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The Valuation of Volatility Options*

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Résumé / Abstract

Cet article examine l'évaluation des options sur volatilité, de type européen ou américain, dans le cadre d'un modèle d'équilibre général avec volatilité stochastique. Certaines propriétés de la région d'exercise optimal et du prix de l'option sont établies lorsque la volatilité suit un processus général de diffusion. Des formules d'évaluation explicites sont ensuite dérivées dans quatre cas particuliers. Nous étudions en détail le cas d'un processus de volatilité de type MRLP (mean-reverting in the log) qui semble être conforme à l'évidence empirique. Les propriétés et le comportement de couverture des options sur volatilité sont examinées dans ce cadre. À l'opposeé d'une option d'achat américaine, le prix d'une option d'achat européenne sur volatilité s'avère être une fonction concave lorsque le niveau de volatilité s'élève.

This paper examines the valuation of European- and American-style volatility options based on a general equilibrium stochastic volatility framework. Properties of the optimal exercise region and of the option price are provided when volatility follows a general diffusion process. Explicit valuation formulas are derived in four particular cases. Emphasis is placed on the MRLP (mean-reverting in the log) volatility model which has received considerable empirical support. In this context we examine the properties and hedging behavior of volatility options. Unlike American options, European call options on volatility are found to display concavity at high levels of volatility.

Mots Clés : Volatilité stochastique, options européennes, options américaines, exercice optimal, prime d'exercice anticipé, couverture de risque, viabilité

Keywords: Stochastic volatility, European options, American options, optimal exercise, early exercise premium, hedging, viability

JEL: G12, G13

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1 Introduction

One well known conclusion of empirical studies pertaining to security markets is that the volatility of asset returns tends to change over time. Changing volatility is apparent in bond and foreign currency markets, but is perhaps most evident in stock markets. The prevalence of volatility fluctuations has prompted the CBOE to introduce, in February 1993, a Market Volatility Index, the VIX, to assist investors in tracking the volatility risk in the stock market. This index provides market participants with minute by minute estimates of the expected volatility of the S&P 100 index (OEX) by averaging the implied volatilities of at-the-money put and call OEX options.

Since the introduction of the VIX, exchanges in several other countries have also launched volatility indices.[2] The usefulness of such indices is predicated on the understanding of the risk of changing volatility in financial assets' returns, risk that may not be spanned by asset returns. As a result, positions in these assets, or in derivative products based on these assets, may not be sufficient to hedge away all the uncertainty embedded in volatility. Contingent claims on the volatility of assets may well be needed to increase the set of hedging instruments available to investors. This is perhaps what prompted several exchanges around the world to consider introducing derivatives written on volatility indices. The CBOE and the AMEX, for instance, have pending options on volatility. The German Futures and Options Exchange has already innovated and markets a futures contract on the VDAX.

In this paper we study the valuation of options on volatility in a general equilibrium stochastic volatility framework. Our starting point is a general class of volatility processes which are known to be viable (see, for instance, Broadie, Detemple, Ghysels and Torres (2000)). Popular members of this class include the geometric Brownian motion process (GBMP), the mean-reverting Gaussian process (MRGP), the mean-reverting square-root process (MRSRP) and the mean-reverting log process (MRLP). We first discuss the structure of the exercise set and the valuation of American volatility options for a general volatility process. Explicit valuation formulas, both for European- and American-style options, are then provided for each of the four popular specifications listed above. Following Bakshi and Madan (1999) we present a unified derivation of some of the components of these option values across models. This unified derivation relies on the explicit formula for the truncated characteristic function of the normal distribution which is the common building block underlying each of the four volatility models.

We then devote more specific attention to the MRLP volatility specification, which has not, heretofore, been explored in the option's literature. The relevance of this model is motivated by (i) substantial empirical evidence supporting the EGARCH model of Nelson (1991) (see Engle and Ng (1993), Danielsson (1994), Hentschel (1995)) and (ii) the fact that EGARCH converges to a gaussian process that is mean reverting *in the log* and thus matches our MRLP specification.[3] In this context we show that the hedging behaviors of European- and American-style options differ quite drastically. It is of particular interest to note that the European call option, unlike its

[2]For example, in December 1995, a volatility index, the VDAX was introduced by the German Futures and Options Exchange.

[3]Some of the most common choices of volatility specification display mean reversion in volatility levels. Recent empirical studies, however, indicate that such models may be misspecified (Nandi (1998)). Related empirical evidence is also reported in Ait-Sahalia and Lo (1998), Bakshi, Cao and Chen (1997), Bates (1996) and Dumas, Fleming and Whaley (1998).

American counterpart, displays *concavity* at high levels of volatility. This property which seems to contradict established results on the convexity of European call option prices (e.g. Bergman, Grundy and Wiener (1996)) is due to the particular dependence of the distribution of the rate of change of volatility on the current level of volatility. Concavity is also a new feature which had not been identified in the prior literature on volatility options. Furthermore its impact proves to be quite important since high volatility tends to occur precisely when stock returns are low.

The idea of derivative securities on volatility is not new. Brenner and Galai (1989), Whaley (1993), and Grunbichler and Longstaff (1996) propose valuation formulas for futures and options on volatility using different models and measures of volatility. So far, efforts have focused on European-style derivatives assuming volatility processes of the GBMP, MRGP and MRSRP types. Grunbichler and Longstaff (1996), in particular, use a CIR model which is a MRSRP.[4] This specification choice has important consequences for the valuation of volatility options, the most notable of which is the implied convexity of the European call price. As discussed above the MRLP specification leads to drastically different behavior since call prices display concavity when volatility becomes large. Evidently, this feature has important consequences for risk management. Moreover, in contrast to the prior literature, we also focus on American options. Considering that traded options are often American-style contracts such emphasis is relevant.

The paper is organized as follows. Section 2 describes several models incorporating stochastic volatility. Section 3 provides valuation formulas for European- and American-style options for each of these models. Section 4 examines the properties of volatility options when the underlying follows an MRLP model. Concluding remarks are formulated in section 5. All proofs are collected in the appendix.

2 Stochastic volatility models

2.1 A class of viable stochastic volatility models.

Suppose that the underlying asset price satisfies

$$\begin{aligned} dS_t &= S_t \left[(r-\delta)dt + \sigma_1(Y_t,t)\left(\rho\, dZ_t^* + \sqrt{1-\rho^2}\, dB_t^* \right) \right] &(1) \\ dY_t &= \left[\mu^Y(Y_t,t) - \sigma^Y(Y_t,t)\rho\sigma_1(Y_t,t) \right] dt + \sigma^Y(Y_t,t)\, dZ_t^* &(2) \end{aligned}$$

where Y represents a volatility state variable, r is the constant interest rate, δ the constant dividend rate and ρ the constant instantaneous correlation between S and Y. The stock price volatility is $\sigma_1(Y_t,t)$. The dynamics of (S,Y) in (1)-(2) are written directly under the risk neutral measure Q. Thus, Z^* and B^* are independent Brownian motion processes under Q. The state variable process has drift $\mu^Y(Y_t,t)$ and volatility $\sigma^Y(Y_t,t)$ under the original probability measure; under the risk neutral measure it involves the risk premium correction $\sigma^Y(Y_t,t)\rho\sigma_1(Y_t,t)$.

[4] Brenner and Galai (1989) show how to construct a binomial tree to price volatility options. In their setting the asset price lives on a binomial (non-recombining) lattice with random up and down returns u_n, d_n. Volatility, which is defined as the difference $u_n - d_n$, is shown to follow a binomial expansion as well. Standard pricing methods can then be applied to price volatility options. Whaley (1993) prices volatility options assuming (i) that volatility follows a lognormal process and (ii) the existence of a futures contract on volatility with futures price equal to the current index level.

This model for (S,Y), with a constant interest rate r and constant dividend rate δ, can be supported in a general equilibrium setting (see, for instance, Broadie, Detemple, Ghysels and Torres (2000)). Note that the volatility structure is very general. Any triplet of functions $(\sigma_1(Y_t,t), \mu^Y(Y_t,t), \sigma^Y(Y_t,t))$ can be selected provided (1)-(2) has a solution. In other words the only restrictions that are required are Lipschitz and Growth conditions on these functions and a regularity condition which ensures the existence of the risk neutral measure. This leaves us with a large class of viable stochastic volatility models out of which empirically relevant candidates can be selected and used to value volatility option.

2.2 General properties of European options.

As we shall see further one of the interesting properties of European volatility options relates to their behavior relative to the underlying, namely volatility. Global properties of European options have been of long-standing interest in the literature. For instance, results of Merton (1973) and Jagannathan (1984) have established that a call price is increasing and convex with respect to the underlying asset price s if the risk-neutralized process

$$ds_t = s_t[r(t)dt + \sigma(t)dZ_t^*] \tag{3}$$

has deterministic coefficients, i.e. if $r(t), \sigma(t)$ are functions of time alone. In fact, as shown by Cox and Ross (1976), this result applies to any contingent claim with increasing and convex payoff function. Recent results of Bergman, Grundy and Wiener (BGW) (1996) extend these properties to more general (univariate or multivariate) diffusion processes modulo certain restrictions. For the univariate case, which is the relevant context for the volatility option models investigated in this paper, they show that the option is increasing and convex if the risk-neutralized process solves the stochastic differential equation (see eq. (4) in BGW)

$$ds_t = s_t[r(t)dt + \sigma(s_t,t)dZ_t^*] \tag{4}$$

where $r(t)$ is a function of time and $\sigma(s_t,t)$ is an arbitrary function of s,t, which satisfies required conditions for the existence of a solution to (4). Note that the drift of the underlying variable retains the proportional structure. If the volatility process under consideration here satisfies (4) then monotonicity and convexity of the volatility call price follow directly from BGW. It bears emphasizing, however, that the volatility models described in (2) may not belong to the same class of processes. This is clear from the general specification (2) and is illustrated in the next set of examples.

2.3 Examples of volatility specifications.

Several useful volatility models are obtained by combining various assumptions for the components of $(\sigma_1(Y_t,t), \mu^Y(Y_t,t), \sigma^Y(Y_t,t))$. We focus on four specific volatility models. Two of these (the mean reverting gaussian process (MRGP) and the mean reverting square root process (MRSRP)) are popular volatility specifications in the option valuation literature. The other two (the geometric Brownian motion process (GBMP) and the log mean reverting gaussian process (MRLP)) are considered to be reasonable descriptors of U.S. stock prices and stock indices.

Let $V_t = \sigma_1(Y_t, t)$ denote the volatility of the underlying asset. Our focal models are

	Type	Volatility process
Model 1	Geometric Brownian motion: GBMP	$dV_t = V_t\left[(2\mu + \sigma^2)dt + 2\sigma dZ_t^*\right]$
Model 2	Mean-reverting Gaussian: MRGP	$dV_t = (\alpha - \lambda V_t)\, dt + \sigma dZ_t^*$
Model 3	Mean-reverting square root: MRSRP	$dV_t = \left(\sigma^2 - 2\lambda V_t\right) dt + 2\sigma\sqrt{V_t}dZ_t^*$
Model 4	Mean-reverting log: MRLP	$d\ln(V_t) = (\alpha - \lambda\ln(V_t))dt + \sigma dZ_t^*$

Note that model 1 has a proportional structure. On the other hand models 2-4 have volatility-dependent drifts and therefore lie outside the class of processes covered by the results of BGW (1996). For model 4 it can be verified that $dV_t = V_t[(\beta - \lambda\ln(V_t))dt + \sigma dZ_t^*]$ where $\beta = \alpha + \frac{1}{2}\sigma^2$. Models 2 and 3 have drifts that are given by

$$V_t\left(\tfrac{\alpha}{V_t} - \lambda\right) \quad \text{and} \quad V_t\left(\tfrac{\sigma^2}{V_t} - 2\lambda\right)$$

respectively. The drift in model 4 in contrast is

$$V_t(\beta - \lambda\ln(V_t)).$$

Inspection of these expressions shows an important difference in drift structure across these models. For models 2 and 3 the drift becomes proportional to a constant as V tends to infinity. For model 4 the drift remains proportional to a function of V as V becomes large and in fact the factor of proportionality, $(\beta - \lambda\ln(V_t))$, tends to $-\infty$ as V tends to infinity. This aspect of the MRLP model distinguishes it from the other models above and in particular from model 3 which has been used in prior studies such as Grunbichler and Longstaff (1996).

