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Regular Population Monotonic Allocation Schemes and the Core

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Abstract

A subclass of games with population monotonic allocation schemes is studied, namely games with regular population monotonic allocation schemes (rpmas). We focus on the properties of these games and we prove the coincidence between the core and both the Davis-Maschler bargaining set and the Mas-Colell bargaining set.

Resum

En aquest article s'estudia una subclase dels jocs cooperatius amb esquemes de distribució monótons des del punt de vista poblacional, i que anomenem jocs amb esquemes de distribució regulars. L'anàlisi es centra en les propietats d'aquests jocs i, com a resultat principal, es demostra que el nucli del joc coincideix amb el conjunt de negociació de Davis-Maschler, així com també amb el conjunt de negociació de Mas-Colell.

1 Introduction

Cooperative games with transferable utility (TU) deal with the problem of freely distributing the profit arising from potential cooperation among agents (players). The idea underlying games with population monotonic allocation schemes (Sprumont, 1990) is that, for such games, it is worthwhile to add new players to a given and previously formed coalition as the individual payoff received will benefit all its members. For instance, consider a convex game (Shapley 1971) and suppose that, if a coalition forms, the Shapley value (Shapley, 1953) will be the allocation rule for distributing potential gains; in this case, any coalition of agents will accept the entrance of new players since the individual profit of already existing players increases if the Shapley value is applied.

Formally, let $N = \{1, 2, \ldots, n\}$ be the set of players. For any coalition $S \subseteq N$, $|S|$ or s denotes the number of players in S. A cooperative game is a pair (N, v) where $v : 2^N \to \mathbb{R}$ is the characteristic function such that $v(S)$ is the worth of coalition $S \subseteq N$ and $v(\emptyset) = 0$. The subgame of (N, v) corresponding to S is denoted by (S, v_S) where v_S is the restriction of v to subsets of S. The set of cooperative games with player set N is denoted by G^N.

Let $\mathcal{P}(N) = \{S \,|\, S \subseteq N, \, S \neq \emptyset\}$ be the set of nonempty subsets of N. An allocation scheme $(x_{iS})_{i \in S, S \in \mathcal{P}(N)}$ is a collection of allocation vectors such that each one corresponds to some coalition S of players with $\sum_{i \in S} x_{iS} = v(S)$ and it is interpreted as the proposed payoff distribution for the members of that coalition. In this context population monotonicity can be expressed by the following condition:

$$\text{for all } S \subset T \subseteq N \Rightarrow x_{iS} \leq x_{iT}, \quad \text{for all } i \in S.$$

Population monotonicity explains, in a way, the fact that the total coalition N will form, because the more players that enter the coalition, the better off they are. Conversely, once a coalition is formed it ensures that no player or set of players has incentives to leave that coalition.

Many authors have worked on the concept of *pmas*: Slikker, Norde and Tijs (2003) have shown that the class of games with *pmas* coincides with the class of *information sharing games*; Voorneveld, Tijs and Grahn (2002) have studied *pmas* for the class of clan games and define a more suitable concept for that model called bi-mas; Slikker (2000)

studies population monotonicity in the context of graph restricted games; Hokari (2000) analyzes population monotonic allocation schemes for convex games. Most of this work has studied whether a solution generates population monotonic allocation schemes, or whether a subclass of games has *pmas*. The aim of this paper is to study a restricted class of games with *pmas* and, in particular, some core properties of these games. Let us introduce this class of games.

Suppose S and T are two non-comparable coalitions with a non-empty intersection (i.e. $S \nsubseteq T$, $T \nsubseteq S$ and $T \cap S \neq \emptyset$). Population monotonicity does not impose restrictions on the allocation corresponding to this pair of coalitions. This fact allows situations in which some players who are in the intersection of the two coalitions might be better off in one of these coalitions, while the rest will prefer to be in the other. The central hypothesis of this paper is that, in such situations, all players that are in both coalitions S and T should be better off in one of these coalitions. Formally,

$$\text{for all } S, T \subseteq N \Rightarrow \text{ either } x_{iS} \leq x_{iT}, \text{ or } x_{iS} \geq x_{iT} \text{ for all } i \in S \cap T.$$

Another way to tackle this situation is to suppose that a coalition $S \subseteq N$ is formed and their members have to decide whether to join coalition Q or coalition Q' outside S. This decision problem turns out to be irrelevant if all players in S will be better off (in terms of the payoff received) by joining, for instance, coalition Q than coalition Q'. In other words, we require that all players in S could rank in the same order of preference potential entrant coalitions. This approach will be the starting point of this work and will motivate the definition of *regular population monotonic allocation scheme*.

Definition 1 *An allocation scheme* $(x_{iS})_{i \in S, S \in \mathcal{P}(N)}$ *is regular population monotonic (in short rpmas) if:*

(i) for all $\emptyset \neq S \subseteq N$ and for all $Q \subseteq N \setminus S$,

$$x_{iS} \leq x_{iS \cup Q} \text{ for all } i \in S;$$

(ii) for all $\emptyset \neq S \subseteq N$ and for all $Q, Q' \subseteq N \setminus S$, $Q \neq Q'$

$$(x_{iS \cup Q} - x_{iS \cup Q'}) \cdot (x_{jS \cup Q} - x_{jS \cup Q'}) \geq 0, \text{ for all } i, j \in S.$$

Notice that condition (i) of regular monotonicity is just population monotonicity while condition (ii) states that, given two players in S, i and j, if player i prefers (does not prefer) to join coalition Q rather than Q', so does (does not) player j. We call this condition *regularity*.

