ante.

4. Some implications

Having seen a typical Bayesian "natural" equilibrium path of the complete game, we can get some implications that sequential auctioning described above has on the competition level and on the expected profits that sellers can obtain, when interdependencies among objects are present. A first result states that the first seller is always non-worse-off in a sequential auction than in an isolated-auctions benchmark.

Result 1:

Consider a BENCHMARK GAME in which winning or losing in the first auction would not have any consequences on second auction distributions, and where symmetric strategies are played. Then, the probability that both bidders participate in the first auction of the original game with the assumed equilibrium is higher or equal than the probability that both bidders take part in the first auction of the benchmark game. Besides, first auction seller is ex ante (non-strictly) better off in the original game than in the benchmark game.

Proof:

In the benchmark game, both the first and the second auctions are identical to the auction that takes place in case c of the second stage in the original game. It is easy to see that $[1-F(\theta^*)]^2$, the probability that both bidders take part in the first auction of the original game, is higher or equal than $[1-F(\theta_F)]^2$, the probability that both bidders take part in the first auction of the benchmark game. This is due to $\theta_F \geq \theta^*$.

Besides, bidding function in the first auction of the original game is $b^O(v) = v + \pi_{win} - \pi_{los}$, while the bidding function in the first auction of the benchmark game is $b^B(v) = v$. Recall that seller's profits in a second-price sealed-bid auction with no reservation price is zero unless more than one bidder takes part in it. Given that both participate, seller's profits equal the minimum of the two bids, a variable that has the following distribution and density functions:

$$H(x) = 1 - [1 - F(x)]^2; h(x) = 2f(x)[1 - F(x)]$$

Hence, first auction seller's ex ante expected profits in the original game are

$$\Pi_0^O = P(v_1 > \theta^*, v_2 > \theta^*) \cdot E(min\{v_1 + \pi_{win} - \pi_{los}, v_2 + \pi_{win} - \pi_{los}\} | v_1 > \theta^*, v_2 > \theta^*) =$$
$$= [1 - F(\theta^*)]^2 \left(\pi_{win} - \pi_{los} + \frac{\int_{\theta^*}^{v^*} 2 f(x)[1 - F(x)] \cdot x dx}{[1 - F(\theta^*)]^2} \right) =$$
$$= [1 - F(\theta^*)]^2 (\pi_{win} - \pi_{los}) + \int_{\theta^*}^{v^*} 2 f(x)[1 - F(x)] \cdot x dx$$

whereas first auction seller's ex ante expected profits in the benchmark game are

$$\Pi_0^B = P(v_1 > \theta_F, v_2 > \theta_F) \cdot E(min\{v_1, v_2\} | v_1 > \theta_F, v_2 > \theta_F) = \int_{\theta_F}^{v^*} 2 f(x)[1 - F(x)] \cdot x dx$$

It is clear that $\Pi_0^O \geq \Pi_0^B$. **QED**

So the first auction seller takes profit of being the first one when it is expected that the result of the auction is going to have effects on future valuation distributions. Notice that when both bidders take part in the auction, seller extracts the whole bidder's future ex ante surplus difference between having won this first auction and not having done so.

It is clear that the first auction seller obtains higher profits when there are interdependencies among objects. However, it is not so clear that the second auction seller in the original game is going to be better off than in the case where the two auctions are not interrelated. In the original game, the second seller faces a probably lesser competition level from the loser of the first auction, but a probably higher competition level from the winner of the first auction. There is a trade-off between these two forces. I have developed some sufficient (not necessary) conditions for yielding a result in which the probability of having zero profit for the second seller is higher in the original game than in the benchmark game. Two of them come from a proposition that can be found in Miralles (2002) (see Proposition 2 in the appendix). Next one is somehow more complicated. Finally, I have depicted in a last result sufficient conditions for the opposite to hold, that is, making the probability of having zero profit for the second seller be lower in the original game than in the benchmark game.

Result 2:

Consider the benchmark case of Result 1. Then, assuming $\theta_F < \theta_G$, *if*

$E_G(v | v > e) = \frac{v^* - c}{1 - G(e)}$ *for some* $e \geq c$, *and*

$$E_G\left(v\middle|\theta \le v \le G^{-1}(c/\theta)\right) > \frac{\frac{c}{\theta}G^{-1}(c/\theta)-c}{\frac{c}{\theta}-G(\theta)} \textit{ for any } \theta \in (e,\theta_G)$$

second-object seller's ex ante expected profits in the original game will be necessarily lower than or equal to the ones in the benchmark game.

