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Max-convex decompositions for cooperative TU games

Francesc Llerena[‡], Carles Rafels[†1]

Adreça correspondència:

†Dep. Matemàtica Econòmica, Financera i Actuarial, Facultat de Ciències Econòmiques i Empresarials, Universitat de Barcelona, Av. Diagonal 690, E-08034 Barcelona
e-mail: crafels@ub.edu

‡Dep. Gestió d'Empreses, Universitat Rovira i Virgili, Avda. Universitat 1, E-43204 Reus
e-mail: fllg@fcee.urv.es

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Abstract

We show that any cooperative TU game is the maximum of a finite collection of convex games. This max-convex decomposition can be refined by using convex games with non-negative dividends for all coalitions of at least two players. As a consequence of the above results we show that the class of modular games is a set of generators of the distributive lattice of all cooperative TU games. Finally, we characterize zero-monotonic games using a strong max-convex decomposition.

Resum

En aquest treball es demostra que tot joc cooperatiu d'utilitat transferible (o joc cooperatiu TU) es pot representar com el màxim d'una col.lecció finita de jocs convexes. Aquest resultat es pot refinar utilitzant jocs quasi-positius. És a dir, jocs convexes on els dividends associats a les coalicions de dos o més jugadors són positius o nuls. Aquests resultats permeten provar que els jocs modulars formen un sistema de generadors del reticle distributiu que formen el jocs cooperatius TU. Finalment, es dona una caracterització dels jocs zero-monòtons imposant una condició més forta: que tots els jocs que intervenen en la descomposició tinguin el mateix conjunt d'imputacions.

Key words: Cooperative TU-game, convex games, modular games, zero-monotonic games, lattice.

JEL Classification: C71, C78

1 Introduction

In classical convex analysis, the maximum of two real-valued convex functions is a convex function. However, in the class of convex cooperative TU games, the maximum as a binary operation does not preserve convexity. This poses a natural question: which games can be expressed as the maximum of a finite collection of convex games?

Representing games as a maximum (minimum) of other games has been discussed by several authors. Rosenmüller and Weidner (1974) provide a maximum decomposition theorem for any convex game in terms of what they call affine set functions. Kalai and Zemel (1982) provide a representation for non-negative totally balanced games as the minimum of a finite collection of modular games. Extending the results of Dubey (1975), Einy (1988) proves that any monotonic game can be uniquely decomposed as the maximum of join-irreducible games (in the sense of lattice theory with the standard order). Curiel and Tijs (1991) characterize concave (convex) games by means of the maximum (minimum) operator applied to a particular type of modular games –those generated by the marginal worth vectors (the so-called additive marginal games).

The main aim of this paper is to study the role of some subclasses of convex games in the lattice of all cooperative TU games. Interestingly, convex games together with the minimum operator characterize the class of totally balanced games. This can be easily proved by combining the results of Kalai-Zemel (1982) and Curiel-Tijs (1991). Our starting point is therefore to analyze the behavior of the maximum operator applied to convex games.

This paper is organized as follows. Section 2 presents the general notation and some definitions. Section 3 contains our main result: any cooperative TU game can be expressed as the maximum of a finite collection of convex games. These games can be selected from a particular class of convex games –those with non-negative dividends (Harsanyi, 1963) associated to coalitions of at least two players. From the above result, we can prove that the class of modular games forms a set of generators of the distributive lattice of cooperative TU games. To end this section, we characterize zero-monotonic games using a (strong) max-convex decomposition. Finally, Section 4 presents some concluding remarks.

2 Notations and terminologies

We denote by $N = \{1, \ldots, n\}$ a finite set of players and by 2^N the set of all subsets (coalitions) of N. We will use $S \subset T$ to indicate strict inclusion, that is $S \subseteq T$ but $S \neq T$. By $|S|$ we will denote the cardinality of the coalition $S \subseteq N$.

A *cooperative game with transferable utility (a game)*, is a pair (N, v) where N is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the characteristic function with $v(\emptyset) = 0$. The set of all games with player set N is denoted by $\mathcal{G}^N$. It is a well-known result that $\mathcal{G}^N$ is a $(2^n - 1)$ linear space, where $n = |N|$. Given a game (N, v) and a coalition $S \subseteq N$, we define the subgame $(S, v_{|S})$ by $v_{|S}(Q) := v(Q)$, for any $Q \subseteq S$.

