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Hat Derivatives

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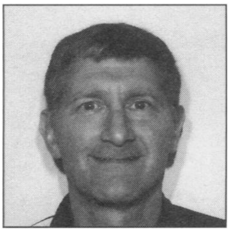

Stephen B Maurer (smaurer1@swarthmore.edu) received his Ph.D. from Princeton in 1972, under A. W. Tucker. He is a mathematics professor at Swarthmore College, where he was also an undergraduate. He is coauthor of *Discrete Algorithmic Mathematics*, with Anthony Ralston. At one time he chaired the MAA high school contests, and he has written guides for undergraduates on writing mathematics. At parties he used to be able to tell people their area codes and their zip codes (to within 5) just by knowing where they lived.

Problem 1. Last year, the population grew by 1% and the average income per person grew by 2%. By what percent did national income grow?

People with number sense know in their hearts, if not by a theorem, that the answer is 3% (but not quite exactly), computed by adding $1 + 2 = 3$. However, many of our students don't know this, and don't have a clue that such problems are easy to do in their heads.

Economists know these things because they apply what is to them a standard result on percentage rates of change, often denoted with hats:

$$\widehat{xy} = \hat{x} + \hat{y}. \tag{1}$$

We mathematicians know (1) too, as soon as we are told that hat derivatives are just logarithmic derivatives:

$$\hat{x} = \frac{x'}{x} = \log' x. \tag{2}$$

It is easy to prove (1) from (2).

Unfortunately, we mathematicians tend to regard logarithmic differentiation merely as a technique for computing certain ugly derivatives that are best done by machine today anyway. To the contrary, it is a tool for generating many useful relationships that allow for mental computation. It is also the right concept for thinking about derivatives in relative (percentage) terms, which is how they are usually thought of outside of mathematics texts.

Despite such importance, I have never found the hat derivative discussed explicitly in calculus books. I'm sure it's in some of them somewhere (tell me at smaurer1@swarthmore.edu if you know where), but not conspicuously, so we don't teach it. The purpose of this article is to put this concept in lights and encourage you to introduce it in your classes. Specifically, I will lay out the definition of hat derivative, state the main results (the hat derivative rules), provide a series of example problems (which I have used with my students), and conclude with some alternative presentations, both less advanced (down to precalculus) and more advanced (up to multivariate).

Percentage rates defined

All quantities x, y, z will be functions of one independent variable t, usually time.

As always, **absolute change** is $\Delta x = x(t + \Delta t) - x(t)$, and the **average rate of absolute change** is the difference quotient defined in all calculus books, $\Delta x/\Delta t$. In contrast, **relative change** is

$$\frac{\Delta x}{x} \tag{3}$$

and the **average rate of relative change** is

$$\frac{\Delta x/x}{\Delta t}. \tag{4}$$

Percentage change and **average rate of percentage change** are 100 times (3) and (4), respectively. However, percents are merely a different unit, so we will usually refer to relative quantities and percentage quantities interchangeably and avoid writing 100. Another word for "relative" is "proportional".

By the **instantaneous rate of relative change** of x, denoted $\hat{x}$, we mean the limit of the average relative rate:

$$\hat{x} = \lim_{\Delta t \to 0} \frac{\Delta x/x}{\Delta t}.$$

Rearranging the fraction, we get

$$\hat{x} = \lim_{\Delta t \to 0} \frac{\Delta x/\Delta t}{x} = \frac{x'}{x} = \log' x,$$

as claimed in (2).

Just as x', the instantaneous rate of absolute change, is often just called the rate of change, so is $\hat{x}$ often called the **rate of relative change** or the **relative rate of change**.

Calculus is the mathematics of change. Calculus courses usually express change in terms of x'. Newspapers are full of discussion of change too, but almost invariably in terms of percentage change. Hat derivatives provide a formalism for such discussions.

To be sure, some changes are more natural in absolute terms. It would seem odd to describe velocity as a percentage rate of change in distance traveled. Distance from where? Choose a different starting point and you get a different percentage. But for many quantities there is a natural starting point—0 population, 0 income, etc., so percentage change is natural. Also, relative change rates have the advantage of being dimensionless except for time. Americans can understand what it means for German national income to grow by 3% per year without knowing the worth of marks or euros.