Proof:

Miralles (2002) proposition 1 (see Proposition 2 in the appendix) can be used to state that if the first conditon holds, there is a cut-off equilibrium in which the loser of the first auction is for sure not going to participate in the second auction. It remains to see if (c,v*) (or alternatively (v*,c)) is the "natural" equilibrium of the second auction whenever there has been a loser and a winner in the first auction. This is what the second condition guarantees for any possible distribution function F that is first order stochastically dominated by G.

Without loss of generality, call "bidder 1" to the bidder that won the first auction in the original game, and "bidder 2" to the other bidder who lost. Concerning the second auction, we have to check that there is no intersection between $\theta_2^{\#}(\theta)$ and $\theta_1^{\#^{-1}}(\theta)$ in the interval $\theta \in (e,\theta_F]$ ($\subset (e,\theta_G)$ given our initial assumption). Given that $\theta_2^{\#}(\theta_F) > \theta_G > \theta_F = \theta_1^{\#^{-1}}(\theta_F)$, the latter curve must be strictly under the former one on the entire interval considered here. Given that $\theta_1^{\#^{-1}}(\theta) \le \theta_2^{\#^{-1}}(\theta)$ always holds, it suffices to check that

$$\theta_2^{\#}(\theta) > \theta_2^{\#^{-1}}(\theta) = G^{-1}(c/\theta), \forall\, \theta \in (e,\theta_F]$$

As $\theta_2^{\#}(\theta) > \theta, \theta_2^{\#^{-1}}(\theta) > \theta, \forall\, \theta < \theta_G$, the former condition could be expressed as

$$G(\theta)\theta + \int_{\theta}^{G^{-1}(c/\theta)} G(x)dx < c$$

Integrating by parts, we get

$$\frac{c}{\theta}G^{-1}(c/\theta) - \int_{\theta}^{G^{-1}(c/\theta)} xg(x)dx < c \text{ , i.e } \frac{c}{\theta}G^{-1}(c/\theta) - c < \int_{\theta}^{G^{-1}(c/\theta)} xg(x)dx$$

Applying the definition of conditional expectation, we obtain

$$E_G\left(v \middle| \theta \le v \le G^{-1}(c/\theta)\right) = \frac{\int_{\theta}^{G^{-1}(c/\theta)} xg(x)dx}{\frac{c}{\theta} - G(\theta)} > \frac{\frac{c}{\theta}G^{-1}(c/\theta) - c}{\frac{c}{\theta} - G(\theta)}$$, as desired.

Hence, if the conditions hold, whenever there is a winner in the first auction, second-auction seller will obtain zero profits for sure as the loser of the first one will never take part in the second, due to implied "natural" second-auction equilibrium. In case of having had a draw in the first auction, in the sense that none of the bidders took part on it, second auction seller faces an auction that is equivalent to the one of the benchmark game, thus yielding same profits. **QED**

The idea of this result is very easy. If conditions guaranteeing $\theta_H = v*$ hold, then the second-auction seller would be in serious trouble compared to a benchmark case in which bidders continue being symmetric. These sufficient conditions have to do with the strength of the first-auction winner's distribution.

However, conditions stated in Result 2 could be thought as rather uninformative, as it does not give a precise idea of which kind of distribution functions accomplish with them. Therefore, I have developed some "easier" sufficient conditions leading to the same result.

Result 3:

Consider the benchmark case of Result 1. Then, assuming $\theta_F < \theta_G$, *if*

$E_G\left(v \middle| v > e\right) = \frac{v*-c}{1-G(e)}$ *for some* $e \ge c$, *and*

F is strictly convex but $\frac{d\left(\frac{g(x)}{G(x)}\right)}{dx} \le 3 \cdot \left(\frac{g(x)}{G(x)}\right)^2$ *for any* $x \in [\theta_G, v*]$

second-object seller's ex ante expected profits in the original game will be necessarily lower than or equal to the ones in the benchmark game.