Example 1 The following table describes the worth of all coalitions in a four-person game and an allocation for each coalition:

S	x_{1S}	x_{2S}	x_{3S}	x_{4S}	$v(S)$
{1}	1	-	-	-	1
{2}	-	3	-	-	3
{3}	-	-	0	-	0
{4}	-	-	-	4	4
{1,2}	1	3	-	-	4
{1,3}	3	-	1	-	4
{1,4}	4	-	-	5	9
{2,3}	-	3	0	-	3
{2,4}	-	3	-	4	7
{3,4}	-	-	1	4	5
{1,2,3}	3	3	1	-	7
{1,2,4}	4	5	-	5	14
{1,3,4}	4	-	1	5	10
{2,3,4}	-	3	1	4	8
{1,2,3,4}	4	5	1	5	15

Notice that in this game, besides population monotonicity, we can check regularity just by comparing the allocation schemes corresponding to three-person coalitions: for instance, if we take $S = \{1,3\}$, $Q = \{2\}$ and $Q' = \{4\}$, notice that (shown inside boxes) $x_{1\{1,2,3\}} \leq x_{1\{1,3,4\}}$ and $x_{3\{1,2,3\}} \leq x_{3\{1,3,4\}}$. The reader may check regularity for other possible pairs of three person coalitions.

Several examples of regular population monotonic allocation schemes can be found in the literature. Some of them are the following:

A Labor-Managed firm (Dutta and Ray, 1989) A firm of n individuals is considered. Output is produced by the combined effort of the individuals. Individual i is capable of

producing α_i units of a unique output, $i \in N$. For any coalition S, the total output is $\sum_{i \in S} \alpha_i$. To set up a firm requires a fixed cost of $c > 0$, $\sum_{i \in N} \alpha_i > c$. Each coalition can choose whether or not to set up a firm. Therefore the worth of a coalition S is given by

$$v(S) := \max\{\sum_{i \in S} \alpha_i - c, 0\}.$$

If we suppose (w.l.o.g.) that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n$, the *egalitarian solution* (Dutta and Ray, 1989) for this model, $\mathcal{E}(v)$, is:

$$\mathcal{E}_i(v) = \tfrac{1}{k} \left[\sum_{r=1}^{k} \alpha_r - c \right] > 0 \quad \text{for } 1 \leq i \leq k$$

$$\mathcal{E}_i(v) = \alpha_i \qquad\qquad\qquad \text{for } i > k,$$

where k is either n, or the smallest integer such that $\tfrac{1}{k} \left[\sum_{r=1}^{k} \alpha_r - c \right] > \alpha_{k+1}$.

The allocation scheme based on the egalitarian solution, i.e. $x_{iS} = \mathcal{E}_i(v_S)$ if $\sum_{i \in S} \alpha_i > c$ and $x_{iS} = 0$ otherwise, is regular population monotonic. To check this, note that, for all $i \in S$, $\mathcal{E}_i(v_S) = \min\{\lambda_S, \alpha_i\}$ where λ_S is such that $\sum_{i \in S} \mathcal{E}_i(v_S) = v(S)$. Given this parametric representation, notice that, if $S \subseteq T \subseteq N \Rightarrow \lambda_S \leq \lambda_T$, which guarantees population monotonicity. On the other hand, given a coalition $S \subseteq N$, and a pair $Q, Q' \subseteq N \setminus S$ such that $\lambda_{S \cup Q} \leq \lambda_{S \cup Q'}$ then $\mathcal{E}_i(v_{S \cup Q}) \leq \mathcal{E}_i(v_{S \cup Q'})$ for all $i \in S$ and so regularity is also met. In another context, but formally identical to this model, we find the class of bankruptcy games (O'Neill, 1982) which are a subclass of convex games.

Average monotonic games (Izquierdo and Rafels, 2001) A game (N, v) is *average monotonic* if (i) it is positive ($v(S) \geq 0$), and (ii) there exists a vector of weights $\alpha \in \mathbb{R}_+^N$, $\alpha \neq 0$ such that
$$\{S \subseteq T \subseteq N\} \Rightarrow \{\alpha(T) \cdot v(S) \leq \alpha(S) \cdot v(T)\}.$$

If $\alpha_i > 0$ for all $i \in N$ this condition reads $\{S \subseteq T \subseteq N\} \Rightarrow \{\frac{v(S)}{\alpha(S)} \leq \frac{v(T)}{\alpha(T)}\}$. In this context, the proportional distribution with respect to the weight α defines an *rpmas* of the game, i.e. $x_{iS} = \alpha_i \cdot \frac{v(S)}{\alpha(S)}$, if $\alpha(S) \neq 0$ and $x_{iS} = 0$, otherwise. Other examples of average monotonic games are *Externality games* (Grafe, Iñarra and Zarzuelo, 1998) and *Clan Games* (Potters, Poos, Tijs and Muto, 1989); the previous model of a *labor-managed*

6

firm and *banruptcy games* are also examples of these kind of games.

In general, games with *rpmas* are not always average monotonic games as example 1 shows. To check this, let us suppose that the game is average monotonic with respect to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Since $v(\{1\}) + v(\{2\}) = v(\{1,2\})$ and $v(\{2\}) = 3v(\{1\}))$, and from the definition of this class of games it follows that $\alpha_2 = 3\alpha_1$; analogously, as $v(\{2\}) + v(\{4\}) = v(\{2,4\})$ and $v(\{4\}) = \frac{4}{3}v(\{2\})$, we have $\alpha_4 = \frac{4}{3}\alpha_2$; and finally, as $v(\{3\}) + v(\{2\}) = v(\{2,3\})$ and $v(\{3\}) = 0$, $\alpha_3 = 0$. Therefore, the vector α should be of the form $(\frac{1}{3}a, a, 0, \frac{4}{3}a)$ with $a > 0$. But then, $\alpha(\{1,2,3,4\}) \cdot v(\{1,3,4\}) > \alpha(\{1,3,4\}) \cdot v(\{1,2,3,4\})$ contradicting the definition of average monotonic game. Furthermore, we can also check that the game is not convex (see definition in next section) as $v(\{1,2,3\}) + v(\{1,2,4\}) > v(\{1,2,3,4\} + v(\{1,2\})$. Thus, games with *rpmas* constitutes a large subclass of games with *pmas* that generalizes *average monotonic games* and are different from *convex games*.

