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**STOCHASTIC VOLATILITY  
AND TIME DEFORMATION:  
AN APPLICATION TO TRADING  
VOLUME AND LEVERAGE EFFECTS**

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# Stochastic Volatility and Time Deformation: An Application to Trading Volume and Leverage Effects\*

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## Abstract / Résumé

*In this paper, we study stochastic volatility models with time deformation. Such processes relate to early work by Mandelbrot and Taylor (1967), Clark (1973), Tauchen and Pitts (1983), among others. In our setup, the latent process of stochastic volatility evolves in a operational time which differs from calendar time. The time deformation can be determined by past volume of trade, past price changes, possibly with an asymmetric leverage effect, and other variables setting the pace of information arrival.*

*The econometric specification exploits the state-space approach for stochastic volatility models proposed by Harvey, Ruiz and Shephard (1994) as well as matching moment estimation procedures using SNP densities of stock returns and trading volume estimated by Gallant, Rossi and Tauchen (1992). Daily data on the price changes and volume of trade of the S&P 500 over a 1950-1987 sample are investigated. Supporting evidence for a time deformation representation is found and its impact on the behaviour of price series and volume is analyzed. We find that increases in volume accelerate operational time, resulting in volatility being less persistent and subject to shocks with a higher innovation variance. Downward price movements have similar effects while upward price movements increase persistence in volatility and decrease the dispersion of shocks by slowing down the operational time clock. We present the basic model as well as several extensions, in particular, we formulate and estimate a bivariate return-volume stochastic volatility model with time deformation. The latter is examined through bivariate impulse response profiles following the example of Gallant, Rossi and Tauchen (1993).*

Nous proposons un modèle de volatilité stochastique avec déformation du temps suite aux travaux par Mandelbrot et Taylor (1967), Clark (1973), Tauchen et Pitts (1983) et autres. La volatilité est supposée être un processus qui évolue dans un temps déformé déterminé par l'arrivée de l'information sur le marché d'actifs financiers. Des séries telles que le volume de transaction et le rendement passé sont utilisées pour identifier la correspondance entre le temps calendrier et opérationnel.

Le modèle est estimé soit par la procédure de pseudo maximum de vraisemblance comme proposé par Harvey *et al.* (1994), soit par des méthodes d'inférence indirecte utilisant la densité SNP de Gallant, Rossi et Tauchen (1992). Dans la partie empirique, nous utilisons des données journalières de la bourse de New York. Un modèle univarié de volatilité stochastique ainsi qu'un modèle bivarié de volume et rendements avec déformation du temps sont analysés.

Key words : stochastic volatility, trading volume.

Mots clés : volatilité stochastique, volume de transaction.

JEL: C13, C22, G12, C12

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## 1. INTRODUCTION

Asset prices respond to the arrival of information. Some days, even some parts of a trading day, very little news, good or bad, is released. Trading is typically slow and prices barely fluctuate. In contrast, when new information changes expectations, trading is brisk and the price process evolves much faster. This observation motivated Mandelbrot and Taylor (1967) and particularly Clark (1973) to suggest modelling asset price processes as subordinated stochastic processes. Instead of studying asset prices as a function of (equally spaced calendar) time, via monthly, weekly, daily or intraday series, they suggested to let asset price movements be a function of information arrival which itself evolves randomly through time. To be slightly more formal, instead of studying say daily returns as  $x(\Delta t) = \log(p(t) / p(t - 1))$ , it was suggested to view  $\log(p(t) / p(t - 1)) = x(T(t))$  where  $T(t)$  is a positive stochastic process, sometimes called directing process though Clark, for instance, deliberately chose the notation  $T(t)$  to indicate he meant the trading volume on day  $t$ . This setup, which is sometimes also called time deformation since the relevant time scale is no longer calendar time  $t$  but *operational* time  $T(t)$ , has several attractive features. For instance, it easily accommodates leptokurtic distributions for asset returns as emphasized by Mandelbrot and Taylor; it is also a convenient framework to study trading volume and asset return comovements as stressed by Clark; and, last but surely not least, it yields a random variance or what nowadays would be called a *stochastic volatility* model. These ideas have been refined and extended in several ways. Particularly, the restrictive assumption made in the early work that  $T(t)$  was an i.i.d. process was relaxed by Tauchen and Pitts (1973). Other contributions include Harris (1987), Lamoureux and Lastrapes (1990), Gallant, Hsieh and Tauchen (1991), Andersen (1993).<sup>1</sup> Moreover, the microstructure foundations for time deformation and the process of price adjustments can be found most explicitly in Easley and O'Hara (1992). It is interesting and at the same time important to note that none of these developments exploited explicitly the continuous time financial modelling approach which has become so widely used since the seminal work of Merton (1973) and many others. Indeed, when one refers to *stochastic volatility*, one typically thinks of models originally constructed for the valuation of options where changes in the volatility were governed by a stochastic differential equation which at least is not explicitly related to the arrival of information through trading volume or other variables. Such models were developed

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<sup>1</sup> There is, of course, also an extensive literature on trading volume, including both theoretical and empirical papers. See, for instance, Foster and Viswanathan (1993a,b), Gallant, Rossi and Tauchen (1992), Hausman and Lo (1991), Huffman (1987), Karpoff (1987), Lamoureux and Lastrapes (1993), Wang (1993), among others.

by Hull and White (1987), Johnson and Shamro (1987), Scott (1987), Wiggins (1987), Chesney and Scott (1989), Stein and Stein (1991) and Heston (1993), among others.

In this paper, we study continuous time stochastic volatility models with time deformation. The setup combines insights borrowed from the earlier literature on subordinated stochastic processes and from the more recently developed diffusion equation stochastic volatility models. Let us briefly return to a more formal discussion and note that the latter class of models typically takes the form :

$$(1.1a) \quad dy(t) = \mu y(t) dt + \sigma(t)y(t)dW_1(t)$$

$$(1.1b) \quad d\log\sigma(t) = a(b - \log\sigma(t))dt + cdW_2(t)$$

where  $W_1(t)$  and  $W_2(t)$  are two standard Wiener processes usually assumed as independent. We will not assume that the volatility process moves continuously and smoothly through calendar time, as is usually assumed and described by (1.1b). The initial motivation for the work of Mandelbrot and Taylor, as well as Clark, was that key variables affecting volatility, like the arrival of information to the market, tend not to evolve continuously and smoothly through time. Therefore, we shall make the volatility process a subordinated stochastic process evolving in a time dimension set by market activity. To make this more explicit, let us assume an operational time scale  $s$  for the volatility process, with  $s = g(t)$ , a mapping between operational and calendar time  $t$ , such that :<sup>2</sup>

$$(1.2a) \quad dy(t) = \mu y(t) dt + \sigma(g(t)) y(t) d\omega_1(t)$$

$$(1.2b) \quad d\log\sigma(s) = a(b - \log\sigma(s)) ds + cd\omega_2(s) .$$

We use the notation  $g(t)$  for the directing process because we prefer to think of some *generic* time deformation, which may include trading volume besides many other series that help determine the pace of the market. Before discussing what might determine  $g(t)$ , we would like to make some observations regarding equations (1.2). Indeed, it should first be noted that the equations collapse to the usual stochastic volatility model if  $g(t) = t$ . This was done on purpose to accommodate econometric hypothesis testing. Obviously, we could have adopted another specification for  $\sigma(g(t))$ .