Let $\mathbb{R}^N$ stand for the space of real-valued vectors indexed by N, $x = (x_i)_{i \in N}$, and for all $S \subseteq N$, $x(S) = \sum_{i \in S} x_i$, with the convention $x(\emptyset) = 0$. In this way, any vector can be viewed as a game.

Given a coalition $T \subseteq N$, $T \neq \varnothing$, the *unanimity game* (N, u_T) is defined by

$$u_T(S) := \begin{cases} 1 & \text{if } T \subseteq S, \\ 0 & \text{otherwise.} \end{cases}$$

The set of unanimity games $\{(N, u_T) \mid \varnothing \neq T \subseteq N\}$ forms a basis of $\mathcal{G}^N$ and the coordinates of a game in this basis are the unanimity coordinates (or dividends) of the game. For any $(N,v) \in \mathcal{G}^N$, $v = \sum_{\varnothing \neq S \subseteq T} \lambda_T \cdot u_T$, where $\lambda_T = \sum_{S \subseteq T} (-1)^{|T|-|S|}\, v(S)$, for all $\varnothing \neq T \subseteq N$.

The *imputation set* of the game (N,v) is defined by $I(N,v) := \{x \in \mathbb{R}^n \,|\, x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}$. The *core* of the game (N,v) is defined by $C(N,v) := \{x \in I(N,v) \mid x(S) \geq v(S) \text{ for all } S \subseteq N\}$.

A game (N,v) is *convex* (Shapley, 1972) if, for every $S,T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$. We denote by $\mathcal{C}^N$ the class of all convex games with player set N, which forms a full-dimensional polyhedral cone in the $(2^n - 1)$ linear space $\mathcal{G}^N$. Each unanimity game (N, u_T) is a convex game.

A subset of convex games is the set of *almost positive* games (Derks, Haller and Peters, 2000), denoted by

$$\mathcal{C}_2^N := \left\{ \sum_{T \subseteq N,\, T \neq \varnothing} \lambda_T \cdot u_T \mid \lambda_T \geq 0, \text{ for all } T \subseteq N \text{ with } |T| \geq 2 \right\}.$$

Notice that there are no sign conditions over the dividends associated to singletons.

A game (N,v) is *zero-monotonic* if for any pair of coalitions S,T, $S \subset T \subseteq N$, it holds $v(S) + \sum_{i \in T \setminus S} v(\{i\}) \leq v(T)$. A game (N,v) is called *modular* if there exists a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$ such that for every $S \subseteq N$, $v(S) = \sum_{i \in S} x_i$. By $\mathcal{G}^N_{mod}$ we denote the class of N-person modular games.

Given the finite set of players $N = \{1, 2, \ldots, n\}$, we shall denote by S_N the set of all permutations over N. Given a game (N,v) and a permutation $\theta \in S_N$, the *marginal worth vector* associated to θ, denoted by $m^v_\theta \in \mathbb{R}^N$, is defined by $(m^v_\theta)_{\theta(k)} = v(\{\theta(1), \ldots, \theta(k)\}) - v(\{\theta(1), \ldots, \theta(k-1)\})$, for $k \in \{2, \ldots, n\}$, and $(m^v_\theta)_{\theta(1)} = v(\{\theta(1)\})$.

3 Max-convex decompositions

To illustrate max-convex decompositions, we shall describe some examples. The first one is the classical model of *assignment games* introduced by Shapley and Shubik (1972). Consider a bilateral market with two types of traders, say sellers $M = \{1, 2, \ldots, m\}$ and buyers $M' = \{1', 2', \ldots, m'\}$, and an assignment non-negative matrix $A \in M_{m \times m'}$, where the element $a_{i\,j'} \geq 0$ is interpreted as the potential gains for each trading pair, $i \in M$ and $j' \in M'$. The value of any coalition $S \subseteq M \cup M'$ is then defined as

$$v(S) := \max \left\{ a_{i_1\,j'_1} + \ldots + a_{i_k\,j'_k} \right\}, \tag{1}$$

where the maximum is taken over all arrangements of $2k$ distinct players $i_1, \ldots, i_k \in S \cap M$ and $j'_1, \ldots, j'_k \in S \cap M'$, with $k = \min\{|S \cap M|, |S \cap M'|\}$.