Proof:

As in the previous Result, we just need to show that when second conditions are met, $\theta_H = v*$ is the high "natural" equilibrium threshold value. Take a two-player auction with the same rules but equal distribution functions G. There are at least two equilibria: (θ_G, θ_G) (by symmetry) and (c,v*) (by the first condition of this result). If we show that this game has no equilibrium in $(c, \theta_G) \times (\theta_G, v^*)$, then we are done. Recall the argument of Result 3: the inverse of the optimal threshold strategy is always higher when a bidder faces a “strong” competitor than when facing a “weak” one. Thus, if the game with “symmetric strong bidders” has no equilibrium in the area under consideration, neither does the game with “strong” and “weak” bidders.

Define y and x as $\theta_G \le y \le x \le v^*$ and $y = \frac{c}{G(x)}$, and define

$$\pi(x) = \frac{c}{G(x)} G\left(\frac{c}{G(x)}\right) + \int_{\frac{c}{G(x)}}^{x} G(x)dx - c$$

If $\pi(x) = (\le) 0$, then (y,x) is a cut-off equilibrium (inequality only applies when x=v*). Notice that both $x = \theta_G$ and x=v* meet this result. We need to show that $\pi(x) < 0 \forall x \in (\theta_G, v^*)$. Then, differentiate the function to obtain

$$\pi'(x) = \frac{1}{G(x)}\left[G(x)^2 - \left(\frac{c}{G(x)}\right)^2 g(x) g\left(\frac{c}{G(x)}\right)\right] \equiv \frac{H(x)}{G(x)}$$

It can be seen that this is strictly negative when $x = \theta_G$ due to convexity of G (see Tan and Yilankaya, 2003). Hence, $\pi(\theta_G + \varepsilon) < 0, \varepsilon > 0, \varepsilon \to 0$ and $\pi(v^*) \le 0$. In order to see $\pi(x) < 0 \forall x \in (\theta_G, v^*)$, it is only sufficient to show that, in the worst case, that is, whenever there is a point where $\pi'(x) = 0$, this point is not a maximum. Denote it as x*, so that $H(x^*) = 0$. Assume that this point exists. We have

$$\pi''(x) = \frac{-g(x)}{G(x)^2} H(x) + \frac{1}{G(x)} H'(x); \pi''(x^*) \ge 0 \Leftrightarrow H'(x^*) \ge 0$$

$$H'(x) = 2g(x)G(x) + 2\left(\frac{c}{G(x)}\right)^2 \frac{g(x) g\left(\frac{c}{G(x)}\right)}{G(x)} g(x) -$$
$$- \left(\frac{c}{G(x)}\right)^2 g'(x) g\left(\frac{c}{G(x)}\right) + \left(\frac{c}{G(x)}\right)^2 g(x) g'\left(\frac{c}{G(x)}\right) \frac{cg(x)}{G(x)^2}$$

Using convexity of G and the fact that $H(x^*) = 0$, we can simplify to

$$H'(x^*) = 2g(x^*)G(x^*) + 2g(x^*)G(x^*) -$$
$$- \frac{g'(x^*)}{g(x^*)} G(x^*)^2 + \left(\frac{c}{G(x^*)}\right)^2 g(x^*) g'\left(\frac{c}{G(x^*)}\right) \frac{cg(x^*)}{G(x^*)^2} \geq$$
$$\geq 4g(x^*)G(x^*) - \frac{g'(x^*)}{g(x^*)} G(x^*)^2$$

This is nonnegative if G meets

$4\frac{g(x)}{G(x)} \geq \frac{g'(x)}{g(x)}, \forall x \in [\theta_G, v^*]$, a condition that is completely equivalent to

$$\frac{d\left(\frac{g(x)}{G(x)}\right)}{dx} \leq 3\left(\frac{g(x)}{G(x)}\right)^2, \forall x \in [\theta_G, v^*].$$ **QED**

So, if G is "strong" enough but "not too much", we have this result for sure. We understand as "not too much" the last inequality condition, which is in fact less restrictive than log-concavity of G, which is defined as

$$\frac{d\left(\frac{g(x)}{G(x)}\right)}{dx} \leq 0$$

We can see as well that there always exists some plausible c such that both three conditions that are stated at Result 3 are compatible.