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<sup>2</sup> The mapping  $s = g(t)$  must satisfy certain regularity conditions which will be discussed later.

Moreover, one could correctly argue that, making changes in volatility a subordinated process amounts to suggesting a more complex law of motion for volatility in comparison to the Ornstein-Uhlenbeck (henceforth O-U) specification appearing in (1.1b). This interpretation is valid, yet it should be noted that, through  $g(t)$ , one can associate many series other than the security price  $y(t)$  to explain volatility; hence, one implicitly deals with a multivariate framework.

What should determine  $g(t)$ ? The perennial problem, of course, is that the flow of information is a latent process. The task of modelling is considerably simplified when we specify the mapping  $s = g(t)$  in terms of observable processes. We propose to use past volume of trade and other variables such as past returns allowing possibly for an asymmetric response to create a leverage effect. Therefore, our setup provides a way of introducing data on trading volume in the specification of stochastic volatility models. Furthermore, it is possible to introduce leverage effects through the specification of asymmetric responses of  $s$  to past price changes, i.e., operational time evolves differently in bull and bear markets. It also appears from the empirical results which were obtained that our specification provides an alternative to a class of processes put forward by Merton (1976a,b) for option pricing, where jumps in the underlying security returns are permitted. Merton suggested to include a Poisson jump process to distinguish between the arrival of *normal* information, modeled as a standard log normal diffusion, and the arrival of *abnormal* information, modeled as a Poisson process. We find that operational time typically moves slowly, but every so often one finds dramatic increases in market speed. In Merton's setup, the information arrival spells are purely exogenous, whereas our approach has the sources of these changes modeled both in a multivariate sense, via the introduction of volume series, and in an endogenous fashion through past price changes. Using daily S&P 500 data and NYSE volume from 1950-1987, we find that increases in volume accelerate operational time, resulting in volatility being higher and less persistent and subject to shocks with a higher innovation variance. Downward price movements have similar effects, while upward price movements increase persistence in volatility and decrease the dispersion of shocks by slowing down the operational time clock.<sup>3</sup>

In order to estimate the subordinated diffusions, we rely on two alternative estimation procedures. The first method involves the Kalman filter and draws upon

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<sup>3</sup> Obviously other series could figure in the specification of  $g(t)$ . Indeed in many instances one can find time deformation arguments in financial modelling. In section 2, we will provide a brief review of examples which appeared in the literature.

results from Harvey, Ruiz and Shephard (1994) on estimating stochastic volatility models in state space form and results from Stock (1988) on estimation of linear processes with time deformation. The second method is based on a "matching moment" principle as presented by Gallant and Tauchen (1994) using SNP densities for stock returns and volume series fitted by Gallant, Rossi and Tauchen (1992).

In section 2, we present the basic model. Estimation and hypothesis testing are discussed in section 3. Empirical results appear in section 4.

## 2. A TIME DEFORMATION APPROACH TO STOCHASTIC VOLATILITY

Stochastic processes used in finance are most often assumed to be generated by a first-order stochastic differential equation of the form :

$$(2.1) \quad dX(s) = a(s, X(s), \theta) ds + b(s, X(s), \theta) dM(s)$$

where  $X(s)$  is a  $n$ -dimensional process adapted to a filtered probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  evolving in some *operational* time. The process is parameterized by  $\theta \in \mathbb{R}^P$  with  $dM(s)$  a  $m$ -dimensional semi-martingale process, while  $a(s, X(s), \theta)$  and  $b(s, X(s), \theta)$  are both bounded predictable processes of dimensions  $n$  and  $n \times m$ , respectively. Equations like (2.1) have been adopted to describe security, bond and derivative prices as well as information flows, mortgage values, inventories and other state variables such as technology. Whenever the assumed operational time scale  $s$  differs from  $t$ , there is so-called time deformation or, alternatively with  $s = g(t)$ , the process  $X(g(t))$  is a subordinated stochastic process. Both expressions will be used throughout the paper.

The idea of time deformation appears in quite a number of finance papers, though not always explicitly. Probably the simplest examples of time deformation are related to the widely documented nontrading day, holiday and weekend effects in asset prices. Bessembinder and Hertzl (1993) are the most recent example of several papers on these so-called stock market anomalies.<sup>4</sup> In foreign exchange markets, there is also a tendency to rely on *activity* scales determined by the number of active markets around the world at any particular moment. Dacorogna et al. (1993) describe explicitly a

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<sup>4</sup> For instance, Lakonishok and Smidt (1988) and Schwert (1990) argue that returns on Monday are systematically lower than any other day of the week, while French and Roll (1986), French, Schwert and Stambaugh (1987) and Nelson (1991) demonstrate that daily return volatility on the NYSE is higher following nontrading days closures.

model of time deformation along these lines for intraday movements of foreign exchange rates. Besides these relatively simple examples, there are a number of more complex ones. The most prominent being the work of Mandelbrot and Taylor as well as Clark and extensions which were mentioned in the introduction. Before elaborating on this further, it is worth mentioning a few other examples as well. For instance, Madan and Seneta (1990) and Madan and Milne (1991) introduced a Brownian motion evaluated at random (exogenous) time changes governed by independent gamma increments as an alternative martingale process for the uncertainty driving stock market returns. Geman and Yor (1993) also used time-changed Bessel processes to compute path-dependent option prices such as is the case with Asian options.<sup>5</sup>

As explained in the introduction, we study a continuous time stochastic volatility model with time deformation. We combine the insights from Mandelbrot and Taylor (1967) and Clark (1973) on subordinated stochastic processes and from the option pricing stochastic volatility models associated with the work of Hull and White (1987) and others mentioned before. Volatility is modeled as a subordinated process driven by a generic directing process  $s = g(t)$ ,  $s$  being an operational time scale, associated with the arrival of information. In particular, we consider the set of equations (1.2) repeated here for convenience :

$$(2.2a) \quad dy(t) = \mu y(t) dt + \sigma(g(t)) y(t) dW_1(t)$$

$$(2.2b) \quad d \log \sigma(s) = a(b - \sigma(s)) ds + c dW_2(s).$$

To enhance our understanding of the mechanics of the process, let us momentarily isolate the volatility equation (2.2b) and discuss its properties as well as its discretization. To simplify this task even further, let us set  $b = 0$  and work with a continuous time AR(1). To describe an investor's information, let us consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and the nondecreasing family  $F = \{ \mathcal{F}_t \}_{t=0}^{+\infty}$  of sub- $\sigma$ -algebras in calendar time. Furthermore, let  $Z_t$  be a  $m$ -dimensional vector process adapted to the filtration  $F$ , i.e.,  $Z_t$  is  $\mathcal{F}_t$ -measurable. The increments of the time deformation mapping  $g$  will be assumed to be  $\mathcal{F}_{t-1}$  measurable via the logistic transformation :