The hat derivative rules

The final advantage of hat derivatives is that it is often easy to compute with them mentally. Specifically, from (2) it is easy to prove the following rules (k is a constant).

$$\text{If } y = kx, \text{ then } \hat{y} = \hat{x}. \tag{5}$$

$$\text{If } z = xy, \text{ then } \hat{z} = \hat{x} + \hat{y}. \tag{6}$$

$$\text{If } y = x^k, \text{ then } \hat{y} = k\hat{x}. \tag{7}$$

$$\text{If } z = x/y, \text{ then } \hat{z} = \hat{x} - \hat{y}. \tag{8}$$

For instance, to prove (8), write

$$\log z = \log(x/y) = \log x - \log y,$$
$$\log' z = \log'(x/y) = \log' x - \log' y,$$
$$\frac{z'}{z} = \frac{x'}{x} - \frac{y'}{y},$$
$$\hat{z} = \hat{x} - \hat{y}.$$

It's a good exercises to get students to prove some of these hat rules, and even to conjecture the statement of, say, (8), having seen the other statements.

We return to Problem 1. Let x be population, y be average income per person, and z be national income, all at time t in years. Then $z = xy$. If we interpret the yearly changes as instantaneous rates, then we seek $\hat{z} = \hat{x} + \hat{y}$, and so $\hat{z}$ in percents is indeed $1 + 2 = 3$.

Of course, yearly changes are not actually instantaneous rates; they are neither rates nor instantaneous! However, it is fair to interpret, say, a yearly population *change* of 1% as an *average rate*, because we may divide by $\Delta t = 1$. Then, replacing this average rate with its instantaneous limit is an approximation, usually a very good one (a point we return to later). Approximating changes with x' is commonplace in calculus books. The point here is that the same goes for relative changes: replace them with hats.

Sample problems

To show what the hat rules (5–8) are good for, and how you might use them with your students, I offer some problems. Most of the calculations are easy. However, setting up the problem sometimes takes thought. Also, in one or two cases the simple rules don't apply, and students must come to understand why.

Problem 2. A leaf is growing in length by 3% a day. At what rate is the surface area growing? (Surface area is proportional to the square of the length.) How is the energy absorbed by the leaf from the sun growing? (Energy absorbed is proportional to surface area.)

Problem 3. Ima Student works irregular hours while trying to pass her courses. Last academic year her total earnings from work were $1400. This year she earned 12% more. But, checking her pay slips, she finds she has worked 8% more hours this year.

a) By what percent did her average hourly wage change from last year to this year?
b) The inflation rate between last year and this year was 5%. How did Ima's real earnings change, where "real" means in "constant" dollars? How did her real average hourly wage change?

Problem 4. A gas satisfies the law $V = kMT/P$, where V is the volume, k is a constant, M is the mass, T is the temperature, and P is the pressure. If at a certain instant the mass is increasing by 2% (more molecules are being pumped in), the temperature is falling by 3%, and the pressure is increasing by 1%, how is the volume changing?

Problem 5. The force of gravity exerted on an object in space by the earth varies as the inverse square of the object's distance from the earth. Suppose that an object's distance from the earth is increasing instantaneously by 2% per hour. How is the the earth's gravitational attraction on the object changing?

Problem 6. This year at University Y, the total fee for undergraduates increased by 7% over last year. The number of undergraduates increased by 1%. The fraction of the total fee that the average student paid decreased by 3% from 60% to 57% (the rest is covered by scholarship, loans and work). How did the total income to University Y from undergraduate fees change? *Caution*: the 3% decrease above is not a percentage change in the sense we have defined. Percents have several standard usages and one must always ask: percent *of what*?

Problem 7. Last year the tuition at College Z increased by 6% and the room and board fee increased by 8%. How did the total fee (tuition, room, and board) change?

Problem 8. Suppose your hourly wage last year was $6 and you worked 35 hours a week. Suppose your hourly wage went up by 10% for this year and you agreed to work 20% more hours (you're hungry).

a) What was your weekly income last year, exactly? This year?
b) Using part a), what was your percentage rate of change (per year) in weekly income?
c) According to the rules of hat derivatives (7–8), what was your percentage rate of change in weekly income?
d) The answers in b) and c) disagree! How can this be?

Alternative presentations

Derivatives of logarithms often aren't introduced until the second semester of calculus. However, hat derivatives can be treated earlier without logarithms. In fact, they can be done without calculus at all—a good thing, since everyone ought to have a sense of how percentage changes combine.

With first semester Calculus. Once you know the usual product rule for derivatives, the hat product rule (6) is easily obtained.

$$(xy)' = x'y + xy',$$

so

$$\widehat{xy} = \frac{(xy)'}{xy} = \frac{x'y + xy'}{xy} = \frac{x'}{x} + \frac{y'}{y} = \hat{x} + \hat{y}.$$

The other hat rules can be proved from the corresponding prime rules similarly.

With no Calculus. We are limited to using average rates of relative change, or just relative change. If $z = xy$, then

$$z + \Delta z = (x + \Delta x)(y + \Delta y)$$

so

$$\Delta z = (\Delta x)y + x(\Delta y) + \Delta x \Delta y$$

and thus

$$\frac{\Delta z}{z} = \frac{(\Delta x)y + x(\Delta y) + \Delta x \Delta y}{xy} = \frac{\Delta x}{x} + \frac{\Delta y}{y} + \left(\frac{\Delta x}{x}\right)\left(\frac{\Delta y}{y}\right). \tag{9}$$

The last term is usually small compared to the others and can be ignored.

