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The Importance of the Loss Function in Option Pricing*

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Résumé / Abstract

Quelle fonction de pertes devrait être utilisée pour l'estimation et l'évaluation des modèles d'évaluation d'options? Plusieurs fonctions différentes ont été suggérées, mais aucune norme ne s'est imposée. Nous ne promouvons aucune fonction, mais soutenons que la cohérence dans le choix des fonctions est cruciale. Premièrement, pour n'importe quel modèle donné, la fonction de pertes utilisée dans l'estimation des paramètres et dans l'évaluation du modèle devrait être la même, sinon on obtient des estimations de paramètres sous-optimales. Deuxièmement, lors de la comparaison de modèles, la fonction de pertes pour l'estimation devrait être la même pour chaque modèle, autrement les comparaisons sont injustes. Nous illustrons l'importance de ces questions dans une application du modèle appelé Black-Scholes du praticien (PBS) aux options de l'index S&P500. Nous trouvons des réductions de plus de 50 pourcent de la racine de l'erreur quadratique moyenne du modèle PBS lorsque les fonctions de pertes d'estimation et d'évaluation sont alignées. Nous trouvons également que le modèle PBS dépasse un modèle de benchmark structurel quand les fonctions de pertes d'estimation sont identiques pour tous les modèles, mais pas dans les autres cas. Le nouveau modèle PBS à fonctions de pertes alignées représente dès lors un benchmark bien plus robuste auquel les futurs modèles structurels pourront être comparés.

Which loss function should be used when estimating and evaluating option pricing models? Many different fucntions have been suggested, but no standard has emerged. We do not promote a partidular function, but instead emphasize that consistency in the choice of loss functions is crucial. First, for any given model, the loss function used in parameter estimation and model evaluation should be identical, otherwise suboptimal parameter estimates will be obtained. Second, when comparing models, the estimation loss function should be identical across models, otherwise unfair comparisons will be made. We illustrate the importance of these issues in an application of the so-called Practitioner Black-Scholes (PBS) model to S&P500 index options. We find reductions of over 50 percent in the root mean squared error of the PBS model when the estimation and evaluation loss functions are aligned. We also find that the PBS model outperforms a benchmark

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structural model when the estimation loss functions are identical across models, but otherwise not. The new PBS model with aligned loss functions thus represents a much tougher benchmark against which future structural models can be compared.

Mots Clés : Évaluation des options, volatilité implicite, approche Black-Scholes du praticien, erreurs d'évaluation, fonctions de perte, prévisions hors-échantillon, stabilité des paramètres

Keywords: Option pricing, implied volatility, practitioner Black-Scholes approach, pricing errors, loss functions, out-of-sample forecasting, parameter stability

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1 Introduction

The literature on option pricing models has expanded dramatically over the last decade. A large number of models have been proposed to address the empirical shortcomings of the classic Black-Scholes (BS) (1973) approach. For instance, an important class of option pricing models specifies the volatility of the underlying asset as a deterministic function of time and the price of the underlying asset (see Derman and Kani (1994), Dupire (1994) and Rubinstein (1994)). Other studies have investigated stochastic volatility models (e.g. see Scott (1987), Hull and White (1987), Heston (1993) and Melino and Turnbull (1990)), jump models (Bates (1996a)) and discrete-time GARCH models (see Duan (1995) and Heston and Nandi (2000)).[1]

The objective of this paper is to contribute to the methodological debate on the estimation and evaluation of these models. A central theme of our work is that all models are to some degree misspecified. The recognition of model misspecification brings to center stage the choice of loss function in model estimation and evaluation. Most standard statistical estimation techniques assume that the model under consideration is correctly specified but simply contains a set of unknown parameters. In this case, most estimators will produce consistent estimates and thus yield the "true" parameter value if a large enough sample is at hand. The choice of estimation technique then largely boils down to finding the most efficient estimator under the prevailing conditions.

However, once model misspecification is recognized, the standard consistency result no longer applies: The parameter estimates obtained will depend on the choice of loss function, even as the sample gets infinitely large. There is no such thing as convergence to the "true" parameter value. Under this scenario, it is crucial to choose a loss function which is relevant for the purpose at hand.

The choice of loss function is particularly important in option pricing. Option pricing models are rarely estimated in order to draw inference about a structural parameter of intrinsic interest. Rather, option pricing models are typically estimated for use in the pricing or hedging of traded options out-of-sample. Thus different purposes, for example hedging, speculating or market making, imply different loss functions on the model errors. Furthermore, option pricing models tend to be based on relatively tightly parameterized models of the underlying asset dynamics and the price of risk, and under the assumption of frictionless markets. These simplifying assumptions have the important benefit of yielding closed-form option pricing formulae, but they quite possibly come at the cost of model misspecification. The multiple potential uses of option pricing models, and the possibility of misspecification, render the choice of loss function crucial.

The academic literature has not been oblivious to the importance of loss functions, but the

[1] For a more complete overview of different approaches used in option pricing see Bakshi, Cao and Chen (1997) and Bates (1996b).

choice of loss function is typically based on statistical and numerical, rather than economic criteria. Whereas some papers in the literature use a loss function based on implied volatility, others calibrate and estimate their models using loss functions based on squared dollar pricing errors (e.g. see Heston and Nandi (2000), Bakshi, Cao and Chen (1997)), relative pricing errors, or a likelihood-based approach (Jacquier and Jarrow (2000)). We do not promote a particular loss function as the choice depends on the end-use of the model, but instead we emphasize that consistency in the choice of loss functions is crucial. Our contribution in this regard is threefold:

First, for any given model, we demonstrate that the loss function used in parameter estimation and model evaluation must be identical in order to minimize the evaluation loss. If the model under consideration is the "true" model then asymptotically any well-behaved loss function will yield the "true" parameter values which in turn will minimize all loss functions. But in a more realistic case where the model is misspecified, the loss function issue is key. We illustrate the importance of the loss function in an application of the benchmark Dumas, Fleming and Whaley (1998) (DFW) implied volatility model, which they refer to as the Practitioner Black Scholes (PBS) model. Using three years of European options on the S&P500 index, we show that correctly aligning the estimation and evaluation loss functions can yield improvements of over 50 percent in the evaluation loss.

Second, when comparing models, the estimation loss function should be identical across models, otherwise unfair comparisons will be made. The PBS model is typically implemented using an implied volatility loss function, which conveniently yields a linear estimator of the parameters. Implemented this way, the PBS model is easily outperformed by more structural stochastic volatility or GARCH option pricing models which are implemented with an estimation loss function which more closely matches the evaluation loss. But our empirical study shows that when the PBS model is implemented fairly, that is, by using the same estimation loss function as the structural model, the PBS model actually outperforms the structural model both in- and out-of-sample.

Third, by implementing the PBS model fairly, we introduce a new PBS model with aligned loss functions, which performs much better than the old. This modified PBS model represents a tougher benchmark against which future structural models can be compared.

The paper proceeds as follows. In Section 2, we first discuss the different loss functions used in empirical option pricing, and we give an overview of their use in the literature. We then present the Practitioner Black-Scholes approach as implemented by DFW and our modification of their approach. Finally, we briefly summarize the relevant theoretical results on estimation under different loss functions. Section 3 describes the data, Section 4 presents our empirical results, and Section 5 briefly discusses the Heston (1993) model and compares it with the PBS model. Section 6 concludes.

2 Methodology

The choice of loss function is particularly important in option pricing. Option pricing models are typically estimated for use in the pricing or hedging of options out-of-sample. Thus different purposes, for example hedging, speculating or market making, might imply different loss functions on the model errors. Even when attention is restricted to simple statistical loss functions, the issue of aligning the evaluation and estimation loss function is key. However, in the extensive and growing literature on option pricing, the specification of the loss function has until now not received much attention compared to other issues, such as model specification and the estimation of continuous-time processes underlying option models. For example, the excellent overview in Campbell, Lo and MacKinlay (1997) does not list any contributions on the importance of the selection of the loss function.