Assumptions on the primitives underlying these specifications are detailed in Appendix A. Let us define the coefficients

$$\begin{aligned}
\phi_t &= e^{-\lambda t} \\
A_t &= \alpha\int_0^t e^{-\lambda(t-s)}ds = \frac{\alpha}{\lambda}(1-\phi_t) \\
a_t &= \sigma\left[\int_0^t e^{-2\lambda(t-s)}ds\right]^{1/2} = \frac{\sigma}{\sqrt{2\lambda}}\left(1-\phi_t^2\right)^{1/2} \\
w_t &\equiv a_t^{-1}\sigma\left[\int_0^t e^{-\lambda(t-s)}dZ_s^*\right].
\end{aligned}$$

Our next table summarizes the solutions for the volatility processes and lists their distributional properties.

	Solution	Distribution of random term
Model 1	$V_t = V_0\exp\left((2\mu - \sigma^2)t + 2\sigma Z_t^*\right)$	$Z_t^* \rightsquigarrow \text{normal}(0,t)$
Model 2	$V_t = V_0\phi_t + A_t + a_t w_t$	$w_t \rightsquigarrow \text{normal}(0,1)$
Model 3	$V_t = [\sqrt{V_0}\phi_t + a_t w_t]^2$	$w_t \rightsquigarrow \text{normal}(0,1)$
Model 4	$V_t = V_0^{\phi_t}\exp\left(A_t + a_t w_t\right)$	$w_t \rightsquigarrow \text{normal}(0,1)$.

Note that the distribution of the normalized volatility V/a^2 in model 3 is a non-central chi-square with non-centrality parameter γ_t and ν degrees of freedom given by

$$\begin{aligned}\gamma_t &= \frac{2\lambda}{\sigma^2(1-e^{-2\lambda t})}e^{-2\lambda t}V_0 \\ \nu &= 1.\end{aligned}$$

However, since V can be written as a function of $Y_t = V_0\phi_t + a_t w_t$ it can be computationally beneficial to view it as a quadratic transformation a standard normal random variable. The fact that a normal distribution underlies all the models above enables us to present a unified derivation of some of the components arising in the valuation of options across specifications (see Lemma 9 in Appendix B).

Option pricing models involving stochastic volatility generally assume one or the other of the above specifications. Hull and White (1987) and Johnson and Shanno (1987) adopt model 1, Stein and Stein (1991) and Scott (1992) focus on model 2, Heston (1993) on model 3, and Wiggins(1987) and Melino and Turnbull (1990) on model 4. In one of the few papers seeking to price volatility options, Grunbichler and Longstaff (1996) also use model 3.

3 Valuation of volatility options.

Our volatility models can all be summarized by a risk neutralized process taking the form

$$dV_t = V_t[(r-\delta_t^V)dt + \sigma_t^V dZ_t^*]$$

for appropriate choices of the coefficients (δ^V, σ^V). Here δ_t^V can be interpreted as an implicit "dividend" rate on the underlying volatility. Let $E^Q[\cdot]$ denote the expectation under the risk neutral measure. The value of a European call option c_0 written on the volatility V, with a strike X and maturity T is

$$c_0(V_0) = E^Q\left[e^{-rt}(V_T - X)^+\right].$$

Standard results state that the value of an American option can be written as the European option value plus an early exercise premium (Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992), Rutkowski (1994)). The early exercise premium is the present value of the gains from early exercise. When the underlying process is the volatility of an asset these gains are given by

$$r(V_t - X) - V_t(r-\delta_t^V) = \delta_t^V V_t - rX.$$

The first term of this expression, $r(V_t - X)$, captures the time value of money (savings realized by reinvesting the exercise proceeds at the riskfree rate) while the second term, $V_t(r-\delta_t^V)$, is the loss incurred by forgoing the natural appreciation of the payoff. The gains from early exercise are naturally collected when immediate exercise is optimal, i.e. in the exercise region $\mathcal{E}$. Standard results can be invoked to show that the exercise region is a closed set for all the models under consideration (see appendix B). Letting $B(\cdot)$ denote the optimal exercise boundary (or boundaries) we can write

$$C_0^A(V_0, B(\cdot)) = c_0(V_0) + \Pi_0(V_0, B(\cdot)) \tag{5}$$

where

$$\Pi_0(V_0, B(\cdot)) = E^Q\left[\int_0^T e^{-rv}(\delta_v^V V_v - rX)1_{\{V_v \in \mathcal{E}_v\}} dv\right] \tag{6}$$

represents the early exercise premium. The set $\mathcal{E}_v$ is the time v-section of the exercise set.

The next four theorems describe the values of volatility options for the models under consideration. For the GBMP model the exercise region is up-connected and thus, the exercise boundary is unique, if the condition $r - 2\mu - \sigma^2 \geq 0$ is satisfied (see Lemma 8 in appendix B). Since the implicit dividend yield equals $\delta = r - 2\mu - \sigma^2$ it is also evident that early exercise is suboptimal when $r - 2\mu - \sigma^2 < 0$. Our next valuation result follows.

Theorem 1 *(GBMP) Consider the geometric Brownian motion volatility specification (Model 1). The value of an American call volatility option is given by (5) with*

$$c_0(V_0, X) = V_0 e^{-\delta T} N(d_{1T}) - X e^{-rT} N(d_{1T} - 2\sigma\sqrt{T})$$

$$\Pi_0(V_0, B(\cdot)) = \int_0^T \left[\delta V_0 e^{-\delta v} N(d_{1v}) - X e^{-rv} N(d_{1v} - 2\sigma\sqrt{v})\right] dv.$$

where $\delta \equiv r - 2\mu - \sigma^2$. *In these expressions* $d_{1T} = d_1(V_0, X, T)$ *and* $d_{1v} = d_1(V_0, B_v, v)$ *with*

$$d_1(V_0, x, v) \equiv \frac{1}{2\sigma\sqrt{v}}\left[\ln(\frac{V_0}{x}) + (r - \delta + 2\sigma^2)v\right].$$

The immediate exercise boundary B *solves the recursive nonlinear integral equation*

$$B_t - X = C_t^A(B_t, B(\cdot))$$

subject to the boundary condition $B_T = \max\{X, \frac{r}{\delta}X\}$.

The MRGP model satisfies $Ee^{-rt}\phi_t \leq 1$. This condition ensures up-connectedness of the exercise region and uniqueness of the optimal exercise boundary. The value of an American option is then as follows.

Theorem 2 *(MRGP) Consider the mean reverting gaussian volatility specification (Model 2). The value of an American call volatility option is given by (5) with*

$$c_0(V_0, X) = e^{-rT}\left[(V_0\phi_T + A_T - X)\, N(-d_{2T}) + a_T\, n(d_{2T})\right]$$

$$\begin{aligned}\Pi_0(V_0, B(\cdot)) &= \int_0^T e^{-rv}\left[(r+\lambda)(V_0\phi_v + A_v) - (\alpha + rX)\right] N(-d_{2v})\, dv \\ &+ \int_0^T e^{-rv}(r+\lambda) a_v n(d_{2v})\, dv\end{aligned}$$

where $d_{2T} = d_2(V_0, X, T)$, $d_{2v} = d_2(V_0, B_v, v)$ *and*

$$d_2(V_0, x, v) \equiv a_v^{-1}\left[x - V_0\phi_v - A_v\right].$$

The immediate exercise boundary B *solves the recursive nonlinear integral equation*

$$B_t - X = C_t^A(B_t, B(\cdot))$$

subject to the boundary condition $B_T = \max\left\{X, \frac{\alpha + rX}{r+\lambda}\right\}$.

As with the MRGP model above the MRSRP model satisfies $Ee^{-rt}\phi_t \leq 1$. Uniqueness of the optimal exercise boundary follows and enables us to conclude

Theorem 3 *(MRSRP) For the mean reverting square root volatility process (Model 3) the value of an American call volatility option is given by (5) with*

$$\begin{aligned} c_0(V_0, X) &= e^{-rT}a_T^2\left(d_{3T}^+ n(d_{3T}^+) - d_{3T}^- n(d_{3T}^-)\right) \\ &\quad +2e^{-rT}\sqrt{V_0}\phi_T a_T\left(n(d_{3T}^+) - n(d_{3T}^-)\right) \\ &\quad +e^{-rT}\left(V_0\phi_T^2 + a_T^2 - X\right)\left(N(d_{3T}^-) + N(-d_{3T}^+)\right) \end{aligned}$$

$$\begin{aligned} \Pi_0(V_0, B(\cdot)) &= \int_0^T e^{-rv}\left[(r+2\lambda)\left(V_0\phi_v^2 + a_v^2\right) - (\sigma + rX)\right]\left(N(d_{3v}^-) + N(-d_{3v}^+)\right)dv \\ &\quad + (r+2\lambda)\int_0^T e^{-rv}a_v^2\left(d_{3v}^+ n(d_{3v}^+) - d_{3t}^- n(d_{3v}^-)\right)dv \\ &\quad +2\sqrt{V_0}\,(r+2\lambda)\int_0^T e^{-rv}\phi_v a_v\left(n(d_{3v}^+) - n(d_{3v}^-)\right)dv. \end{aligned}$$

where $d_{3T}^{\pm} = d_3^{\pm}(V_0, X, T)$, $d_{3v}^{\pm} = d_3^{\pm}(V_0, B_v, v)$ *and*

$$d_3^{\pm}(V_0, x, t) = a_t^{-1}\left(\pm\sqrt{x} - \sqrt{V_0}\phi_t\right).$$

The immediate exercise boundary B *solves the recursive nonlinear integral equation*

$$B_t - X = C_t^A(B_t, B(\cdot))$$

subject to the boundary condition $B_T = \max\{X, \frac{\sigma + rX}{r+2\lambda}\}$.

For our last specification, the MRLP model, up-connectedness is obtained under the condition $r - \alpha - \frac{1}{2}\sigma^2 \geq 0$; the exercise boundary is again unique.