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<sup>5</sup> Time deformation is also used for a variety technical reasons in, for instance, Detemple and Murthy (1993) to characterize intertemporal asset pricing equilibria with heterogeneous beliefs. Nelson and Foster (1993, 1994) use changes in time scales to study ARCH models as filters for diffusion models.



$$(2.3) \quad \frac{dg(\tau; Z_{t-1})}{d\tau} \equiv \dot{g}(\tau; Z_{t-1}) \equiv \exp(c'Z_{t-1}) / \left\{ \frac{1}{T} \sum_{t=1}^T \exp(c'Z_{t-1}) \right\}$$

for  $t - 1 \leq \tau < t$ .<sup>6</sup> Equation (2.3), setting the speed of change of operational time as a measurable function of calendar time process  $Z_{t-1}$ , is complemented with additional identification assumptions :

$$(2.4a) \quad 0 < \dot{g}(\tau; Z_{t-1}) < \infty$$

$$(2.4b) \quad g(0) = 0$$

$$(2.4c) \quad \frac{1}{T} \sum_{t=1}^T \Delta g(t) = 1 .$$

These three technical conditions, which will not be discussed at length here as they are covered in detail in Stock (1988), guarantee that the operational time clock progresses in the same direction as calendar time without stops or jumps.<sup>7</sup> Given that  $\dot{g}$  is constant between successive calendar time observations via (2.3), its discrete time analogue  $\Delta g(t) \equiv g(t) - g(t - 1)$  takes the same logistic form appearing in (2.3). At this point, we have not yet discussed what series should enter the vector  $Z_{t-1}$ . A detailed discussion will be delayed until later in the section, but it may be worth pointing out at those variables like past trading volume or any other processes linked to information arrival will be candidate series to enter equation (2.3). Proceeding with the discussion of equation (2.2b), we note that the solution in operational time of a first-order linear process can be expressed as :

$$(2.5) \quad \log \sigma(s) = e^{a(s-s')} \log \sigma(s') + \int_{s'}^s e^{a(s-r)} dW_2(r)$$

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<sup>6</sup> The fact that the denominator in (2.3) contains a sample average may suggest that  $\sigma(g(t))$  is not measurable with respect to the filtration  $\mathcal{F}_t$  in calendar time. However, the denominator in (2.3) is there for reasons of numerical stability of the algorithms described in the next section. Since it is only a scaling factor, its presence is of no conceptual importance.

<sup>7</sup> Excluding jumps for the time deformation process must not be confused with the presence of jumps in the stock return process, as proposed by Merton (1976a, b). The time deformation will govern the (stochastic) volatility of the return process. Arbitrarily large (yet finite) changes in operational time will make the stock return process extremely volatile through the conditional variance.

where  $s' < s$ . To recover the solution in calendar time, we let  $s = g(t)$  and  $s' = g(t - 1)$  and obtain :

$$(2.6a) \quad h_t = e^{\Lambda \Delta g(t)} h_{t-1} + v_t \quad t = 1, \dots, T$$

$$(2.6b) \quad v_t \sim N(0, -\Sigma(1 - \exp(2A \Delta g(t))) / 2A)$$

$$(2.6c) \quad \Delta g(t) = \exp(c'Z_{t-1}) / \frac{1}{T} \{\sum_{t=1}^T \exp(c'Z_{t-1})\}$$

Hence, the process where  $h_t = \log \sigma(g(t))$  while linear in operational time becomes a random coefficient model, also called doubly stochastic process, in calendar time also featuring conditional heteroskedasticity governed by  $\Delta g(t)$ .<sup>8</sup>

The brief digression on time deformation facilitates the presentation of the process of main interest, which is a SV model with time deformation. Suppose now that  $\{y_t\}$  represents a discrete time sample of the process in (2.2). A standard Euler approximation to (2.2a) yields :

$$(2.7) \quad \log y_t = \lambda + \log y_{t-1} + \sigma_t \varepsilon_t \quad \varepsilon_t \sim \text{i.i.d. } N(0, 1).$$

Let us combine this expression with the volatility equation. If we assume again that  $\beta \neq 0$  in (2.2b) and furthermore observe that the stock returns used in our empirical application exhibit small yet significant autocorrelation at lag 1, we obtain the following discrete time representation :

$$(2.8a) \quad \Delta \log y_t - a_1 \Delta \log y_{t-1} - \lambda = \sigma_t \varepsilon_t$$

$$(2.8b) \quad h_t = [(1 - \exp(A \Delta g(t)))] \beta + \exp(A \Delta g(t)) h_{t-1} + v_t$$

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<sup>8</sup> Doubly stochastic processes have been discussed in detail by Tjøstheim (1986). Stability conditions and existence of moments have been studied for cases where  $\Delta g(t)$  is Markovian. It may be worth noting at this point that the  $Z_{t-1}$  process need not be exogenous. Indeed, Stock (1988) showed that by setting  $Z_{t-1}$  equal to the square of the process appearing in the mean equation, one obtains an ARCH-like process having the additional feature of a random coefficient model.

where the variance of  $v_t$  is given by (2.6b). Equations (2.8a) and (2.8b) are the basic set of equations of the discrete time representations of the SV model with a subordinated volatility process which evolves at a speed set by  $\Delta g(t)$ . The set of equations (2.8) will be used for a simulated method of moments estimation procedure which will be discussed in the next section. We will also use a quasi-maximum likelihood estimation algorithm, however, based on a Kalman filter state space representation. For this, we rely on Harvey, Ruiz and Shephard (1994) and write equations (2.8a) as :

$$(2.9) \quad \log[\Delta \log y_t - a_1 \Delta \log y_{t-1} - \lambda]^2 = h_t + \log \varepsilon_t^2$$

where  $E \log \varepsilon_t^2 = -1.27$  and  $\text{Var} \log \varepsilon_t^2 = \pi^2/2$ . We can rewrite equation (2.9) adding (2.8b), as follows :

$$(2.10a) \quad \log[\Delta \log y_t - a_1 \Delta \log y_{t-1} - \lambda]^2 = -1.27 + h_t + \zeta_t$$

$$(2.10b) \quad h_t = [(1 - \exp(A\Delta g(t))) \beta + \exp(A\Delta g(t))] h_{t-1} + v_t.$$

Apart from the parameter  $\lambda$ , whose treatment is discussed, for instance, by Gouriéroux, Monfort and Renault (1993), we obtain a state-space model with time-varying coefficients similar to that obtained by Stock (1988), except for the properties of the  $\zeta_t$  process which is no longer Gaussian.<sup>9</sup> Consequently, the estimation procedure based on the Kalman filter will result here in a quasi-maximum likelihood estimator, similar to Harvey, Ruiz and Shephard (1994).

