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Coalitionally Monotonic Set-solutions for Cooperative TU Games

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Abstract

A static comparative study on set-solutions for cooperative TU games is carried out. The analysis focuses on studying the compatibility between two classical and reasonable properties introduced by Young (1985) in the context of single valued solutions, namely core-selection and coalitional monotonicity. As the main result, it is showed that coalitional monotonicity is not only incompatible with the core-selection property but also with the bargaining-selection property. This new impossibility result reinforces the trade-off between these kinds of interesting and intuitive economic properties. Positive results about compatibility between desirable economic properties are given replacing the core-selection requirement by the core-extension property.

Resum

En aquest article es realitza un estudi comparatiu de solucions conjuntistes per a jocs cooperatius d'utilitat transferible. L'anàlisi es centra en la compatibilitat entre dues propietats introduides per Young (1985) en el contexte de solucions puntuals: monotonia coalicional i selecció dins el nucli del joc. Com a principal resultat, es mostra que la monotonia coalicional no és només incompatible amb què es seleccionin solucions dins el nucli sinò que també és incompatible amb la selecció dins el conjunt de negociació. Aquest nou resultat d'incompatibilitat reforça la confrontació entre aquesta mena de propietats econòmiques que són interessants i també molt intuïtives. En l'article també es donen resultats positius respecte a la compatibilitat entre propietats, si es canvia l'exigència de seleccionar solucions dins el nucli per la selecció de solucions que continguin el nucli

Key words: set-solution, coalitional monotonicity, core-selection, bargaining-selection, core-extension.

JEL Classification: C71

1 Introduction

Referring to cooperative games of transferable utility, Young (1985) proved that, for general games with at least five players, no core allocation is coalitionally monotonic. Coalitional monotonicity assumes that if the worth of a given coalition increases while the worth of all other coalitions remains the same then the payoff of every member of that coalition does not decrease. This is the known as Young's impossibility theorem. From an economic point of view, this is surprising and interesting as it shows that there is a trade-off between two important and intuitive properties: core selection and coalitional monotonicity. Maschler[2] pointed out the importance of this fact:

> "*... if you want a* **unique** *point outcome in the core you must face some undesirable non-monotonicity consequences. On the other hand, if you feel that monotonicity is essential, say, because it "provides incentives" if imposed on a society then you should sometimes discard the core, and the nucleolus is not a solution concept that you should recommend*".

Housman and Clark (1998) analyze and prove the above single point impossibility result for the case of four-player games. Furthermore, they show there exists an infinite class of core allocations which are coalitionally monotonic for three-player games.

As a first result of the present paper we will extend Young's impossibility result to a set-solution framework and find that, for $n \geq 4$, *no core-selection (single or multiple) is coalitionally monotonic (Theorem 1).* Coalitional monotonicity for set-solution concepts requires: that for any selected allocation in the solution set, if the worth of a coalition increases while the worth of all other coalition remains the same, then there exists an allocation in the new solution set where the payoff of every member of that coalition does not decrease.

The core is an important set-solution concept supported by a wide range of applications. An alternative to the core is the bargaining set *à la* Davis and Maschler (1963, 1967). It is remarkable that this bargaining set is always non-empty and includes any core element. In some situations, the bargaining set makes more sense than the core itself, as Maschler (1976) pointed out in an economic example of a game with a non-empty core.

[2] See Maschler (1992), p. 614. The boldface in the quoted paragraph is due to the authors.

Section 2 analyzes whether the new set-version of Young's impossibility result still holds if we consider the bargaining set instead of the core. Our main result in this section is that for $n \geq 4$ *no selection (single or multiple) in the Davis-Maschler bargaining set is coalitionally monotonic* (Theorem 2). It is important to point out that there does not exist any logic dependence between these two new theorems.

The bargaining impossibility result has a connection with the first and seminal paper of Meggido (1974) showing an undesirable non-monotonicity property for the nucleolus, the kernel and the bargaining set. Meggido showed there is a nine-player cooperative game where the above set or point solution concepts are not aggregate-monotonic. Recently Hokari (2000) has shown that the nucleolus is not aggregate-monotonic for $n \geq 4$ in the domain of convex games. We have extended this result to $n \geq 4$ for any non-empty subset of the bargaining set but imposing coalitional monotonicity instead of aggregate monotonicity, which is the coalitional monotonicity property but only for the grand coalition.

In this section we also show that the bargaining set *à la* Mas-Colell (Mas-Colell,1989) does not satisfy the coalitional monotonicity property for $n \geq 4$.

The above bargaining sets are core catchers. Hence, we examine wheter there is any source of incompatibility for a non-empty set-solution between being a core-extension and satisfying the coalitional monotonicity property. In Section 3 we look for some set-solutions that simultaneously meet both properties. In particular, we show that the Weber set and the Weber set of order k are examples of this type. As a consequence we obtain a new interpretation for the core. The core, in spite of not being coalitionally monotonic, it is always the intersection of all its coalitionally monotonic extensions.

Due to the importance of the core as a set-solution, the last part of the paper is devoted to analyzing the restrictions or conditions under which the core keeps the desired monotonicity property. We will show that under some restrictions on the incremented game, the core would meet the coalitional monotonicity property. Two important classes of games are analyzed: the class of convex games and the class of average monotonic games.

An n-player cooperative game is a real-valued function v defined on all coalitions $S \subseteq N = \{1, \dots, n\}$ such that $v(\emptyset) = 0$. The set of all cooperative games will be denoted by G^N. A preimputation for a game v is an allocation vector of the worth of the grand coalition, i.e. $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^N$ such that $\sum_{i=1}^{n} x_i = v(N)$. This latter condition is called efficiency and $\mathbf{x}$ is said to be an efficient vector. The set of all efficient vectors will be denoted by $I^*(v)$. Adding individual rationality, i.e. $x_i \geq v(\{i\})$, for all $i \in N$, we obtain the imputation set of the game v denoted by

$$I(v) = \{\mathbf{x} \in I^*(v) \mid x_i \geq v(\{i\}), \quad \text{for all} \quad i = 1, 2, \ldots, n\}.$$

We will denote by $\mathbf{x}_S$ the restriction of the vector $\mathbf{x} \in \mathbb{R}^N$ to the coalition $S \subseteq N$. A game with a non-empty imputation set is called essential (recall that v is essential if and only if $\sum_{i \in N} v(\{i\}) \leq v(N)$). The core of a game, denoted by $C(v)$ is the set of those efficient vectors satisfying coalitional rationality, $x(S) = \sum_{i \in S} x_i \geq v(S)$, for all $\emptyset \neq S \subseteq N$. Notice that $C(v) \subseteq I(v)$ for any $v \in G^N$. A balanced game $v \in B^N$ is a game with a non-empty core. The core of a cooperative game satisfies the dummy player property, which means that if $i \in N$ is a dummy player of v then $x_i = v(i)$ for all $\mathbf{x} \in C(v)$. A dummy player $i \in N$ of a game $v \in G^N$ is a player such that all his marginal contributions to any coalition are equal to his individual worth, i.e. $v(S \cup \{i\}) - v(S) = v(i)$, for all $S \subseteq N \setminus \{i\}$.

A set-solution concept is a rule α that assigns to any cooperative game $v \in G^N$ a subset of its preimputation set, $\alpha(v) \subseteq I^*(v)$. If $\alpha(v)$ is a singleton, $\alpha(v) = \{\mathbf{x}\}$, for any $v \in G^N$, we say α is a one-point solution or single-valued solution. A set solution is **non-empty**, if for any $v \in G^N$, we have $\alpha(v) \neq \emptyset$.

A set-solution α is a **core-selection** if for any balanced game $v \in B^N$ (i.e. $C(v) \neq \emptyset$) we have $\alpha(v) \subseteq C(v)$.

A set-solution α is **coalitionally monotonic** if for any two games v, $v' \in G^N$ such that, for some coalition $\emptyset \neq S \subseteq N$, $v(S) \leq v'(S)$ and $v(T) = v'(T)$, for any $T \neq S$ (we will denote this by $v \leq_S v'$), it holds that for any $\mathbf{x} \in \alpha(v)$ there exists a vector $\mathbf{x'} \in \alpha(v')$ satisfying $x_i \leq x'_i$ for all players $i \in S$. This coalitional monotonicity property is the natural extension to set-solution rules of the one defined in Young (1985).