Result 4:

Consider the benchmark case of Result 1. Assume that $\theta_F \leq \frac{3}{2}c, \theta_H < v^*$. *Then, if F hazard-rate dominates the uniform distribution between c and v* on* $[\theta_F, \theta_H]$, *i.e.*

$$\frac{f(x)}{F(x)} \geq \frac{1}{x-c}, \forall x \in [\theta_F, \theta_H]$$

then the ex ante probability of second-auction seller having zero profits is (non-strictly) higher in the original case than in the benchmark case.

Proof:

The probability of having zero profits for the seller is equal to the probability of not having more than one bidder finally in the auction. Thus, we have to check that whenever the hazard-rate condition is met, and given our assumptions, then

$$\left[1-F(\theta_F)\right]^2 \geq F(\theta^*)^2\left[1-F(\theta_F)\right]^2+\left[1-F(\theta^*)^2\right]\cdot\left[1-F(\theta_H)\right]\cdot\left[1-G(\theta_L)\right], \text{ i.e}$$

$$\left[1-F(\theta_F)\right]^2 \geq \left[1-F(\theta_H)\right]\cdot\left[1-G(\theta_L)\right]$$

The first equation only states that the probability of having two final bidders in the second auction is (non-strictly) higher in the benchmark game than in the original game.

Given that $F(\theta_F)\theta_F = c$, we can see that $1-F(\theta_F)=\frac{\theta_F-c}{\theta_F}=F(\theta_F)\frac{\theta_F-c}{c}$, so that

$$\left[1-F(\theta_F)\right]^2=\left[1-F(\theta_F)\right]F(\theta_F)\frac{\theta_F-c}{c}$$

On the other hand, as $\theta_H < v^*$

$$G(\theta_L)\theta_L+\int_{\theta_L}^{\theta_H}G(x)dx=c \Leftrightarrow G(\theta_L)=\frac{c-\int_{\theta_L}^{\theta_H}G(x)dx}{\theta_L}\geq\frac{c-(\theta_H-\theta_L)}{\theta_L}=1-\frac{\theta_H-c}{\theta_L}$$

Having that $F(\theta_H)\theta_L = c$, we can follow as

$$\left[1-F(\theta_H)\right]\left[1-G(\theta_L)\right]\leq\left[1-F(\theta_H)\right]\frac{\theta_H-c}{\theta_L}=\left[1-F(\theta_H)\right]F(\theta_H)\frac{\theta_H-c}{c}$$

It is then sufficient to check that

$$\left[1-F(\theta_F)\right]F(\theta_F)\frac{\theta_F-c}{c}\geq\left[1-F(\theta_H)\right]F(\theta_H)\frac{\theta_H-c}{c}$$

which holds for sure if the function $H(x)=\left[1-F(x)\right]F(x)(x-c)$ is non-strictly decreasing on x over the range $\left[\theta_F,\theta_H\right]$. The condition that makes $H(x)$ be decreasing is

$$\frac{f(x)}{F(x)}\frac{2F(x)-1}{1-F(x)}\geq\frac{1}{x-c}$$

$\theta_F \leq \frac{3}{2}c \Leftrightarrow F(\theta_F)=\frac{c}{\theta_F}\geq\frac{2}{3}\Leftrightarrow F(x)\geq\frac{2}{3}\forall x\geq\theta_F \Leftrightarrow 2F(x)-1\geq 1-F(x)\forall x\geq\theta_F$ and the hazard-rate dominance condition becomes sufficient. **QED**

Result 5:

Consider the benchmark case of Result 1. Assume that $\theta_H < v^*$. *If*

$$\frac{f(x)}{1-F(x)} \le \frac{c}{x}\frac{1}{x-c}, \forall x \in [\theta_F, \theta_H]$$

then the ex ante probability of second-auction seller having zero profits is (non-strictly) lower in the original case than in the benchmark case.

Proof:

Opposite to Result 4, in this case we want to check that

$$[1-F(\theta_F)]^2 \le [1-F(\theta_H)] \cdot [1-G(\theta_L)]$$

Given that $\theta_H < v^*$ and that any distribution function is non-decreasing:

$$G(\theta_L)\theta_L + \int_{\theta_L}^{\theta_H} G(x)dx = c \Leftrightarrow G(\theta_L)\theta_L + G(\theta_L)(\theta_H - \theta_L) \le c \Leftrightarrow G(\theta_L) \le c/\theta_H$$

so that $[1-F(\theta_H)] \cdot [1-G(\theta_L)] \ge [1-F(\theta_H)] \cdot (1-c/\theta_H)$.