Obviously, the SV model with time deformation can be viewed simply as a model with a doubly stochastic process for  $h_t$  which replaces the usual linear or Ornstein-Uhlenbeck process. Yet, the stochastic variation in the autoregressive coefficient has a very specific interpretation through the specification of the mapping  $g(t)$ . Let us, therefore, turn our attention now to a description of the functional form that will be adopted. The work by Clark (1973), Tauchen and Pitts (1983) and Gallant, Rossi and Tauchen (1992) suggests that  $Z_{t-1}$  should include both past volume and price movements. With respect to price movements, we will adopt a functional form which

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<sup>9</sup> The innovations  $v_t$  and  $\zeta_t$  are assumed i.i.d. Correlation between the two processes would create asymmetries in the conditional variance [see Harvey and Shephard (1993)]. We do not need to assume such a correlation since the asymmetry will come through the time deformation (as will be discussed later in the text).

can allow for asymmetries in the time deformation when prices move upward or downward. Such asymmetry allows us to investigate so-called leverage effects in the conditional variance [cfr. Black (1976) and Christie (1982)]. Several recent empirical studies, including Gallant, Rossi and Tauchen (1992), Nelson (1989, 1991), and Pagan and Schwert (1990) indeed suggest asymmetries in the conditional variance function. Finally, we could also include a set of predetermined processes denoted  $d_t$  to account for nontrading day effects and possibly other periodic patterns discussed, for instance, by Bollerslev and Ghysels (1994). The logistic function emerging from the above discussion would be :<sup>10</sup>

$$(2.11) \quad \exp(c' Z_{t-1}) \equiv \exp(c'_d d_t + c_v \log \text{Vol}_{t-1} + c_p \Delta \log y_{t-1} + c_\ell |\Delta \log y_{t-1}|).$$

where  $\log \text{Vol}_t$  represents a trading volume series and  $\Delta \log y_t$  the return series. The specification of the time deformation function is chosen in light of certain existing stylized facts we would like the model to fit. Other specifications can be chosen, however. The general model we develop holds for any process  $Z_{t-1}$ , which is assumed to capture the flow of information. The specification in (2.11) is just one of possibly many, yet is directly related to the existing literature on conditional variance models. Further research may find other series appropriate as well.

It might be useful to describe the stochastic behavior of the process obtained so far. Referring to some of the empirical results, discussed later, we must first observe that coefficient  $A$  in (2.8b) is found to be negative. Therefore, when  $c_v > 0$ , the model predicts that increases in volume make  $\Delta g(t)$  increase. This acceleration in operational time results in a decline in  $a_t \equiv \exp A\Delta g(t)$  and an increase in  $\sigma_{vt}^2$  defined in (2.6b). These two effects imply that the  $h_t$  process becomes more erratic since its persistence declines and it is subject to larger shocks. Thus, trading volume increases are paired with volatility increases, an empirical fact documented via SNP fitting by Gallant, Rossi and Tauchen. If we find  $c_p < 0$  combined with  $c_\ell > 0$ , while  $|c_\ell| > |c_p|$  to ensure  $\Delta g(t) > 0$ , then a change in price of the same magnitude but of the opposite sign will result in  $\Delta g(t)$  to be smaller with upward price movements and larger with falling prices. Consequently, declining stock prices have an effect of making the volatility

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<sup>10</sup> Note that the timing of  $d_t$  differs from the other processes. Since the variables entering  $d_t$  are predetermined, they are measurable with respect to  $\mathcal{F}_{t-1}$  and, therefore, legitimate for setting the pace of operational time changes  $\Delta g(t)$ . Moreover, it should be observed that  $c_d$  is a vector of parameters since  $d_t$  may be multivariate.

process more erratic (i.e.,  $a_t$  declines and  $\sigma_{v_t}^2$  increases), while a positive price move of the same size has an opposite effect, namely,  $a_t$  increases and  $\sigma_{v_t}^2$  decreases.

When the subordinated stochastic volatility model involves trading volume through  $\Delta g(t)$  it would be natural to consider a bivariate model of stock returns and trading volume since both series are jointly determined by the arrival of information. It is indeed a major point stressed by Clark (1973), Tauchen and Pitts (1983) and many others. The framework developed so far lends itself easily to extensions which take into account the laws of motion of trading volume. Such model would be as follows :

$$(2.12a) \quad \begin{bmatrix} (\Delta \log y_t - a_t \Delta \log y_{t-1}) \\ \log \text{Vol}_t \quad v_t \end{bmatrix} = \begin{bmatrix} \mu_p \\ \mu_v \end{bmatrix} + \begin{bmatrix} \sigma_t \varepsilon_t \\ v_t \end{bmatrix}$$

$$(2.12b) \quad \begin{bmatrix} h_t \\ v_t \end{bmatrix} = [I - \exp(A\Delta g(t))] \begin{bmatrix} \beta_p \\ \beta_v \end{bmatrix} + \exp(A\Delta g(t)) \begin{bmatrix} h_{t-1} \\ v_{t-1} \end{bmatrix} + \begin{bmatrix} v_{1t} \\ v_{2t} \end{bmatrix}$$

where :

$$(2.13) \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$(2.14) \quad E v_t v_t' = \Sigma [(I - \exp(A\Delta g(t))) A^{-1/2}]$$

$$(2.15) \quad \Sigma = \begin{bmatrix} \sigma_p^2 & \rho \\ \rho & \sigma_r^2 \end{bmatrix}$$

Equation (2.14) is the bivariate extension of (2.6b) now involving the matrix  $A$  defined in (2.13) and the covariance matrix  $\Sigma$ . Note also that the time deformation  $\Delta g(t)$  is common to both processes. While the specification of  $\Delta g(t)$  remains the same, it is clear that the arrival of information which drives jointly  $h_t$  and  $v_t$  will be identified differently in comparison to the univariate model only involving  $h_t$ . It should also be noted the past volume affects  $h_t$  directly through the term  $a_{12} \Delta g(t) v_{t-1}$  as well as through  $\Delta g(t)$ . The dynamics of the joint process (2.12) will be more difficult to analyze via description. Instead, our discussion will revolve around the analysis of impulse response functions for the nonlinear bivariate system along the lines suggested by Gallant, Rossi and Tauchen (1993) and Potter (1991).