Let us denote by $\mathcal{M}(M, M')$ the set of all possible arrangements among players of both sides of the market (or matchings), i.e. all possible arrangements of $2k$ distinct players, where $k = \min\{|M|, |M'|\}$. It is easy to verify that the previously described assignment game can be rewritten in a more compact way as

$$v = \max_{\mu \in \mathcal{M}(M,M')} \left\{ \sum_{(i,j') \in \mu} a_{i\,j'} \cdot u_{\{i\,j'\}} \right\}. \tag{2}$$

Hence, an assignment game is no more than the maximum of a finite collection of some particular convex games: positive linear combinations of the unanimity games associated to the mixed-pair coalitions.

Another example of max-convex decomposition is the case of *monotonic simple games*, which have been widely used to formalize voting situations. Formally, $(N, v) \in \mathcal{G}^N$ is a monotonic simple game if $v(N) = 1$, $v(S) \in \{0, 1\}$ for all $S \subseteq N$, and $v(S) \leq v(T)$ for any $S \subseteq T \subseteq N$.

Any monotonic simple game can be written as

$$v = \max_{S \in \mathcal{M}_v} \{u_S\}\,, \tag{3}$$

where $\mathcal{M}_v$ denotes the minimal winning coalitions of (N, v), namely, those coalitions with value 1 (winning) without proper winning subcoalitions.

Let us point out a difference between the above two max-convex decompositions. In the case of monotonic simple games, all convex games involved (expression (3)) have the same efficiency level as the original game: $u_S(N) = v(N) = 1$, $S \in \mathcal{M}_v$. Notice that this is not the case for the max-convex decomposition associated to the assignment game (expression (2)). However, it easy to modify the max-convex decomposition to achieve the same efficiency level in all components. Indeed, let $v = \max\{v_1, \ldots, v_k\}$ be a max-convex decomposition for the game (N, v), where $(N, v_1), \ldots, (N, v_k)$ are convex games, not necessarily with the same efficiency. Then, the game (N, v) can be written as

$$v = \max_{i=1,2,\ldots,k} \{v_i + (v(N) - v_i(N)) \cdot u_N\}\,. \tag{4}$$

Notice that the games involved in this decomposition are convex with the same efficiency as (N, v), since $v(N) - v_i(N) \geq 0$ for all $i = 1, 2, \ldots, k$, and the class of convex games forms a cone. Then, a max-convex decomposition can we rewritten with the same efficiency for all games. This feature may be useful to connect solutions of the original game with solutions of the games involved in the decomposition.

We will now prove that any game (even if it is not balanced, or monotonic,...) has a max-convex decomposition (Theorem 1). To prove this result we use an induction argument. In this way, we have a method to find at least one max-convex decomposition for any game.

Theorem 1 *For any game* (N, v) *there exists a finite collection of convex games,* $(N, v_1), \ldots, (N, v_k) \in \mathcal{C}^N$*, all of which have the same efficiency as* (N, v)*, such that* $v = \max\{v_1, \ldots, v_k\}$.

PROOF: We use an induction argument over the number of players.

- Let $(N, v) \in \mathcal{G}^N$ with $|N| = 2$ and $(c_1, c_2) \in \mathbb{R}^2$ be a vector such that, for any $i \in \{1, 2\}$, $c_i = \min_{S \subseteq N, i \in S} \{v(S) - v(S \backslash \{i\})\}$. Now, define the 2-person games (N, v_1) and (N, v_2), as:

$$v_1 := c_1 \cdot u_{\{1\}} + v(\{2\}) \cdot u_{\{2\}} + [v(N) - v(N \backslash \{1\}) - c_1] \cdot u_N,$$

$$v_2 := v(\{1\}) \cdot u_{\{1\}} + c_2 \cdot u_{\{2\}} + [v(N) - v(N \backslash \{2\}) - c_2] \cdot u_N.$$

  Clearly, (N, v_1) and (N, v_2) are convex games with the same efficiency as (N, v) and $v = \max\{v_1, v_2\}$.