Theorem 4 *(MRLP) Consider the log mean reverting gaussian volatility specification (Model 4) and suppose that* $r - \beta \geq 0$ *where* $\beta \equiv \alpha + \frac{1}{2}\sigma^2$. *The value of the American call volatility option is given by (5) with*

$$c_0(V_0, X) = e^{-rT}\left[V_0^{\phi_T}\exp\left\{\frac{\alpha}{\lambda}(1 - \phi_T) + \frac{1}{2}a_T^2\right\}N(d_{4T} + a_T) - XN(d_{4T})\right]$$

$$\begin{aligned}\Pi_0(V_0, B(\cdot)) \quad = \quad & (r-\beta)\int_0^T V_0^{\phi_v} e^{\left\{\frac{\alpha}{\lambda}(1-\phi_v)+\frac{1}{2}a_v^2-rv\right\}} N(d_{4v}+a_v)dv \\ & +\lambda\int_0^T V_0^{\phi_v} e^{\left\{\frac{\alpha}{\lambda}(1-\phi_v)+\frac{1}{2}a_v^2-rv\right\}}\left(\phi_v \ln(V_0)+\frac{\alpha}{\lambda}(1-\phi_v)+a_v^2\right)N(d_{4v}+a_v)dv \\ & +\lambda\int_0^T V_0^{\phi_v} e^{\left\{\frac{\alpha}{\lambda}(1-\phi_v)+\frac{1}{2}a_v^2-rv\right\}} a_v n(d_{4v}+a_v)dv \\ & -rX\int_0^T e^{-rv}N(d_{4v})\,dv\end{aligned}$$

where $d_{4T} = d_4(V_0, X, T)$, $d_{4v} = d_4(V_0, B_v, v)$ *and*

$$d_4(V_0, x, v) \equiv \frac{1}{a_v}\left[\phi_v \ln(V_0) - \ln(x) + \frac{\alpha}{\lambda}(1-\phi_v)\right].$$

The immediate exercise boundary B *solves the recursive nonlinear integral equation*

$$B_t - X = C_t^A(B_t, B(\cdot))$$

subject to the boundary condition $B_T = \max\{X,\, B^*\}$, *where* B^* *solves*

$$r(B^* - X) - B^*\left(\alpha + \frac{1}{2}\sigma^2 - \lambda\ln(B^*)\right) = 0.$$

The option pricing formulas given in theorems 1-4 illustrate the dependence on the specification of the volatility process. This is clearly not only true of option prices, but also of the comparative statics and hedging behavior associated with these different prices. Empirical evidence, however, suggests that an EGARCH volatility specification provides a good description of stock returns (Engle and Ng (1993), Danielson (1994), Hentschel (1995)). Since the MRLP volatility process is the continuous time limit of a discrete time EGARCH this is the model that we focus upon in the next section.

4 The MRLP volatility option model.

Let t denote the current time. Recall from model 4 that for a mean reverting log process, the time T volatility is related to current volatility by

$$V_T = V_t^{\phi_{T-t}} \exp\left\{\frac{\alpha}{\lambda}(1-\phi_{T-t}) + a_{T-t}w_{T-t}\right\} \tag{7}$$

where

$$\begin{cases} \phi_{T-t} \equiv e^{-\lambda(T-t)} \\ a_{T-t} \equiv \sigma\left[\int_t^T e^{-2\lambda(T-s)}ds\right]^{1/2} = \frac{\sigma}{\sqrt{2\lambda}}\left(1-\phi_{T-t}^2\right)^{1/2} \\ w_{T-t} \equiv a_{T-t}^{-1}\sigma\left[\int_t^T e^{-\lambda(T-s)}dZ_s^*\right] \end{cases}$$

The long run mean of volatility is $\exp\{(\alpha + \frac{1}{4}\sigma^2)/\lambda\}$. The speed of reversion λ impacts the volatility levels through the log of volatility. As evidenced by the volatility formula above this means that future volatility is a multiplier of a power function of current volatility. Moreover, since $\phi_{T-t} \leq 1$, the relationship is concave. Our next results will illustrate the importance of this feature for the properties of call options.

4.1 Properties of European options.

The MRLP volatility European call option has the following limiting properties.

Proposition 5 *The European call option price $c_t(V_t)$ satisfies*

(i) $\lim_{t\to T} c_t(V_t) = \max(V_T - X, 0)$

(ii) $\lim_{T\to\infty} c_t(V_t) = 0$

(iii) $\lim_{V_t\to 0} c_t(V_t) = 0.$

The first part of the proposition is the usual convergence of the option price to the payoff at maturity. The second part says that an infinite maturity European option on volatility will be worthless. The third part says that the value of the option approaches zero as volatility goes to zero. This is in contrast to the model of Grunbichler and Longstaff in which such an option still has value. The reason for this difference is the multiplicative impact of uncertainty on future volatility. Empirical evidence suggests that this impact is more representative of asset returns' volatility than the additive impact of mean reversion implied by the mean reverting square root process.

Our next proposition details some key derivatives of the pricing function.

Proposition 6 *The European volatility option price satisfies*

$$\frac{\partial c_t(V_t)}{\partial V_t} = e^{-r(T-t)}\phi_{T-t}V_t^{\phi_{T-t}-1}e^{\frac{\alpha}{\lambda}(1-\phi_{T-t})+\frac{1}{2}a^2_{T-t}}N(d_{4T}+a_{T-t}) \geq 0$$

$$\frac{\partial^2 c_t(V_t)}{(\partial V_t)^2} = e^{-r(T-t)}\phi_{T-t}V_t^{\phi_{T-t}-2}e^{\frac{\alpha}{\lambda}(1-\phi_{T-t})+\frac{1}{2}a^2_{T-t}}\left[(\phi_{T-t}-1)N(d_{4T}+a_{T-t}) + n(d_{4T}+a_{T-t})\frac{\phi_{T-t}}{a_{T-t}}\right]$$

$$\begin{aligned}\frac{\partial c_t(V_t)}{\partial(T-t)} &= -rc_t(V_t) \\ &\quad + e^{-r(T-t)}V_t^{\phi_{T-t}}e^{\frac{\alpha}{\lambda}(1-\phi_{T-t})+\frac{1}{2}a^2_{T-t}}\left(\phi_{T-t}(\alpha - \frac{\lambda}{V_t}) + \frac{1}{2}\phi^2_{T-t}\sigma^2\right)N(d_{4T}+a_{T-t}) \\ &\quad + e^{-r(T-t)}V_t^{\phi_{T-t}}e^{\frac{\alpha}{\lambda}(1-\phi_{T-t})+\frac{1}{2}a^2_{T-t}}n(d_{4T}+a_{T-t})\left(-\frac{\sigma}{\sqrt{2\lambda}}\frac{\lambda\phi^2_{T-t}}{\sqrt{1-\phi^2_{T-t}}}\right).\end{aligned}$$

Note that the second derivative of the option may be positive or negative. When V is large the first component in bracket dominates and the derivative is negative, i.e. the price is concave. The underlying reason for price concavity is the mean reversion behavior embedded in the MRLP model. In the limit as $V \to \infty$ the probability of a decrease in volatility approaches 1. The option is then worth significantly less than its intrinsic value.

4.2 Model calibration.

We calibrate the MRLP model by estimating the corresponding EGARCH model and then taking the limit. Let θ_t denote the risk premium of the stock and consider an EGARCH-M model adjusted by θ_t for a positive shift in the news impact curve (see Hentschel 1995)

$$\begin{aligned}
\ln\left(\frac{S_t}{S_{t-1}}\right) &= \mu - \delta - \frac{1}{2}\sigma_t^2 + \sigma_t \varepsilon_t \\
\varepsilon_t &\sim N(0,1) \\
\ln(\sigma_t^2) &= \alpha_0 + \alpha_1 \ln(\sigma_{t-1}^2) + \alpha_2 \left(|\varepsilon_{t-1} + \theta_{t-1}| + \beta_1(\varepsilon_{t-1} + \theta_{t-1})\right).
\end{aligned}$$

The weak limit of this model converges to the unique strong solution of the MRLP stochastic volatility diffusion model (Model 4). Represented under the $Q-$measure the limiting process is

$$\begin{aligned}
d\ln(S_t) &= \left(r - \delta - \frac{1}{2}V_t^2\right) dt + V_t \left(\rho dZ_t^* + \sqrt{(1-\rho^2)} dB_t^*\right) \\
d\ln(V_t) &= (\alpha - \lambda \ln(V_t))\, dt + \sigma dZ_t^*
\end{aligned}$$

where the parameters α, λ, σ, and ρ are functions of the parameters of the EGARCH process and are given by (see Nelson (1990) or Duan (1997))

$$\begin{aligned}
\alpha &= \frac{\alpha_0}{2} + \frac{\alpha_2}{\sqrt{2\pi}} \\
\lambda &= 1 - \alpha_1 \\
\sigma &= \frac{|\alpha_2|}{2}\sqrt{\beta_1^2 + \left(\frac{\pi - 2}{\pi}\right)} \\
\rho &= \frac{\left(\frac{\alpha_2 \beta}{2}\right)}{\sigma}.
\end{aligned}$$

Here ρ represents the correlation between Z_1 and Z_2. Table 1 (appendix C) gives the results of the estimated parameters of the above EGARCH-M model fitted to the S&P 500 daily index series from July 2 ,1962 to December 29, 1989. The resulting stochastic volatility parameters are $\alpha = -0.1020$, $\lambda = 0.0215$, $\sigma = 0.1031$, and $\rho = -0.3544$. Note that negative correlation between Z_1 and Z_2 corresponds to the asymmetric relationship between returns and changes in volatility, that is, the leverage effect. Also, α being negative implies mean reversion with a long run mean for $\log(V)$ of $\alpha/\lambda = -4.7442$ which in turn implies a long run mean annualized volatility (based on 365 days) of 16.63 percent. The speed of reversion λ, is small, indicating strong autocorrelation in volatility which in turn implies volatility clustering. Parameter values are therefore consistent with observed empirical regularities regarding the volatility of asset returns.

4.3 Numerical results and discussion.

Using the estimated parameters from above, Figures 1 and 2 show how the price of the European call option changes as volatility and maturity change. The non-tradedness of volatility suggests that the usual rational restrictions of Merton (1973) may not always hold since arbitrage strategies involving positions in volatility cannot be constructed. Thus we observe from Figure 1 that European option prices are not bounded below by $V-X$. This result is similar to that obtained by Grunbichler and Longstaff and will be true for options on any variable that cannot be duplicated. An interesting aspect of these volatility call options is that at times of low (high) volatility, long maturity options have a higher (lower) value than short maturity options. The sources of this behavior can be traced to the mean reversion and the volatility clustering properties. Indeed, when volatility is low, it is likely to remain low in the short-term and increase toward the long-run mean in the long-term, and conversely. We can also see from Proposition 5(iii), that contrary to Grunbichler and Longstaff, the value of European volatility call options converge to zero as volatility goes to zero.

Figure 1 also shows that at (relatively) high volatilities the European option displays concavity with respect to current volatility levels. This turns out to be an important result which, on the surface, appears to contradict BGW (1996). This property will be discussed further below and also when we examine American options.

From Figure 2, we observe that for European volatility call options of all moneyness, as maturity increases option prices initially increase, and then decrease. In fact, as maturity goes to infinity, the price of the option will again converge to zero, as long as the riskfree rate is positive. The intuitive explanation for this behavior is that due to mean reversion, the drift of the volatility process cannot sufficiently compensate for discounting over time.

The hedging properties of volatility options are closely related to the delta of the option i.e. the partial derivative of the option with respect to the underlying variable (in this case $delta = \partial c(V)/\partial V$). Let Δ_V, Δ_S, and Υ_S be respectively the delta of a volatility option, the delta of a stock option, and the vega of the same stock option ($\Upsilon_S = \partial c\,(S,V)\,/\partial V$). Then it is easy to show that a portfolio consisting of Δ_S shares of the stock long, one stock option short, and $n = \Upsilon_S/\Delta_V$ volatility options long, will be immune to risk. An investor wishing to hedge volatility risk can do so by buying or selling $n = \Upsilon_S/\Delta_V$ volatility options depending on the direction of his exposure.[5],[6]

Figures 3 and 4 show how Δ_V changes as initial volatility and maturity change. For European volatility options of all three maturities, delta first increases then decreases. Again, this implies that the option is convex at low levels of volatility, and concave at high levels of volatility. As maturity increases, this effect is reduced as the curvature flattens out. Three regions can be identified from Figure 3 in which the delta of each maturity is highest. The implication is that at low levels of volatility, the long maturity option has the highest delta and will be the most effective hedging tool, at medium levels of volatility, the intermediate maturity option has the highest delta and will be most effective, and at high levels of volatility, the short maturity option has the highest delta and will be most effective.[7]

[5]An alternative hedging strategy is to use the bond, the risky asset and an option written on the risky asset. In fact any pair of securities with nondegenerate diffusion matrix will suffice to hedge all the uncertainty in the model.