The use of different loss functions at the estimation and evaluation stage is generally accepted and widely used in the literature. For example, Bakshi, Cao and Chen (1997) use $\$MSE$ in estimation and $\%MSE$ as well as $\$MSE$ in the evaluation stage. Rosenberg and Engle (2000) use $\$MSE$ in estimation, and % hedging error in evaluation. Hutchinson, Lo and Poggio (1994) use an MSE based on option price divided by exercise price, and evaluate the model out-of-sample using hedging errors, among other things. Also, several papers estimate model parameters from option prices using an estimation loss function based on the statistical properties of the underlying process or the statistical structure of the measurement errors (see Renault (1997), Jacquier and Jarrow (2000)) and proceed to evaluate the models out-of-sample using a different loss function. Pan (2000) uses a GMM loss function in estimation and $IVMSE$ in evaluation. Chernov and Ghysels (2000) estimate parameters using EMM, and evaluate models using $\$MSE$ and $\%MSE$ loss functions. Benzoni (1998) estimates parameters using both EMM and $\$MSE$ (normalized by the index value) and proceeds to evaluate the model using $\$MSE$ (again normalized). Finally, whereas most recent papers estimate option pricing parameters using option data or option data as well as returns data, until recently many option pricing studies were conducted by estimating option pricing parameters from asset returns and inserting these parameters in option pricing formulae out-of-sample. Again, this amounts to using a different loss function in-sample and out-of-sample.

Problems may arise when one compares out-of-sample errors generated in this way with the errors from other models where in-sample and out-of sample loss functions are identical. For example, DFW (1998) compare the out-of-sample performance of the PBS model with the out-of-sample performance of the implied tree models implemented with identical in- and out-of-sample loss functions. The conclusion of DFW is that the pricing performance of the PBS model compares favorably with that of the implied tree models. Because the implementation of the PBS model proposed in this paper will certainly not deteriorate the model's performance, the conclusions of DFW will therefore be reinforced when the PBS model is implemented properly. This may not be the case for the studies by Heston and

Nandi (2000) and Garcia, Luger and Renault (2000). Both papers use the $\$MSE$ loss functions for the out-of-sample comparison, but use the implied-volatility based loss function for the PBS model in estimation. Heston and Nandi (2000) then compare the PBS model with a GARCH model which has identical in-sample and out-of-sample loss functions. They find that the GARCH model improves upon the performance of the PBS model. Garcia, Luger and Renault (2000) compare PBS to a new Generalized Black-Scholes model, which is also implemented with aligned loss functions, and which is also found to dominate the PBS model. The potential problem is that both studies use the PBS model as a benchmark, but the performance of the benchmark is not as good as it would be if it were implemented using the appropriate loss function.

We now proceed to analyze impact of the loss function in more detail. First, we describe the loss functions most commonly used to estimate and calibrate parameters in empirical option pricing. Then we introduce the PBS model which can be viewed as an ad-hoc model of the well-known "smile" and "smirk" patterns exhibited by standard derivatives prices. We also discuss relevant results from the econometrics literature on the estimation of misspecified models, which in turn motivates the ensuing empirical study.

2.1 Option Pricing Model Evaluation

The performance of different option pricing models is often evaluated using mean-squared dollar errors, that is, the loss function is given by

$$\$MSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}(C_i - C_i(\theta))^2 \tag{1}$$

where C_i and $C_i(\theta)$ are the data and model option prices respectively, and n is the number of option contracts used.[2] The $\$MSE$ loss function has the advantage that the errors are easily interpreted as \$-errors once the square root is taken of the mean-squared-error. However, the relatively wide range of option prices across moneyness and maturity raises the problem of heteroskedasticity for $\$MSE$-based parameter estimation.

Also, because the $\$MSE$ loss function implicitly assigns a lot of weight to options with high valuations (in-the-money and long time-to-maturity contracts) and therefore high \$-errors, some researchers instead favor the relative or percent loss function,[3] defined as

$$\%MSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}((C_i - C_i(\theta))/C_i)^2 \tag{2}$$

[2] Although we estimate new parameters each day, we omit the time subscript, t, on the parameters in this section in order to save on notation.

[3] Note that the %-sign below is just a convenient short-hand for relative loss. We do not in fact multiply the relative loss by 100 anywhere, and so the losses are not actually expressed in percent but in decimals.

The $\%MSE$ loss function has the advantage that a \$1 error on a \$50 dollar option carries less weight than a \$1 error on a \$5 option, which is sensible from a rate-of-return perspective. The disadvantage is that short-time to maturity out-of-the money options with valuations close to zero will implicitly get assigned a lot of weight and can create numerical instability if the $\%MSE$ loss function is used in estimation.

Partly based on the above considerations of heteroskedasticity, and partly based on the market convention of quoting option prices in terms of volatility, some researchers favor estimating option pricing models minimizing the MSE of the implied Black-Scholes volatility from the option. We therefore define the implied volatility MSE as

$$IVMSE(\theta) = \frac{1}{n}\sum_{i=1}^{n}(\sigma_i - \sigma_i(\theta))^2, \tag{3}$$

where the implied volatilities are obtained as

$$\sigma_i = BS^{-1}(C_i^{Mkt}, T_i, X_i, S, r), \text{ and } \sigma_i(\theta) = BS^{-1}(C_i(\theta), T_i, X_i, S, r), \tag{4}$$

with BS^{-1} being the inverse of the Black-Scholes formula, C_i^{Mkt} the market price of option i, $C_i(\theta)$ the model price for option i, T_i the time-to-maturity, X_i the strike price, S the price of the underlying stock and r the riskless interest rate.

This paper only considers the loss functions in (1), (2) and (3). A number of other estimation loss functions are used in the literature as discussed above. Functions based on hedging or speculation loss could potentially be more interesting, but we focus on the three functions listed here as they are arguably the most prevalent in previous work.

2.2 The Practitioner Black-Scholes Model

We illustrate the importance of the estimation loss function using the simplest model possible, namely the so-called Practitioner Black-Scholes (PBS) model. In the PBS model, implementation is done in three steps. First the Black-Scholes implied volatility is calculated for each observed option. Second, the implied volatilities are regressed on different polynomials in T and X using simple ordinary least squares. Third, the fitted values for volatility are plugged back into the Black-Scholes formula to get the practitioner model price.

DFW consider different implied volatility functions. We limit our attention to the most general model they investigate, which is of the form,[4]

$$\sigma = \theta_0 + \theta_1 X + \theta_2 X^2 + \theta_3 T + \theta_4 T^2 + \theta_5 XT + \varepsilon \tag{5}$$

[4]DFW consider switching between specifications based on the number of maturities available in their data set on any given day. Our data set is sufficiently rich that the specification used in (5) would always be chosen by DFW's switching model.

and where the fitted value of the implied volatility is then

$$\sigma(\theta) = \theta_0 + \theta_1 X + \theta_2 X^2 + \theta_3 T + \theta_4 T^2 + \theta_5 XT \tag{6}$$

Notice that estimating (5) by OLS amounts to letting the estimation loss function be $IVMSE$. OLS solves

$$\theta_{IV} = Arg\min_{\theta} IVMSE(\theta) \equiv Arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}(\sigma_i - \sigma_i(\theta))^2 = (Z'Z)^{-1}Z'\sigma \tag{7}$$

where $Z = \begin{bmatrix} 1 & X & X^2 & T & T^2 & XT \end{bmatrix}$ is the matrix of regressors from the implied volatility model. To evaluate the model, the estimate of the parameter vector, θ_{IV}, is then plugged back into the implied volatility model, which in turn is plugged into the Black-Scholes formula. The model is typically assessed using

$$\$MSE(\theta_{IV}) = \frac{1}{n}\sum_{i=1}^{n}\left(C_i - C_i^{BS}(\sigma_i(\theta_{IV}))\right)^2 \tag{8}$$

or

$$\%MSE(\theta_{IV}) = \frac{1}{n}\sum_{i=1}^{n}\left(\left(C_i - C_i^{BS}(\sigma_i(\theta_{IV}))\right)/C_i\right)^2 \tag{9}$$