The paper is organized as follows. In section 2 we study properties of games having *rpmas* and we provide (proposition 3) a sufficient condition in terms of positive linear combination of simple monotonic games for a game to have an *rpmas*. Finally, in section 3 we prove - for this subclass of games - the coincidence between the core and both the Mas-Colell bargaining set (Mas-Colell, 1989) and the classical bargaining set (David and Maschler, 1963, 1967).

2 Preliminaries

A game (N, v) is *superadditive* if for all $S, T \subseteq N$, $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$. A game (N, v) is *zero-monotonic*, if $v(S) - \sum_{i \in S} v(\{i\}) \leq v(T) - \sum_{i \in T} v(\{i\})$, whenever $S \subseteq T \subseteq N$. A game (N, v) is said to be *convex* if, for every $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. A game is a simple monotonic game if $v(S) \in \{0, 1\}$ with $v(N) = 1$ and $v(S) \leq v(T)$, whenever $S \subseteq T \subseteq N$. A simple *unanimity game* corresponding to $T \subseteq N$, (N, u_T) is defined as $u_T(S) := 1$ if $S \supseteq T$, and $u_T(S) := 0$ otherwise; if $T = \{i\}$ we simply denote the game by u_i.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors $x = (x_i)_{i \in N}$. Given $x \in \mathbb{R}^N$, $x(S)$ denotes $\sum_{i \in S} x_i$, with $x(\emptyset) = 0$ and $x_S \in \mathbb{R}^S$ is the restriction of x to S. The *set of*

preimputations of a game (N, v) is defined by:

$$I^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}.$$

The *set of imputations* of a game (N, v) is defined by:

$$I(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

The core of the game (N, v) is defined by:

$$C(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subseteq N\}.$$

A game with a non-empty core is called balanced, and is called totally balanced if each subgame (S, v_S) is also balanced.

A game with *rpmas* keeps the basic properties of games with *pmas*, this is, zero-monotonicity, superadditivity, balancedness and relative invariance under S-equivalence. This last property means that for any game v with *rpmas*, any vector $d \in \mathbb{R}^N$ and $\delta \in \mathbb{R}_+$, the game $\delta \cdot v + d$, where $(\delta \cdot v + d)(S) := \delta \cdot v(S) + d(S)$ also has *rpmas*.

A notable property of games with *pmas* which is no longer preserved for games with *rpmas* refers to the sum of games: given two games v and w with *rpmas* the sum game $v + w$ defined as $(v + w)(S) := v(S) + w(S)$ is not necessarily a game with *rpmas*. The next example shows this point.

Example 2 Let $N = \{1, 2, 3, 4\}$ and let $(N, u_{\{1,2\}})$ and $(N, u_{\{3,4\}})$ be the simple unanimity games corresponding to coalitions $\{1, 2\}$ and $\{3, 4\}$ respectively. These games are games with *rpmas* (as can be seen by computing the Shapley value for the corresponding games and subgames). Nevertheless, the sum game $v = u_{\{1,2\}} + u_{\{3,4\}}$ does not have *rpmas*. To check this, let $(z_{iS})_{i \in S, S \in \mathcal{P}(N)}$ be an arbitrary allocation scheme of the game v. First, if population monotonicity must hold we have that $z_{iS} \geq 0$. Notice also that $v(\{1, 2\}) = v(\{3, 4\}) = 1$ and so $z_{1\{1,2\}} + z_{2\{1,2\}} = 1$ and $z_{3\{3,4\}} + z_{4\{3,4\}} = 1$. Furthermore and without loss of generality, we will suppose that, on the one hand, $z_{1\{1,2\}} \geq 0$ and $z_{2\{1,2\}} > 0$ and, on the other hand, $z_{3\{3,4\}} \geq 0$ and $z_{4\{3,4\}} > 0$. In this case, since $v(\{1, 2, 4\}) = 1$ it should hold that $z_{4\{1,2,4\}} = 0$ and, since $v(\{2, 3, 4\}) = 1$ we have $z_{2\{2,3,4\}} = 0$. Moreover again, if population monotonicity must hold, we have $z_{2\{1,2,4\}} \geq z_{2\{1,2\}} > 0$ and $z_{4\{2,3,4\}} \geq z_{4\{3,4\}} > 0$. But then regularity is not satisfied

8

for $S = \{2,4\}$, $Q = \{1\}$ and $Q' = \{3\}$. For other cases, similar reasoning leads to an analagous incompatibility.

Finally, in line with the characterization result given in Sprumont (1990, theorem 1) for games with *pmas*, we state a sufficient condition for a game to have an *rpmas* in terms of simple monotonic games with veto-power (i is a veto-player if $v(S) = 0$ for all S such that $i \notin S$).

Proposition 1 *Let $(N, v_1), (N, v_2), \ldots, (N, v_m)$ a family of simple monotonic games such that $v_1 \geq v_2 \geq \ldots \geq v_m$. Then, if*

$$v = \sum_{j=1}^{m} \sum_{i \in N} \lambda_i^j \cdot \min\{v_j, u_i\}, \quad \lambda_i^j \geq 0, \text{ for all } i \in N \text{ and } j = 1, \ldots, m,$$

then the game (N, v) has an rpmas.