Of course, this is very close to the main part of the proof of the usual calculus product rule, but it won't hurt students to see it in a useful context before calculus.

A further advantage of carrying out the noncalculus proof is that it shows why the hat derivative rules are only approximations for relative changes and average relative rates, and it indicates how to tell just how good the approximations are. For instance, based on (9) you can solve the following problem.

Problem 9. Suppose x increases by r percent and y increases by s percent. What are the smallest values of r and s so that $z = xy$ increases by $r + s + 1$ percent instead of $r + s$ percent?

Problem 9 is a little open ended (what does it mean for a pair of variables to be smallest?), but it is easy to come up with good answers.

With more Calculus. To analyze how average relative rates differ from instantaneous rates, we can use Taylor series. Suppose we are interested in a/x as x varies from b. If your students agree that, by scaling, it is sufficient to consider $y = 1/x$ when x varies from 1, then a solution is easy. The geometric series gives

$$\frac{1}{x} = \frac{1}{1+\Delta x} = 1 - \Delta x + (\Delta x)^2 - (\Delta x)^3 + \cdots,$$

and so

$$\Delta y = \frac{1}{x} - 1 = -\Delta x + (\Delta x)^2 - \cdots. \tag{10}$$

Thus, to the first approximation, the percent change in y is negative the percent change in x, but in reality the percent change in y is greater (less negative).

If claims about scaling are not accepted, you can take the Taylor series of $y = a/x$ around $x = b$. After some algebra, we get from

$$y = \frac{a}{x} = \frac{a}{b} - \frac{a}{b^2}(x-b) + \frac{a}{b^3}(x-b)^3 + \cdots$$

to

$$\frac{\Delta y}{y} = -\frac{\Delta x}{b} + \left(\frac{\Delta x}{b}\right)^2 + \cdots,$$

which gives the same conclusion about percentage changes as (10).

With multivariate methods. Consider how the next problem differs from previous ones.

Problem 10. Suppose $f(x, y) = x^2 + 2xy^2$. Suppose we measure x and y to be 1 and -1, respectively, but there might be a 1% error in each case. We compute $f(1, -1)$ to be 3. By what percent might our value of f be off?

This problem is not about functions of time, but rather about two independent quantities that may have been mismeasured. Of course, we could pretend that these errors occurred over time, which would allow us to solve this problem by first-year calculus methods. However, multivariate concepts allow us to show that every problem of the form $z = f(x, y)$ (or more variables) can be analyzed for percentage growth. Namely, using total derivative notation this time, we have

$$dz = f_x dx + f_y dy.$$

Uglifying, we get

$$\frac{dz}{z} = \frac{xf_x}{z}\left(\frac{dx}{x}\right) + \frac{yf_y}{z}\left(\frac{dy}{y}\right). \tag{11}$$

In other words, for *every* differentiable function z of x, y, the relative rate $\hat{z}$ is a *linear combination* of $\hat{x}$ and $\hat{y}$ (but unlike for the simple functions treated previously, the coefficients of $\hat{x}$ and $\hat{y}$ may change from point to point).

For Problem 10, (11) reduces to

$$\frac{dz}{z} = \frac{4}{3}\left(\frac{dx}{x}\right) + \frac{4}{3}\left(\frac{dy}{y}\right).$$

Therefore, the greatest percentage error possible (as a first-order approximation) in our particular value of f is $\pm\frac{8}{3}\%$.

Problem 11. If you did Problem 7, you concluded that it can't be answered. How is this possible given the general answer in (11)?

To analyze the difference between average and instantaneous rates when there are several variables, one may again use Taylor expansions, this time multivariate (not shown).

Literature

I first learned about hat-derivatives when writing some papers with economists in the 1980s. Hat rules are commonplace for them, and the notation is familiar, if not standard. A well-known text devotes a section to the concept [**1**, Sec. 10.7], including the rules. Interestingly, the author does not call $\hat{x}$ the relative rate or use hats. He calls it the "instantaneous rate of growth." He contrasts this (on pp. 291–2) with the usual derivative, which he calls the "instantaneous rate of change." Do both phrases mean x' to you, as they do to me?

Plea

I've tried to make a case that hat derivatives are important, interesting, and easy to introduce. Dear calculus book writers: add a brief section on them to your next edition! By all means, steal my problems.

Reference

1. Alpha C. Chiang, *Fundamental Methods of Mathematical Economics*, 2nd ed., McGraw-Hill, 1974.