In this framework, the estimation loss function, which is defined on implied volatilities, is different from the evaluation loss function, which is defined on dollar or percent pricing errors. Whereas this is a convenient and easily implemented procedure, it has potentially serious costs if the model is assessed in terms of a $\$MSE$ or $\%MSE$ loss function as we shall see below. The appropriate procedure is instead to use nonlinear least squares (NLS) to directly estimate θ, as follows

$$\theta_{\$} = Arg\min \$MSE(\theta) \equiv Arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}\left(C_i - C_i^{BS}(\sigma_i(\theta))\right)^2 \tag{10}$$

or

$$\theta_{\%} = Arg\min \%MSE(\theta) \equiv Arg\min_{\theta} \frac{1}{n}\sum_{i=1}^{n}\left(\left(C_i - C_i^{BS}(\sigma_i(\theta))\right)/C_i\right)^2 \tag{11}$$

This approach amounts to estimating the model parameters using the relevant loss function. Baring potential numerical problems in the nonlinear estimation, this should generate better in-sample fit. The key question is how the different estimates fare out-of-sample, which will be assessed below.

2.3 Parameter Estimation

As the three loss functions above are all well-behaved, and as the Black-Scholes pricing formula and the implied volatility function are continuous in the parameters, the estimates will in each case asymptotically converge to the values which minimize the estimation loss function. More explicitly, in population, let

$$\theta_X^* = Arg \min_{\theta} \lim_{n \to \infty} XMSE(\theta), \text{ for } X = \$, \%, IV \tag{12}$$

Then, as the Black-Scholes pricing function is continuous in the parameter vector, θ, and using the results in White (1981), we get

$$\lim_{n \to \infty} \theta_X = \theta_X^*, \tag{13}$$

where, depending on the loss function, θ_X is either θ_{IV}, $\theta_\$$ or $\theta_\%$, that is, the loss-function specific, finite sample estimates from above.[5]

Notice that we can view $\theta_\%$ as a weighted least squares (WLS) estimate of $\theta_\$$ and vice versa, where the weights are $1/C_i^2$ and C_i^2 respectively.[6] Under the general assumption that the model being estimated is misspecified, White (1981) shows that the WLS estimate will converge to a limit which depends on the weights, and which will therefore be different from the unweighted least squares estimate.[7] As we are estimating the practitioner version of the Black-Scholes model, which clearly violates the underlying assumption of constant volatility, model misspecification is indeed built-in from the start. Furthermore, it is probably reasonable to assume that the tightly parameterized structural models put forth in the literature are all to some degree misspecified, but the quantitative importance of this from a loss function perspective is of course an empirical question.

For our purposes, it suffices to notice that as the implied volatility function estimated is clearly misspecified, the estimates from each loss function will converge to different limits, that is

$$\theta_{IV}^* \neq \theta_\$^* \neq \theta_\%^* \tag{14}$$

Thus even asymptotically, the estimates minimizing $IVMSE$ will not minimize $\$MSE$ etc. Consequently, under the realistic assumption that the model is misspecified, matching the estimation loss function to the evaluation loss function will matter–even asymptotically. How important this is in a realistic finite-sample, out-of-sample setting will be the focus of the empirical analysis below.

[5] Thus θ_{IV} is from an OLS regression, and $\theta_\$$ and $\theta_\%$ are from NLS estimation.

[6] Note that the loss functions mentioned above which normalize the option pricing error by the strike price or the stock price can be viewed as WLS estimates of the $\$MSE$ loss as well.

[7] Thus we could construct tests for model misspecification based on the differences between $\theta_\%$ and $\theta_\$$. This idea was first put forward in the context of linear models by Hausman (1978).

We close this section by noting that we are not promoting any particular loss function but instead stressing the importance of being consistent in the choice of loss function when the model is potentially misspecified. In order to obtain the best possible fit, the loss function used in out-of-sample evaluation should also be used in in-sample estimation. If a researcher is interested in several different loss functions when evaluating and comparing models, care should be taken that the estimation loss function is identical across models, otherwise unfair comparisons will be made.

3 The Data

We analyze the methodological issues outlined above using a very standard data set on S&P 500 call option prices for 755 days in the period from June 1, 1988 through May 31, 1991. The data were graciously provided to us by Gurdip Bakshi and are practically identical to the ones used in the Bakshi, Cao and Chen (1997) study. We limit ourselves to a few important features of the data here and refer the reader to that study for further details. The data set is well suited for our empirical analysis because options written on the S&P 500 are the most actively traded European style contracts. Particular care was taken to adjust the S&P 500 spot index series for dividend payments and to obtain synchronous recording of stock and option prices. The resulting data set contains a wide variety of option quotes for different values of moneyness and maturity. Table 1 lists the number of contracts for a set of maturity and moneyness bins, where S/X denotes the option's moneyness and DTM stands for days to maturity.[8]

Table 1: Number of Contracts Across Moneyness and Maturity

	DTM < 60	60 < DTM < 180	180 < DTM	Total
S/X < .94	674	3075	2553	6302
.94 < S/X < .97	2058	2049	1014	5121
.97 < S/X < 1.00	2604	1978	963	5545
1.00 < S/X < 1.03	2445	1744	803	4992
1.03 < S/X < 1.06	2206	1501	731	4438
1.06 < S/X	4665	4734	2690	12089
Total	14652	15081	8754	38487

Table 2A reports the average price for option contracts with different moneyness and maturity. For our purpose, the most important observation in Table 2A is the large differences

[8]To be exact, we are sorting the data by $(S - D_i)/X_i$, where D_i is the present value of dividends accruing to option i until its expiration.

in option prices for different maturities and moneyness. As a result, expensive contracts would implicitly receive much more weight in the $\$MSE$ loss function than cheap contracts.

Table 2A: Average Call Prices Across Moneyness and Maturity

	DTM < 60	60 < DTM < 180	180 < DTM
S/X < .94	1.53	5.09	10.32
.94 < S/X < .97	2.60	9.58	18.81
.97 < S/X < 1.00	5.29	14.87	25.00
1.00 < S/X < 1.03	11.02	21.25	31.32
1.03 < S/X < 1.06	18.44	28.06	37.19
1.06 < S/X	39.59	49.85	62.41

In Table 2B we report the average implied Black-Scholes volatilities from the call prices in Table 2A. Notice that in general the implied volatilities are much less variable across entries in the table than are the call prices themselves. A noticeable exception is the short-maturity, deep out of the money calls. Notice also that the well known post-crash smirk is apparent in every column, but that it is most apparent at the shortest maturity.

Table 2B: Average Implied Volatility Across Moneyness and Maturity

	DTM < 60	60 < DTM < 180	180 < DTM
S/X < .94	0.0591	0.1719	0.1678
.94 < S/X < .97	0.1664	0.1719	0.1767
.97 < S/X < 1.00	0.1713	0.1820	0.1854
1.00 < S/X < 1.03	0.1899	0.1936	0.1966
1.03 < S/X < 1.06	0.2161	0.2038	0.1950
1.06 < S/X	0.3122	0.2349	0.2178

As mentioned above, we investigate the importance of the choice of loss function by estimating the relevant parameters for every one of the 755 daily cross sections. Figure 1 indicates that the optimization problem under study can be substantially different for different days. To illustrate the variation over the sample, we depict the average Black-Scholes implied volatility calculated for each of the 755 days in the sample. Notice that implied volatility changes through time but that swings in average implied volatility seem relatively persistent through time. This finding suggests that the out-of-sample performance of some models may actually turn out to be fairly satisfactory if the parameters are appropriately estimated.