Proof. Define $x_{iS} = \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S), u_i(S)\}$. Notice that

$$\sum_{i \in S} x_{iS} \quad = \sum_{i \in S} \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S), u_i(S)\} = [\text{ as } u_i(S) = 0 \text{ for all } i \in N \setminus S]$$

$$= \sum_{i \in S} \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S), u_i(S)\} + \sum_{i \in N \setminus S} \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S), u_i(S)\}$$

$$= v(S).$$

This allocation scheme is population monotonic, since for any $S \subseteq T \subseteq N$ we have $v_j(S) \leq v_j(T)$ and so, for all $i \in S$ it holds that

$$x_{iS} = \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S), u_i(S)\} \leq \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(T), u_i(T)\} = x_{iT}.$$

Moreover, for all $S \subseteq N$ it holds that

$$v_1(S) \geq v_2(S) \geq \ldots \geq v_m(S).$$

Therefore, as these games are simple monotonic games, let us define j_S as follows:

$$j_S = card\{j \in \{1, \ldots, m\} | v_j(S) = 1\}.$$

Hence, let $S \subseteq N$ and $Q, Q' \subseteq N \setminus S$. If $j_{S \cup Q} = 0$ then $v(S \cup Q) = 0$ and trivially $0 = x_{iS \cup Q} \leq x_{iS \cup Q'}$ for all $i \in S$. If $j_{S \cup Q} > 0$, let us suppose, without loss of generality, that $j_{S \cup Q} \leq j_{S \cup Q'}$. In this case we have that, for all $i \in S$,

$$
\begin{aligned}
x_{iS \cup Q} &= \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S \cup Q), u_i(S \cup Q)\} = \text{ (as } i \in S \text{ and } u_i(S \cup Q) = 1) \\
&= \sum_{j=1}^{m} \lambda_i^j \cdot v_j(S \cup Q) = \sum_{j=1}^{j_{S \cup Q}} \lambda_i^j \leq \text{ (as } j_{S \cup Q} \leq j_{S \cup Q'})
\end{aligned}
$$

$$
\leq \sum_{j=1}^{j_{S \cup Q'}} \lambda_i^j = \sum_{j=1}^{m} \lambda_i^j \cdot v_j(S \cup Q') = \sum_{j=1}^{m} \lambda_i^j \cdot \min\{v_j(S \cup Q'), u_i(S \cup Q')\} = x_{iS \cup Q'}.
$$

Hence, regularity is also satified. $\qquad\square$

This proposition describes a way to generate games with *rpmas* as a positive linear combination of monotonic simple games with veto-power (notice that the game $\min\{v_j, u_i\}$ is a monotonic simple game where i is a veto player).

Example 3 *The game of example 1 can be decomposed as follows.*

Let $v_1 = \max\{u_{\{1\}}, u_{\{2\}}, u_{\{3\}}, u_{\{4\}}\}$, $v_2 = \max\{u_{\{4\}}, u_{\{1,3\}}\}$ and $v_3 = u_{\{1,4\}}$. Notice that $v_1 \geq v_2 \geq v_3$. If we denote by $v_j^i = \min\{v_j, u_i\}$, then

$$
v = v_1^1 + 3v_1^2 + v_1^4 + 2v_2^1 + v_2^3 + 3v_2^4 + v_3^1 + 2v_3^2 + v_3^4.
$$

It remains an open question whether this condition is also necessary.

3   The bargaining sets

It is known that in several classes of totally balanced games the core coincides with different concepts of bargaining set: convex games, average monotonic games and assignment games are some examples. Until now, there is no answer to the same question with respect to the general class of games with *pmas*. The aim of this section is to prove that the core of games with *rpmas* coincides with both the Mas-Colell bargaining set (Mas-Colell, 1989) and the classical bargaining set (Davis and Maschler, 1963, 1967). Let us recall these bargaining sets.

The classical bargaining set includes the imputations that survive a bargaining process comprising objections and counter-objections. Given a game (N, v) and y an imputation of this game, an *objection* of player k against player l at y is a pair (T, u), where T is a coalition containing k but not l and u is a vector in $\mathbb{R}^T$ such that

$$u(T) = v(T),$$

$$u_i > y_i \text{ for all } i \in T.$$

In addition, let (T, u) be an objection of k against l at y. A *counter-objection* to this objection is a pair (M, z), where M is a coalition containing l but not k, and z is a vector in $\mathbb{R}^M$ sucht that

$$z(M) = v(M),$$

$$z_i \geq u_i \text{ for all } i \in M \cap T,$$

$$z_i \geq y_i \text{ for all } i \in M \setminus T.$$

An objection is *justified* if there is no counter-objection to it. The classical bargaining set of (N, v) is the set:

$$\mathcal{M}_1^{(i)}(N, v) = \{x \in I(N, v) \mid \text{no player has a justified objection at } x \text{ against any other player}\}$$

Next, we define the Mas-Colell bargaining set. Let (N, v) be a game and let y be a preimputation. An *objection* at y is a pair (T, u), where T is a non-empty coalition and u is a vector in $\mathbb{R}^T$ such that

$$u(T) = v(T),$$

$$u_i \geq y_i \text{ for all } i \in T, \text{ and at least one of the inequalities is strict.}$$

Let (T, u) be an objection at y. A *counter-objection* to this objection is a pair (M, z) where M is a non-empty coalition and z is in $\mathbb{R}^M$ and satisfies

$$z(M) = v(M),$$

$z_i \geq u_i$ for all $i \in M \cap T,$
$z_i \geq y_i$ for all $i \in M \setminus T,$ and at least one of the inequalities is strict.