- Assume that the statement of the theorem holds for any game (N, v), with $|N| \leq r - 1$, and let us prove that it also holds for $|N| = r$.

  Let $(N, v) \in \mathcal{G}^N$ and for $i \in N$ consider $(N \backslash \{i\}, v_{|N \backslash \{i\}})$, the subgame associated to the coalition $N \backslash \{i\}$. By the induction hypothesis, there exists a finite number of convex games, $(N \backslash \{i\}, w_1^i), \ldots, (N \backslash \{i\}, w_{k_i}^i) \in \mathcal{C}^{N \backslash \{i\}}$, such that

$$w_1^i(N \backslash \{i\}) = \ldots = w_{k_i}^i(N \backslash \{i\}) = v_{|N \backslash \{i\}}(N \backslash \{i\}) = v(N \backslash \{i\})$$

  and

$$v_{|N \backslash \{i\}} = \max\{w_1^i, \ldots, w_{k_i}^i\}. \tag{5}$$

  Now, let $j \in \{1, \ldots, k_i\}$ and define the game $(N, \bar{w}_j^i)$ as follows:

$$\bar{w}_j^i(S) := \begin{cases} w_j^i(S) & \text{if } i \notin S, \\ w_j^i(S \backslash \{i\}) & \text{otherwise.} \end{cases} \tag{6}$$

From the convexity of the game $(N\backslash\{i\}, w_j^i)$, it follows that $(N, \bar{w}_j^i) \in \mathcal{C}^N$. Notice that, if $\{\lambda_T\}_{T\subseteq N\backslash\{i\}, T\neq\varnothing}$ are the unanimity coordinates of $(N\backslash\{i\}, w_j^i)$ in $\mathcal{G}^{N\backslash\{i\}}$, then $\bar{w}_j^i = \sum_{T\subseteq N\backslash\{i\}, T\neq\varnothing} \lambda_T \cdot u_T$.

In order to have a set of convex games with the same efficiency of (N, v), define, for any $i \in N$, $c_i = \min_{S\subseteq N, i\in S}\{v(S) - v(S\backslash\{i\})\}$, and the games $\{(N, \tilde{w}_j^i)\}_{j=1,\ldots,k_i}$ as

$$\tilde{w}_j^i := c_i \cdot u_{\{i\}} + \bar{w}_j^i + [v(N) - v(N\backslash\{i\}) - c_i] \cdot u_N. \tag{7}$$

Obviously $\tilde{w}_j^i(N) = v(N)$ and $(N, \tilde{w}_j^i) \in \mathcal{C}^N$, for any $i \in N$ and $j \in \{1, \ldots, k_i\}$, since $v(N) - v(N\backslash\{i\}) - c_i \geq 0$.

At this point we list two properties of these games, $(N, \tilde{w}_j^i)$, which can be derived directly from expressions (5) and (7):

P1 For any $i \in N$, any $j \in \{1, \ldots, k_i\}$ and any $S \subseteq N\backslash\{i\}$, $\tilde{w}_j^i(S) \leq v(S)$ and there exists $j_i \in \{1, \ldots, k_i\}$ such that $\tilde{w}_{j_i}^i(S) = v(S)$.

P2 For any $i \in N$, any $j \in \{1, \ldots, k_i\}$ and any $S \subset N$, $i \in S$, $\tilde{w}_j^i(S) \leq c_i + v(S\backslash\{i\})$ and there exists $j_i \in \{1, \ldots, k_i\}$ such that $\tilde{w}_{j_i}^i(S) = c_i + v(S\backslash\{i\})$.

Finally, to end the proof we only have to check that $(N, v) = (N, w)$ where

$$w := \max_{i=1,\ldots,n} \left\{ \max_{j=1,\ldots,k_i} \{\tilde{w}_j^i\} \right\}.$$

Clearly $w(N) = v(N)$. Let $S \subset N$ be a proper coalition of N and let $i \in N\backslash S$. By **P1**, we know that there exists $j_i \in \{1, \ldots, k_i\}$ such that $\tilde{w}_{j_i}^i(S) = v(S)$ and, for any $j \in \{1, \ldots, k_i\}$, $\tilde{w}_j^i(S) \leq v(S)$.