4 Empirical Results

The main results of the paper are contained in Figures 2A and 2B and Table 3A. For each of the 755 days in the sample, we repeat the following exercise: First, we estimate the parameter θ characterizing the implied volatility function (5) using three different loss functions. We will refer to these three estimates of θ as $\theta_{t,\$}$, $\theta_{t,\%}$ and $\theta_{t,IV}$ respectively, where the t subscript indicates that the estimate was obtained using the t-th day (or cross-section) in the sample. The first estimate, $\theta_{t,\$}$, is obtained by minimizing the loss function (10). The second estimate, $\theta_{t,\%}$, is obtained by minimizing the loss function (11). The third estimate, $\theta_{t,IV}$, is obtained by minimizing the loss function (7). Subsequently we use these estimates to evaluate the model's pricing performance in- and out-of-sample using different loss functions.[9] Possibly the most interesting exercise is to evaluate the different loss functions one day out of sample. Consider the two middle pictures in Figure 2A. Both pictures show the square root of MSE's (RMSE) for the dollar-based loss function (1) evaluated at $t+1$ using a parameter estimate obtained at t. However, in the left panel the estimate used is $\theta_{t,IV}$, which is obtained by minimizing the "wrong" loss function (7). In contrast, in the right panel the estimate $\theta_{t,\$}$ is obtained by minimizing the "correct" loss function (10). The differences in RMSE between the two panels are striking. Whereas on a few occasions (especially in the second half of the sample) the RMSE is quite large in the right panel, the RMSE is often minuscule compared to the left panel.

The other panels in Figure 2A contain results for related exercises. In the two top panels we present the same exercise using estimates $\theta_{t,IV}$ and $\theta_{t,\$}$, but now the RMSE is computed for the same day t. Again we observe that the RMSE in the left panel is much larger than the RMSE in the right panel.[10] Finally, the two bottom panels again use the estimates $\theta_{t,IV}$ and $\theta_{t,\$}$, but now the RMSE is evaluated at day $t+5$. We see that whereas RMSEs in the left panel are not much different from the ones in the top left panel, the RMSEs for the bottom right panel are considerably higher than the ones for the top right panel. Nevertheless, it is clear that the RMSEs in the bottom right panel are again much lower than those in the bottom left panel.

Figure 2B presents results for an exercise that is analogous to the one presented in Figure 2A, except that the RMSEs are now for the percentage-based loss function (2) evaluated at t, $t+1$ and $t+5$. The estimates used are the ones using the "wrong" loss function $\theta_{t,IV}$ and

[9] Our exercise is similar to Dumas, Fleming and Whaley (1998). Hull and Suo (2000) alternatively investigate the usefulness of the PBS model fitted to standard European options in-sample for pricing exotic options out-of-sample.

[10] The fact that the in-sample losses are so different indicates the degree of misspecification of the model. If we were working with the true model, then asymptotically the parameters would be identical across loss functions, and the in-sample losses would therefore be the same across loss functions. Besides model misspecification issues, the finite-sample properties of the estimators from the different loss functions will of course play a role as well.

the one using the "correct" loss function $\theta_{t,\%}$. The conclusion from Figure 2B is identical to the one obtained from Figure 2A: the use of the wrong loss function in estimation leads to dramatic under-performance in the in-sample and out-of-sample RMSE.

Table 3A: Average RMSE Losses for Various Estimates

	$IVRMSE$-Loss	$\$RMSE$-Loss	$\%RMSE$-Loss
In-Sample			
IV Estimates	0.0576	0.6778	0.0967
\$-Estimates	0.0645	0.2977	0.0622
%-Estimates	0.0657	0.5292	0.0329
1-Day Out			
IV Estimates	0.0618	0.8738	0.1204
\$-Estimates	0.0651	0.5025	0.0781
%-Estimates	0.0664	0.6642	0.0605
5-Days Out			
IV Estimates	0.0645	1.0486	0.1461
\$-Estimates	0.0655	0.6924	0.1059
%-Estimates	0.0673	0.8269	0.0935

Table 3A summarizes the information in Figures 2A and 2B by presenting the average RMSE computed over the 750 days in the sample.[11] The diagonal on each part of the table corresponds to the loss from using the relevant loss function in estimation. Off-diagonal entries correspondingly report the losses from using estimates which minimize a loss function different from the relevant one. For example, for the results in Figure 2A, we look in the middle column. We see that on average, when using the appropriate loss function to obtain the estimates $\theta_{t,\$}$, the one-day out-of-sample RMSE is 0.5025, whereas if the wrong loss function is used to obtain $\theta_{t,IV}$, the average RMSE is 0.8738. For 5-days out-of-sample, the corresponding RMSEs are 0.6924 and 1.0486, respectively. The data in Figure 2B is summarized in the right column. Again we see large average improvements in RMSEs from using the appropriate loss function. Finally, in the left column we present evidence on an estimation exercise that was not reported in Figure 2. Whereas most studies cited above use the dollar-based loss function (1) or percentage-based loss function (2) for out-of-sample

[11] In order to compare the in-sample and one through five days out-of-sample numbers we only report results for the last 750 out of 755 days.

performance evaluation, we investigate which average RMSEs one would obtain when evaluating the square root of the $IVMSE$ (3) out-of-sample. It can be seen that regardless of whether we evaluate the RMSEs at t, $t+1$ or $t+5$, we always obtain the lowest RMSEs by using the appropriate in-sample loss function. Notice however that the differences across estimates in $IVRMSE$ loss are much smaller than the differences across estimates in $\$RMSE$ and $\%RMSE$ loss.

In order to facilitate the comparison of different RMSEs, Table 3B reports average RMSEs from Table 3A but normalized by the RMSE from the relevant loss function. Notice that normalized loss is always at least one. Notice also that the IV estimates fare particularly poorly when used in the other loss functions. These tables therefore illustrate our main point that it is critically important to use the correct loss function. We again stress that we are not advocating any particular loss function, but simply cautioning researchers to be consistent in their choice: The estimation loss function should ideally be the same as the evaluation loss function, or at a minimum, the estimation loss function should be identical across models. We view the existing discussion about the relative merits of certain loss functions is to some extent a moot one: The choice of loss function should be driven by user objectives. Even though some loss functions have obvious econometric problems associated with them, such as heteroskedasticity and numerical stability issues, our analysis indicates that these concerns are outweighed by the gains from matching loss functions. Regardless of the loss function of interest in model evaluation, it should also be used in estimation as long as it is reasonably well-behaved.

Table 3B: Normalized Average RMSE Losses for Various Estimates

	$IVRMSE$-Loss	$\$RMSE$-Loss	$\%RMSE$-Loss
In-Sample			
IV Estimates	1.0000	2.2764	2.9387
$-Estimates	1.1206	1.0000	1.8898
%-Estimates	1.1409	1.7774	1.0000
1-Day Out			
IV Estimates	1.0000	1.7388	1.9910
$-Estimates	1.0528	1.0000	1.2908
%-Estimates	1.0744	1.3217	1.0000
5-Days Out			
IV Estimates	1.0000	1.5143	1.5629
$-Estimates	1.0165	1.0000	1.1328
%-Estimates	1.0440	1.1942	1.0000

Is it possible to provide some intuition for why the $IVMSE$-based estimates seem so much poorer than the \$-based and %-based estimates? Figures 3A and 3B graph the estimates of the coefficients in the implied volatility relation (5) for all 755 cross-sections used, when using different loss functions. The panels on the left use the IV loss function (3). The panels in the middle use the dollar based loss function (1) and the panels on the right use the percentage loss function (2). The problem with the $IVMSE$ estimates seems to be that, although they are obtained from linear regression, they are much more volatile than the estimates from the other loss functions, which are obtained using nonlinear estimation.

Table 4 complements Figure 3 by reporting the 5th, 50th, and 95th percentile of the daily estimates for the six parameters. The parameters have been rescaled to take on reasonable values and the magnitudes are therefore not directly interpretable. This table confirms the variability in the IV estimates when compared with the dollar and percentage-based estimates. It appears that the dollar and percentage based estimates are more robust over time than the IV estimates, despite their nonlinear features. As a direct result their out-of-sample performance is much better.