On the other hand, $F(\theta_F)\theta_F = c \Leftrightarrow [1-F(\theta_F)]^2 = [1-F(\theta_F)] \cdot (1-c/\theta_F)$

That means that it is sufficient to check that

$$[1-F(\theta_H)](1-c/\theta_H) \ge [1-F(\theta_F)](1-c/\theta_F)$$

Define the function $H(x) = [1-F(x)](1-c/x)$. If this function is non-strictly increasing over the range $[\theta_F, \theta_H]$, we are done. And this happens if and only if the condition mentioned at the result statement is met. **QED**

It is readily seen, and easily understood, that the conditions of Results 4 and 5 are non-compatible. As they have opposite consequences, they cannot be simultaneously met. We see it mathematically. At some point of the proof of Result 4, we had

$$\frac{f(x)}{F(x)}\frac{2F(x)-1}{1-F(x)} \ge \frac{1}{x-c} \text{ over the range } x \in [\theta_F, \theta_H]$$

This is compatible with the condition of Result 5 if and only if

$$\frac{2F(x)-1}{F(x)} \ge \frac{x}{c}$$

, which only happens when $x = \theta_H = v^* = c$, something that is impossible given the definition of *c*.

5. Some conclusions

In this paper, I have developed a simple sequential game in order to find out what the effects of contracting out local services on privatized markets competition could be if there are positive geographical externalities (interdependencies in terms of Gandal, 1997) between municipalities, due mainly to geographical scale economies. This sequential game could be applied to other cases where wining a contract or an object give the new owner advantage when trying to get another one. I use an auction theory perspective. Other approaches such as contract theory or bargaining games could give a different light to this issue, but auction theory applied to this field is good in explaining effective competition patterns among private firms.

I present a model of sequential weakly efficient auctions with participation costs, with two potential bidders in each auction and two objects to be sold. Bidders are ex ante identical in the first auction. There is a positive externality between objects in the sense that acquiring the first one gives the buyer a higher expected second object valuation. The model follows the IPV assumption as it focuses on specific uncorrelated individual features. It also assumes that each bidder learns his valuation in each auction before taking the participation decision, as in Menezes and Monteiro (2000), which is quite realistic concerning firm’s cost structure.

I derive as a result of the model that first-object seller takes profit of this positive externality. Both bidders participate in the first auction with higher probability than if no positive externality existed. They also bid more aggressively, as they advance future earnings. If both bidders participate, first-object seller extracts the whole bidder’s expected second-auction profit differential between being a winner in the first auction and being a loser. All this is shown in Result 1.

Consequences of the positive externality on second-object seller’s profit are less clear. There are two contradictory effects. On one hand, there is a bidder who is “stronger”, hence raising expected price if there is a competitive environment. On the other hand, the other bidder becomes relatively “weaker”, hence reducing potential competition. I have derived some conditions that make one effect outweigh the other.

The distribution function of the winner could be “strong” enough to entirely deter competition, as in Results 2 and 3, which is a frankly bad result for the second-object seller. If this distribution function is not that “strong”, I derive conditions that make competition intensity, measured as the probability of having two final bidders, be lower (higher) with respect to a no-externality benchmark. This is found in Result 4 (Result 5).

An alternative research project could be devoted to explaining why some municipalities are pioneering in privatizing services and other ones are followers, from the auction theory perspective as well as from other approaches. Also, changes in the assumptions underlying this model can be assessed. For instance, auction mechanism could be like in Levin and Smith (1994), in the sense that bidders have to pay participation costs before learning own valuations. However, this assumption is, from my point of view, less realistic concerning local service contracting out.

Appendix

First of all, I present an extension of Tan and Yilankaya (2003) proposition 4, which is in turn a generalization of Lemma 1 here.