On the other hand, by **P2**, for any $h \in S$, there exists $j_h \in \{1, \ldots, k_h\}$ such that $\tilde{w}_{j_h}^h(S) = c_h + v(S\backslash\{h\})$ and, for any $j \in \{1, \ldots, k_h\}$, $\tilde{w}_j^h(S) \leq c_h + v(S\backslash\{h\})$.

Hence, we can conclude that

$$w(S) = \max\left\{\max_{h\in S}\{c_h + v(S\backslash\{h\})\}, v(S)\right\} = v(S),$$

since $c_h \leq v(S) - v(S\backslash\{h\})$, for any $S \subset N$ and $h \in S$. ■

Notice that c_i could be chosen arbitrarily as long as $c_i \leq \min_{S\subseteq N, i\in S}\{v(S) - v(S\backslash\{i\}\}$, for all $i \in N$, and thus the decomposition is not unique. The max-convex decomposition obtained in the above theorem can be refined by using only almost positive games, i.e. by using convex games with non-negative dividends for all non-singleton coalitions, as the next corollary shows.

Corollary 1 *Any game (N, v) can be expressed as the maximum of a finite collection of almost positive games, all of which have the same efficiency.*

PROOF: It is sufficient to replace the induction hypothesis used in the proof of Theorem 1 by the following: any game (N, v), with $|N| \leq r - 1$, can be decomposed as the maximum of a finite collection of almost positive games, and to check that the case $|N| = 2$ also holds under this condition. ■

This general decomposition result cannot be refined by using the modular subclass of convex games. Indeed, consider the following example.

EXAMPLE: Let (N, v) be a 3-person monotonic game where $v(\{i\}) = 0$ for all $i \in N$, $v(\{1, 2\}) = v(\{1, 3\}) = 2$, $v(\{2, 3\}) = 5$ and $v(N) = 6$.

Suppose that $v = \max\{m_1, \ldots, m_k\}$, where $\{(N, m_i)\}_{i=1,\ldots,k} \in \mathcal{G}^N_{mod}$ with $m_1(N) = \ldots = m_k(N) = 6$.

For any $i \in N$, $v(\{i\}) = \max\{m_1(\{i\}), \ldots, m_k(\{i\})\} = 0$ and so, for some $h^* \in \{1, \ldots, k\}$, $m_{h^*}(\{i\}) = 0$. But then, by efficiency, $m_{h^*}(N\backslash\{i\}) = 6$, which implies $\max\{m_1(N\backslash\{i\}), \ldots, m_k(N\backslash\{i\})\} \geq 6 > v(N\backslash\{i\})$, which leads to a contradiction.

Notice that even if we drop the condition that the efficiency in the decomposition in the modular games is the same, we still do not get all the class of cooperative games since, if $v = \max\{m_1, \ldots, m_k\}$, then $-v = \min\{-m_1, \ldots, -m_k\}$, and this game is always totally balanced (Kalai and Zemel, 1982). Therefore, our general result (Theorem 1) can be summarized saying that the class of convex games describes the whole class of cooperative games from a lattice point of view.

Once we have proved our general max-convex decomposition result, it seems natural to connect it with the decomposition reached by Curiel and Tijs (1991). These authors prove that any convex game is the minimum of its marginal worth vectors taken as modular games, and vice-versa. By combining these two results, we find that any game is the max-min or the min-max of a suitable family of modular games, all of which have the same efficiency (Corollary 2). This result has a nice interpretation in terms of the lattice structure of games, since it says that modular games are a set of generators of all cooperative games.