We can now state and prove the first impossibility result, which can be viewed as an extension of the Young's classical result to the class of set-solution concepts. The proof is not just a counterexample showing that the core of a carefully selected game shrinks into one single point(this leading to a contradiction) but it is a counterexample based on a parametric family of assignment games with relatively large cores.

Theorem 1 *For $n \geq 4$, non-emptiness, core-selection, and coalitional monotonicity are incompatible.*

Proof Let α be a non-empty set-solution rule which is a core-selection and coalitionally monotonic. Let $N = \{1, \ldots, n\}$ be the set of and let (N, v) be the assignment game

(Shapley and Shubik, 1972) associated to the matrix

$$\begin{array}{c} \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{cccccc} 3 & 4 & 5 & 6 & \dots & n \end{array} \\ \begin{pmatrix} a & b & 0 & 0 & \dots & 0 \\ c & a & 0 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

where $a \geq b > 0$, $a \geq c > 0$ and $a < b + c$.

Notice that for all $S \subseteq N$, we have $v(S) = v(S \cap N_1)$ where $N_1 = \{1, 2, 3, 4\}$. It follows that all players in $N_2 = N \setminus N_1$ are dummy players. As the core meets the dummy player property and α is a core-selection, we know that for any $\mathbf{x} \in \alpha(v)$

$$x_j = 0, \text{ for any } j \in N_2. \tag{1}$$

.

In fact, the four-player subgame[3] (N_1, v_{N_1}) where

$$\begin{array}{lll} v(1) = 0 & v(12) = 0 & v(123) = a \\ v(2) = 0 & v(13) = a & v(124) = a \\ v(3) = 0 & v(14) = b & v(134) = a \\ v(4) = 0 & v(23) = c & v(234) = a \\ & v(24) = a & \\ & v(34) = 0 & v(N_1) = 2a, \end{array}$$

fully determines the core of v, i.e. $C(v) = \{(\mathbf{x}, \mathbf{0}_{N_2}) \mid \mathbf{x} \in C(v_{N_1})\}$.

Now, let v_1 be the same game as v but increasing the worth of the coalition $S = \{1, 2, 3\}$ to $v_1(123) = 2a$. It is easy to check that the core of the game v_1 is non-empty and is equal to

$$C(v_1) = \{(a - x_3, a, x_3, 0, \mathbf{0}_{N_2}) \mid 0 \leq x_3 \leq a - b\}.$$

Hence, by coalitional monotonicity it holds that for all $\mathbf{x} \in \alpha(v)$ there exists $\mathbf{x}' \in \alpha(v_1) \subseteq C(v_1)$ such that

$$x_3 \leq x'_3 \leq a - b. \tag{2}$$

Repeating the same argument, step by step, for coalitions $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$ we obtain respectively, that for any $\mathbf{x} \in \alpha(v)$,

[3] We will omit commas and curly brackets and write, for instance, $v(123)$ instead of $v(\{1, 2, 3\})$.

$$\begin{aligned} x_4 &\leq a - c \\ x_1 &\leq a - c \\ x_2 &\leq a - b. \end{aligned} \tag{3}$$

Adding (1), (2), and (3) we get $x(N) \leq 4a - 2b - 2c$. Furthermore, by efficiency, it holds that $x(N) = v(N) = 2a$, and so $b + c \leq a$ which contradicts the initial hypothesis on the parameters a,b, and c. □

As the reader may guess, the Young's classical result is now a direct consequence of the above theorem, simply applied to point-solution rules. What is more important, the above result states that, coalitionally monotonic solutions can only be found if we discard the core and any of its non-empty subsets. Hence, what we have just proved allows us to drop the word "unique" in the comment by Maschler quoted in our introduction. Consequently, some well-known non-empty core-selections as the nucleolus, the prenucleolus, and the least-core (for definitions see Driessen, 1988) will not be coalitionally monotonic.

Corollary 1 *For $n \geq 4$, the prenucleolus $\eta^*(v)$, the nucleolus $\eta(v)$, and the least-core $LC(v)$ are not coalitionally monotonic.*

The negative result stated in Theorem 1 leads to search for alternatives. A first alternative to the core is the bargaining set and so we analyze whether we could escape the impossibility result replacing the core-selection property by looking for selections in the bargaining set. This work is presented in the next section. A second alternative is to analyze under which conditions the core itself preserves the coalitional monotonicity property. This work is shown in the last section of the paper.

2 The Bargaining set impossibility theorems

The main purpose of this section is to prove that for $n \geq 4$, no solution in the bargaining set is coalitionally monotonic. We will analyze two definitions of a bargaining set: one by Davis and Maschler (1963, 1967) and the other by Mas-Colell (1989).

Let us start with the Davis and Maschler definition. Let v be an essential cooperative game. Let $\mathbf{x}$ be an imputation and $i \neq j \in N$. We say that a pair $(S, \mathbf{y})$ is an objection of player i against player j at $\mathbf{x}$ if $S \in \Gamma_{ij} = \{R \subseteq N \mid i \in R, j \notin R\}$, $\mathbf{y} \in \mathbb{R}^S$, $y(S) = v(S)$ and $y_k > x_k$, for all $k \in S$. A counterobjection of player j to the objection $(S, \mathbf{y})$ of player

i at $\mathbf{x}$ is a pair $(T, \mathbf{z})$ where $T \in \Gamma_{ji}$, $\mathbf{z} \in \mathbb{R}^T$, $z(T) = v(T)$, $z_k \geq y_k$, for all $k \in T \cap S$, and $z_k \geq x_k$, for all $k \in T \setminus S$.

The bargaining set (for the grand coalition) $\mathcal{M}(v)$ is defined as the set of all imputations of v at which every objection can be countered. The bargaining set is non-empty for any essential game and contains the core.

A **bargaining-selection** will be a set-solution α such that for any essential game, i.e. $I(v) \neq \emptyset$, it assigns a subset of the bargaining set, $\alpha(v) \subseteq \mathcal{M}(v)$.

The proof of incompatibility between bargaining selection and coalitional monotonicity is done by first checking the impossibility result for four-player games and extending it to the general case, $n > 4$. Refering to this second part, it is interesting to point out that we have not used the standard dummy player technique as, in general, the Davis-Maschler bargaining set does not satisfy the dummy player property (see Granot and Maschler, 1997). Hence, we have had to prove and use an alternative extension method to obtain the general result (Lemma 1). Roughly speaking, it says that adding players to a 0-normalized game and giving zero worth to the new emerging coalitions basically does not affect the bargaining set. Later on we will use this result to extend the impossibility theorem from $n = 4$ to an arbitrary number of players. It is important to point out that in the extended game new players are not dummies and the game does not need to be superadditive. Nevertheless, new players will obtain zero payoff in any allocation in the bargaining set, which is crucial for our purpose.

Lemma 1 *Let $N = N_1 \cup N_2 = \{1, 2, \ldots, n_1\} \cup \{n_1 + 1, \ldots, n\}$ be the set of players and let $v \in G^{N_1}$ be an essential 0-normalized game (i.e. $v(\{i\}) = 0, \forall i \in N_1$ and $v(N_1) \geq 0$). Then,*

$$\mathcal{M}(\hat{v}) = \{(\mathbf{x}, \mathbf{0}_{N_2}) \mid \mathbf{x} \in \mathcal{M}(v)\},$$

where $\hat{v}(S) = v(S)$, for all $S \subseteq N_1$, $\hat{v}(N) = v(N_1)$, otherwise, $\hat{v}(S) = 0$.

Proof $\subseteq$). Let $\hat{\mathbf{x}} \in \mathcal{M}(\hat{v})$. First, we shall prove that for any $j \in N_2$, $\hat{x}_j = 0$. If $\hat{x}_j > 0$ for some $j \in N_2$ then $\hat{x}(N_1) < \hat{x}(N) = \hat{v}(N) = v(N_1)$. Hence any player $i \in N_1$ can make an objection against the player $j \in N_2$ by using coalition $S = N_1$. But then player $j \in N_2$ could not counter-object as for any coalition $T \in \Gamma_{ji}$ - the ones available for counter-objecting - it holds that $T \cap N_2 \neq \emptyset$ and so $\hat{v}(T) = 0$. Hence, since $x_j > 0$, for any eventual counter-objection $(\mathbf{z}, T)$ we would find that $z(T) \geq x(T) > 0 = \hat{v}(T)$ which is not allowed in any valid counter-objection. Therefore we conclude that $\hat{\mathbf{x}} = (\mathbf{x}, \mathbf{0}_{N_2})$.