Proposition 1:

Consider a second-price sealed-bid auction of an indivisible object, where there are N potential, risk-neutral bidders. There is no reservation price. Participating in the auction has a known cost c ($0 < c < v^*$) for each final participant. Each bidder learns his valuation before deciding whether to participate or not in the auction. Each ex ante i^{th}-bidder's valuation is believed by other bidders to follow a distribution F_i over the support $[0,v^*]$. *Valuations are independent among bidders. Then, if bidders can be ordered in a first order stochastic dominance ranking, such that*

$$F_1(v) \le F_2(v) \le ... \le F_N(v), \forall v \in [0, v^*]$$

, then this game has at least a cut-off equilibrium $\Theta^* = (\theta_1^*, \theta_2^*, ..., \theta_N^*)$ *that meets*

$$\theta_1^* \le \theta_2^* \le ... \le \theta_N^*$$

Proof:

We assume, without loss of generality, that bidders are numbered according to the first order stochastic dominance ranking. Now, if N=2, the proof is done in Lemma 1, so we focus on the cases where N>2. First of all, define an ordered (k)-semi-equilibrium

$$\Theta^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N) = (\theta_1^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N), ..., \theta_k^k(\theta_{k+1}, \theta_{k+2}, ..., \theta_N))$$

as a k-first-players cut-off equilibrium that would appear if bidders k+1, k+2,…, N kept their threshold strategy constant at the values above indicated. Define continuos ordered k-semi-equilibrium line as a function that returns for each possible value $x \in [0, v^*]$ and some fixed vector $(\theta_{k+2}, ..., \theta_N)$ an ordered k-semi-equilibrium

$$\Theta^k(x, \theta_{k+2}, ..., \theta_N) = (\theta_1^k(x, \theta_{k+2}, ..., \theta_N), ..., \theta_k^k(x, \theta_{k+2}, ..., \theta_N))$$

in such a way that the resulting line is continuos and defined on $[0,v^*]^N$. Semi-equilibria always exist in the same way as equilibrium does. A continuous semi-equilibrium path can always be found as optimal threshold strategies are continuous and hence a little change in one of the parameters change the function values smoothly. Then the following claims follow:

First claim: If for any parameters $(\theta_{k+2},...,\theta_N) \in [0, v^*]^{N-k-1}$ there is a continuos k-semi-equilibrium line such that for any $x \in [0, v^*]$

$$\theta_1^k(x,\theta_{k+2},...,\theta_N) \le \theta_2^k(x,\theta_{k+2},...,\theta_N) \le ... \le \theta_k^k(x,\theta_{k+2},...,\theta_N) \qquad (1)$$

, then for the previously fixed parameters $(\theta_{k+3},...,\theta_N)$ and whenever $k+2 \le N$ there exists a continuous (k+1)-semi-equilibrium line such that for any $y \in [0, v^*]$

$$\theta_1^{k+1}(y,\theta_{k+3},...,\theta_N) \le \theta_2^{k+1}(y,\theta_{k+3},...,\theta_N) \le ... \le \theta_{k+1}^{k+1}(y,\theta_{k+3},...,\theta_N) \qquad (2)$$

Proof: It can be shown, given weak efficiency and the revenue equivalence theorem, that bidder i's expected profits, once he knows his valuation v_i and chooses to bid in the auction, is equal to

$$\int_0^{v_i} \prod_{j \ne i} F_j(max\{\theta_j, \varepsilon\}) d\varepsilon - c$$

Each (k)-semi-equilibrium meets then

$$\int_0^{\theta_i^k(x,\theta_{k+2},...,\theta_N)} \prod_{j \ne i}^{j \le k} F_j(max\{\theta_j^k(x,\theta_{k+2},...,\theta_N), \varepsilon\}) \cdot F_{k+1}(max\{x, \varepsilon\}) \cdot \prod_{j>k+1} F_j(max\{\theta_j, \varepsilon\}) d\varepsilon -$$
$$- c = (\le) 0, \forall i \in \{1,...,k\}$$

The less-or-equal inequality only applies when the upper bound of the integral is equal to v^*, as in further equations where the symbol appears in brackets. We define the function $\pi_{k+1}(x)$ as equal to

$$\int_0^x \prod_{j \le k} F_j(max\{\theta_j^k(x,\theta_{k+2},...,\theta_N), \varepsilon\}) \cdot \prod_{j>k+1} F_j(max\{\theta_j, \varepsilon\}) d\varepsilon - c$$

If this function equals zero in x (or is non-positive for x=v^*), then we have a (k+1)-semi-equilibrium $(\theta_1^k(x,\theta_{k+2},...,\theta_N),...,\theta_k^k(x,\theta_{k+2},...,\theta_N), x)$ for the vector $(\theta_{k+2},...,\theta_N)$.