As before, an objection is justified if there is no counter-objection to it. The *Mas-Colell bargaining set* of (N, v) is the set:

$$\mathcal{MB}(N, v) = \{x \in I^*(N, v) \mid \text{no non-empty coalition has a justified objection at } x\}.$$

To prove the main result of this section, we use reduced games as a tool. Specifically, we define a slightly modified version of the classical reduced game given in Davis and Maschler (1965).

Definition 2 *Given a cooperative game (N, v), a vector $y \in \mathbb{R}^N$, and a proper coalition $S \subseteq N$, $S \neq \emptyset$, the reduced game on S at y, (S, v_y^S) is defined as,*

$$v_y^S(\emptyset) := 0,$$

$$v_y^S(R) := \max_{\emptyset \subseteq Q \subseteq N \setminus S} \{v(R \cup Q) - y(Q)\}, \quad \text{for all } \emptyset \neq R \subseteq S.$$

Notice that, in contrast to the Davis and Maschler reduced game, the worth of the grand coalition follows the same definition as that of other subcoalitions. The following lemma states sufficient conditions for the non-emptiness of the core of this reduced game. To make it easier to prove these results let us denote the reduced game as maximum of games $(S, v_y^{S,Q})$:

$$v_y^S(R) = \max_{\emptyset \subseteq Q \subseteq N \setminus S} \{v_y^{S,Q}(R)\}, \text{ for all } R \subseteq S.$$

where $v_y^{S,Q}(R) = v(R \cup Q) - y(Q)$, for all $\emptyset \neq R \subseteq S$ and $v_y^{S,Q}(\emptyset) = 0$.

Lemma 1 *Let (N, v) be a TU game, $x = (x_{iS})_{i \in S, S \in \mathcal{P}(N)}$ an rpmas of this game, $y \in I^*(N, v) \setminus C(N, v)$, and let $S \subseteq N$ be such that $y(Q) \geq \sum_{i \in Q} x_{iS \cup Q}$ for all $Q \subseteq N \setminus S$. Then, to each $Q \subseteq N \setminus S$ there exists an allocation scheme $z^Q = \left(z_{iT}^Q\right)_{i \in T, T \in \mathcal{P}(S)}$ of the game $(S, v_y^{S,Q})$ such that*

(a) $z^Q = \left(z_{iT}^Q\right)_{i \in T, T \in \mathcal{P}(S)}$ is population monotonic (define a pmas of the game $(S, v_y^{S,Q})$);

(b) the following conditions holds:

$$x_{iT \cup Q} \geq z_{iT}^Q, \text{ for all } T \subseteq S, \text{ all } Q \subseteq N \setminus S \text{ and all } i \in T; \tag{1}$$

(c) for any coalition $Q^M \subseteq N \setminus S$ such that $v(S \cup Q^M) - y(Q^M) \geq v(S \cup Q) - y(Q)$ for all $Q \subseteq N \setminus S$ it holds:

(c1) $z_{iS}^{Q^M} \geq z_{iS}^Q$ for all $Q \subseteq N \setminus S$ and all $i \in S$.

(c2) $z_S^{Q^M} = \left(z_{iS}^{Q^M} \right)_{i \in S} \in C(S, v_y^S)$ and so the reduced game is balanced.

Before proving this lemma let me remark that a direct consequence of item (a) is that the reduced game (S, v_y^S) is a maximum of games with *pmas*. Item (b) states that the payoffs according to the new population allocation schemes are, in some sense, below the original ones. Finally, item (c1) and (c2) gives a point in the core of the reduced game and so states that the core of this game is nonempty. This point is precisely the payoff vector to S corresponding to an allocation scheme mentioned in item (a). This lemma is necessary to prove the main result in this section.

Proof of the lemma. First of all notice that, as x is an *rpmas* of the game v, we can describe $2^{N \setminus S}$ as an ordered set of elements $\{Q_1, Q_2, \ldots, Q_m\}$ such that $Q_k \subseteq N \setminus S$, $k = 1, 2, \ldots, m$ and $x_{iS \cup Q_k} \leq x_{iS \cup Q_{k+1}}$ for all $i \in S$.
As a consequence, notice that

$$\sum_{j \in S} x_{jS \cup Q_k} \leq \sum_{j \in S} x_{jS \cup Q_{k+1}}, \quad \text{for all } k \in \{1, \ldots, m-1\}. \tag{2}$$

Moreover, for all $Q \subseteq N \setminus S$, we have that

$$v(S \cup Q) = \sum_{j \in S} x_{jS \cup Q} + \sum_{j \in Q} x_{jS \cup Q} \leq \sum_{j \in S} x_{jS \cup Q} + y(Q)$$

and so

$$v(S \cup Q) - y(Q) \leq \sum_{j \in S} x_{jS \cup Q}. \tag{3}$$

Hence, for any $Q \subseteq N \setminus S$, we will define the allocation scheme $z^Q = \left(z_{iT}^Q \right)_{i \in T, T \in \mathcal{P}(S)}$ in two steps:

1. Taking into account the inequalities (2) and (3) and the description of $2^{N \setminus S}$ we first define $\left(z_{iS}^Q \right)_{i \in S}$ as follows:

(i) if $\sum_{j\in S} x_{jS\cup Q_k} < v(S\cup Q) - y(Q) \le \sum_{j\in S} x_{jS\cup Q_{k+1}}$ then

$$z_{iS}^Q := \lambda^Q\, x_{iS\cup Q_k} + (1-\lambda^Q)\, x_{iS\cup Q_{k+1}}, \qquad (4)$$

where $\lambda^Q \in [0,1]$ and $v(S\cup Q) - y(Q) = \lambda^Q \sum_{j\in S} x_{jS\cup Q_k} + (1-\lambda^Q) \sum_{j\in S} x_{jS\cup Q_{k+1}}$;