[6]Volatility options can also be used to span other derivatives with volatility-based payoffs. In fact, as shown by Bakshi and Madan (1999), any L_1-integrable payoff written on a given state variable can be written as an integral (infinite sum) of call and put options written on the same underlying state variable.

[7]This is particularly true in the presence of transaction costs where high deltas mean small hedge ratios and lower hedging cost.

Why does concavity occur, and of what importance is it? As discussed earlier BGW have shown that convexity holds in univariate models with stochastic volatility when the drift is proportional to the process underlying the call option contract. Merton (1973), however, pointed out that if the distribution of the (stochastic) rate of change through time of a random process is not independent of the current value, then convexity of an option on this random variable (with respect to this variable) is not assured. Indeed, for volatility dynamics described by *any* mean reverting process, the distribution of the volatility rate of change (dV/V) *will* depend on current volatility. This, however, does not necessarily imply concavity in call option prices. Loosely speaking, concavity in option prices will only occur when the distribution is not just dependent but such that the incremental upside potential becomes smaller and smaller as the underlying increases. This is precisely the situation with the MRLP process (see (7)). As volatility increases there is a stronger pull toward the mean. In fact, in the limit, the expected rate of growth of volatility converges to negative infinity. This means that the call option gains less and less value from a unit increase in volatility as volatility increases. Concavity of the price formula naturally results. This feature of the volatility process is unique to the MRLP model, among the specifications that have been analyzed here and elsewhere in the options' literature. For instance, the MRSRP model of Grunbichler and Longstaff (1996) has finite expected rate of volatility growth as volatility increases and therefore implies convexity of call option prices, as predicted by the theoretical results of BGW.

The importance of concavity at high levels of volatility is twofold. First, mispricing resulting from the use of another model failing to embed sufficient mean-reversion will be more severe, and second, hedge ratios obtained from that mispriced formula will be incorrect. The negative correlation between volatility and stock returns (the leverage effect) means that this will tend to occur precisely at those times when volatility options become important as hedging tools.

As Figure 4 shows, at very long maturities, deltas of options of all moneyness approach zero, implying that they become less and less responsive to volatility changes and lose their effectiveness as hedging tools.

4.4 American-style options.

4.4.1 Numerical implementation.

The numerical scheme that we implement follows Kim (1990) and is based on the EEP representation and the recursive equation for the boundary displayed in Theorem 4. It involves the following steps:

1. Discretize the time horizon $[0, T]$ by dividing it into n subintervals of length $h = T/n$. Let $t_0, ..., t_n$ denote the set of time points straddling these intervals. In steps 3-5 below we use the standard approximation of integrals of the form $\int_0^T f(t)dt$ by $\sum_{k=0}^{n-1} f(t_k)h$.

2. Starting at time $T = hn$ solve for B^* using (say) Newton's method applied to the boundary condition. Set $B(nh) = \max(X, B^*)$.

3. At time $t_{n-1} = h \times (n-1)$ the boundary point $B((n-1)h)$ is obtained by using Newton's method on the equation $B((n-1)h) - X = c(B((n-1)h), (n-1)h) + \Pi(B((n-1)h, B(nh), (n-1)h)$.

4. Proceed recursively through the set of time points. At time t_k the estimate of the boundary is determined by solving $B(t_k)-X = c(B(t_k), t_k)+\Pi^n(B(t_k), B(\cdot), t_k)$ where $\Pi^n(B(t_k), B(\cdot), t_k)$ represents the approximated value of the EEP (using the approximation of integrals described in step 1). Newton's method is used to solve this nonlinear equation.

5. The American option value at date 0 is computed based on the approximated boundary.

It is easy to show that this procedure converges to the true boundary as the number of time points increases. Note also that alternative approaches are available to compute estimates of the exercise boundary. Examples that have been discussed in the literature include piecewise-exponential approximations (Ju (1998)) and approximations derived by optimizing over a class of (simple) suboptimal stopping times (Broadie and Detemple (1996)).

The numerical results reported below are based on a discretization involving 100 time points. Experiments involving 500 time points were also performed and were found to have little effect on the results.

4.4.2 Properties.

Figure 5 provides an illustration of the immediate exercise boundary and the corresponding exercise region. A volatility realization which lies in the exercise region implies that immediate exercise of the volatility option is optimal. The immediate exercise boundary increases as maturity increases, for all strike prices. The exercise boundary is seen to be lower as the strike price increases, which can be explained by the lower payoff at exercise of higher strike options.

The price function displays the following properties.

Proposition 7 *The American option price has the following properties*
(i) $C^A(V,t)$ *is continuous on* $\mathbb{R}^+ \times [0,T]$.
(ii) $C^A(\cdot,t)$ *is nondecreasing on* $\mathbb{R}^+$ *and* $C^A(V,t) = V - X$ *for* $V \geq B_t$*, for all* $t \in [0,T]$,
(iii) $\frac{\partial C^A(V,t)}{\partial V}$ *is continuous on* $\mathbb{R}^+$ *for all* $t \in [0,T]$.

Figure 6 demonstrates that early exercise has value, as seen by the fact that $C^A(V,t) \geq c(V,t)$ for all $V \in \mathbb{R}^+,\ \ t \in [0,T]$. In addition, C^A, unlike c is bounded below by $V-X$. Evidently, the lower bound is a consequence of the possibility of immediate exercise.

Figure 7 shows that the early exercise premium not only increases with volatility but also that it is higher, at all levels of volatility, for options with lower strike prices. In addition, it is convex for large values of volatility. Figure 6 illustrates the linearity of the American call price above the exercise boundary (property (ii) in the proposition above). This implies that the convexity of the early exercise premium compensates exactly for the concavity of the European option and illustrates a major difference in hedging behavior between European and American options. Figure 7 is also indication that long maturity American options are still responsive to changing levels in volatility as opposed to long maturity European options and are therefore still useful hedging tools.

5 Conclusion

Empirical studies demonstrate that the volatility of stock prices changes over time. This suggests a need for financial instruments that can be used to manage this type of risk. As a result indices of volatility are springing up around the world and so are volatility option contracts. In this paper, we have valued European- and American-style options on volatility in the context of a general equilibrium model incorporating stochastic volatility. Explicit formulas have been proposed for a set of popular volatility specifications. Special attention has been devoted to the MRLP model based on its empirical appeal. In this context we have demonstrated an unusual property of the European call option, which unlike its American counterpart, displays concavity at high levels of volatility. These and other results provide important guidelines for volatility risk management engineered through the use of volatility options.

6 Appendix.

6.1 Appendix A: model specifications.

The volatility models in section 2.3 are obtained from the following assumptions on the primitives:

	$\sigma_1(Y_t, t)$	$\sigma^Y(Y_t, t)$	$\mu^Y(Y_t, t)$
Model 1	Y_t^2	σY_t	$\mu Y_t + \rho\sigma Y_t^3$
Model 2	Y_t	σ	$\alpha + \rho\sigma Y_t - \lambda Y_t$
Model 3	Y_t^2	σ	$\rho\sigma Y_t^2 - \lambda Y_t$
Model 4	e^{Y_t}	σ	$\alpha + \rho\sigma e^{Y_t} - \lambda Y_t.$

Note that the associated state variable processes are

Model 1	$dY_t = Y_t \left[\mu dt + \sigma dZ_t^* \right]$
Models 2, 3 and 4	$dY_t = (\alpha - \lambda Y_t)\, dt + \sigma dZ_t^*$

with $\alpha = 0$ in the case of model 3.

6.2 Appendix B: proofs.

Let $\mathcal{E} = \{(V, t) : C(V, t) = V - K\}$ denote the exercise region. Continuity of the option price with respect to V implies that $\mathcal{E}$ is a closed set. To establish certain properties of $\mathcal{E}$ we are led to consider the following condition

$$Ee^{-rt}(V_t' - V_t) \leq V_0' - V_0 \text{ for all } V_0' \geq V_0. \tag{8}$$

The following are important features of the exercise region.

Lemma 8 *The exercise region $\mathcal{E}$ has the following properties*

(i) right-connectedness: for all $s \geq t$, $(V, t) \in \mathcal{E} \Longrightarrow (V, s) \in \mathcal{E}$.

(ii) up-connectedness: suppose that condition (8) holds. Then $(V, t) \in \mathcal{E} \Longrightarrow (V', t) \in \mathcal{E}$ for all $V' \geq V$.

Proof of Lemma 8: Property (i) follows from the fact that the holder of a long maturity option can implement any exercise policy chosen by the holder of a shorter maturity option. For property (ii) note that

$$C(V',t) \leq C(V,t) + \sup_{\tau \in \mathcal{S}(t,T)} E_t e^{-r\tau}(V'_\tau - V_\tau)$$

where $\mathcal{S}(t,T)$ denotes the set of stopping times of the filtration with values in $[t,T]$, which follows from the inequality $(a+b)^+ \leq a^+ + b^+$. Under condition (8) the expectation on the right hand side is bounded above by $V' - V$. Now let $V' \geq V$ and suppose that immediate exercise is optimal at (V,t) but not at (V',t). The inequality above then implies

$$V' - X < C(V',t) \leq V - X + V' - V = V' - X,$$

a contradiction. Thus we must have $(V',t) \in \mathcal{E}$. ■

For the GBMP model condition (8) holds if $r - (2\mu + \sigma^2) \geq 0$ since

$$Ee^{-r\tau}V_\tau = V_0 E e^{-(r-(2\mu+\sigma^2))\tau - \frac{1}{2}(2\sigma)^2\tau + 2\sigma Z^*_\tau} \leq V_0 E e^{-\frac{1}{2}(2\sigma)^2\tau + 2\sigma Z^*_\tau} = V_0$$

under this condition. Furthermore, inspection of the volatility process shows that $r - (2\mu + \sigma^2)$ is the implicit dividend rate δ. Standard results can be invoked to conclude that exercise prior to maturity is suboptimal when $\delta < 0$.

For the MRGP model (8) follows from $Ee^{-rt}\phi_t \leq 1$. For the MRSRP model we can use Ito's lemma to write

$$d\left(e^{-rt}(V'_t - V_t)\right) = -(r+2\lambda)e^{-rt}(V'_t - V_t)dt + 2\sigma e^{-rt}\left(\sqrt{V'_t} - \sqrt{V_t}\right)dZ^*_t$$

where V_t and V'_t are the solutions associated respectively with the initial conditions V_0 and V'_0 with $V'_0 \geq V_0$. It follows that

$$\begin{aligned} & Ee^{-rt}(V'_t - V_t) \\ = \;& V'_0 - V_0 - (r+2\lambda)E\int_0^t e^{-rs}(V'_s - V_s)ds + 2\sigma E\int_0^t e^{-rs}\left(\sqrt{V'_s} - \sqrt{V_s}\right)dZ^*_s \\ \leq \;& V'_0 - V_0 \end{aligned}$$

where the inequality in the second line is obtained by using the martingale property of the stochastic integral and the pathwise inequality $V'_s \geq V_s$ which follows from the comparison theorem for solutions of SDEs.

For the MRLP process we can write

$$d(V'_t - V_t) = \left[(\alpha + \frac{1}{2}\sigma^2)(V'_t - V_t) - \lambda\left(\ln(V'_t)V'_t - \ln(V_t)V_t\right)\right]dt + \sigma(V'_t - V_t)dZ^*_t$$

Using Ito's lemma we then obtain

$$Ee^{-rt}(V'_t - V_t) \;=\; V'_0 - V_0 + (-r + \alpha + \frac{1}{2}\sigma^2)E\int_0^t e^{-rs}(V'_s - V_s)ds$$

$$
\begin{aligned}
& -\lambda E \int_0^t e^{-rs}\left(\ln(V_s')V_s' - \ln(V_s)V_s\right) ds + E\int_0^t e^{-rs}\sigma(V_s' - V_s)dZ_s^* \\
\leq\ & V_0' - V_0 + (-r + \alpha + \frac{1}{2}\sigma^2)E\int_0^t e^{-rs}(V_s' - V_s)ds
\end{aligned}
$$

where the inequality uses the comparison theorem and the fact that the function $\log(V)V$ is increasing. Condition (8) holds if $r - \alpha - \frac{1}{2}\sigma^2 \geq 0$.