Now, it remains to be proved that $\mathbf{x} \in \mathcal{M}(v)$. Notice first that $x(N_1) = \hat{x}(N) = \hat{v}(N) = v(N_1)$ and so $\mathbf{x}$ is efficient and, therefore, an imputation of v, $\mathbf{x} \in I(v)$. Let us suppose that $\mathbf{x} \notin \mathcal{M}(v)$. This means that there is an objection $(S, \mathbf{y})$ of player i against player j ($i \neq j$, $S \subseteq N_1$, $i \in S$, and $j \in N_1 \setminus S$) without counter-objection using any subcoalition in N_1. Notice that x_j should be strictly positive, $x_j > 0$; otherwise, since the game is zero-normalized, player j could counter-object using the coalition $\{j\}$. But in this case the same pair $(S, \mathbf{y})$ could also be used as a justified objection of player i against player j at $\hat{x} = (\mathbf{x}, \mathbf{0}_{N_2})$ in the game $\hat{v}$. On the other hand, as $x_j > 0$, for any coalition $S \subseteq N$ such that $S \cap N_2 \neq \emptyset$ and $j \in S$, we have $\hat{v}(S) = 0$ and this invalidates coalitions of this type for making counter-objections. Hence, any counter-objection may be made only via a coalition in N_1 and this is not possible by hypothesis. Therefore we conclude that if $\mathbf{x} \notin \mathcal{M}(v)$, then $\hat{\mathbf{x}} = (\mathbf{x}, \mathbf{0}_{N_2}) \notin \mathcal{M}(\hat{v})$, which leads to a contradiction.

$\supseteq$). Let $\mathbf{x} \in \mathcal{M}(v)$ and define $\hat{\mathbf{x}} = (\mathbf{x}, \mathbf{0}_{N_2})$. We shall prove that $\hat{\mathbf{x}} \in \mathcal{M}(\hat{v})$. Let $(S, \mathbf{y})$ be an arbitrary objection from player i to j, $i \neq j$ at $\hat{\mathbf{x}}$ in the game $\hat{v}$. Therefore, $S \in \Gamma_{ij}$, $y_k > \hat{x}_k$, for all $k \in S$ and $y(S) = \hat{v}(S)$. First notice that $S \subseteq N_1$; otherwise $\hat{v}(S) = 0$ and therefore $\sum_{k \in S} \hat{x}_k < 0$ which involves a contradiction as $\hat{\mathbf{x}} \in I(\hat{v})$. As a consequence, $i \in N_1$. At this point, we have to analyze two cases.
Case 1: $j \in N_2$. In this case player j can counter-object using the single coalition $(\{j\}, z_j = 0)$ as $z_j = \hat{v}(j) = 0 \geq \hat{x}_j = 0$.
Case 2: $j \in N_1$. As $S \subseteq N_1$, $i \neq j \in N_1$ we can see $(S, \mathbf{y})$ as an objection from player i to j at $\mathbf{x}$ in v. Moreover, since $\mathbf{x} \in \mathcal{M}(v)$, this objection can be countered and this counter-objection can be used to make a counterobjection in the original game $\hat{v}$. Therefore, $\hat{\mathbf{x}} = (\mathbf{x}, \mathbf{0}_{N_2}) \in \mathcal{M}(\hat{v})$, for all $\mathbf{x} \in \mathcal{M}(v)$. □

Theorem 2 *For $n \geq 4$, non-emptiness, bargaining-selection, and coalitional monotonicity are incompatible.*

Proof For $n = 4$ take $N = \{1, 2, 3, 4\}$ and v the assignment game associated to the matrix

$$\begin{array}{cc} & \begin{array}{cc} 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \left(\begin{array}{cc} a & b \\ c & a \end{array} \right) \end{array}$$

where $a \geq b > 0$, $a \geq c > 0$ and $a < b + c$. The description of the characteristic function was given in the proof of Theorem 1. Similarly, let us define the game v_1 identical to v but increasing the worth of coalition $S = \{1, 2, 3\}$ to $v_1(123) = 2a$. The four-player game v_1 is balanced and superadditive (i.e. for any $S \cap T = \emptyset$, $v(S) + v(T) \leq v(S \cup T)$). Therefore, from Solymosi (1999), we know that

$$\mathcal{M}(v_1) = C(v_1) = \{(a - x_3, a, x_3, 0) \mid 0 \leq x_3 \leq a - b\}.$$

Now, by following the same argument as that presented in Theorem 1 we obtain the desired incompatibility.

For $|N| \geq 5$, let us denote by $\hat{v}$ and $\hat{v}_1$ the extensions of the previous games in the sense of Lemma 1. In particular, denoting by $N = N_1 \cup N_2 = \{1, 2, 3, 4\} \cup \{5, 6, \ldots, n\}$ we have

$$\hat{v}_1(S) := \begin{cases} v_1(S) & \text{if } S \subseteq N_1 \\ v_1(N_1) & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

By Lemma 1 we obtain that

$$\mathcal{M}(\hat{v}_1) = \mathcal{M}(v_1) \times \{\mathbf{0}_{N_2}\} \; = \; \{(a - x_3, a, x_3, 0, \mathbf{0}_{N_2}) \mid 0 \leq x_3 \leq a - b\}.$$

and the desired incompatibility is now straightforward just increasing the worth of three-person coalitions to the value $2a$. □

From now, coalitionally monotonic solutions can only be found if we discard the bargaining set and any subset of it, as it is the kernel and also the reactive bargaining set (Granot and Maschler, 1997).

With respect to Mas-Colell's bargaining set, we state a partial result: we only prove the incompatiblity of the Mas-Colell bargaining set $\mathcal{M}_{MC}$, and the monotonicity property. Whether there are point or set-selections in this bargaining set that are coalitionally monotonic remains to be determined.

For Mas-Colell, an objection to a given preimputation $\mathbf{x} \in I^*(v)$ is a pair $(S, \mathbf{y})$, $S \subseteq N$, $S \neq N$, $\mathbf{y} \in \mathbf{R}^S$ such that: *i)* $y_i \geq x_i$ for all $i \in S$, and *ii)* $x(S) < y(S) \leq v(S)$. On the other hand, a counter-objection with respect to $(S, \mathbf{y})$ is a pair $(T, \mathbf{z})$, $T \subseteq N$,$T \neq N$, $z \in \mathbf{R}^T$ such that: *i)* $z_i \geq x_i$ for all $i \in T \setminus S$, $z_i \geq y_i$ for all $i \in T \cap S$, and *ii)* $x(T \setminus S) + y(T \cap S) < z(T) \leq v(T)$. Notice that objections and counterobjection can only be made via coalitions with strictly positive excess, i.e. $e^v(R, \mathbf{x}) = v(R) - x(R) > 0$ where $e^v(R, \mathbf{x})$ is the excess of coalition $R \subset N$ at $\mathbf{x}$ in the game v.

Theorem 3 *For $n \geq 4$, the Mas-Colell bargaining set is not coalitionally monotonic.*

Proof We prove the result showing a generic game with an arbitrary number of players $n \geq 4$, where the Mas-Colell bargaining set is not coalitionally monotonic.