### 3. ECONOMETRIC ANALYSIS

Estimating SV models represents some stiff challenges for econometricians. In recent years, several estimation principles were proposed involving the use of simulated method of moments, Kalman filter and Bayesian procedures. Recent contributions include Duffie and Singleton (1993), Gallant and Tauchen (1994), Gouriéroux, Monfort and Renault (1993), for the method of moments estimators while Harvey, Ruiz and Shephard (1994) and Jacquier, Polson and Rossi (1994) discuss, respectively, the Kalman filter and Bayesian methods. To estimate the SV models with time deformation, we shall adopt two methods : one using the Kalman filter and one relying on the "matching moments" approach described by Gallant and Tauchen (1994). A subsection will be devoted to each method. Before turning to the specifics, it is worth making several observations. Both estimation procedures should be viewed as complementary especially with regard to estimating subordinated processes. The Kalman filter estimator is a quasi-maximum likelihood procedure, henceforth QMLE, and therefore has the disadvantage of being asymptotically inefficient. Simulation evidence reported in Andersen and Sorensen (1994) and Jacquier, Polson and Rossi (1994) suggests that the state space QMLE may be quite inefficient, depending on the circumstances. This setup has certain advantages, however, in comparison to the simulated methods of moments procedure. Indeed, there is a greater flexibility with the Kalman filter in formulating  $\Delta g(t)$  without having to match the moments of all the series involved in the time scale transformation. Hence, there are certain trade-offs between the two estimation procedures which we will discuss. Therefore, we turn our attention now to the specific details to clarify these observations.

#### 3.1 Quasi-Maximum Likelihood Estimation of SV Models with Time Deformation

This method consists of maximizing the quasi-likelihood function of a nonlinear SV model written in a form of linear discrete-time state space system as specified in equations (2.10a and b). The Gaussian quasi-likelihood function is evaluated in the (calendar) time domain using a Kalman filter with time varying filter parameters that depend on  $\Delta g(t)$ . This algorithm is described in Stock (1988) and summarized in this section. We cast the presentation in a general multivariate context since we also want to cover the bivariate model involving volume described by equation (2.12).

The evolution of the state is described by the transition equation. In operational time  $s$ , the  $r^{\text{th}}$ -order linear differential equation representing a  $n$ -dimensional O-U process can be written in a stacked form as :

$$(3.1.1) \quad d\psi^*(s) = A[R\beta - \psi^*(s)] ds + R d\eta(s) ,$$

where

$$\psi^*(s) = \begin{bmatrix} \xi(s) \\ D\xi(s) \\ \vdots \\ D^{r-1}\xi(s) \end{bmatrix}, \quad R = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & & & \\ \vdots & & & \\ 0 & & I & \\ A_r & A_{r-1} & \dots & A_1 \end{bmatrix} .$$

Here  $\xi(s)$  may represent any subordinated process of interest while the vector  $\psi^*(s)$  is of dimension  $nr \times 1$  and the matrix  $R$  is  $nr \times n$ . The matrix of coefficients  $A$  is of dimension  $nr \times nr$ , its elements being  $n \times n$ , while the mean vector  $\beta$  is  $n \times 1$ . We denote the mean-square differential operator by  $D$ . The innovation process  $\eta(s)$  is Gaussian with zero-mean increments and covariance matrix  $E[d\eta(s) d\eta(s')] = \Sigma ds$  for  $s = s'$  and 0 otherwise. The real parts of the roots of matrix  $A$  are required to be negative for stability. We will also assume that they are distinct in order to adopt a useful eigenvalue decomposition  $A = G\Lambda G^{-1}$ , where  $\Lambda$  is a diagonal matrix of eigenvalues of  $A$ , which are, in general, complex numbers, while  $G$  is a matrix of eigenvectors of  $A$ . Following Stock (1988), we set  $\psi(s) = G^{-1}\psi^*(s)$  and observe that in operational time the transformed variable satisfies the following equation :

$$(3.1.2) \quad \psi(s) = [I - e^{\Lambda(s-s')}] G^{-1}R\beta + e^{\Lambda(s-s')} \psi(s') \\ + \int_{\tau=s'}^s e^{\Lambda(s-\tau)} G^{-1}R d\eta(\tau) ,$$

where  $s > s'$ . Let the calendar time state vector be denoted  $S(\tau) = \psi(g(\tau))$ . Evaluating the previous equation at  $s = g(\tau)$  and  $s' = g(\tau - 1)$ , we find that  $S(\tau)$  satisfies :

$$(3.1.3) \quad S(\tau) = [I - e^{\Lambda(g(\tau)-g(t-1))}] G^{-1} R \beta + e^{\Lambda(g(\tau)-g(t-1))} S(t-1) \\ + \int_{r=g(t-1)}^{g(\tau)} e^{\Lambda(g(\tau)-r)} G^{-1} R d\eta(r).$$

Developing the first term on the r.h.s of (3.1.3), we obtain :

$$(3.1.4) \quad S(\tau) = G^{-1} R \beta - e^{\Lambda(g(\tau)-g(t-1))} G^{-1} R \beta \\ + e^{\Lambda(g(\tau)-g(t-1))} S(t-1) + \int_{r=g(t-1)}^{g(\tau)} e^{\Lambda(g(\tau)-r)} G^{-1} R d\eta(r),$$

and hence,

$$(3.1.5) \quad S(\tau) - G^{-1} R \beta = e^{\Lambda(g(\tau)-g(t-1))} [S(t-1) - G^{-1} R \beta] \\ + \int_{r=g(t-1)}^{g(\tau)} e^{\Lambda(g(\tau)-r)} G^{-1} R d\eta(r).$$

Now, set  $\tilde{S}(\tau) = S(\tau) - G^{-1} R \beta$ . It is easy to note that equation (3.1.5) can be written as :

$$(3.1.6) \quad \tilde{S}(\tau) = e^{\Lambda(g(\tau)-g(t-1))} \tilde{S}(t-1) + \int_{r=g(t-1)}^{g(\tau)} e^{\Lambda(g(\tau)-r)} G^{-1} R d\eta(r).$$

Equation (3.1.6) evaluated at  $\tau = t$  yields the final representation of the transition equation :

$$(3.1.7) \quad \tilde{S}_t = T_t \tilde{S}_{t-1} + v_t,$$

where  $T_t = \exp(\Lambda \Delta g(t))$  and  $v_t = \int_{r=g(t-1)}^{g(t)} \exp[\Lambda(g(t) - r)] G^{-1} R d\eta(r)$ .

The multivariate measurement equation can be written, in terms of the state vector  $\tilde{S}_t$ , as :



$$(3.1.8) \quad Y_t = -1.27t + G\bar{S}_t + R\beta + \zeta_t,$$

where  $Y_t$  and  $\zeta_t$  are  $n \times 1$  vectors with elements  $y_{it} = \log[\Delta \log y_{it} - \lambda]^2$ ,  $\zeta_{it} = \log \varepsilon_{it}^2 + 1.27$ ,  $i = 1, \dots, n$ , and  $t$  is a  $n \times 1$  vector of ones. The equations (3.1.7) and (3.1.8) form a linear state space system. We suppose that the disturbances in both equations are uncorrelated since in our setup any eventual impact of prices on volatility is channelled through the time deformation term  $\Delta g(t)$ .