(ii) if $v(S\cup Q) - y(Q) \le \sum_{j\in S} x_{jS\cup Q_1}$,

$$z_{iS}^Q := x_{iS\cup Q_1} - \frac{\sum\limits_{j\in S} x_{jS\cup Q_1} - [v(S\cup Q) - y(Q)]}{|S|}. \qquad (5)$$

Notice that $z_{iS}^Q \le x_{iS\cup Q}$, for all $Q \subseteq N\setminus S$ and that, by definition,

$$\sum_{i\in S} z_{iS}^Q = v(S\cup Q) - y(Q), \quad \text{for all } Q \subseteq N\setminus S. \qquad (6)$$

2. For all $Q \subseteq N\setminus S$ and for all $i \in S$ let us first define $\delta_{iS}^Q := x_{iS\cup Q} - z_{iS}^Q \ge 0$. Notice that, being fixed $Q \subseteq N\setminus S$, for all $T \subseteq S$, $T \ne \emptyset$, we have

$$\sum_{i\in T} \delta_{iS}^Q \le \sum_{i\in S} \delta_{iS}^Q = \sum_{i\in S}[x_{iS\cup Q} - z_{iS}^Q] = \sum_{i\in S} x_{iS\cup Q} - [v(S\cup Q) - y(Q)]$$

$$= \sum_{i\in S} x_{iS\cup Q} - [\sum_{i\in S\cup Q} x_{iS\cup Q} - y(Q)] = -\sum_{i\in Q} x_{iS\cup Q} + y(Q)$$

$$\le -\sum_{i\in Q} x_{iT\cup Q} + y(Q).$$

Hence, for any $T \subseteq S$, $T \ne \emptyset$, let us define $\left(\delta_{iT}^Q\right)_{i\in T}$ as follows:

$$\delta_{iT}^Q := \delta_{iS}^Q + \frac{-\sum\limits_{i\in Q} x_{iT\cup Q} + y(Q) - \sum\limits_{i\in T} \delta_{iS}^Q}{|T|}. \qquad (7)$$

Note that, for all $T \subseteq T' \subseteq S$, and all $i \in T$,

$$\delta_{iT}^Q \ge \delta_{iT'}^Q \ge 0. \qquad (8)$$

14

You may check easily this inequality upon definition (7) since $-\sum_{i \in Q} x_{iT \cup Q} \geq -\sum_{i \in Q} x_{iT' \cup Q}$, $\delta_{iS}^{Q} \geq 0$ and $-\sum_{i \in T} \delta_{iS}^{Q} \geq -\sum_{i \in T'} \delta_{iS}^{Q}$.

Finally, for all $T \subsetneq S$, $T \neq \emptyset$, we define $\left(z_{iT}^{Q} \right)_{i \in T}$ as:

$$z_{iT}^{Q} := x_{iT \cup Q} - \delta_{iT}^{Q}. \tag{9}$$

Note that, since inequality (8) holds, the allocation schemes $\left(z_{iT}^{Q} \right)_{i \in T, T \in \mathcal{P}(S)}$ defined in (4), (5) and (9) are population monotonic (hence condition (a) of the lemma is satisfied). Moreover, since $\delta_{iT}^{Q} \geq 0$, it also holds that $x_{iT \cup Q} \geq z_{iT}^{Q}$, for all $T \subseteq S$ and all $i \in T$ (hence condition (b) of the lemma also holds). Finally, it follows from (4) and (5) that, for any coalition $Q^{M} \subseteq N \setminus S$ such that $v(S \cup Q^{M}) - y(Q^{M}) \geq v(S \cup Q) - y(Q)$ for all $Q \subseteq N \setminus S$, $z_{iS}^{Q^{M}} \geq z_{iS}^{Q}$ for all $Q \subseteq N \setminus S$ and all $i \in S$. Thus condition (c1) of the lemma is satisfied. Item (c2) is straightforward from (a) and (c1). $\qquad\square$

Now we will use two results from Holzman (2001). The first establishes (theorem 2.1) the inclusion of the classical bargaining set in the Mas-Colell bargaining set for the class of superadditive games:

if (N, v) is a superadditive game, then $\mathcal{M}_{1}^{(i)}(N, v) \subseteq \mathcal{MB}(N, v)$.

The second proves that a preimputation y in the Mas-Colell bargaining set but not in the core of the game is characterized by the emptiness of the core of the so-called excess game. That is,

$$y \in \mathcal{MB}(N, v) \setminus C(N, v) \iff C(N, w_{y}) = \emptyset,$$

where $w_{y}(S) := \max_{T \subseteq S}\{v(T) - y(T)\}$ for all $S \subseteq N$. Combining these two results and the fact that the core is included in both bargaining sets, it holds that for superadditive games:

if $C(N, w_{y}) \neq \emptyset$ for all $y \in I^{*}(N, v)$, then $\mathcal{M}_{1}^{(i)}(N, v) = \mathcal{MB}(N, v) = C(N, v).$ (10)

This result will be the argument of the proof of the main theorem of this paper.