Under property (ii) of lemma 8 the optimal exercise boundary is unique and can be defined as

$$
B_t = \inf\{V : (V,t) \in \mathcal{E}\} = \inf\{V : C(V,t) = V - X\}.
$$

We can then use the early exercise representation formula (5)-(6) substituting the event $\{V_v \geq B_v\}$ in place of $\{V_v \in \mathcal{E}_v\}$ in the expression for (6). With this representation we now prove Theorems 1-4.

In order to unify the derivations of the formulas in Theorems 1-4 we follow Bakshi and Madan (1999) and use the characteristic function of the state variables under the risk neutral measure. Since all of our models involve transformations of normally distributed random variables the relevant characteristic function is particularly simple.

Lemma 9 *Let $I = [a,b]$ denote an arbitrary closed interval in $\mathbb{R}$ and consider a normally distributed random variable w. The I-truncated characteristic function $f(k;I) = \int_I e^{ikw}n(w)dw$ is*

$$
f(k;I) = e^{-\frac{1}{2}k^2}\left[N(b - ik) - N(a - ik)\right]
$$

where $n(w), N(w)$ are respectively the standard normal density and the cumulative normal distribution function. The truncated moments of the distribution are

$$
\int_I w^n n(w)dw = i^{-n}\frac{\partial^n}{\partial k^n} f(k;I)|_{k=0}
$$

for $n \in \mathbb{N}$. In particular

$$
\int_I n(w)dw = N(b) - N(a)
$$

$$
\int_I wn(w)dw = -n(b) + n(a)
$$

$$
\int_I w^2 n(w)dw = N(b) - N(a) - bn(b) + an(a)
$$

$$
\int_I e^{cw}n(w)dw = e^{\frac{1}{2}c^2}\left[N(b - c) - N(a - c)\right]
$$

$$
\int_I e^{cw}wn(w)dw = ce^{\frac{1}{2}c^2}\left[N(b - c) - N(a - c)\right] + e^{\frac{1}{2}c^2}\left[n(a - c) - n(b - c)\right].
$$

The results above also hold for open and semi-open intervals.

Proof of Lemma 9: Straightforward computations give

$$\begin{aligned} f(k;I) &= \int_I e^{ikw} n(w)dw \\ &= e^{-\frac{1}{2}k^2} \frac{1}{\sqrt{2\pi}} \int_I e^{-\frac{1}{2}(w-ik)^2} dw \\ &= e^{-\frac{1}{2}k^2} \int_{I-ik} e^{-\frac{1}{2}z^2} dw \\ &= e^{-\frac{1}{2}k^2} \left[N(b-ik) - N(a-ik) \right]. \end{aligned}$$

Since

$$\frac{\partial^n}{\partial k^n} f(k;I)|_{k=0} = \int_I (iw)^n e^{ikw} n(w) dw|_{k=0} = i^n \int_I w^n n(w)dw$$

the expression for the truncated moments follows.

In order to derive the last two moments define the function $g(k;I) = f(k + c/i; I)$ and note that

$$g(k;I) = f(k+c/i;I) = \int_I e^{i(k+c/i)w} n(w)dw = \int_I e^{ikw} e^{cw} n(w)dw$$

$$\frac{\partial^n}{\partial k^n} g(k;I) = \frac{\partial^n}{\partial k^n} f(k+c/i;I) = i^n \int_I e^{ikw} e^{cw} w^n n(w)dw$$

which implies

$$g(k;I)|_{k=0} = \int_I e^{cw} n(w)dw$$

$$\frac{\partial^n}{\partial k^n} g(k;I)|_{k=0} = i^n \int_I e^{cw} w^n n(w)dw.$$

Using the expressions for $f(\cdot;I)$ gives

$$g(k;I) = f(k+c/i;I) = e^{-\frac{1}{2}(k+c/i)^2} \left[N(b - i(k+c/i)) - N(a - i(k+c/i)) \right]$$

$$\begin{aligned} \frac{\partial}{\partial k} g(k;I) &= -(k+c/i)\, e^{-\frac{1}{2}(k+c/i)^2} \left[N(b - i(k+c/i)) - N(a - i(k+c/i)) \right] \\ &\quad -ie^{-\frac{1}{2}(k+c/i)^2} \left[n(b - i(k+c/i)) - n(a - i(k+c/i)) \right] \end{aligned}$$

and therefore

$$\int_I e^{cw} n(w)dw = g(k;I)|_{k=0} = e^{\frac{1}{2}c^2} \left[N(b-c) - N(a-c) \right]$$

$$\begin{aligned} \int_I e^{cw} w n(w)dw &= \frac{1}{i} \frac{\partial}{\partial k} g(k;I)|_{k=0} \\ &= ce^{\frac{1}{2}c^2} \left[N(b-c) - N(a-c) \right] + e^{\frac{1}{2}c^2} \left[n(a-c) - n(b-c) \right]. \end{aligned}$$

A limiting argument can be used to prove the results for arbitrary intervals in $\mathbb{R}$. ■

Proof of Theorem 1: When the underlying volatility follows the GMBP model and the interest rate is constant we are back in the Black-Scholes framework. Defining the implicit dividend rate $\delta \equiv r-(2\mu+\sigma^2)$ and using the standard formulas (or the results of Lemma 9) leads to the results. ■

Proof of Theorem 2: For the MRGP specification let

$$d_{2t} \equiv d_2(V_0, X\,, t) \equiv a_t^{-1}\left[X - V_0\phi_t - A_t\right].$$

Since $V_T = V_0\phi_T + A_T + a_T w$ where w has a standard normal distribution the event $\{V_T \geq X\}$ is equivalent to $I = \{w \geq d_2(V_0, X, T)\}$. Using Lemma 9 enables us to write the value of a European call option as

$$\begin{aligned}
c_0 &= \int_{d_{2T}}^{\infty} e^{-rT}(V_0\phi_T + A_T + a_T w - X)n(w)dw \\
&= e^{-rT}(V_0\phi_T + A_T - X)\int_{d_{2T}}^{\infty} n(w)dw + e^{-rT}a_T\int_{d_{2T}}^{\infty} wn(w)dw \\
&= e^{-rT}\left[(V_0\phi_T + A_T - X)N(-d_{2T}) + a_T n(d_{2T})\right]
\end{aligned}$$

For this volatility specification the implicit dividend rate is

$$\delta_t^V = r + \lambda - \frac{\alpha}{V_t}.$$

Taking $I = \{w \geq d_2(V_0, B_v, v)\}$ gives the early exercise premium

$$\begin{aligned}
\Pi_0(V_0, B\,(\cdot)) &= E^Q\left[\int_0^T e^{-rv}\left((r+\lambda)V_v - (\alpha + rX)\right)1_{\{V_v \geq B_v\}}dv\right] \\
&= \int_0^T e^{-rv}\left[\int_{d_{2v}}^{\infty}\left((r+\lambda)(V_0\phi_v + A_v + a_v w) - (\alpha + rX)\right)n(w)dw\right]dv \\
&= \int_0^T e^{-rv}\left((r+\lambda)(V_0\phi_v + A_v) - (\alpha + rX)\right)\int_I n(w)dwdv \\
&\quad + \int_0^T e^{-rv}(r+\lambda)a_v\int_I wn(w)dwdv.
\end{aligned}$$

Substituting $f(k;I)|_{k=0} = N(-d_{2v})$ and $\frac{\partial}{\partial k}f(k;I)|_{k=0} = n(d_{2v})$ gives the formula in the theorem. To get the equation for the exercise boundary use the fact that immediate exercise is optimal at the point $V = B$. The boundary condition follows since the limiting value of the exercise premium as $t \to T$, equals $(r+\lambda)V - (\alpha + rX)$. Setting this expression equal to zero and solving for V at the point $V = B$ leads to the condition stated. ■

Proof of Theorem 3: For the MRSRP the event $\{V_T \geq X\}$ is equivalent to $\{\sqrt{V_0}\phi_T + a_T w_T \geq \sqrt{X}\} \cup \{\sqrt{V_0}\phi_T + a_T w_T \leq -\sqrt{X}\}$. Taking

$$d_{3t}^{\pm} = d_3^{\pm}(V_0, X, T) = a_t^{-1}\left(\pm\sqrt{X} - \sqrt{V_0}\phi_T\right)$$

we get $I \equiv \{w_t \geq d_{3T}^{+}\} \cup \{w_t \leq d_{3T}^{-}\}$. The European call option value is

$$\begin{aligned} c_0 &= \int_I e^{-rT}\left[V_0\phi_T^2 + a_T^2 w^2 + 2\sqrt{V_0}\phi_T a_T w - X\right] n(w)dw \\ &= e^{-rT}\left[a_T^2 \int_I w^2 n(w)dw + 2\sqrt{V_0}\phi_T a_T \int_I wn(w)dw + \left(V_0\phi_T^2 - X\right)\int_I n(w)dw\right] \end{aligned}$$

where, from Lemma 9,

$$\int_I n(w)dw = N(-d_{3t}^{+}) + N(d_{3t}^{-})$$

$$\int_I wn(w)dw = n(d_{3t}^{+}) - n(d_{3t}^{-})$$

$$\int_I w^2 n(w)dw = d_{3t}^{+} n(d_{3t}^{+}) - d_{3t}^{-} n(d_{3t}^{-}) + N(-d_3^{+}) + N(d_3^{-})$$

Substituting gives

$$\begin{aligned} c_0 &= e^{-rT}a_T^2\left(d_{3T}^{+}n(d_{3T}^{+}) - d_{3T}^{-}n(d_{3T}^{-})\right) + 2e^{-rT}\sqrt{V_0}\phi_T a_T\left(n(d_{3T}^{+}) - n(d_{3T}^{-})\right) \\ &\quad + e^{-rT}\left(V_0\phi_T^2 + a_T^2 - X\right)\left(N(d_{3T}^{-}) + N(-d_{3T}^{+})\right) \end{aligned}$$

For the American option, we can write the early exercise premium as

$$\begin{aligned} \Pi_0(V_0, B(\cdot)) &= E^Q\left[\int_0^T e^{-rv}\left((r+2\lambda)V_v - (\sigma^2 + rX)\right) 1_{\{V_v \geq B_v\}} dv\right] \\ &= \int_0^T e^{-rv}\left[(r+2\lambda)\left(V_0\phi_v^2 - \left(\sigma^2 + rX\right)\right]\int_I n(w)dwdv \right. \\ &\quad + (r+2\lambda)\int_0^T e^{-rv}a_v^2 \int_I w^2 n(w)dwdv \\ &\quad + 2\sqrt{V_0}\,(r+2\lambda)\int_0^T e^{-rv}\phi_v a_v \int_I wn(w)dwdv. \end{aligned}$$

Substituting the relevant expressions for the truncated moments gives the premium formula announced. The exercise boundary B solves the recursive integral equation $B_t - X = C_t(B_t, B(\cdot))$. The boundary condition is

$$B_T = \max\left\{X, \frac{\sigma + rX}{r + 2\lambda}\right\}.$$

This completes the proof of the Theorem. ■

Remark 1 *Since V_t has a noncentral chi-square distribution for the MRSRP we could also have developed our valuation formula based directly on that distribution. Instead we have chosen to exploit the fact that chi-square is a quadratic of a normal variate and hence to work directly with normal densities. This enables us to unify the derivations for our 4 models since all of them involve transformations of normals.*

Proof of Theorem 4: For this case let

$$d_{4t} \equiv d_4 \left(V_0, X, t\right) \;\equiv\; \frac{1}{a_t}\left[\phi_t \ln(V_0) - \ln(X) + \frac{\alpha}{\lambda}(1-\phi_t)\right].$$