Table 4: Parameter Estimates from Various Loss Functions: Percentiles of Daily Estimates

$IVMSE$-Estimates	Constant	Strike	Strike^2	Maturity	Maturity^2	Strike$*$Maturity
5th Percentile	0.0970	-1.7932	-0.6234	-0.2496	-0.1451	0.0433
Median	1.0584	-0.3214	0.1897	-0.0824	0.0239	0.2483
95th Percentile	3.6234	0.1988	2.4775	-0.0085	0.4012	0.6828
\$-Estimates						
5th Percentile	0.3543	-0.4984	-0.2678	-0.0789	-0.1205	0.0447
Median	0.7983	-0.2012	0.0000	-0.0422	-0.0287	0.1427
95th Percentile	1.2843	0.0035	0.5349	-0.0106	0.0561	0.2417
%-Estimates						
5th Percentile	0.2061	-0.9323	-0.0000	-0.0666	-0.2113	-0.0641
Median	0.7572	-0.1808	0.0000	-0.0173	-0.0559	0.0762
95th Percentile	2.1000	-0.0117	1.2503	0.0241	0.1092	0.2310

5 Comparison with a Structural Model

So far the empirical analysis has focused on documenting the improvement in the evaluation loss when the appropriate estimation loss is used. We now ask if the loss function issue is important enough to reverse existing empirical rankings of models. Interestingly, it is.

To document this, we compare the PBS model's pricing performance with the pricing performance of the classic stochastic volatility model proposed by Heston (1993). Heston's model assumes that the stock price under risk neutrality evolves according to

$$dS(t) = rSdt + \sqrt{v(t)}Sdz_1(t) \tag{15}$$

with the variance process

$$dv(t) = \kappa(\varphi - v(t))dt + \sigma\sqrt{v(t)}dz_2(t) \tag{16}$$

with $z_1(t)$ and $z_2(t)$ standard Brownian motion with correlation coefficient ρ.[12] The Heston model is attractive because it yields an analytical solution for the option price (up to a numerical integral that can be evaluated fast and accurately). This solution can be found in Heston (1993) and Bakshi, Cao and Chen (1997). We implement this model by estimating the four parameters κ, φ, σ and ρ. Also, because we estimate the model on a day-by-day basis, we follow the example of Bakshi, Cao and Chen (1997) and estimate the initial conditional volatility, $v(0)$, as a fifth parameter each day. We estimate these parameters for $\$MSE$ as well as $\%MSE$ loss functions. We then proceed to evaluate the model in-sample and out-of-sample and to compare the pricing errors with those of the PBS model.

Table 5 presents average RMSE losses for the Heston model using the same 750 days of options contracts that were used to generate the empirical results in Table 3A. To facilitate comparisons between the models we repeat certain entries from Table 3A in Table 5. More detailed evidence on the day-by-day performance of the Heston model is reported in Figure 4.

Table 5 clearly illustrates that the use of the appropriate loss function is of critical importance. For instance, consider the performance of the Heston model when the $\$RMSE$ loss function is used. For the in-sample evaluation the average $\$RMSE$ over 750 days is 0.3858 (middle column). Consider comparing this with the PBS model as implemented in DFW (left column). We would conclude that the Heston model beats the benchmark PBS model, because 0.3858 is lower than 0.6778. However, when the PBS model is implemented using the appropriate loss function (right column), the average $\$RMSE$ is 0.2977. Therefore, the structural model actually does not perform better than the modified PBS model.[13] Similarly, inspecting the $\$RMSE$ loss functions evaluated out-of-sample, we see that the average 1-day and 5-day out-of-sample $\$RMSE$s are 0.5769 and 0.8253, higher than 0.5025 and 0.6924

[12] It is well-known from a number of recent papers that the empirical performance of this model is not entirely satisfactory and that extending the model can improve its pricing performance (see Andersen, Benzoni and Lund (1999), Bakshi, Cao and Chen (1997), Benzoni (1998), Chernov and Ghysels (2000), Jones (2001) and Pan (2000)). We simply want to compare different implementations of the PBS model to a mainstream structural model that is easy to implement and which fits the data reasonably well.

[13] Notice that while the Heston model nests the original Black and Scholes (1973) model it does not nest the PBS model. Therefore, it is possible for the PBS model to have a better fit than the Heston model.

respectively for the PBS model. If we had used the standard $IVMSE$-implementation of the model we would have concluded that 0.5769 and 0.8253 are lower than 0.8738 and 1.0486 respectively.

Table 5: Average RMSE Losses for Heston and PBS Models

$\$RMSE$ Loss	PBS $IVMSE$ Estimates	Heston $\$MSE$ Estimates	PBS $\$MSE$ Estimates
In-Sample	0.6778	0.3858	0.2977
1-Day Out	0.8738	0.5769	0.5025
5-Days Out	1.0486	0.8253	0.6924
$\%RMSE$ Loss	PBS $IVMSE$ Estimates	Heston $\%MSE$ Estimates	PBS $\%MSE$ Estimates
In-Sample	0.0967	0.0361	0.0329
1-Day Out	0.1204	0.0654	0.0605
5-Days Out	0.1461	0.1046	0.0935

Inspection of the second set of rows in Table 5 shows that identical conclusions obtain when evaluating $\%RMSE$ loss functions. For the in-sample evaluation, the average Heston $\%RMSE$ of 0.0361 is lower than 0.0967 but higher than 0.0329. For the 1-day out-of-sample exercise the average Heston $\%RMSE$ of 0.0654 is lower than 0.1204 but higher than 0.0605. Finally, for the 5-day out-of-sample exercise the average Heston $\%RMSE$ of 0.1046 is lower than 0.1461 but higher than 0.0935. In summary, the conclusions from the comparison of the Heston model with the PBS model are robust. Regardless of whether one uses $\$RMSE$ or $\%RMSE$ loss functions, and regardless of whether one evaluates the loss functions in-sample or out-of-sample, the PBS model performs better than the Heston model when implemented using the appropriate loss function. However, when the PBS model is implemented using the $IVMSE$ loss function to estimate the parameters, which is the standard in the literature, it cannot improve upon the performance of the Heston model.

6 Conclusions

This paper raises an important methodological issue concerning the estimation of parameters for use in option pricing models. Until now, the literature has mainly focused on the choice of option pricing model. Once the model is chosen, the main concern is the consistent and efficient econometric estimation of the parameters characterizing the model. The focus is thus much more on estimation as opposed to evaluation of the model. We argue that the relevant discussion should not start out by discussing how to estimate the parameters, but rather by stating what the evaluation (typically out-of-sample) loss function is. This loss function is dictated by the purpose of the empirical exercise and can be related to a hedging problem or a risky investment strategy for example.

It may be difficult at first to intuitively grasp why this issue is of more than philosophical importance. The key is that one should stop thinking about the search for a "true" model. Instead, we are searching for parsimonious models that are inherently misspecified and that fit certain features of the data better than others. If the model under investigation is the "true" model, it does not matter which loss function we use. We always obtain the same "true" parameters. This is the mind-set inspiring much of current practice, which puts a lot of emphasis on efficient econometric estimation. We argue that the focus should be on the minimization of the loss function used in evaluation. The standard econometric solution to this problem when working with misspecified models is to use an in-sample estimation loss function identical to the out-of-sample evaluation loss function.

The paper then demonstrates that this approach is of great practical importance. To do this, we focus on the simplest model available in the literature that attempts to account for the well-known biases in the Black-Scholes model, namely the Practitioner Black-Scholes (PBS) model. The PBS model is typically implemented with an estimation loss function which is different from the evaluation loss function, and this constitutes a problem. Our analysis shows that this problem is quantitatively important, with the implementation used in the literature leading to out-of-sample RMSEs that are more than twice the lowest possible RMSE using the proper estimation loss function. This finding has serious implications for future studies and for papers that have implemented this technique the traditional way. To demonstrate these implications, we compare the empirical performance of the PBS model to the performance of a well-known stochastic volatility model due to Heston (1993). We find that when the PBS model is implemented using an inappropriate loss function, it performs much worse than the Heston model. However, the PBS model performs somewhat better than the Heston model when the appropriate loss function is used. Thus our modified PBS model represents a new and tougher standard against which the performance of future structural models can be measured.