Theorem 1 *Let (N, v) be a game with rpmas. Then*

$$\mathcal{M}_1^{(i)}(N, v) = \mathcal{MB}(N, v) = C(N, v).$$

Proof. Let $(x_{iS})_{i \in S, S \in \mathcal{P}(N)}$ be an *rpmas* of the game (N, v). If $y \in C(N, v)$ it trivially holds that $C(N, w_y)$ is non-empty as the null vector belongs to it. If $y \in I^*(N, v) \setminus C(N, v)$, let us denote by $\bar{S}$ a largest coalition of maximal excess at y. That is,

$$v(\bar{S}) - y(\bar{S}) \geq v(S) - y(S) \quad \forall S \subseteq N, \text{ and}$$

$$v(\bar{S}) - y(\bar{S}) > v(S) - y(S) \quad \forall S \supseteq \bar{S}, \ S \neq \bar{S}.$$

Hence, let us define

$$\mathcal{A}_y(v) := \left\{ S \subseteq \bar{S} \mid y(Q) \geq \sum_{i \in Q} x_{iS \cup Q}, \quad \text{for all } Q \subseteq N \setminus S \right\}.$$

This set is nonempty as $\bar{S} \in \mathcal{A}_y(v)$. To check this, let us suppose that for some coalition $Q \subseteq N \setminus \bar{S}$ we have $y(Q) < \sum_{i \in Q} x_{i\bar{S} \cup Q}$. But then we will have that $v(\bar{S}) - y(\bar{S}) < \sum_{i \in \bar{S}} x_{i\bar{S}} - y(\bar{S}) + \sum_{i \in Q} x_{i\bar{S} \cup Q} - y(Q) \leq \sum_{i \in \bar{S}} x_{i\bar{S} \cup Q} - y(\bar{S}) + \sum_{i \in Q} x_{i\bar{S} \cup Q} - y(Q) = v(\bar{S} \cup Q) - y(\bar{S} \cup Q)$, and this contradicts $\bar{S}$ to be a coalition of largest excess.

Take a minimal element with respect to the inclusion in $\mathcal{A}_y(v)$, say S^* ($S^* \neq \emptyset$ as in other case $y \in C(N, v)$), and notice that

$$v_y^{S^*}(S^*) = \max_{\emptyset \subseteq Q \subseteq N \setminus S^*} \{v(S^* \cup Q) - y(Q)\} = v(S^* \cup Q^*) - y(Q^*), \text{ where } Q^* = \bar{S} \setminus S^* \quad (11)$$

Moreover, by lemma 1 taking $S = S^*$, it holds that for each game $(S^*, v_y^{S^*, Q})$, $\emptyset \neq Q \subseteq N \setminus S^*$, there exists an allocation scheme $(z_{iT}^Q)_{i \in T, T \in \mathcal{P}(S^*)}$ meeting items (a), (b), (c1) and (c2) of the lemma (for (c1) and (c2) we can take $Q^M = Q^*$).

Furthermore, it can be proved that

$$z_{iS^*}^{Q^*} > y_i, \text{ for all } i \in S^*. \quad (12)$$

To check this, we will show that if there exists a player $j \in S^*$ such that $z_{jS^*}^{Q^*} \leq y_j$, then $S^* \setminus \{j\} \in \mathcal{A}_y(v)$ contradicting the minimality of S^*. For this purpose, take an arbitrary coalition $Q \subseteq N \setminus (S^* \setminus \{j\})$ and consider two cases:

16

- $j \notin Q$. Then, it is straightforward that

$$y(Q) \geq \sum_{i \in Q} x_{iS^* \cup Q} \geq \sum_{i \in Q} x_{i(S^* \setminus \{j\}) \cup Q}.$$

- $j \in Q$. In this case,

$$y_j \geq z_{jS^*}^{Q^*} \geq \text{[by lemma 1 (c1)]}$$

$$\geq z_{jS^*}^{Q \setminus \{j\}} = x_{jS^* \cup (Q \setminus \{j\})} - [x_{jS^* \cup (Q \setminus \{j\})} - z_{jS^*}^{Q \setminus \{j\}}] \geq \text{[by lemma 1 (b)]}$$

$$\geq x_{jS^* \cup (Q \setminus \{j\})} - [x_{jS^* \cup (Q \setminus \{j\})} - z_{jS^*}^{Q \setminus \{j\}}] - [\sum_{i \in S^* \setminus \{j\}} (x_{iS^* \cup (Q \setminus \{j\})} - z_{iS^*}^{Q \setminus \{j\}})]$$

$$= x_{jS^* \cup (Q \setminus \{j\})} - [\sum_{i \in S^*} (x_{iS^* \cup (Q \setminus \{j\})} - \sum_{i \in S^*} z_{iS^*}^{Q \setminus \{j\}}] = \text{[by (6)]}$$

$$= x_{jS^* \cup (Q \setminus \{j\})} - [\sum_{i \in S^*} x_{iS^* \cup (Q \setminus \{j\})} - v(S^* \cup (Q \setminus \{j\})) + y(Q \setminus \{j\})]$$

$$= x_{jS^* \cup (Q \setminus \{j\})} - [\sum_{i \in S^*} x_{iS^* \cup (Q \setminus \{j\})} - \sum_{i \in S^* \cup (Q \setminus \{j\})} x_{iS^* \cup (Q \setminus \{j\})} + y(Q \setminus \{j\})]$$

$$= x_{jS^* \cup (Q \setminus \{j\})} - [- \sum_{i \in Q \setminus \{j\}} x_{iS^* \cup (Q \setminus \{j\})} + y(Q \setminus \{j\})]$$

$$= [\sum_{i \in Q} x_{iS^* \cup (Q \setminus \{j\})} - y(Q \setminus \{j\})] \geq \sum_{i \in Q} x_{i(S^* \setminus \{j\}) \cup Q} - y(Q \setminus \{j\}),$$

and thus $y(Q) \geq \sum_{i \in Q} x_{i(S^* \setminus j) \cup Q}$, for all $Q \subseteq N \setminus (S^* \setminus \{j\})$ where $j \in Q$.