The next step of the procedure consists of applying the Kalman filter algorithm. Note, however, that  $\zeta_{it}$  in (3.1.8) are not normally distributed and, hence the linear filtering method can only be approximate while estimation will be asymptotically inefficient (see Andersen and Sorensen (1994) and Jacquier et al. (1994) for further discussion of this issue as well as simulation evidence).

Following Stock (1988), we initialize the Kalman filter by taking unconditional expectations and assuming that prior to the sample  $\Delta g(t) = 1$ . The one-step ahead forecast of the state,  $a_{1|0}$  is equal to zero and its covariance matrix  $P_{1|0} = \sum_{i=0}^{\infty} T^i Q \bar{T}^i$  can be easily obtained by computing  $T = T_t$ , and  $Q = Q_t = E(v_t \bar{v}_t')$  evaluated at  $\Delta g(t) = 1$ . Moreover, the  $(i - j)$  the element of the matrix  $Q_t$  is known to be equal to

$$q_{ij} \int_{r=0}^{\Delta g(t)} \exp[(\lambda_i + \bar{\lambda}_j)(\Delta g(t) - r)] dr = -q_{ij}(1 - T_{it} \bar{T}_{jt}') / (\lambda_i + \bar{\lambda}_j), \text{ where } q_{ij} \text{ is the}$$

$(i - j)$  element of the matrix  $G^{-1}R \Sigma R' \bar{G}^{-1}$ .

### 3.2 Simulated Method of Moments Estimation of Subordinated Stochastic Processes

Gallant, Rossi and Tauchen (1992) have analyzed stock returns and volume of transactions data and estimated semi-nonparametric densities, henceforth SNP, of the joint process. These SNP densities will be the setting for the simulated method of moments procedure described in this section. The moment matching procedure involving SNP densities, dubbed by Gallant and Tauchen (1994) as efficient GMM (henceforth EMM), allows one to avoid problems related to the appropriate choice of moments in a standard GMM setup. The choice of moments is indeed particularly cumbersome in cases of highly nonlinear models, such as SV models. The EMM procedure relies on moment conditions which are generated in a first step using the score function of the auxiliary SNP model. To facilitate the presentation, let us denote the parameter vector describing the SNP density as  $\theta$  while the vector  $\alpha$  describes the

parameters of the SV model. For the EMM method, we are interested in generating the vector of moment conditions using expectations under the SV model of the score from an auxiliary SNP model. The task of computing this expectations vector is facilitated since it is obtained easily by simulating the realizations, for a given value of the parameter vector  $\alpha$ , of the SV model with deformation of time. To be more formal, consider the mapping obtained through simulation :

$$(3.2.1) \quad \alpha \rightarrow \left\{ \hat{y}_\tau(\alpha), \hat{x}_{\tau-1}(\alpha) \right\}_{\tau=1}^N,$$

where  $\hat{y}_\tau$  and  $\hat{x}_\tau$  denote respectively the set of simulated endogeneous variables and the set of lagged endogeneous variables both generated by the time deformation SV model. It is worth pointing out that we no longer rely on the normal approximation, as appearing in (2.10a), but instead use directly (2.8a). In our case,  $\hat{y}_\tau(\alpha)$  would typically contain stock returns and trading volume while  $\hat{x}_\tau(\alpha)$  consists of their past realisations. The estimation is performed in two steps. First, the estimation of the auxiliary model (called the score generator) yields :

$$(3.2.2) \quad \bar{\theta}_n = \underset{\theta \in \Theta}{\text{Argmax}} \frac{1}{n} \sum_{t=1}^n \ln f_t(\tilde{y}_t | \tilde{x}_{t-1}, \theta),$$

where  $\{\tilde{y}_t, \tilde{x}_{t-1}\}_{t=1}^n$  denotes the set of observed data from a sample of size  $n$  (the simulated data set is of size  $N$ ). In the second step, the following moment criterion is computed :

$$(3.2.3) \quad m_n(\alpha, \bar{\theta}_n) = \frac{1}{n} \frac{1}{N} \sum_{\tau=1}^N \frac{\partial}{\partial \theta} \ln f_t(\hat{y}_\tau(\alpha) | \hat{x}_{\tau-1}(\alpha), \bar{\theta}_n).$$

Finally, the estimation of the time deformation SV model is given as :

$$(3.2.4) \quad \hat{\alpha}_n = \underset{\alpha}{\text{Argmin}} m_n'(\alpha, \bar{\theta}_n) (\tilde{I}_n)^{-1} m_n(\alpha, \bar{\theta}_n),$$

where  $\tilde{I}_n$  is a weighting matrix. [For a discussion of the appropriate choice of the  $\tilde{I}_n$  estimator, see Gallant and Tauchen (1994)]. The efficiency of EMM depends, of course, on the choice of the auxiliary model. A score generator nesting the time

deformation SV model would allow to attain the maximum likelihood efficiency. However, even a score generator that only closely approximates the actual distribution of the data is nearly fully efficient. As noted before, the score function selected for the time deformation SV model is the derivative of the log density estimated by a semi-nonparametric method (SNP) proposed by Gallant, Rossi and Tauchen (1992). The SNP density function is based on a hermite expansion of the form :

$$(3.2.5) \quad h(z) \propto [P(z)]^2 \phi(z),$$

where  $z$  denoted a  $M$ -dimensional vector,  $P(z)$  is a multivariate polynomial of degree  $K_z$  and  $\phi(z)$  denotes the density function of the multivariate Gaussian distribution with mean zero and an identity covariance matrix. The constant of proportionality  $1 / \int [P(s)]^2 \phi(s) ds$  makes  $h(z)$  integrate to one. A more complex specification can easily be handled by means of a change of variables  $y = Rz + \mu$ , where  $R$  is defined as an upper triangular matrix and  $\mu$  is a vector of dimension  $M$ . In consequence, we obtain the following expression :

$$(3.2.6) \quad f(y | \theta) \propto \{P[R^{-1}(y - \mu)]\}^2 \{\phi[R^{-1}(y - \mu)] / |\det(R)|\},$$

where the leading term is now proportional to the multivariate Gaussian density function with mean  $\mu$  and covariance matrix  $\Sigma = RR'$ . Hence, by setting  $K_z$  equal to zero, a multivariate normal density can be estimated. Nonzero values of  $K_z$  result in shape modifications that can accommodate fat tails and skewness. Other modifications can be done by adjusting values of the remaining turning parameters  $L_\mu, L_p, L_r, K_x, I_z$  and  $I_x$ . The turning parameter  $L_\mu$  determines the number of lags in the location shift  $\mu$  considered as a linear function of  $L$  past values of  $y$  for accommodating a Gaussian VAR specification. To approximate a conditionally heterogeneous process, each coefficient of the polynomial  $P(z)$  can be defined as a polynomial of degree  $K_x$  in past values of  $y$ . The new polynomial  $P(z, x)$ , where  $x$  is the vector of lagged values of  $y$  is hence of degree  $K_z + K_x$ . The conditional heteroskedasticity can also be captured by letting  $R$  be a linear function of past values of  $y$ . The number of lags in the scale shift  $R_x$  is determined by the parameter  $L_r$ . The number of lags in the  $x$  part of the polynomial  $P(z, x)$  is controlled by  $L_p$ . Finally, the parameters  $I_z$  and  $I_x$  allow to suppress an excessive number of cross product terms in case of multivariate series.