Let $\mathcal{N} = \{1, 2, 3, 4, 5, ...\}$ be the set of potential players, let $N_1 = \{1, 2, 3, 4\}$ and let $N_2 \subseteq \mathcal{N} \setminus N_1$. Then, we define the game $(N_1 \cup N_2, v)$ as

$$\begin{array}{lll} v(1) = 0 & v(12) = 0 & v(123) = 2 \\ v(2) = 0 & v(13) = 1 & v(124) = 1 \\ v(3) = 0 & v(14) = 1 & v(134) = 1 \\ v(4) = 0 & v(23) = 1 & v(234) = 1 \\ & v(24) = 1 & \\ & v(34) = 0 & v(1234) = 2 \end{array}$$

otherwise, $v(S) = v(S \cap N_1)$. The core of this game consists on a single element $\mathbf{p}$, where $p_1 = p_2 = 1$, otherwise, $p_i = 0$. Nevertheless, the bargaining set of Mas-Colell contains other elements. For instance, take $\mathbf{x} \in I^*(v)$ such that $x_1 = x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = \frac{1}{4}$ and $x_i = 0$ for all $i \in N_2$. For this preimputation, the only coalitions with positive excesses are $\{1, 2, 3\} \cup D$, $\{1, 4\} \cup D$ and $\{2, 4\} \cup D$ for all $D \subseteq N_2$. In particular,

$$e^v(\{1, 2, 3\} \cup D, \mathbf{x}) = \tfrac{1}{4},$$

$$e^v(\{1, 4\} \cup D, \mathbf{x}) = e^v(\{2, 4\} \cup D, \mathbf{y}) = \tfrac{1}{4},$$

To check that $\mathbf{x} \in \mathcal{M}_{MC}(v)$ we must prove that for any objection there exists a counter-objection. In general, we will denote an objection by $(S, \mathbf{y})$ and a counter-objection by $(T, \mathbf{z})$. As objections and counter-objections can only be made via coalitions with positive excess, we will analyze the three types of coalitions described above.

[Case 1: $S = \{1, 2, 3\} \cup D$] A general objection via this coalition is

$$\begin{aligned} y_1 &= \tfrac{1}{2} + \epsilon_1 \\ y_2 &= \tfrac{1}{2} + \epsilon_2 \\ y_3 &= \tfrac{3}{4} + \epsilon_3 \\ y_i &= \epsilon_i, \quad \text{for any } i \in D \end{aligned}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_i \geq 0$ and $\epsilon_1 + \epsilon_2 + \epsilon_3 + \sum_{i \in N_2} \epsilon_i \leq \frac{1}{4}$. Then we can built a counter-objection depending on whether $\epsilon_1 > 0$ or $\epsilon_1 = 0$. If $\epsilon_1 > 0$ then $\epsilon_2 < \frac{1}{4}$, and so we can define a counterobjection $(\mathbf{z}, \{2, 4\})$ as

$$z_2 = \frac{3}{4}$$
$$z_4 = \frac{1}{4}$$

where $z_2 > y_2$ and $z_4 = x_4$ and $z_2 + z_4 = v(24) = 1$. If $\epsilon_1 = 0$ we will make the counter-objection via coalition $\{1,4\}$ and

$$z_1 = \frac{3}{4}$$
$$z_4 = \frac{1}{4}$$

[Case 2: $S = \{1,4\} \cup D$] A general objection via this coalition is

$$y_1 = \frac{1}{2} + \epsilon_1$$
$$y_4 = \frac{1}{4} + \epsilon_4$$
$$y_i = \epsilon_i, \quad \text{for any } i \in D$$

where $\epsilon_1, \epsilon_4, \epsilon_i \geq 0$ and $\epsilon_1 + \epsilon_4 + \sum_{i \in N_2} \epsilon_i \leq \frac{1}{4}$. Then we can construct a counter-objection also depending on whether $\epsilon_1 > 0$ or $\epsilon_1 = 0$. If $\epsilon_1 > 0$ then $\epsilon_4 < \frac{1}{4}$, and so we can define a counterobjection $(\mathbf{z}, \{2,4\})$ as

$$z_2 = \frac{1}{2}$$
$$z_4 = \frac{1}{2}$$

where $z_2 = x_2$ and $z_4 > y_4$ and $z_2 + z_4 = v(24) = 1$. If $\epsilon_1 = 0$, we will made the counterobjection via coalition $\{1,2,3\}$ and

$$z_1 = \frac{1}{2}$$
$$z_2 = \frac{1}{2}$$
$$z_3 = 1$$

where $z_1 = y_1$, $z_2 = x_2$, $z_3 > x_3$ and $z_1 + z_2 + z_3 = v(123) = 1$.

[Case 3: $S = \{2,4\} \cup D$] It is symmetric to Case 2, exchanging player 1 by player 2.

Therefore $\mathbf{x} \in \mathcal{M}_{MC}(v)$. Now, take the game $(N_1 \cup N_2, v')$ defined as $v'(124) = 2$ $(> v(124) = 1)$, otherwise, $v'(S) = v(S)$. This game is an average monotonic game with respect to $\mathbf{p} = (1,1,0,\ldots,0)$. Average monotonic games is a class of cooperative TU games introduced by Izquierdo and Rafels (2001). They are totally balanced and the core coincides with both the bargaining set *à la* Davis-Maschler and the bargaining set *à la* Mas-Colell. The definition of this class of games and the definition of an intuitive core allocation are also introduced in the next section of this paper (definition 3). Hence, $C(v') = \mathcal{M}_{MC}(v')$. If the bargaining set of Mas-Colell would meet the coalitional monotonicity property we should be able to take a vector $\mathbf{x}' \in C(v')$ such that $x'_1 \geq x_1 = 1/2$, $x'_2 \geq x_2 = 1/2$ and $x'_4 \geq x_4 = 1/4$. But the core (and so the bargaining set) of v' contains a unique element $\mathbf{x}' = (1, 1, 0, 0, ..., 0)$, and therefore $x'_4 = 0 < x_4 = \frac{1}{4}$. These statements prove the non-monotonicity of the Mas-Colell bargaining set. $\square$

3 Core-extensions and coalitional monotonicity

The bargaining sets studied in the previous section are not coalitional monotonic. In spite of being **core-extensions**, that is, non-empty set solutions which include the core, we have not been able to escape the impossibility result; in some sense they are still too restrictive. The question whether there exists coalitional monotonic core-extensions becomes now relevant. The answer to this question is affirmative and in this section we present some examples of it (the Weber set, the Weber set of order k,...). Furthermore, from this study we can state a new interpretation of the core of a cooperative TU game: the core, not being coalitionally monotonic, is the intersection of all its extensions which are coalitionally monotonic. Due to the importance of the core and to end this section, we will focus on relaxing the monotonicity axiom and analyzing restricted coalitional monotonicity properties. Let us start analyzing some core-extensions.

A well-known core catcher is the Weber set. For each permutation $\theta = (i_1, i_2, \dots, i_n)$ on the player set N the corresponding marginal worth vector $m^v_\theta \in \mathbf{R}^N$ is defined as

$$\begin{aligned}
&m^v_\theta(i_1) = v(i_1)\\
&m^v_\theta(i_2) = v(i_1\, i_2) - v(i_1)\\
&\vdots\\
&m^v_\theta(i_n) = v(i_1\, i_2 \dots i_n) - v(i_1\, i_2 \dots i_{n-1}).
\end{aligned}$$

The Weber set, $W(v)$, is the convex hull of all these efficient vectors.

$$W(v) := conv\{m^v_\theta\}_{\theta \in \mathcal{S}_n},$$

where $\mathcal{S}_n$ denotes the set of all permutations over N. Originally, the Weber set was introduced by Weber (1988) who proved it was a core catcher. Later on, Derks (1992) gave a short proof of this result by using classical real convex analysis. Other properties on this set can be observed in Shapley (1971), Ichiishi (1981), Rafels and Ybern (1995), Rafels and Tijs (1997) and Martínez-de-Albéniz and Rafels (2001). The Weber set will be the first non-empty set solution satisfying the desired monotonicity property.

Proposition 1 *The Weber set, $W(v)$, is a non-empty core-extension which satisfies coalitional monotonicity.*

Proof Let $\mathbf{x} \in W(v)$ be an arbitrary element of the Weber set and let us suppose $v \leq_S v'$. From the definition of the Weber set, $\mathbf{x} = \sum_{\theta \in \mathcal{S}_n} \lambda_\theta \cdot m_\theta^v$, $\lambda_\theta \geq 0$ for all $\theta \in \mathcal{S}_n$ and $\sum_{\theta \in \mathcal{S}_n} \lambda_\theta = 1$. Take now $\mathbf{x}' = \sum_{\theta \in \mathcal{S}_n} \lambda_\theta \cdot m_\theta^{v'}$. Obviously $\mathbf{x}' \in W(v')$ and for any $i \in S$ we have $m_\theta^v(i) \leq m_\theta^{v'}(i)$ since $m_\theta^v(i) = v(P_{\theta,i} \cup \{i\}) - v(P_{\theta,i})$ where $P_{\theta,i}$ is the set of predecessors of player i in the ordering θ (excluding player i). Since $i \notin P_{\theta,i}$ we have $P_{\theta,i} \neq S$ and then, for any $i \in S$ we have,

$$x_i = \sum_{\theta \in \mathcal{S}_n} \lambda_\theta \cdot m_\theta^v(i) \leq \sum_{\theta \in \mathcal{S}_n} \lambda_\theta \cdot m_\theta^{v'}(i) = x_i'$$

and the proof is done. □

The above proof has a direct consequence which is to show that coalitional monotonicity condition is preserved by some algebraic operations as the union, the convex hull of a set – which will be denote by $conv(\alpha(v))$ – and the convex combination of set-solutions.