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Figure 1: Mean Implied Volatility Each Day

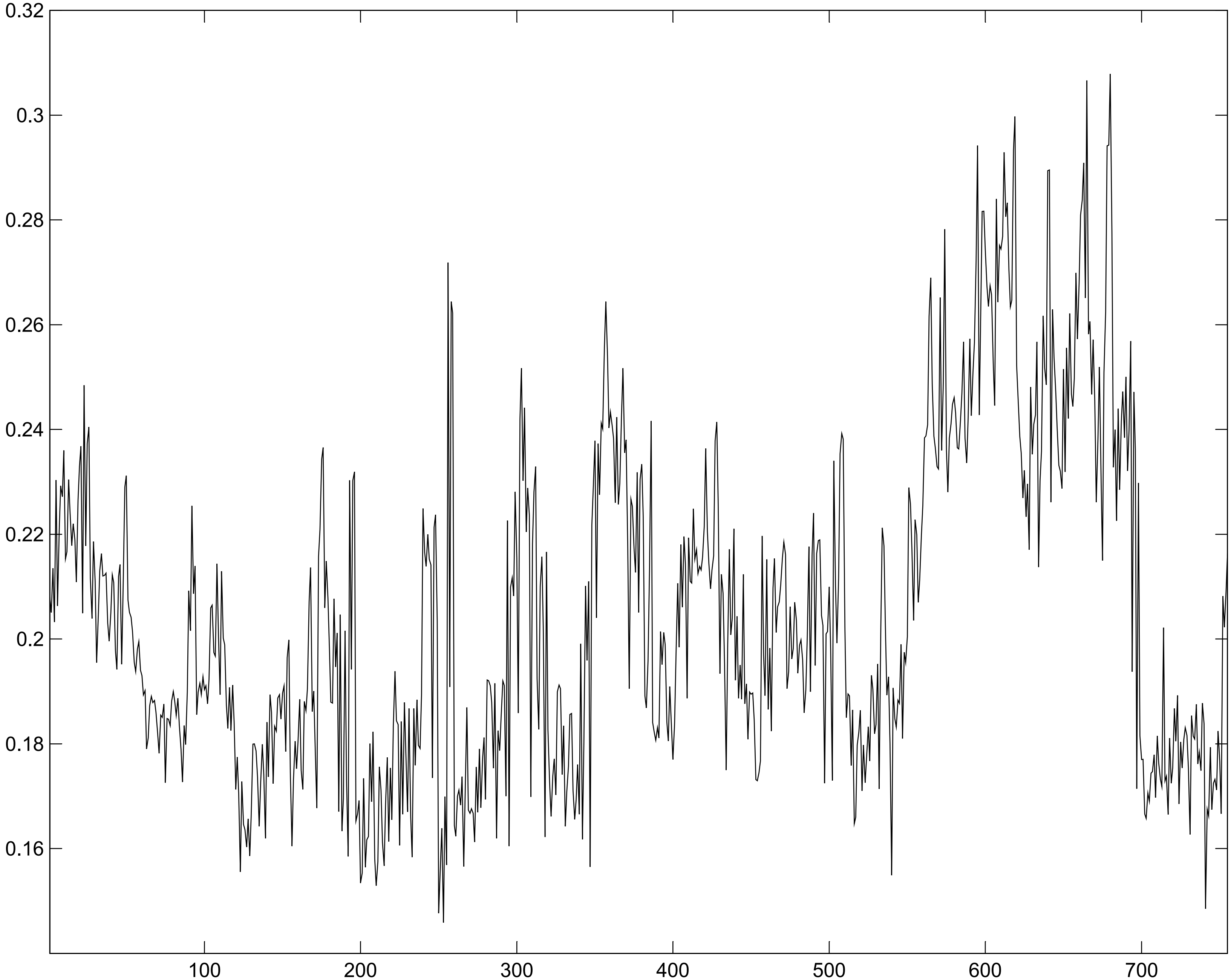


Figure 2A: $-Root-Mean-Squared-Error Losses From IV and $ Estimates

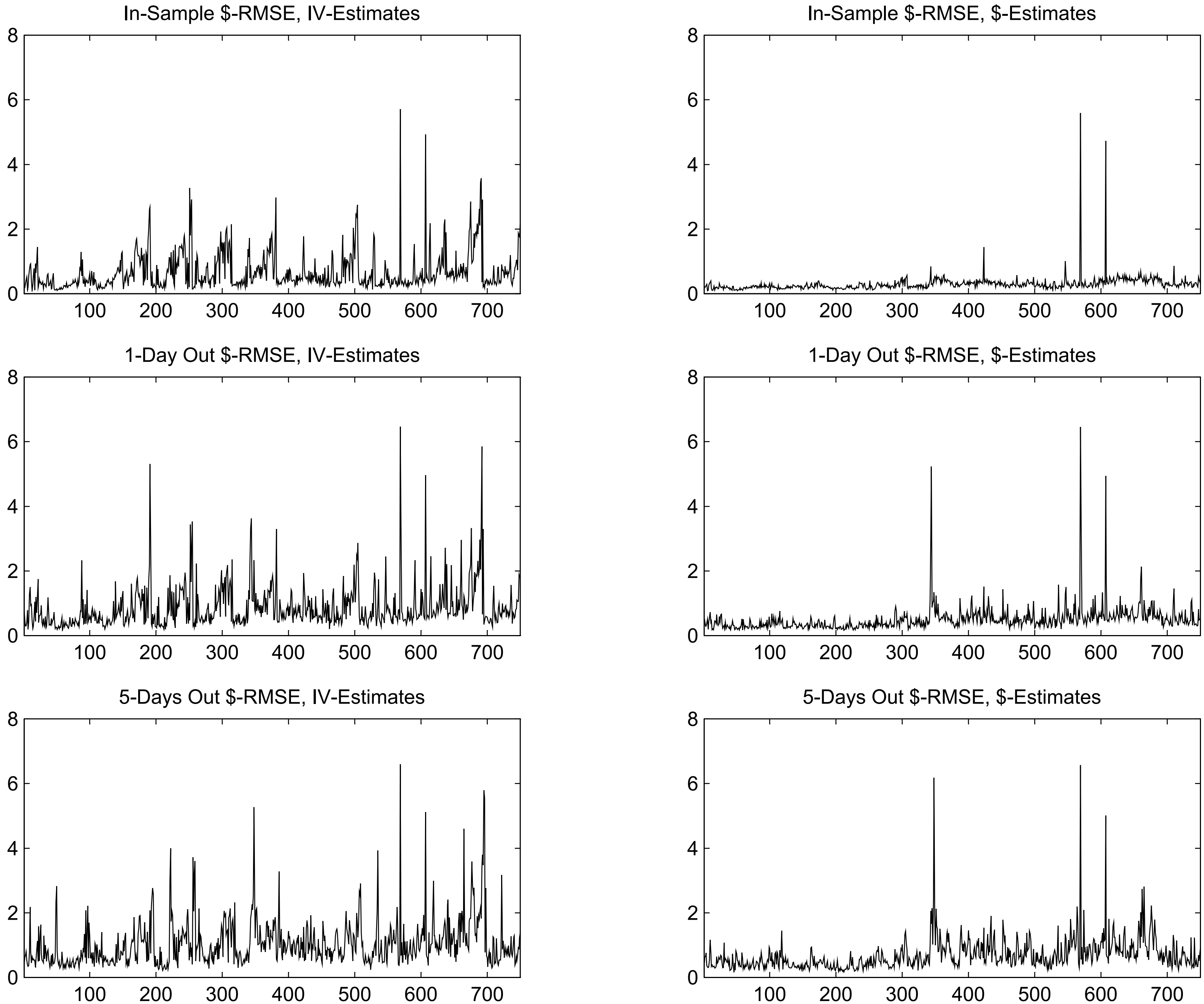