The optimal score selection strategy is summarized in Table 3.1. The data used consist of the daily closing value of the S&P composite stock index and the daily volume of shares traded on the NYSE. The data set is identical to that used by Gallant, Rossi and Tauchen (1992), who described its sources in detail.<sup>11</sup> To determine the best fit, we computed three criteria : AIC, BIC and HQ. AIC is the Akaike criterion defined as :

$$\text{AIC} = S_n(\hat{\theta}) + P_{\theta} / n ,$$

where  $S_n(\hat{\theta})$  is the mean of the log likelihood and  $P_{\theta}$  denotes the number of parameters. The more conservative criterion of Schwartz (BIC), which penalizes specifications involving too many parameters is computed as :

$$\text{BIC} = S_N(\hat{\theta}) + \frac{1}{2}(P_{\theta} / N) \log(N) .$$

Finally, the Hannan-Quinn criterion lies in between the last two and is given by the following expression :

$$\text{HQ} = S_n(\hat{\theta}) + (P_{\theta} / n) \log[\log(n)] .$$

We retained the one of the two best performing specifications under the Schwartz criterion. Our preferred SNP model is described by the following set of the turning parameter values :  $\{L_{\mu} = 2, L_r = 16, L_p = 2, K_z = 4, K_x = 1, I_z = I_x = 0\}$  with total number of 35 parameters.

Before turning to the empirical results, it may be worth pointing out that the fitted SNP densities reported in Table 3.1 only involve stock returns. A consequence of this is that the matching moment SV estimates will be limited to a time deformation based on past returns, since trading volume does not figure in the SNP density. We have to formulate a *bivariate* SNP involving both returns and volume to fit a SV model with  $\Delta g(t)$  including both series. This is, of course, different from the QMLE setup described in the previous section. A bivariate SNP involving both returns and volume has been estimated by Gallant, Rossi and Tauchen (1992). We relied on an optimal score defined by the following values of the tuning parameters :  $\{L_{\mu} = 2, L_r = 18,$

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<sup>11</sup> Besides the description of data sources, they also describe several transformations.

Table 3.1  
Univariate Price Changes 1950–1987 Optimized Likelihood

$L_{\mu}$	$L_r$	$L_p$	$K_z$	$I_z$	$K_x$	$I_x$	$S_N$	$P_{\Theta}$	BIC	HQ	AIC
1	16	2	4	0	0	0	1.2866	24	1.2975	1.2919	1.2890
2	16	2	4	0	0	0	1.2849	25	1.2963	1.2904	1.2874
1	16	2	4	0	1	0	1.2792	34	1.2949	1.2868	1.2826
2	16	2	4	0	1	0	1.2793	35	1.2955	1.2871	1.2828
1	16	4	4	0	0	0	1.2866	24	1.2975	1.2919	1.2890
2	16	4	4	0	0	0	1.2849	25	1.2963	1.2904	1.2874
1	16	4	4	0	1	0	1.2750	44	1.2964	1.2859	1.2804
2	16	4	4	0	1	0	1.2759	45	1.2969	1.2861	1.2805
1	16	6	4	0	1	0	1.2748	54	1.3000	1.2870	1.2803
2	16	6	4	0	1	0	1.2746	55	1.3003	1.2871	1.2802
1	16	8	4	0	1	0	1.2736	64	1.3036	1.2881	1.2802
2	16	8	4	0	1	0	1.2735	65	1.3040	1.2882	1.2802

Notes :  $L_{\mu}$  : number of lags in VAR part  $\mu_x$ ;  $L_r$  : number of lags in the ARCH part  $R_x$ ;  $L_p$  : number of lags in the polynomial part  $P(z, x)$ ;  $P(z, x)$  is of degree  $K_z$  in  $z$  with cross product terms exceeding  $K_z - I_z$  set to zero; same for  $K_x$  and  $I_x$ ;  $S_N$  : negative of the log likelihood divided by sample size (9636);  $P_{\Theta}$  : total number of parameters; BIC, HQ, AIC : respectively : Schwartz, Hannan-Quinn and Akaike criterions.

$L_p = 2, K_z = 4, I_z = 1, K_x = 2, I_x = 1$ ).<sup>12</sup> Since the bivariate SNP involves a very large number of parameters (368), we extended the sample period and considered data on prices and trading volume from a 1928–1987 sample.

#### 4. EMPIRICAL RESULTS

In this section, we turn our attention to an empirical study of SV models subject to time deformation. As noted in the previous section, the data used consist of the daily closing value of the S&P composite stock index and the daily volume of shares traded on the NYSE. The data are plotted in Figure 4.1 which consists of two parts : namely, 4.1a displays the return series, while 4.1b contains volume. The empirical results reported here are based exclusively on the series appearing in Figure 4.1. A first subsection will be devoted to SV models not involving the laws of motion of trading volume  $\log V_t$ . The second subsection will be based on the joint volatility–volume specification.

##### 4.1 Empirical Subordinated SV Models

We fitted two continuous time SV models : the first one contains a simplified volatility equation and is based on the assumption that in a long run of operational time, the Ornstein–Uhlenbeck process is pulled towards zero. The second model corresponds exactly to the setup presented in section 2, where the stochastic volatility is allowed to tend towards any finite level. Two estimation methods, namely the QMLE and EMM presented in the previous section, were applied. The Kalman filter parameter estimates of the zero drift model appear in Table 4.1, while the second volatility process specification is covered in Table 4.2. The EMM estimates appear in Table 4.3.

A total of six variants of each model were evaluated, with the sixth being a SV model without time deformation, i.e., imposing  $c_\ell = c_v = c_p = 0$ . The other five specifications involve time deformation, yet with different functional forms. The most general specification is the unconstrained model with  $\Delta g(t)$  as a function of past volume and returns with a leverage effect. The second model involves only volume; the third, only prices with leverage effect; the fourth, prices and volume without leverage; and, finally, the fifth model has  $\Delta g(t)$  determined by past price changes.

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<sup>12</sup> For further details see Gallant, Rossi and Tauchen (1992).