Therefore $S^* \setminus \{j\}$ will be in $\mathcal{A}_y(v)$ and S^* will not be a minimal coalition of this set, contradicting the hypothesis on S^*. Now, we will prove that the excess game (N, w_y) is balanced where $w_y(S) := \max_{T \subseteq S} \{v(T) - y(T)\}$ for all $S \subseteq N$. In fact, we will show that the vector $r = (z_{S^*}^{Q^*} - y_{S^*}; 0_{N \setminus S^*}) \in C(N, w_y)$.

First note that it is efficient since

$$\sum_{i \in N} r_i = \sum_{i \in S^*} r_i = \sum_{i \in S^*} z_{iS^*}^{Q^*} - \sum_{i \in S^*} y_i = [\text{by (6)}] = v(S^* \cup Q^*) - y(S^* \cup Q^*) = w_y(N).$$

Then, for all $S \subsetneq N$, $S \cap S^* = \emptyset$ and for all $T \subseteq S$, note that $y(T) \geq \sum_{i \in T} x_{iS^* \cup T} \geq \sum_{i \in T} x_{iT} = v(T)$ where the first inequality holds since $S^* \in \mathcal{A}_y(v)$ and $T \subseteq N \setminus S^*$. Therefore, $w_y(S) = 0$ and thus $\sum_{i \in S} r_i = 0 \geq w_y(S)$, for all $S \subsetneq N$, $S \cap S^* = \emptyset$.

On the other hand, for all $S \subsetneq N$, $S \cap S^* \neq \emptyset$ we have that

$$\sum_{i \in S} r_i = \sum_{i \in S \cap S^*} r_i = \sum_{i \in S \cap S^*} (z_{iS^*}^{Q^*} - y_i) = [\text{as } z_{iS^*}^{Q^*} > y_i, \text{ see (12)}]$$

$$= \max_{\emptyset \subseteq R \subseteq S \cap S^*} \left\{ \sum_{i \in R} (z_{iS^*}^{Q^*} - y_i) \right\} = \max_{\emptyset \subseteq R \subseteq S \cap S^*} \left\{ \sum_{i \in R} z_{iS^*}^{Q^*} - y(R) \right\} \geq [\text{as } z_{S^*}^{Q^*} \in C(S^*, v_y^{S^*})]$$

$$\geq \max_{\emptyset \subseteq R \subseteq S \cap S^*} \{ v_y^{S^*}(R) - y(R) \}$$

$$= \max_{\emptyset \subseteq R \subseteq S \cap S^*} \left\{ \max_{\emptyset \subseteq Q \subseteq N \setminus S^*} \{ v(R \cup Q) - y(Q \cup R) \} \right\}$$

$$\geq \max_{\emptyset \subseteq R \subseteq S \cap S^*} \left\{ \max_{\emptyset \subseteq Q \subseteq S \setminus S^*} \{ v(R \cup Q) - y(Q \cup R) \} \right\}$$

$$= \max_{\emptyset \subseteq T \subseteq S} \{ v(T) - y(T) \} = w_y(S).$$

Hence, we conclude that the excess game (N, w_y) is balanced for any $y \in I^*(N, v)$ and so, by (10), that $C(N, v) = \mathcal{M}_1^{(i)}(N, v) = \mathcal{MB}(N, v)$. $\qquad\square$

References

[1] Davis M. and Maschler M.(1963). Existence of stable payoff configurations for cooperative games. *Bull. Amer. Math. Soc.* 69, 106–108.

[2] Davis M. and Maschler M.(1965). The Kernel of a Cooperative Game. *Naval Res. Logist. Quart.* 12, 223–259.

[3] Davis M. and Maschler M.(1967). Existence of stable payoff configurations for co-operative games. In: Shubik M.(ed.), *Essays in mathematical economics in honor of Oskar Morgenstern.* Princeton University Press, 39–52.

[4] Dutta B. and Ray D.(1989). A Concept of Egalitarianism under Participation Constraints. *Econometrica* 57, 615–635.

[5] Grafe F., Iñarra E. and Zarzuelo J.M. (1998). Population Monotonic Allocation Schemes on Externality Games. *Mathematical Methods of Operation Research* 48, 71–80.

[6] Hokari T.(2000). The nucleolus is not aggregate-monotonic onthe domain of convex games. *International Journal of Game Theory* 29, 133–137.

[7] Holzman R. (2001). The comparability of the classical and the Mas-Colell bargaining sets. *International Journal of Game Theory* 29, 543–553.

[8] Izquierdo J.M. and Rafels C. (2001). Average Monotonic Cooperative Games. *Games and Economic Behavior* 36, 174–192.

[9] Mas-Colell A. (1989). An Equivalence Theorem for a Bargaining Set. *J. Math. Econ.* 18, 129–139.

[10] O'Neill B. (1982). A Problem of Rights Arbitration from the Talmud. *Math. Soc. Sci.* 2, 345–371.

[11] Potters J., Poos J. Tijs S.H. and Muto S. (1989). Clan Games. *Games and Economic Behavior* 1, 275–293.

[12] Shapley L.S. (1953). A Value for n-Person Games. In: Kuhn H.and Tucker A. (eds.), *Contributions to the theory of games II*, Princeton University Press, 307–317.

[13] Shapley L.S. (1971). Cores of Convex Games. *International Journal of Game Theory* 1, 11–26.

[14] Slikker M.(2000). Inheritance of Properties in Communications Situations. *International Journal of Game Theory* 29, 241–268.

[15] Slikker M., Norde H. and Tijs S. (2003). Information Sharing Games. *International Game Theory Review* 5, 1–12.

[16] Sprumont Y.(1990). Population Monotonic Allocation Schemes for Cooperative Games with Transferable Utility. *Games and Economic Behavior* 3, 378–394.

[17] Voorneveld M., Tijs S. and Grahn, S. (2002). Monotonic Allocation Schemes in Clan Games. *Mathematical Methods of Operation Research* 56, 439–449.