Figure 2B: %-Root-Mean-Squared-Error Losses From IV and % Estimates

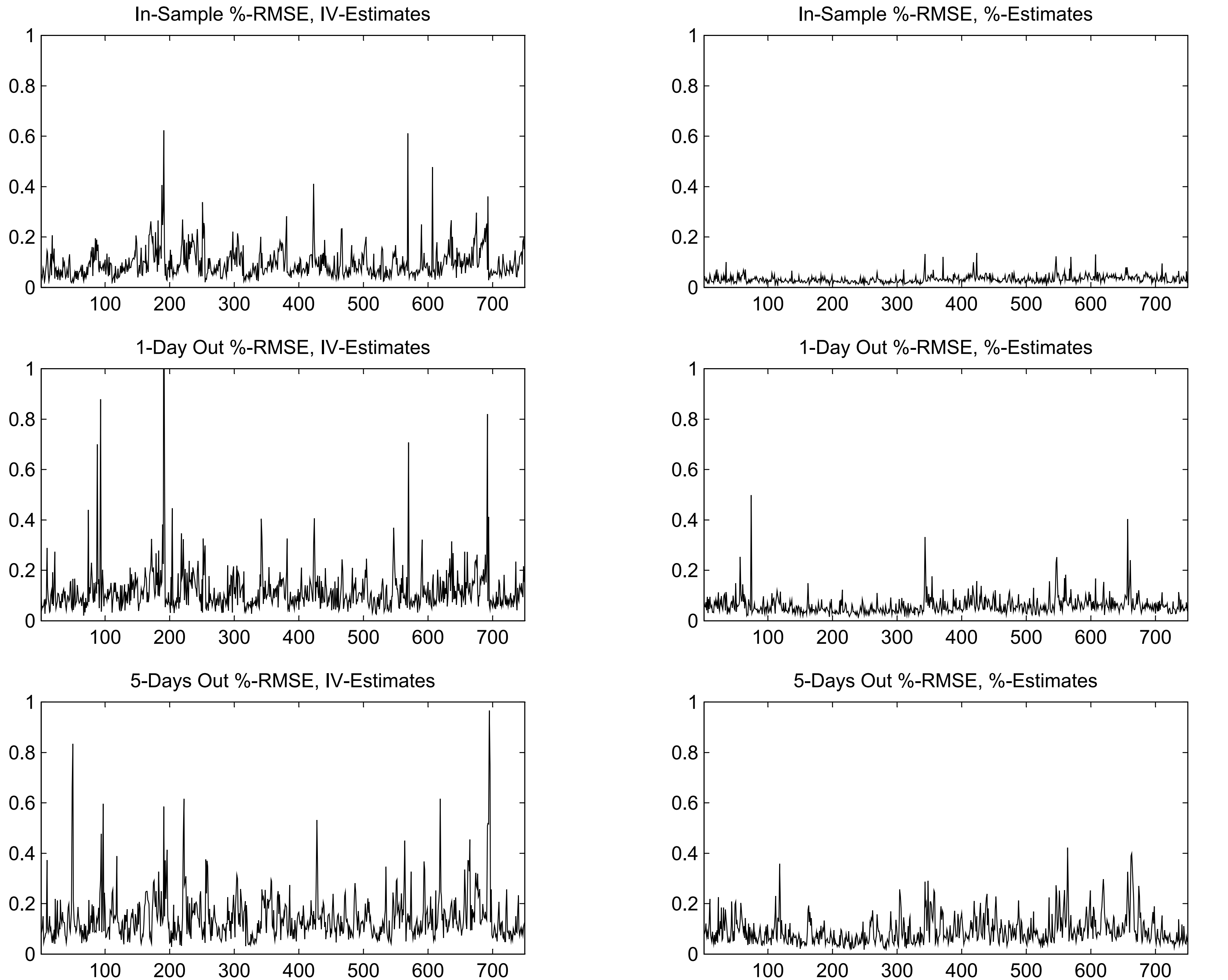


Figure 3A: Parameter Estimates Across Loss Functions

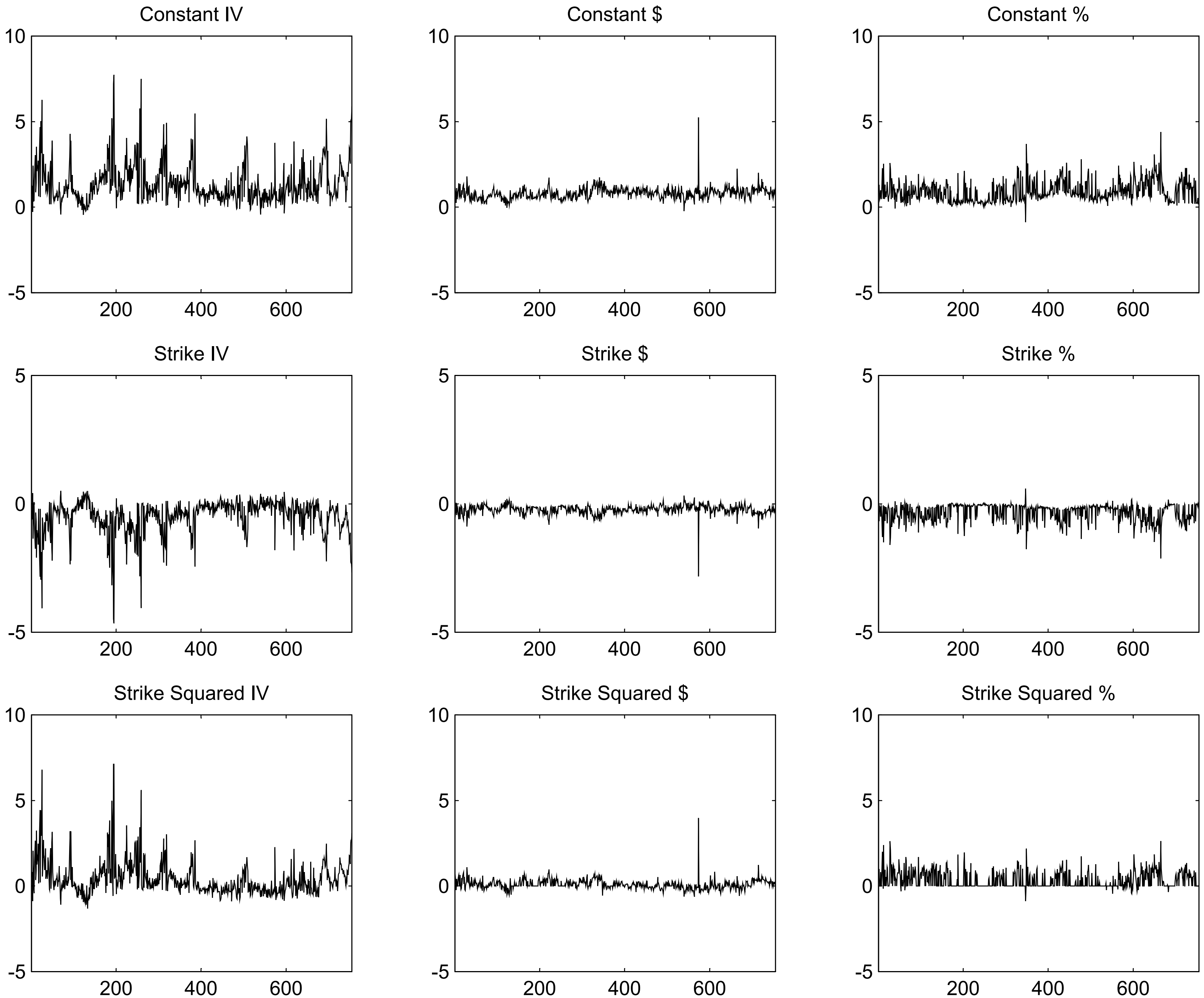