Corollary 2 *Let $\{\alpha_i\}_{i \in I}$ be an arbitrary family of coalitional monotonic solutions on G^N, then*

1. *$\beta_1(v) = conv(\alpha_i(v))$, $i \in I$ is coalitionally monotonic.*
2. *$\beta_2(v) = \cup_{i \in I} \alpha_i(v)$ is coalitionally monotonic.*
3. *$\beta_3(v) = \sum_{i \in I} \lambda_i \cdot \alpha_i(v)$, $\lambda_i \geq 0$, $i \in I$ and $\sum_{i \in S} \lambda_i = 1$ is coalitionally monotonic if I is finite.*

We can generalize proposition 1 to obtain new non-empty core-extensions satisfying the coalitional monotonicity property. We only have to take the Weber set of an appropriate associated game.

Let v be an arbitrary cooperative TU game on $N = \{1, 2, \ldots, n\}$ and let $k \in \{1, 2, \ldots, n-1\}$. We associate a new game denoted by v_k as

$$v_k(S) := \begin{cases} \sum_{i\in S} v(i) & \text{if } |S| < k \\ \\ v(S) & \text{if } |S| \geq k. \end{cases}$$

The game v_k can be interpreted as a game where for some exogenous reasons (partial information, costly valuations) changes the true valuations of the coalitions of sizes smaller than k by the aggregate worth of its individuals. Notice that for $k = 1$ we obtain $v_1 = v$. Let us define a set-solution concept on the class of all cooperative games as the Weber set of v_k. Formally,

$$\alpha_k(v) := W(v_k).$$

We will name this solution as the Weber set of order $k \in \{1, 2, \ldots, n-1\}$ and we denote it by $W_k(v)$. It is interesting to notice that by definition $W_k(v) \neq \emptyset$, for all $k = 1, 2, \ldots, n-1$ and for all $v \in G^N$. Moreover, if the initial game v is essential, then $W_{n-1}(v) = I(v)$. This solution concept is analyzed deeply by Martínez-de-Albéniz and Rafels (2001).

Proposition 2 *For any $k \in \{1, \ldots, n-1\}$ the Weber set of order k, $W_k(v)$ is a non-empty core-extension which satisfies coalitional monotonicity.*

Proof The Weber set of order $k \in \{1, \ldots, n-1\}$ is the convex hull of the marginal worth vectors of the game v_k. Therefore, $W_k(v) := conv\{m_\theta^{v_k}\}_{\theta\in\mathcal{S}_n}$ where for each permutation $\theta = (i_1, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_n)$ we have

$$\begin{aligned}
m_\theta^{v_k}(i_1) \quad &= v(i_1) \\
m_\theta^{v_k}(i_2) \quad &= v(i_2) \\
\vdots \\
m_\theta^{v_k}(i_{k-1}) \quad &= v(i_{k-1}) \\
m_\theta^{v_k}(i_k) \quad &= v(i_1\, i_2 \ldots i_k) - (v(i_1) + v(i_2) + \ldots + v(i_{k-1})) \\
m_\theta^{v_k}(i_{k+1}) \quad &= v(i_1\, i_2 \ldots i_k, i_{k+1}) - v(i_1\, i_2 \ldots i_k) \\
\vdots \\
m_\theta^{v_k}(i_n) \quad &= v(N) - v(i_1\, i_2 \ldots i_{n-1}).
\end{aligned}$$

It is straightforward to see that all of these vectors are coalitionally monotonic solutions and so is their convex hull. Moreover, the Weber set of order $k \in \{1, \ldots, n-1\}$ is a core-extension since $C(v) \subseteq C(v_k) \subseteq W(v_k)$. $\square$

We have just illustrated and proved the existence of set-solutions which are non-empty, core-extensions and coalitionally monotonic. It is interesting to observe that the intersection of these kinds of set-solutions may lose the coalitional monotonicity property. In fact the core itself, which is not a coalitionally monotonic solution, is the intersection of all of its non-empty extensions which are coalitionally monotonic. For this purpose, we introduce what we call the individual core. This is the set of those **preimputations** satisfying the inequalities of the core, but only for those coalitions containing a fixed individual.

Definition 1 *Given a cooperative game (N, v), the individual core associated to player $i \in N$ is defined as*

$$C_i(v) := \{\mathbf{x} \in I^*(v) \mid x(S) \geq v(S), \ \textit{for all } S \subseteq N, \ \textit{such that } i \in S\}.$$

It is interesting to note that the individual core $C_i(v)$ is always non-empty for any game $v \in G^N$. Obviously, it extends the core of the game and the intersection of all individual cores is the core of the original game. Moreover, the individual cores satisfy the coalitional monotonicity property (the proof is left to the reader). As a consequence the core is the intersection of a finite family of non-empty core-extensions which are coalitionally monotonic . In a more general context, if we denote by ξ the set of non-empty set-solutions which are core-extension and coalitionally monotonic, we can state the following proposition.

Proposition 3 *The core is the intersection of the family of all its non-empty extensions which are coalitionally monotonic. Formally,*

$$C(v) = \bigcap_{\beta \in \xi} \beta(v), \qquad \textit{for all } v \in G^N.$$

Due to the fact that the core of a cooperative TU game is an important solution concept, we will analyze some special cases where the core has the monotonicity property. Actually, we will require the classical monotonicity only if the incremented game belongs to a fixed class of cooperative games.

Definition 2 *Given a class of cooperative games $\mathcal{A} \subseteq G^N$, we say that a set-solution satisfies* **$\mathcal{A}$-coalitional monotonicity** *if for any $v \in G^N$ and any $\emptyset \neq S \subseteq N$ if $v \leq_S v'$ and $v' \in \mathcal{A}$ then for each $\mathbf{x} \in \alpha(v)$ there exists $\mathbf{x}' \in \alpha(v')$ such that $x_i \leq x'_i$, for all $i \in S$.*

The interpretation of this property is as follows: if the worth of a coalition grows (all the others remaining unchanged, $v \leq_S v'$), and the new game belongs to a certain class of games $\mathcal{A}$, then the set-solution analyzed satisfies the classic coalitional monotonicity property. We are making restrictions on the classical property and this will enable us to escape the impossiblity theorem for the core. Notice that if we take $\mathcal{A} = G^N$ we obtain the original monotonicity property. In this sense, two important classes of games will be analyzed: convex games (Shapley, 1971) and average monotonic games (Izquierdo and Rafels, 2001).

A TU cooperative game $v \in G^N$ is convex if for any $S, T \in 2^N$ it satisfies $v(S)+v(T) \leq v(S \cup T) + v(S \cap T)$. A remarkable characterization of convex games is that a game is convex if and only if the core and the Weber set coincide (Shapley, 1971, and Ichiishi, 1981). Hence, it is easy to show that the core is convex-coalitionally monotonic by using the result stated in Proposition 1. To see this, take v and v', being this latest game a convex game, such that $v \leq_S v'$. Then, by Proposition 1, for any allocation $\mathbf{x}$ in the core of v, as well as in the Weber set of v, there exists a vector $\mathbf{x}'$ in the Weber set of v' such that $x_i \leq x_i'$, for all $i \in S$. Nevertheless, as v' is a convex game, $\mathbf{x}'$ will be in the core of v'. This result is formally stated as follows.

Proposition 4 *The core of a cooperative TU game satisfies the* **convex-coalitional monotonicity** *property.*

We also prove this restricted monotonicity property for the class of average monotonic games.