Then, since $V_T = V_0^{\phi_t} \exp\left(\frac{\alpha}{\lambda}(1-\phi_t) + a_t w_t\right)$ where w has a standard normal distribution, the event $\{V_T \geq X\}$ is equivalent to $\{w \geq -d_{4T}\}$ and we obtain a closed form expression for the European call option.

$$c_0 = e^{-rT}\left[V_0^{\phi_T} \exp\left(\frac{\alpha}{\lambda}(1-\phi_T) + \frac{1}{2}a_T^2\right) N(d_{4T} + a_T) \;-\; X\, N(d_{4T})\right]$$

Define $\beta \equiv \alpha + \frac{1}{2}\sigma^2$. Then, for the American option, the early exercise premium is given by

$$\begin{aligned}
\Pi_0(V_0, B\left(\cdot\right)) &= E^Q\left[\int_0^T e^{-rv}\left(V_v\left(r-\beta+\lambda\ln(V_v)\right) - rX\right) 1_{\{V_v \geq B_v\}} dv\right] \\
&= (r-\beta)E^Q \int_0^T e^{-rv} V_v 1_{\{V_v \geq B_v\}} dv \\
&\quad +\lambda E^Q \int_0^T e^{-rv} V_v \ln(V_v) 1_{\{V_v \geq B_v\}} dv \\
&\quad -rXE^Q \int_0^T e^{-rv} 1_{\{V_v \geq B_v\}} dv.
\end{aligned}$$

Substituting the expression for volatility enables us to write

$$\begin{aligned}
&\Pi_0(V_0, B\left(\cdot\right)) \\
= \;& (r-\beta)\int_0^T e^{-rv} V_0^{\phi_v} e^{\frac{\alpha}{\lambda}(1-\phi_v)}\left(\int_{-d_{4v}}^{\infty} e^{a_v w} n(w)dw\right) dv \\
&+\lambda \int_0^T e^{-rv} V_0^{\phi_v} e^{\frac{\alpha}{\lambda}(1-\phi_v)}\left(\int_{-d_{4v}}^{\infty} e^{a_v w}\left(\phi_v \ln(V_0) + \frac{\alpha}{\lambda}(1-\phi_v) + a_v w\right) n(w)dw\right) dv \\
&-rX\int_0^T e^{-rv}\left(\int_{-d_{4v}}^{\infty} n(w)dw\right) dv.
\end{aligned}$$

An application of Lemma 9 now shows

$$\int_{-d_{4v}}^{\infty} n(w)dw = N(d_{4v})$$

$$\int_{-d_{4v}}^{\infty} e^{a_v w} n(w) dw = e^{\frac{1}{2}a_v^2}\left[1 - N(-d_{4v} - a_v)\right] = e^{\frac{1}{2}a_v^2} N(d_{4v} + a_v)$$

$$\begin{aligned} \int_{-d_{4v}}^{\infty} e^{a_v w} w n(w) dw &= a_v e^{\frac{1}{2}a_v^2}\left[1 - N(-d_{4v} - a_v)\right] + e^{\frac{1}{2}a_v^2} n(-d_{4v} - a_v) \\ &= a_v e^{\frac{1}{2}a_v^2} N(d_{4v} + a_v) + e^{\frac{1}{2}a_v^2} n(d_{4v} + a_v). \end{aligned}$$

It then follows that $\Pi_0(V_0, B(\cdot))$ equals

$$\begin{aligned} &(r-\beta)\int_0^T e^{-rv} V_0^{\phi_v} e^{\frac{\alpha}{\lambda}(1-\phi_v)} e^{\frac{1}{2}a_v^2} N(d_{4v} + a_v) dv \\ &+\lambda \int_0^T e^{-rv} V_0^{\phi_v} e^{\frac{\alpha}{\lambda}(1-\phi_v)} \left(\phi_v \ln(V_0) + \frac{\alpha}{\lambda}(1-\phi_v)\right) e^{\frac{1}{2}a_v^2} N(d_{4v} + a_v) dv \\ &+\lambda \int_0^T e^{-rv} V_0^{\phi_v} e^{\frac{\alpha}{\lambda}(1-\phi_v)} a_v \left(a_v e^{\frac{1}{2}a_v^2} N(d_{4v} + a_v) + e^{\frac{1}{2}a_v^2} n(d_{4v} + a_v)\right) dv \\ &-rX \int_0^T e^{-rv} N(d_{4v}) dv. \end{aligned}$$

Rearranging the terms produces the expression in the theorem. Clearly the exercise boundary B solves the recursive nonlinear integral equation $B_t - X = C_t^A(B_t, B(\cdot))$. The boundary condition is

$$B_T = \max\{X,\, B^*\}$$

where B^* solves $r(B^* - X) - B^*\left(\alpha + \frac{1}{2}\sigma^2 - \lambda \ln(B^*)\right) = 0$. ■

Proof of Proposition 5: the results follow directly by taking limits. ■

Proof of Proposition 6: taking partial derivatives and using

$$V_0^{\phi_v} e^{\left\{\frac{\alpha}{\lambda}(1-\phi_v) + \frac{1}{2}a_v^2\right\}} n\left(d_4(V_0, B_v, v) + a_v\right) - Xn\left(d_4(V_0, B_v, v)\right) = 0$$

leads to the result. ■

Proof of Proposition 7:
(i) This follows from the continuity of the option payoff function and the continuity of the flow of the stochastic differential equation in Model 4 relative to the initial values.
(ii) This follows from the monotonicity (increasing) of the flow and the increasing structure of the payoff. In the exercise region immediate exercise is optimal: the option value is the payoff function.
(iii) This follows from (i) and the differentiability of C^A (which can easily be established). ■

6.3 Appendix C: estimation.

Table: EGARCH estimation of S&P500 index (without dividends).
Estimation conducted by maximum likelihood.

Parameters	Estimates	Std. Err.	Est./S.E.	Prob
$\mu - \delta$	0.0003	0.0001	5.034	0.0000
α_0	-0.3527	0.0286	-376.447	0.0000
α_1	0.9785	0.0025	-45.677	0.0000
α_2	0.1864	0.0110	24.320	0.0000
β_1	-0.3921	0.0403	0.025	0.4901

Table: Correlation Matrix of parameters.

Parameters	$(\mu - \delta)$	α_0	α_1	α_2	β_1
$\mu - \delta$	1.0000				
α_0	-0.0057	1.0000			
α_1	0.0507	0.9618	1.0000		
α_2	0.0771	-0.6000	-0.3656	1.0000	
β_1	0.1926	-0.0989	0.0501	0.4616	1.0000

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Figure 1

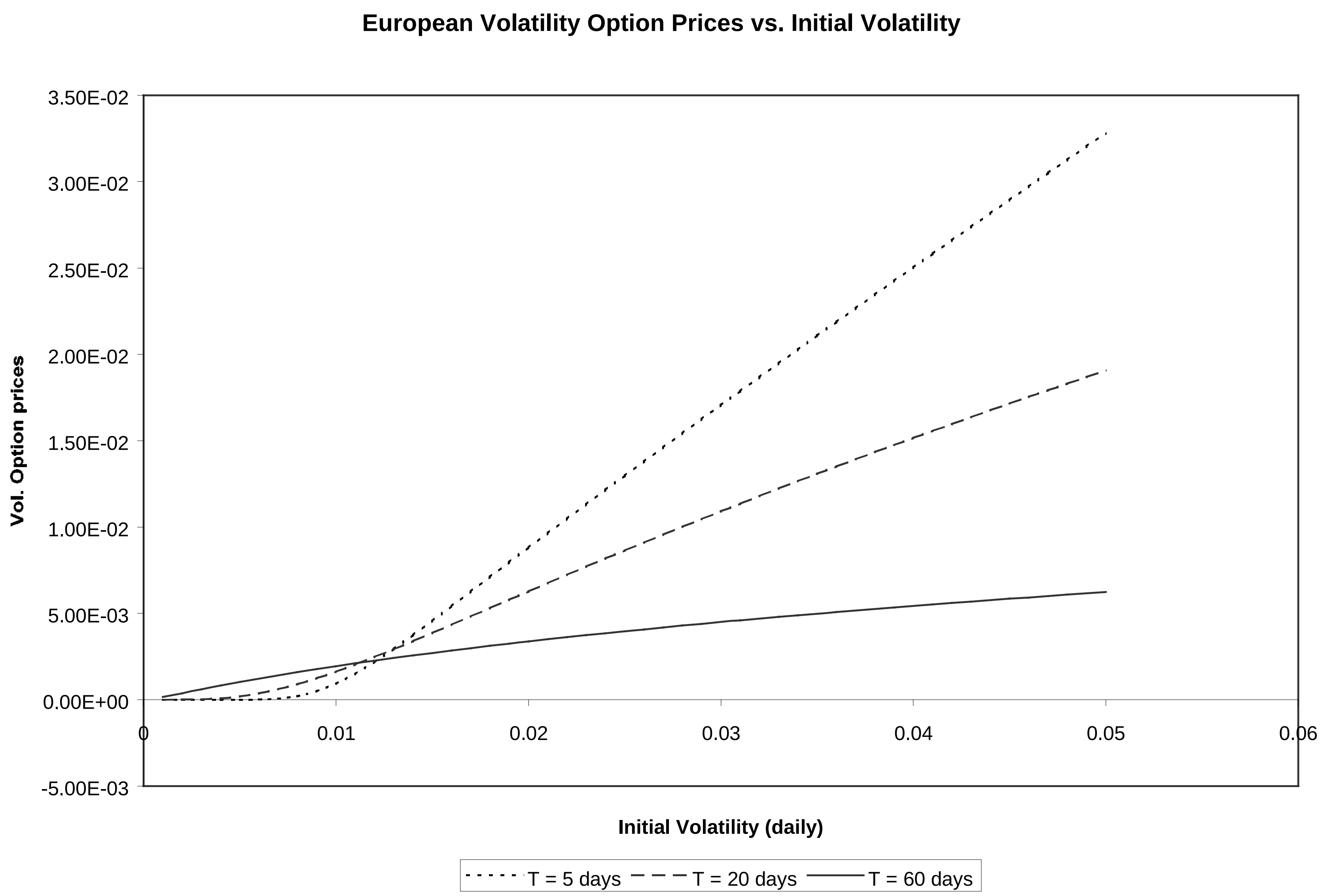

European Volatility Option Prices vs. Initial Volatility
Vol. Option prices
3.50E-02
3.00E-02
2.50E-02
2.00E-02
1.50E-02
1.00E-02
5.00E-03
0.00E+00
-5.00E-03
0
0.01
0.02
0.03
0.04
0.05
0.06
Initial Volatility (daily)
T = 5 days
T = 20 days
T = 60 days