Figure 3B: Parameter Estimates Across Loss Functions

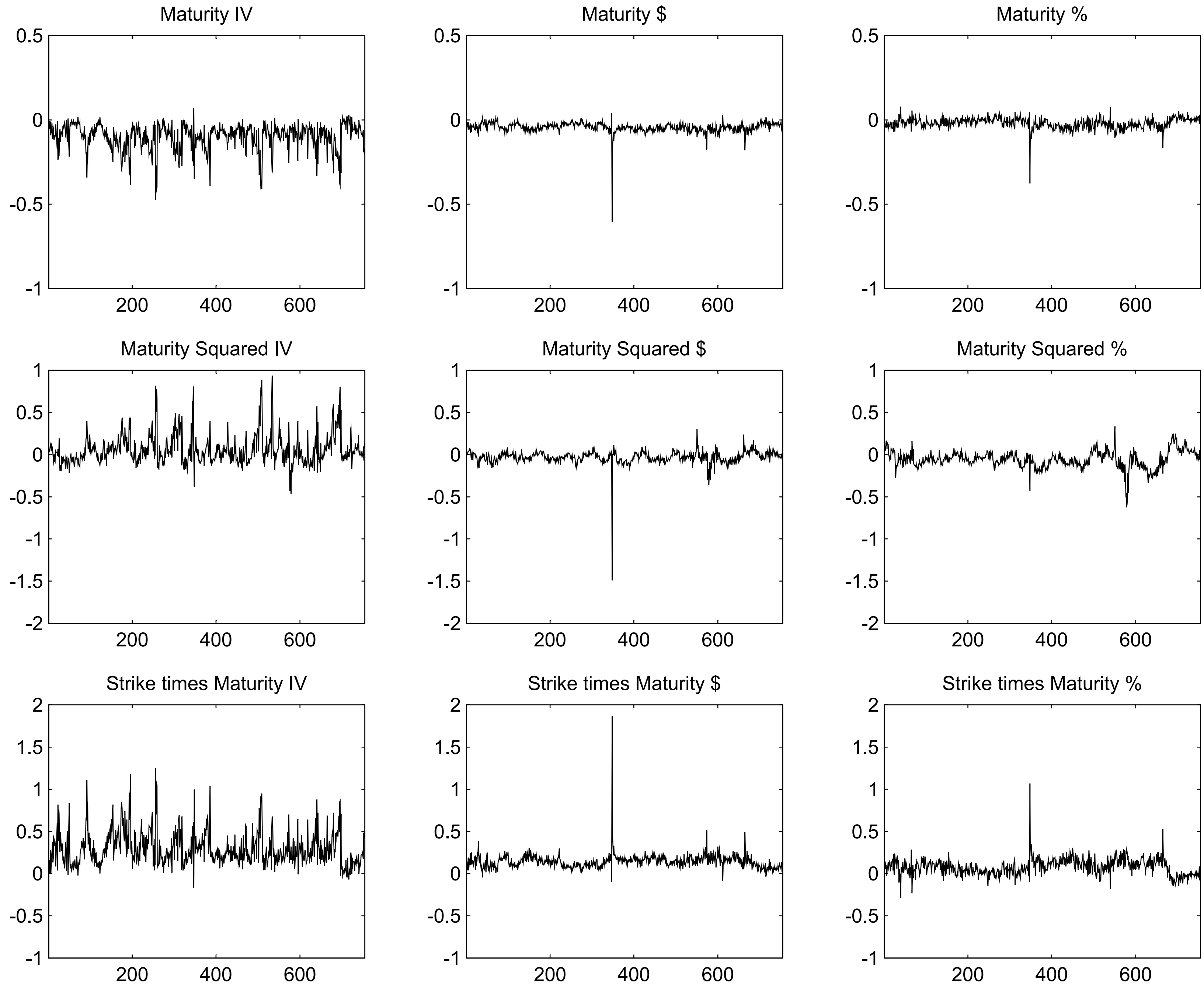
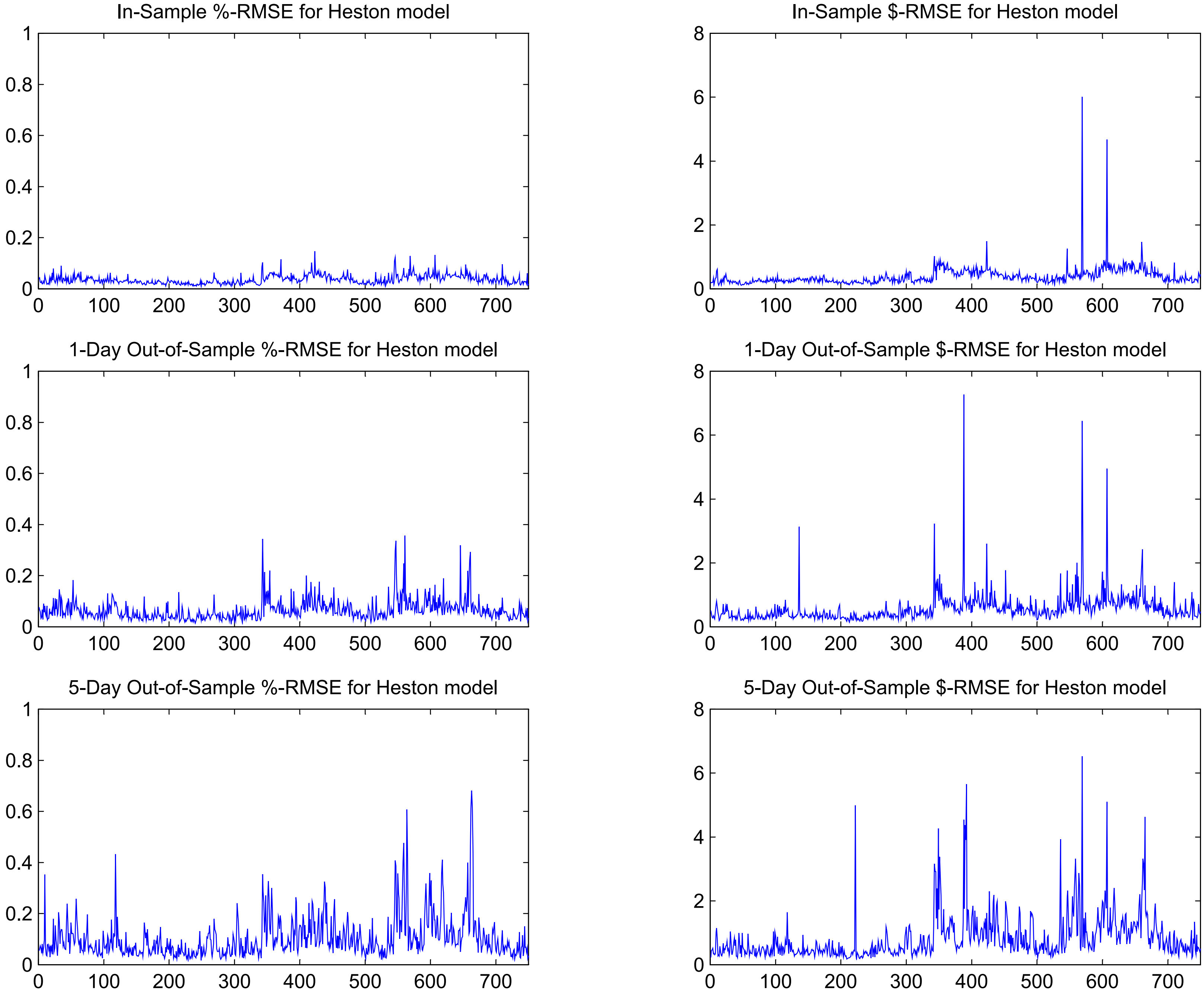


Figure 4 : %- and $-Root-Mean-Squared-Error Losses for the Heston model



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