Definition 3 A game (N, v) is *average monotonic* if and only if

i) $v(S) \geq 0$ for all $S \subseteq N$, and

ii) there exists a vector $\alpha \in \mathbf{R}^N_+ \setminus \{\mathbf{0}\}$ such that, for all $S \subseteq T \subseteq N$,

$$\alpha(T) \cdot v(S) \leq \alpha(S) \cdot v(T) \tag{4}$$

where $\alpha(S) = \sum_{i \in S} \alpha_i$ and $\alpha(T) = \sum_{i \in T} \alpha_i$.

If conditions *i)* and *ii)* hold we will say that (N, v) is an average monotonic game with respect to the vector α. To give an interpretation to these games, notice that if $\alpha_i > 0$ for all $i \in N$ we have $\frac{v(S)}{\alpha(S)} \leq \frac{v(T)}{\alpha(T)}$, for all $S \subset T \subseteq N$, which means that the average worth of coalitions with respect to some exogenous and fixed vector $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ grows as players are added to a coalition. For instance, an application of this class of games is the *single-input, increasing returns production function* (Mas-Colell et al., 1996) where it is analyzed the case of a one input-one output joint production situation. Each player owns a positive amount ω_i of input. The corresponding characteristic function is defined as $v(S) := f(\sum_{i \in S} \omega_i)$ where f is the production function, $f(0) = 0$, and $f(z)/z$ is nondecreasing in $\mathbf{R}_+ \setminus \{\mathbf{0}\}$. The game is then average monotonic with respect to ω. Other applications are *bankruptcy games* (O'Neill, 1982), *financial cooperative games* (Izquierdo, 1996) or *veto rich games* (Arin and Feltkamp, 1997). These games are always totally balanced and it is easy to check that the proportional distribution with respect to the vector α is a distinguished core element.

Definition 4 *Let (N, v) be an average monotonic game with respect to $0 \neq \alpha \in \mathbb{R}^N_+$. We define the proportional distribution $\mathbf{p}(v) = (p_i(v))_{i \in N}$ as*

$$p_i(v) := \alpha_i \cdot \tfrac{v(N)}{\alpha(N)} \text{ for all } i \in N.$$

The proof that shows that the core is **average monotonic-coalitionally monotonic** is not so straighforward. We will use as a tool the so called reduced games. Given a game $v \in G^N$, a preimputation $\mathbf{x} \in I^*(v)$ and a coalition $S \subseteq N$, the reduced game on $\mathbf{x}$ at S, $(S, v^S_{\mathbf{x}})$, is defined (Davis and Maschler, 1965) as

$$\begin{aligned} v^S_{\mathbf{x}}(\emptyset) &:= 0, \\ v^S_{\mathbf{x}}(T) &:= \max\nolimits_{\emptyset \subseteq Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\} \text{ for all } \emptyset \neq T \subseteq S,\ T \neq S \\ v^S_{\mathbf{x}}(S) &:= v(N) - x(N \setminus S). \end{aligned}$$

Given a preimputation $\mathbf{x}$ and a coalition $S \subseteq N$ the reduced game aims to evaluate the worth of coalitions in S taking into account that the payoff to players in $N \setminus S$ is fixed and cannot be modified in order to revise the payoff of players in S. Hence, the worth of a coalition S is the maximum worth that this coalition could attain by adding players from outside S and rewarding them by the payoff fixed by $\mathbf{x}$. Notice that $v^N_{\mathbf{x}} = v$ and for all $S_2 \subset S_1 \subseteq N$,

$$v^{S_2}_{\mathbf{x}} = \left[v^{S_1}_{\mathbf{x}}\right]^{S_2}_{\mathbf{x}_{S_1}} \tag{5}$$

Theorem 4 *Let v be an arbitrary game and let v' be an average monotonic game such that for a certain non-empty coalition $S \subseteq N$ we have that $v \leq_S v'$. Then, for any $\mathbf{x} \in C(v)$ there exists a vector $\mathbf{x}' \in C(v')$ such that $x_i \leq x'_i$ for all $i \in S$.*

Proof

If $S = N$ then we can define $\mathbf{x}'$ as $x'_i = x_i + \frac{v'(N)-v(N)}{n}$ and so $x_i < x'_i$ for all $i \in N$. If $\emptyset \neq S \subseteq N$, $S \neq N$, let us define $\mathcal{D}_S$ as

$$\mathcal{D}_S := \left\{ T \subseteq N \text{ such that } \begin{array}{ll} (i) & N \setminus S \subseteq T,\ N \setminus S \neq T \text{ and} \\ (ii) & v'^{T}_{\mathbf{x}} \text{is an average monotonic game} \end{array} \right\} \tag{6}$$

Notice that this set is always non-empty as $N \in \mathcal{D}_S$. Let T^* be a minimal coalition with respect to the inclusion in $\mathcal{D}_S$. Since $N \setminus S \subseteq T^*$ and $T^* \neq N \setminus S$ we know $S \cap T^* \neq \emptyset$. Hence, we will study two cases: (i) $x_i < p_i(v'^{T^*}_{\mathbf{x}})$ for all $i \in S \cap T^*$, or (ii) there exists a player $i_* \in S \cap T^*$ such that $x_{i_*} \geq p_{i_*}(v'^{T^*}_{\mathbf{x}})$.

In case (i) we can define $\mathbf{x}'$ as $x'_i = x_i$ for all $i \in N \setminus T^*$ and $x'_i = p_i(v'^{T^*}_{\mathbf{x}})$ for all $i \in T^*$, where $p_i(v'^{T^*}_{\mathbf{x}})$ is the proportional payoff to player i in the game $v'^{T^*}_{\mathbf{x}}$. Notice that $x'_i \geq x_i$ for all $i \in S$. Moreover, the vector $\mathbf{x}'$ is in the core of the game v' as, for all $R \subseteq N \setminus T^*$, $x'(R) = x(R) \geq v(R) = v'(R)$ and for all R such that $R \cap T^* \neq \emptyset$, $x'(R) = x'(R \cap T^*) + x'(R \setminus T^*) = p(v'^{T^*}_{\mathbf{x}})(R \cap T^*) + x(R \setminus T^*) \geq v'^{T^*}_{\mathbf{x}}(R \cap T^*) + x(R \setminus T^*) = \max_{\emptyset \subseteq Q \subseteq N \setminus T^*}\{v((R \cap T^*) \cup Q) - x(Q)\} + x(R \setminus T^*) \geq (Q = R \setminus T^*) \geq v(R) - x(R \setminus T^*) + x(R \setminus T^*) = v(R)$. Furthermore, notice that $v'^{T^*}_{\mathbf{x}}(T^*) = v'(N) - x(N \setminus T^*) = v(N) - x(N \setminus T^*) = x(T^*)$. Hence it trivially holds that $x'(N) = x'(T^*) + x(N \setminus T^*) = v'^{T^*}_{\mathbf{x}}(T^*) + x(N \setminus T^*) = v(N) - x(N \setminus T^*) + x(N \setminus T^*) = v(N) = v'(N)$.

In case (ii), let us see that $S \cap T^* = \{i_*\}$ since T^* is a minimal coalition in $\mathcal{D}_S$. If the contrary holds, that is, if $|S \cap T^*| \geq 2$, since $v'^{T^*}_{\mathbf{x}}$ is an average monotonic game with respect to a vector $\alpha \in \mathbf{R}^{T^*}_{+} \setminus \{\mathbf{0}\}$, we will prove that $v'^{T^* \setminus \{i_*\}}_{\mathbf{x}}$, where $i_* \in S \cap T^*$ will be also an average monotonic game, contradicting the minimality of T^* in $\mathcal{D}_S$.