Corollary 2 *For any game (N, v), there exists a finite collection of modular games, $(N, m_j^i) \in \mathcal{G}_{mod}^N$, with $i = 1, \ldots, k$ and $j = 1, \ldots, h$, all of which have the same efficiency as (N, v), such that*

$$v = \max_{i=1,\ldots,k} \left\{ \min_{j=1,\ldots,h} \{m_j^i\} \right\} = \min_{j=1,\ldots,h} \left\{ \max_{i=1,\ldots,k} \{m_j^i\} \right\}.$$

Proof: Let (N, v) be a game. By Theorem 1, we know that there exists a finite collection of convex games with the same efficiency as (N, v): $(N, v_j) \in \mathcal{C}^N$ with $j = 1, \ldots, k$, such that $v = \max\{v_1, \ldots, v_k\}$. On the other hand, since any convex game is the minimum of its extreme core points, namely, its marginal worth vectors, taken as modular games (Curiel and Tijs (1991)), we can write $v_j = \min_{\theta \in S_N} \{m_\theta^{v_j}\}$, for any $j \in \{1, \ldots, k\}$, and obtain the decomposition $v = \max_{j=1,\ldots,k} \left\{ \min_{\theta \in S_N} \{m_\theta^{v_j}\} \right\}$.

Now define the game (N, w), where $w := \min_{\theta \in S_N} \left\{ \max_{j=1,\ldots,k}\{m_\theta^{v_j}\} \right\}$. Notice that $v(N) = w(N)$. Let $\varnothing \neq S \subset N$. By convexity of (N, v_j), $m_\theta^{v_j}(S) \geq v_j(S)$, for any $j \in \{1, \ldots, k\}$ and any $\theta \in S_N$. In addition, there exists $\theta^* \in S_N$ such that $m_{\theta^*}^{v_j}(S) = v_j(S)$, for any $j \in \{1, \ldots, k\}$. Hence, $w(S) = \min_{\theta \in S_N} \left\{ \max_{j=1,\ldots,k}\{m_\theta^{v_j}(S)\} \right\} = \max_{j=1,\ldots,k}\{m_{\theta^*}^{v_j}(S)\} = \max_{j=1,\ldots,k}\{v_j(S)\} = v(S)$. ■

Finally, let us analyze the behavior of a strong max-convex representation by imposing the additional condition that the games involved in the decomposition have the same imputation set. This question is interesting because many of the solutions for cooperative games are defined as imputations: if the original game and the convex games of the decomposition have the same imputation set, we can compare and analyze solutions among them. In Corollary 3, we see that this strong max-convex decomposition characterizes the class of zero-monotonic games.

Corollary 3 *A game (N, v) is zero-monotonic if and only if there exists a finite collection of convex games, $(N, v_1), \ldots, (N, v_k) \in \mathcal{C}^N$, such that $v = \max\{v_1, \ldots, v_k\}$, with $I(N, v) = I(N, v_1) = \ldots = I(N, v_k)$. Moreover, this decomposition can be taken in $\mathcal{C}_2^N$.*

Proof: To prove the "only if" part, it is sufficient to notice that, in the proof of Theorem 1, if (N, v) is a zero-monotonic game, then $\min_{S \subseteq N, i \in S}\{v(S) - v(S \backslash \{i\})\} = v(\{i\})$, for all $i \in N$, and all the subgames preserve the zero-monotonicity. From Corollary 2, the games involved in the decomposition can be taken in $\mathcal{C}_2^N$.

The converse part trivially holds from the fact that any convex game is zero-monotonic and the class of zero-monotonic games with the same imputation set is closed under the maximum operation (recall that this is not true if the games do not have the same imputation set). ■

4 Concluding remarks

The above results are an attempt to clarify the lattice structure of cooperative games and the special position of some classes within it. These results should help us to find relationships between the games involved.

For example, an interesting application is to find sufficient conditions for guaranteeing that the core of one of the convex games used in a max-convex decomposition is a stable set for the original game. To illustrate this idea, consider a monotonic simple game (N, v) and expression (3): $v = \max_{S \in \mathcal{M}_v} \{u_S\}$. It is well-known that the core of each unanimity game involved in this decomposition is a stable set for the game (N, v). In this way, the famous counterexamples on the framework of stability (see Lucas, 1992) can be reanalyzed using a max-convex decomposition. Moreover, the existence of several max-convex decompositions could help us to find different stable sets for the original game.

Finally, from Theorem 1 and Corollary 2, the core of a game can be expressed as the intersection of the cores of convex games which form part of a max-convex decomposition. This property may be useful for characterizing the core and other set-solutions from a lattice point of view.

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