Figure 2

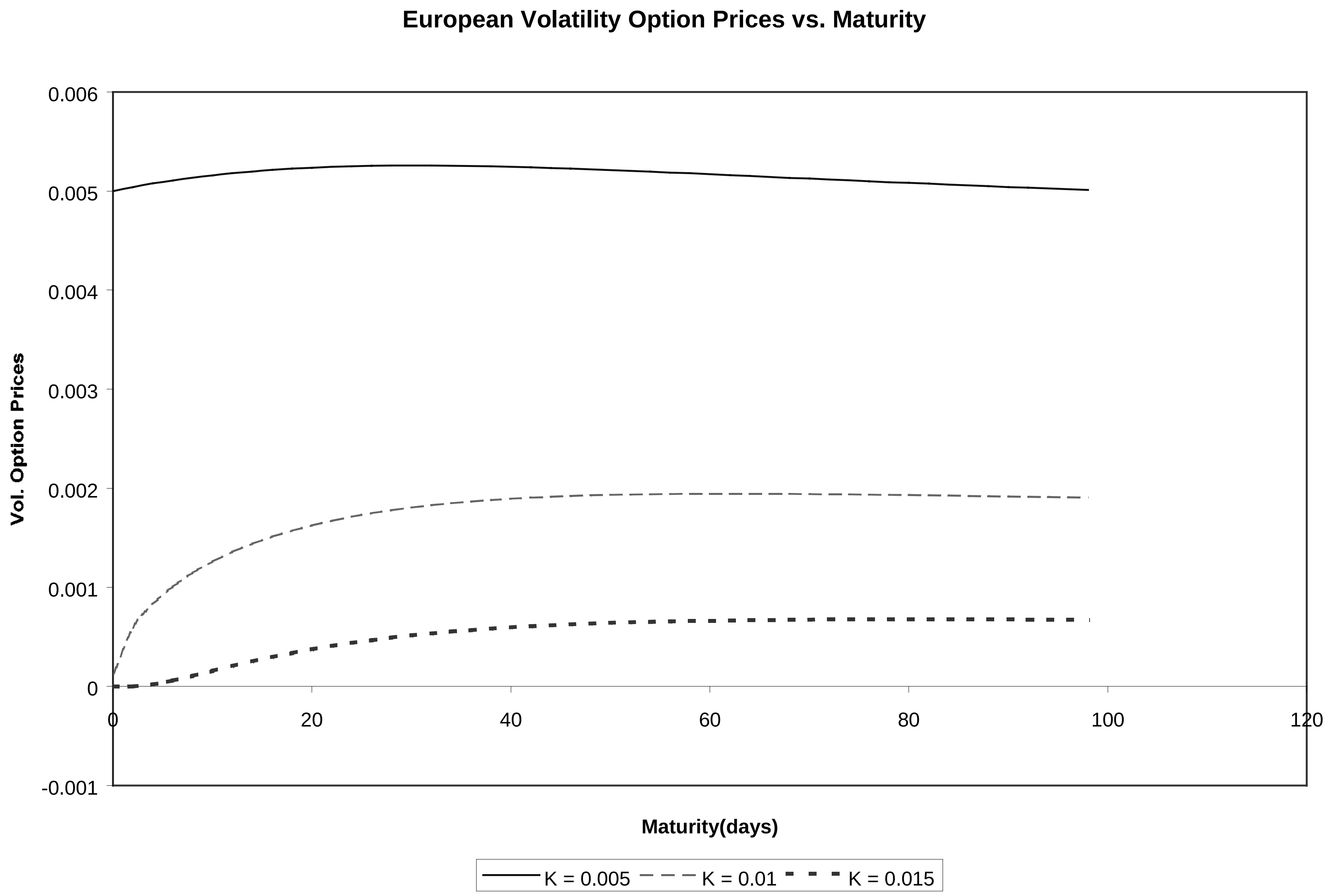

European Volatility Option Prices vs. Maturity
Vol. Option Prices
0.006
0.005
0.004
0.003
0.002
0.001
0
-0.001
0
20
40
60
80
100
120
Maturity(days)
K = 0.005
K = 0.01
K = 0.015

Figure 3

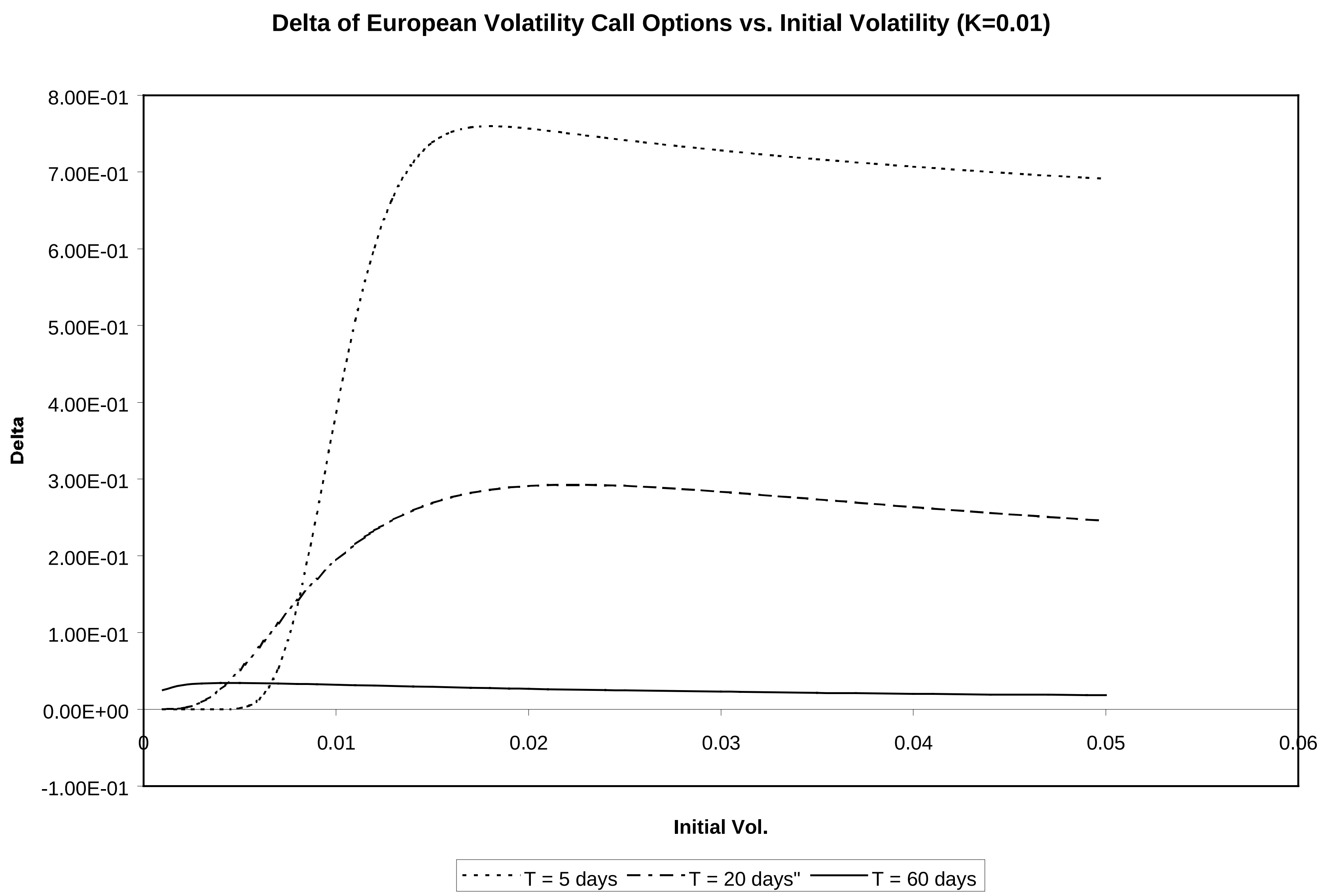

Delta of European Volatility Call Options vs. Initial Volatility (K=0.01)
8.00E-01
7.00E-01
6.00E-01
5.00E-01
4.00E-01
3.00E-01
2.00E-01
1.00E-01
0.00E+00
-1.00E-01
Delta
0
0.01
0.02
0.03
0.04
0.05
0.06
Initial Vol.
T = 5 days
T = 20 days"
T = 60 days

Figure 4

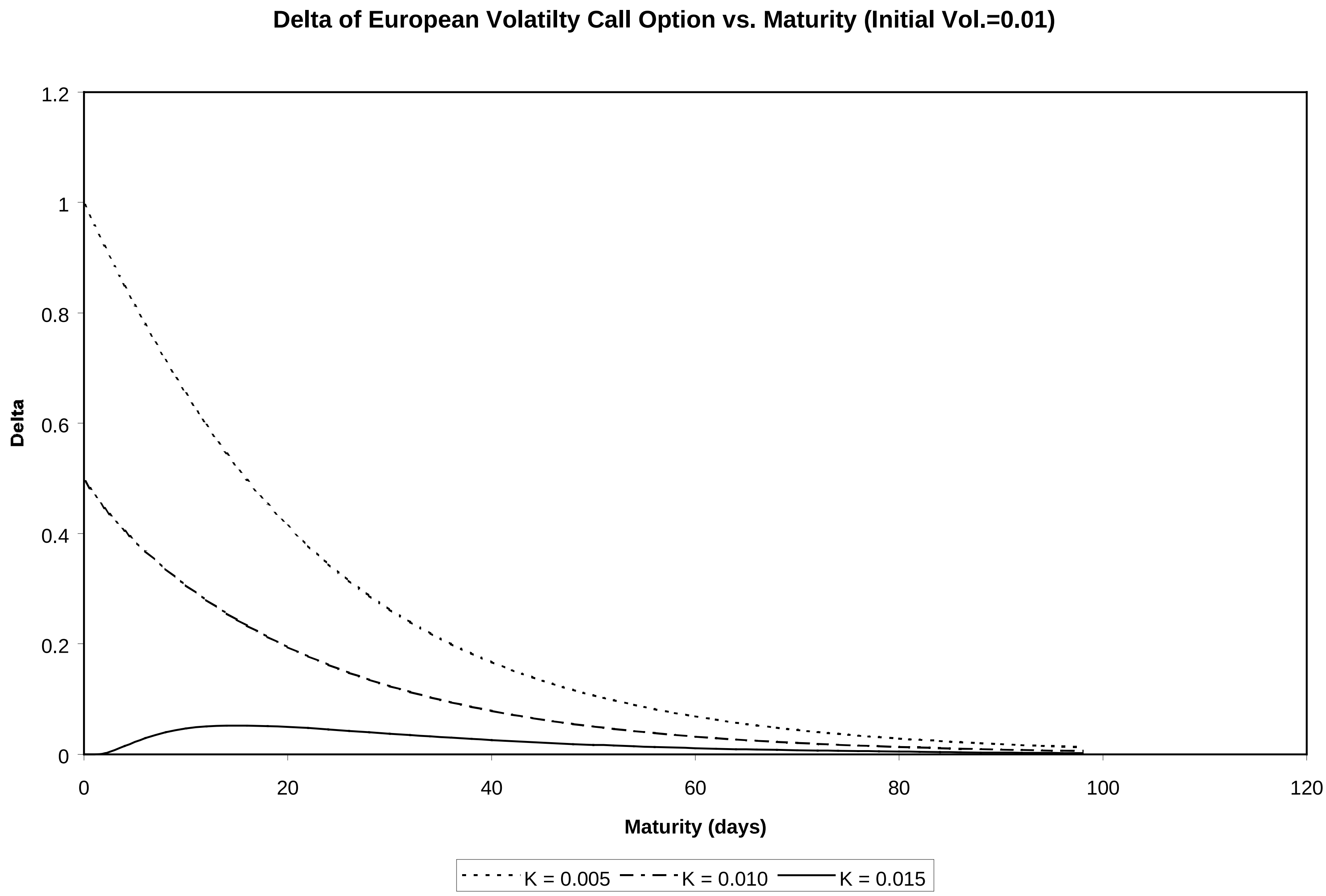
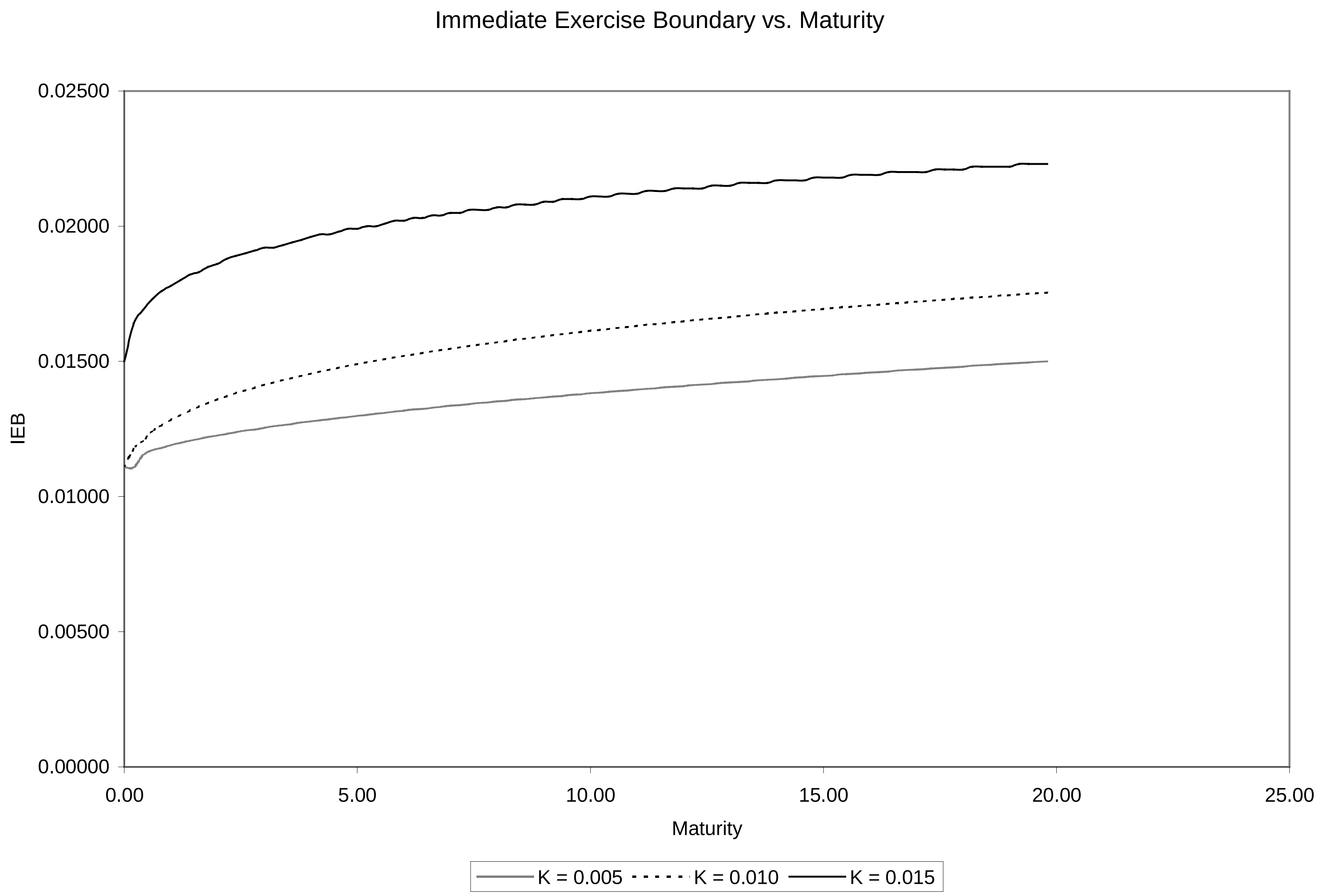