From the definition of the reduced game and using property (5), we have

$$v'^{T^* \setminus \{i_*\}}_{\mathbf{x}}(R) = \max\{v'^{T^*}_{\mathbf{x}}(R),\, v'^{T^*}_{\mathbf{x}}(R \cup \{i_*\}) - x_{i_*}\}$$

for all $\emptyset \neq R \subseteq T^* \setminus \{i_*\}$, $R \neq T^* \setminus \{i_*\}$

$$v'^{T^* \setminus \{i_*\}}_{\mathbf{x}}(T^* \setminus \{i_*\}) = x(T^* \setminus \{i_*\}).$$

Since $v'^{T^*}_{\mathbf{x}}$ is an average monotonic game, we know its positiveness, and so, $v'^{T^* \setminus \{i_*\}}_{\mathbf{x}}(R)$ for all $\emptyset \neq R \subseteq T^* \setminus \{i_*\}$, $R \neq T^* \setminus \{i_*\}$. Let us see the positiveness of the worth of the grand coalition. First of all, since $\mathbf{x} \in C(v)$ we know $v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(T^* \setminus \{i_*\}) = x(T^* \setminus \{i_*\}) \geq v(T^*\setminus\{i_*\})$. Secondly, since $T \neq T^*\setminus\{i_*\}$ and $v \leq_S v'$ we know $v(T^*\setminus\{i_*\}) = v'(T^*\setminus\{i_*\})$. Finally, since v' is an average monotonic game we have $v' \geq 0$, which implies that

$$v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(T^* \setminus \{i_*\}) = x(T^* \setminus \{i_*\}) \geq v(T^* \setminus \{i_*\}) \geq 0.$$

Therefore positiviness of $v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}$ is proved.

To prove condition *ii)* in the definition of average monotonic game, let $S_1 \subseteq S_2 \subseteq T^* \setminus \{i^*\}$, where $S_1 \neq S_2$. Notice first that if $\alpha_{|T^*\setminus\{i_*\}} = 0$ then $v_{\mathbf{x}}^{\prime T^*}(R) = 0$ for all $\emptyset \neq R \subseteq T^* \setminus \{i_*\}$ and $v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(T^* \setminus \{i_*\}) = x(T^* \setminus \{i_*\}) \geq 0$. Hence, $v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}$ will be an average monotonic game. Otherwise, i.e. $\alpha_{|T^*\setminus\{i_*\}} \neq 0$ we will study two cases.

Case 1: $\alpha(S_1) = 0$.

Since $v_{\mathbf{x}}^{\prime T^*}$ is an average monotonic game w.r.t. $\alpha \in \mathbf{R}_+^{T^*} \setminus \{\mathbf{0}\}$ we know

$$0 \leq \alpha(T^*) \cdot v_{\mathbf{x}}^{\prime T^*}(S_1) \leq \alpha(S_1) \cdot v_{\mathbf{x}}^{\prime T^*}(T^*), \text{ and then } v_{\mathbf{x}}^{\prime T^*}(S_1) = 0$$

Moreover, by hypothesis of the theorem,

$$\begin{aligned} x_{i_*} &\geq p_{i_*}(v_{\mathbf{x}}^{\prime T^*}) = \alpha_{i_*} \cdot \tfrac{v_{\mathbf{x}}^{\prime T^*}(T^*)}{\alpha(T^*)} = (\alpha(S_1) = 0) \\ &= \alpha(S_1 \cup \{i_*\}) \cdot \tfrac{v_{\mathbf{x}}^{\prime T^*}(T^*)}{\alpha(T^*)} \geq (v_{\mathbf{x}}^{\prime T^*} \text{ is average monotonic }) \\ &\geq v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\}). \end{aligned}$$

Therefore,

$$\begin{aligned} v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(S_1) &= \max\{v_{\mathbf{x}}^{\prime T^*}(S_1), v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\}) - x_{i_*}\} \\ &= \max\{0, v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\}) - x_{i_*}\} = 0 \end{aligned}$$

which implies that

$$0 = \alpha(S_2) \cdot v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(S_1) \leq \alpha(S_1) \cdot v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(S_2) = 0.$$

Case 1: $\alpha(S_1) > 0$.

In this case we have to prove $\frac{v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(S_1)}{\alpha(S_1)} \leq \frac{v_{\mathbf{x}}^{\prime T^*\setminus\{i_*\}}(S_2)}{\alpha(S_2)}$. We know, since $v_{\mathbf{x}}^{\prime T^*}$ is average monotonic with respect to α,

$$\frac{v_{\mathbf{x}}^{\prime T^*}(S_1)}{\alpha(S_1)} \leq \frac{v_{\mathbf{x}}^{\prime T^*}(S_2)}{\alpha(S_2)} \tag{7}$$

Let us prove that

$$\frac{v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\}) - x_{i_*}}{\alpha(S_1)} \leq \frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\}) - x_{i_*}}{\alpha(S_2)} \tag{8}$$

Notice that

$$\begin{aligned}
\frac{v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\}) - x_{i_*}}{\alpha(S_1)} \quad &= \frac{v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\})}{\alpha(S_1 \cup \{i_*\})} + \frac{\frac{v_{\mathbf{x}}^{\prime T^*}(S_1 \cup \{i_*\})}{\alpha(S_1 \cup \{i_*\})} \cdot \alpha_{i_*} - x_{i_*}}{\alpha(S_1)} \\
&\leq (\text{since } v_{\mathbf{x}}^{\prime T^*} \text{ is average monotonic w.r.t. } \alpha) \\
&\leq \frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\})}{\alpha(S_2 \cup \{i_*\})} + \frac{\frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\})}{\alpha(S_2 \cup \{i_*\})} \cdot \alpha_{i_*} - x_{i_*}}{\alpha(S_1)} \\
&\leq (\text{since } 0 < \alpha(S_1) \leq \alpha(S_2) \text{ and} \\
&\qquad\qquad \frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\})}{\alpha(S_2 \cup \{i_*\})} \cdot \alpha_{i_*} \leq p_{i_*}(v_{\mathbf{x}}^{\prime T^*}) \leq x_{i_*}) \\
&\leq \frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\})}{\alpha(S_2 \cup \{i_*\})} + \frac{\frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\})}{\alpha(S_2 \cup \{i_*\})} \cdot \alpha_{i_*} - x_{i_*}}{\alpha(S_2)} \\
&= \frac{v_{\mathbf{x}}^{\prime T^*}(S_2 \cup \{i_*\}) - x_{i_*}}{\alpha(S_2)}
\end{aligned}$$

From (7) and (8) we obtain,

$$\frac{v_{\mathbf{x}}^{\prime T^* \setminus \{i_*\}}(S_1)}{\alpha(S_1)} \leq \frac{v_{\mathbf{x}}^{\prime T^* \setminus \{i_*\}}(S_2)}{\alpha(S_2)}$$

and so we conclude that $v_{\mathbf{x}}^{\prime T^* \setminus \{i_*\}}$ is an average monotonic game, contradicting the minimality of T^* in $\mathcal{D}_S$. Therefore, in case (ii) it holds that $|S \cap T^*| = 1$ or equivalently $S \cap T^* = \{i_*\}$.

To finish the proof of the theorem we have to check $\mathbf{x}' \in C(v')$ where $x_i \leq x'_i$, for all $i \in S$ (recall that $v \leq_S v'$ and v' is average monotonic). As $S \cap T^* = \{i_*\}$ and, by hypothesis of the case (ii), $x_{i_*} \geq p_{i_*}(v_{\mathbf{x}}^{\prime T^*}) \geq v_{\mathbf{x}}^{\prime T^*}(i_*) = \max_{\emptyset \subseteq Q \subseteq N \setminus T^*}\{v'(\{i_*\} \cup Q) - x(Q)\} \geq (Q = S \setminus \{i_*\}) \geq v'(S) - x(S \setminus \{i_*\})$ and so $x(S) \geq v'(S)$. Therefore we have deduced that $\mathbf{x}' = \mathbf{x}$ is one of the desired vectors.□

This latest result has an interesting application in order to study monotonicity properties in applied models. For instance, consider the one input-one output joint production situation described by Mas-Colell et al. (1996) that coud be defined by the pair (ω, f) where $\omega \in \mathbf{R}^n_{++}$ is the vector of inputs and f is the production function, $f(0) = 0$, and $f(z)/z$ is nondecreasing in $\mathbf{R}_+ \setminus \{\mathbf{0}\}$, i.e. if $0 < z_1 \leq z_2$ then $\frac{f(z_1)}{z_1} \leq \frac{f(z_2)}{z_2}$. As we have seen before we can associate a cooperative game $v(S) := f(\sum_{j \in S} \omega_j)$. The core of this game is nonempty. Take now an element of the core and suppose that a player i_* decides to increase his input contribution, that is $\omega'_{i_*} > \omega_{i_*}$ being the rest of inputs fixed $\omega'_i = \omega_i$, for all $i \neq i_*$ (we denote this fact by $\omega' >_{i_*} \omega$). The natural question is whether there exist a core allocation $\mathbf{x}'$ in the new game $v'(S) := f(\sum_{j \in S} \omega'_j)$ such that $x'_{i_*} > x_{i_*}$. The answer is yes and this result is proved in the next proposition.