Define $\omega_k(\theta_{k+2},...,\theta_N)$ as

$$\int_0^{\omega_k(\theta_{k+2},...,\theta_N)} \prod_{j \ne k}^{j \le k+1} F_j(max\{\omega_k(\theta_{k+2},...,\theta_N), \varepsilon\}) \prod_{j>k+1} F_j(max\{\theta_j, \varepsilon\}) d\varepsilon - c = (\le) 0$$

Given (1), it is seen that $\theta_k^k(\omega_k(\theta_{k+2},...,\theta_N), \theta_{k+2},...,\theta_N) \ge \omega_k(\theta_{k+2},...,\theta_N)$. On the other hand, it is obvious that $\theta_k^k(v^*, \theta_{k+2},...,\theta_N) \le v^*$. We deduce that

$$H_k \equiv \{x \in [\omega_k(\theta_{k+2},...,\theta_N), v^*]: \theta_k^k(x,\theta_{k+2},...,\theta_N) = x\} \neq \varnothing$$

Call h_k to the maximum element of H_k. Then, by first order dominance between bidders k and k+1, $\pi_{k+1}(h_k) \leq 0$. Hence, if $\pi_{k+1}(v^*) \leq 0$, we have a (k+1)-semi-equilibrium

$$\Theta^{k+1}(\theta_{k+2},...,\theta_N) = (\theta_1^k(v^*,\theta_{k+2},...,\theta_N),...,\theta_k^k(v^*,\theta_{k+2},...,\theta_N), v^*)$$

which meets (2). And if $\pi_{k+1}(v^*) > 0$, there exists some point $x^* \in [h_k, v^*)$ that yields the following (k+1)-semi-equilibrium

$$\Theta^{k+1}(\theta_{k+2},...,\theta_N) = (\theta_1^k(x^*,\theta_{k+2},...,\theta_N),...,\theta_k^k(x^*,\theta_{k+2},...,\theta_N), x^*)$$

Notice that this semi-equilibrium also meets (2). The reason is that for any $x \in [h_k, v^*]$, we have for sure $\theta_k^k(x,\theta_{k+2},...,\theta_N) \leq x$, due to the definition of h_k and the fact that $\theta_k^k(v^*,\theta_{k+2},...,\theta_N) \leq v^*$.

Finally, it only remains to generalize (2) to all possible cases of θ_{k+2}, and to state that the (k+1)-semi-equilibrium line obtained is also continuos, which is always possible as argued before. DONE

<u>Second claim:</u> For any $(\theta_4,...,\theta_N) \in [0, v^*]^{N-3}$, whenever these parameters make sense, and in any case when N=3, there exists a continuos (2)-semi-equilibrium line such that for any $x \in [0, v^*]$(just ignore $\theta_4,...,\theta_N$ if N=3)

$$\theta_1^2(x,\theta_4,...,\theta_N) \leq \theta_2^2(x,\theta_4,...,\theta_N) \qquad (3)$$

Proof: Define $\omega_1(x,\theta_4,...,\theta_N)$ and $\omega_2(x,\theta_4,...,\theta_N)$ as

$$\int_0^{\omega_1(x,\theta_4,...,\theta_N)} F_2(max\{\omega_1(x,\theta_4,...,\theta_N),\varepsilon\}) \cdot F_3(max\{x,\varepsilon\}) \cdot \prod_{j>3} F_j(max\{\theta_j,\varepsilon\})d\varepsilon -$$
$$-c = (\leq)0$$

$$\int_0^{\omega_2(x,\theta_4,...,\theta_N)} F_1(max\{\omega_2(x,\theta_4,...,\theta_N),\varepsilon\}) \cdot F_3(max\{x,\varepsilon\}) \cdot \prod_{j>3} F_j(max\{\theta_j,\varepsilon\})d\varepsilon -$$
$$-c = (\leq)0$$