Delta of European Volatilty Call Option vs. Maturity (Initial Vol.=0.01)
1.2
1
0.8
0.6
0.4
0.2
0
Delta
0
20
40
60
80
100
120
Maturity (days)
K = 0.005
K = 0.010
K = 0.015

Figure 5



Immediate Exercise Boundary vs. Maturity
0.02500
0.02000
0.01500
0.01000
0.00500
0.00000
IEB
0.00
5.00
10.00
15.00
20.00
25.00
Maturity
K = 0.005
K = 0.010
K = 0.015

Figure 6

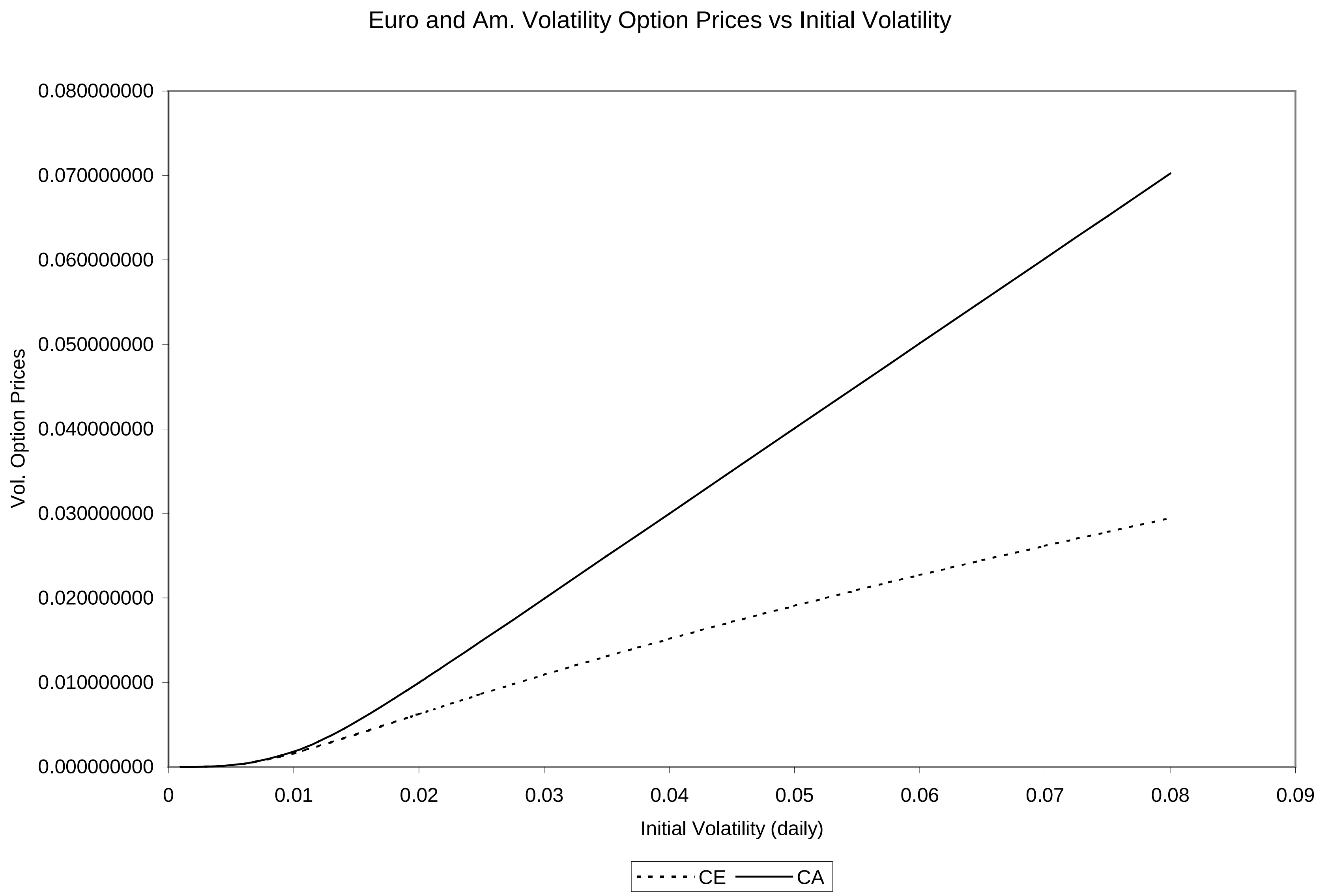
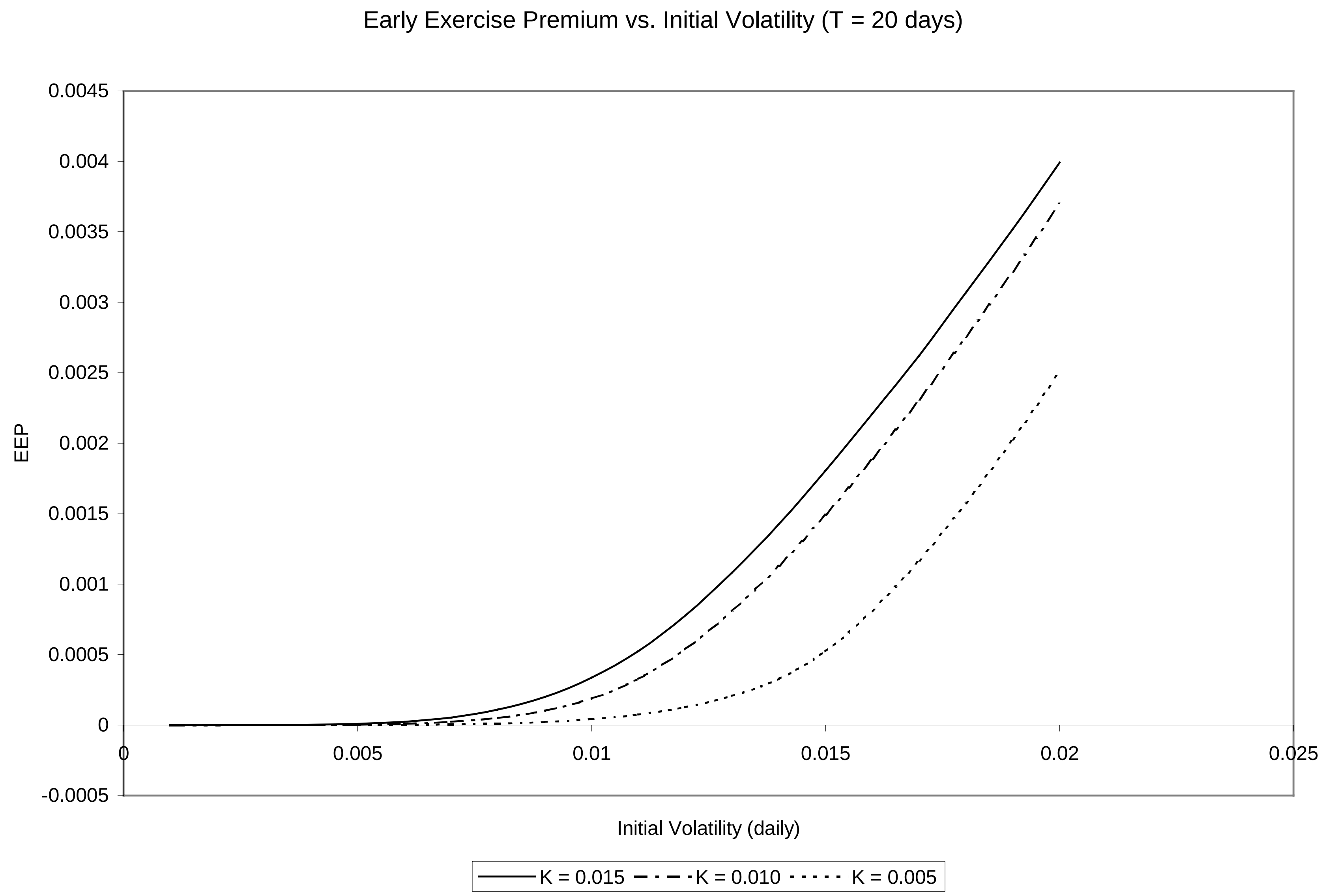

Euro and Am. Volatility Option Prices vs Initial Volatility
Vol. Option Prices
0.080000000
0.070000000
0.060000000
0.050000000
0.040000000
0.030000000
0.020000000
0.010000000
0.000000000
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09
Initial Volatility (daily)
CE
CA

Figure 7


Early Exercise Premium vs. Initial Volatility (T = 20 days)
EEP
0.0045
0.004
0.0035
0.003
0.0025
0.002
0.0015
0.001
0.0005
0
-0.0005
0
0.005
0.01
0.015
0.02
0.025
Initial Volatility (daily)
K = 0.015
K = 0.010
K = 0.005

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