Proposition 5 *Let (ω, f) and (ω', f) be two one input-one output joint production problems such that $\omega' >_{i_*} \omega$, and let v and v' the respective associated games. Then, for any $\mathbf{x} \in C(v)$ there exists $\mathbf{x}' \in C(v')$ such that $x'_{i_*} > x_{i_*}$.*

Proof As $\omega'_{i_*} > \omega_{i_*}$ and $\omega'_i = \omega_i$ for all $i \neq i_*$, we have to study two cases depending whether some coalitions have increased their average worth or not. Let $\mathcal{S}$ be the set of coalitions that have increased their average worth. Formally, $\mathcal{S} = \{S \subseteq N \text{ such that } \frac{f(\omega'(S))}{\omega'(S)} > \frac{f(\omega(S))}{\omega(S)}\}$. Notice that the set $\mathcal{S}$ is finite, $|\mathcal{S}| = m$, and $i_* \in S$, for any $S \in \mathcal{S}$.

Case 1 $\mathcal{S} = \emptyset$.
In that case $v'(S) = \omega'(S) \cdot \frac{f(\omega(S))}{\omega(S)}$ since, for any $S \subseteq N$, $\frac{f(\omega'(S))}{\omega'(S)} = \frac{f(\omega(S))}{\omega(S)}$.

Hence, let us define $\mathbf{x}' \in \mathbf{R}^n$

$$x'_i := \begin{cases} x_i, & \text{for all } i \neq i_*, \\ \\ x_{i_*} + (\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(N))}{\omega(N)}, & \text{if } i = i_* \end{cases}$$

First notice that $x'_{i_*} > x_{i_*}$. Moreover, the vector $\mathbf{x}' \in C(v')$ since $x'(N) = x(N) + (\omega'_{i_*} - \omega_{i_*})\frac{f(\omega(N))}{\omega(N)} = \frac{f(\omega(N))}{\omega(N)} \cdot (\omega(N \setminus \{i_*\}) + \omega'_{i_*}) = \frac{f(\omega'(N))}{\omega'(N)} \cdot \omega'(N) = f(\omega'(N)) = v'(N)$ and, for any $S \subseteq N$, if $i_* \notin S$ then $x'(S) = x(S) \geq v(S) = v'(S)$ and, if $i_* \in S$, $x'(S) = x(S) + (\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(N))}{\omega(N)} \geq v(S) + (\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(N))}{\omega(N)} = \omega(S) \cdot \frac{f(\omega(S))}{\omega(S)} +$

$(\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(N))}{\omega(N)} \geq \omega(S) \cdot \frac{f(\omega(S))}{\omega(S)} + (\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(S))}{\omega(S)} \geq (\omega(S \setminus \{i_*\}) + \omega'_{i_*}) \cdot \frac{f(\omega(S))}{\omega(S)} = \omega'(S) \cdot \frac{f(\omega'(S))}{\omega'(S)} = f(\omega'(S)) = v'(S)$.

Case 2 $\mathcal{S} \neq \emptyset$.

Let us denote $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$ where

$$\frac{f(\omega'(S_1))}{\omega'(S_1)} \leq \frac{f(\omega'(S_2))}{\omega'(S_2)} \leq \ldots \leq \frac{f(\omega'(S_m))}{\omega'(S_m)}.$$

We have ordered in an average increasing way the coalitions of $\mathcal{S}$. Notice that player i_* belongs to all these coalitions, that is, $i_* \in S_1 \cap S_2 \cap \ldots \cap S_{m-1} \cap S_m$. Furthermore, let us define the game v_0 and the vector $\mathbf{x}_0 \in \mathbb{R}^n$ as

$v_0(S) = \omega'(S) \cdot \frac{f(\omega(S))}{\omega(S)}$, for all $S \subseteq N$,

$$x_{0,i} := \begin{cases} x_i & \text{for all } i \neq i_*, \\ \\ x_{i_*} + (\omega'_{i_*} - \omega_{i_*}) \cdot \frac{f(\omega(N))}{\omega(N)} & \text{if } i = i_* \end{cases}$$

Following the same reasoning that in the first case, we can prove $\mathbf{x}_0 \in C(v_0)$ and $x_{0,i_*} > x_{i_*}$. Let us now consider a increasing finite sequence $v_1, v_2, \ldots, v_m$ $(m = |\mathcal{S}|)$ of average monotonic games with respect to the vector $\omega' \in \mathbb{R}^n_{++}$ defined as follows:

$$v_1(S) = \begin{cases} v_0(S) & S \neq S_1 \\ v'(S_1) & S = S_1 \end{cases}$$

$$v_2(S) = \begin{cases} v_1(S) & S \neq S_2 \\ v'(S_2) & S = S_2 \end{cases}$$

$$\vdots$$

$$v_m(S) = \begin{cases} v_{m-1}(S) & S \neq S_m \\ v'(S_m) & S = S_m \end{cases}$$

Notice first that $v_0 \leq_{S_1} v_1 \leq_{S_2} v_2 \leq \ldots \leq_{S_m} v_m$ (recall that $v \leq_S v'$ means that only the worth of coalition S has increased and so, only one coalition has increased its worth in each step). Moreover, $v_m = v'$ since $v_m(S) = v_0(S) = \omega'(S)\frac{f(\omega(S))}{\omega(S)} = f(\omega'(S)) = v'(S)$, for all $S \notin \mathcal{S}$, and $v_m(S_k) = v_k(S_k) = v'(S_k)$, for all $S_k \in \mathcal{S}$.

It is straightforward to see that all the games of the sequence v_k, $k = 1, 2, \ldots, m$ are average monotonic w.r.t. $\omega' \in \mathbb{R}^n_{++}$. This is due to the fact that v_0 is an average monotonic game with respect to ω' and the way we have ordered coalitions in $\mathcal{S}$. Therefore, the situation is

$$v_0 \leq_{S_1} v_1 \leq_{S_2} v_2 \leq \ldots \leq_{S_m} v_m = v'$$

and $\mathbf{x}_0 \in C(v_0)$ where $x_{i_*} < x_{0,i_*}$. Applying reiteratively theorem 4 we obtain there exists a sequence of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ such that $\mathbf{x}_1 \in C(v_1), \mathbf{x}_2 \in C(v_2), \ldots, \mathbf{x}_m \in C(v_m)$ and $x_{k-1,i} \leq x_{k,i}$ for all $i \in S_k$, $k = 1, 2, \ldots, m$. Since $i_* \in S_1 \cap S_2 \cap \ldots S_{m-1} \cap S_m$ we obtain

$$x_{i_*} < x_{0,i_*} \leq x_{1,i_*} \leq \ldots \leq x_{m,i_*}$$

and $x_m \in C(v_m) = C(v')$. Hence, the proof is finished. □

At first sight, it seems not difficult and natural to extend this result to the case of two or more players increasing their contribution at the same time. Surprisingly, next example shows the contrary.

Example 1 *Let* $\omega = (1, 2, 3)$ *and* $f(x) := \begin{cases} x & 0 \leq x < 5 \\ 1.5 \cdot x & 5 \leq x \end{cases}$. *The associated game is*

$$v(1) = 1,\ v(2) = 2, v(3) = 3,\ v(12) = 3,\ v(13) = 4,\ v(23) = 7.5,\ v(N) = 9$$

Let us take the point $\mathbf{x} = (1, 5, 3)$ *in the core of the game* v *and suppose that players 1 and 2 increase their initial contribution in one unit, that is* $\omega' = (1+1, 2+1, 3) = (2, 3, 3)$. *The new associated game is*

$$v'(1) = 2,\ v'(2) = 3, v'(3) = 3,\ v'(12) = v'(13) = 7.5,\ v'(23) = 9,\ v'(N) = 12$$

Notice that for any core element $\mathbf{x}'$ *of* v' *we know* $x'_2 \leq v'(N) - v'(\{13\}) = 4.5$. *Therefore it will be not possible to find a core element* $\mathbf{x}'$ *of* v' *such that* $x'_1 \geq x_1$ *and* $x'_2 \geq x_2 = 5$.

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