Because of first order dominance, $\omega_1(x,\theta_4,...,\theta_N) \leq \omega_2(x,\theta_4,...,\theta_N)$. Notice that

$$\omega_1(x,\theta_4,...,\theta_N) = \theta_1^{\#}(\omega_1(x,\theta_4,...,\theta_N), x,\theta_4,...,\theta_N)$$

$$\omega_2(x,\theta_4,...,\theta_N) = \theta_2^{\#}(\omega_2(x,\theta_4,...,\theta_N),x,\theta_4,...,\theta_N)$$

, where # refers to the optimal threshold strategy. In the plane $(x,\theta_4,...,\theta_N)$, bidder 1's optimal threshold strategy is closed under $[c,\omega_1(x,\theta_4,...,\theta_N)]\times[\omega_1(x,\theta_4,...,\theta_N),v^*]$ and bidder 2's one is closed under $[c,\omega_2(x,\theta_4,...,\theta_N)]\times[\omega_2(x,\theta_4,...,\theta_N),v^*]$. Hence, both functions are closed in this plane under the compact intersection set $[c,\omega_1(x,\theta_4,...,\theta_N)]\times[\omega_2(x,\theta_4,...,\theta_N),v^*]$. By Brower's fixed-point theorem, this implies that an intersection between both optimal threshold functions, i.e. a 2-semi-equilibrium, must be in this set. In such equilibrium, condition (3) is met for sure. A trivial generalization to any possible *x* holds, and, by analogous argumentation to the one of the end of claim 1 proof, we can find a continuos (2)-semi-equilibirum line that meets the condition. DONE

By claims two and one, we see that we can find a continuos (*N*-1)-semi-equilibrium line $\Theta^{N-1}(x)$ that meets the condition

$$\theta_1^{N-1}(x) \le \theta_2^{N-1}(x) \le ... \le \theta_{N-1}^{N-1}(x), \forall x \in [0,v^*]$$

It remains to check that this line intersects with N^{th}-bidder's optimal threshold strategy in a point $\Theta^* = (\theta_1^*,\theta_2^*,...,\theta_N^*)$ that meets $\theta_1^* \le \theta_2^* \le ... \le \theta_N^*$. But mimicking claim 1 proof except for its last paragraph readily does this. **QED**

Finally, I present a proof of a version of Miralles (2002) proposition 1.

Proposition 2:

Consider a second-price sealed-bid auction of an indivisible object, where there are N potential, risk-neutral bidders. There is no reservation price. Participating in the auction has a known cost c ($0 < c < v^*$*) for each final participant. Each bidder learns his valuation before deciding whether to participate or not in the auction. Each ex ante* i^{th}*-bidder's valuation is believed by other bidders to follow a differentiable distribution* F_i *over the support* [0,*v**]. *Valuations are independent among bidders. Then, if (and only if) there exists some potential bidder k and some number* $e \ge c$ *such that*

$$E_{F_k}(v|v>e) = \frac{v^*-c}{1-F_k(e)}$$

, the auction has a cut-off equilibrium in which the rest of bidders never participate and bidder k participates if and only if his value is greater than c.

Proof:

$$E_{F k}\left(v \middle| v > e\right) = \frac{v^{*} - c}{1 - F_k(e)} \Leftrightarrow \frac{\int_e^{v^*} x f_k(x)dx}{1 - F_k(e)} = \frac{v^{*} - c}{1 - F_k(e)} \Leftrightarrow v^{*} - \int_e^{v^*} x f_k(x)dx = c \Leftrightarrow$$

$$v^{*} - \left[F_k(v^*)v^{*} - F_k(e)e - \int_e^{v^*} F_k(x)dx \right] = c \Leftrightarrow F_k(e)e + \int_e^{v^*} F_k(x)dx = c \Rightarrow$$

$$\Rightarrow F_k(c)c + \int_c^{v^*} F_k(x)dx \leq c$$

This implies that, if bidder k plays a threshold strategy c and the rest of bidders except for one of them play a threshold strategy of v^* (hence they do not participate in any case), the remaining bidder would obtain non-positive expected profits in case of taking part in the auction. His best response would be a threshold strategy of v^* (so never participating as well). If all bidders except for k never participate, k would participate in the auction whenever his valuation is higher than c. Hence, his optimal threshold strategy would be precisely c. This completes the equilibrium. **QED**

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