Table 4.1  
Stochastic Volatility with Time Deformation Determined by Past Trading Volume  
and Prices with Leverage Effects

Sample : 1950-1987, QMLE, nonzero drift

	(1)			(2)			(3)		
	Est	SE	P	Est	SE	P	Est	SE	P
$c_v$	0.5164	0.9152	0.5726	1.2475	0.3793	0.0010	-	-	-
$c_p$	-0.2043	0.1734	0.2386	-	-	-	-0.1874	0.0906	0.0386
$c_\ell$	0.2461	0.3011	0.4138	-	-	-	0.3333	0.1118	0.0028
$\Sigma$	0.0108	0.0034	0.0014	0.0143	0.0037	0.0000	0.0129	0.0045	0.0038
$\Lambda$	-0.0098	0.0040	0.0146	-0.0146	0.0034	0.0000	-0.0105	0.0036	0.0036
$\beta$	-0.2163	0.1133	0.0562	-0.2742	0.0853	0.0014	0.1007	0.0752	0.1808
$\lambda$	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100
$a_1$	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000

	(4)			(5)			(6)		
	Est	SE	P	Est	SE	P	Est	SE	P
$c_v$	1.3210	0.6898	0.0554	-	-	-	-	-	-
$c_p$	-0.3724	0.0816	0.0000	-0.2252	0.0830	0.0066	-	-	-
$c_\ell$	-	-	-	-	-	-	-	-	-
$\Sigma$	0.0150	0.0054	0.0052	0.0114	0.0039	0.0032	0.0126	0.0037	0.0008
$\Lambda$	-0.0133	0.0048	0.0052	-0.0117	0.0044	0.0074	-0.0133	0.0037	0.0004
$\beta$	-0.1591	0.1168	0.1730	-0.2471	0.0980	0.0118	-0.2379	0.0772	0.0020
$\lambda$	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100
$a_1$	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000

Note : The standard errors reported are based on corrected QMLE asymptotic covariance matrix.

**Table 4.2**  
**Stochastic Volatility with Time Deformation Determined by Past Trading Volume**  
**and Prices with Leverage Effects**

Sample : 1950-1987, QMLE, zero drift

	(1)			(2)			(3)		
	Est	SE	P	Est	SE	P	Est	SE	P
$c_v$	1.0098	0.6236	0.1054	0.9790	0.7012	0.1626	-	-	-
$c_p$	-0.1674	0.0959	0.0810	-	-	-	-0.1565	0.0979	0.1098
$c_\ell$	0.2562	0.1356	0.0588	-	-	-	0.3542	0.1152	0.0022
$\Sigma$	0.0149	0.0051	0.0038	0.0123	0.0035	0.0002	0.0125	0.0034	0.0002
$\Lambda$	-0.0121	0.0040	0.0024	-0.0115	0.0035	0.0008	-0.0107	0.0031	0.0006
$\lambda$	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100
$a_1$	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000

	(4)			(5)			(6)		
	Est	SE	P	Est	SE	P	Est	SE	P
$c_v$	1.2559	0.5759	0.0292	-	-	-	-	-	-
$c_p$	-0.3742	0.0767	0.0000	-0.5101	0.0472	0.0000	-	-	-
$c_\ell$	-	-	-	-	-	-	-	-	-
$\Sigma$	0.0138	0.0037	0.0002	0.0155	0.0058	0.0039	0.0117	0.0033	0.0004
$\Lambda$	-0.0117	0.0032	0.0002	-0.0132	0.0047	0.0026	-0.0113	0.0030	0.0002
$\lambda$	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100	0.0281	0.0109	0.0100
$a_1$	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000	0.1768	0.0100	0.0000

Note : The standard errors reported are based on corrected QMLE asymptotic covariance matrix.



Table 4.3  
 Stochastic Volatility with time Deformation Determined by Past Prices  
 with Leverage Effects

Sample 1950-1987, EFGMM, nonzero drift

	(3)		(5)		(6)
	EFGMM	NOISE ADDED	EFGMM	NOISE ADDED	EFGMM
$c_p$	-0.8875	0.5562	-2.2543	0.4292	-
$c_\ell$	3.2260	1.5634	-	-	-
$c_n$	-	0.0101	-	-0.1524	-
$\Sigma$	0.0002	0.0001	0.0022	0.0005	0.0159
$A$	-0.0003	-0.0002	-0.0025	-0.0003	-0.0221
$\beta$	0.2173	0.0491	-0.0447	-0.5198	-0.5543
$\lambda$	0.0445	0.0447	0.0447	0.0448	0.0529
$a_1$	0.2125	0.2117	0.2125	0.2127	0.2143
$\chi^2$	113.846	127.082	123.726	129.985	126.563

Note : QMLE results are from Table 4.2 repeated here for comparison. The "noise added" column corresponds to the model of volatility specified in equation (4.1).

A total of seven coefficients in the zero drift O-U and eight in the other case were estimated. The parameter  $\lambda$  was obtained as a sample average of  $\Delta \log y_t$  following the suggestion of Gouriéroux, Monfort and Renault (1993). Moreover, as there appears to be some minor autocorrelation left in  $\Delta \log y_t$ , we first fitted first-order autoregressive models to  $\Delta \log y_t$  and replaced  $\Delta \log y_t$  by the residuals to estimate the SV models. The autoregressive coefficients appear as  $a_1$  in both tables. The standard errors reported in Tables 4.1 and 4.2 are based on a QMLE covariance matrix estimator. The EMM method based on simulations allows to estimate only three out of six variants, namely, the model without the deformation of time and with time deformation either determined by past returns only or returns with leverage effects. The eight parameters (we considered only the nonzero drift volatility specification) were estimated simultaneously, as opposed to the two-step procedure adopted in the QML approach involving the estimation of  $\lambda$  separately. The parameter estimates are presented in Table 4.3, where we compare them to the results of the Kalman filter.

The parameter values all appear to agree with the stochastic process behavior described in section 2. To evaluate the significance of individual coefficients, we rely on the QML t ratios. In particular, the basic continuous time parameters  $A$  and  $\Sigma$  are significantly different from zero throughout all specifications and  $A$  takes only negative values. The mean coefficient  $\beta$  yields mixed results since it is significant in four out of six specifications. This would mean that we should have a preference for the nonzero drift model if we were to choose between the two volatility specifications. Moreover  $\beta$  always takes negative values, except for the model (3) estimated by EMM, where the long run mean of the volatility process in operational time is much higher than elsewhere. The parameters appearing in the return equation,  $\lambda$  and  $a_1$  are significant, though their values vary depending on the estimation method. In general, the QML estimates of  $\lambda$  and  $a_1$  are larger than the EMM estimates in a one-step procedure, while the continuous time parameters  $A$  and  $\Sigma$  resulting from the simulation-based method are much lower.

Let us now discuss briefly the estimates of the time deformation parameters  $c_v$ ,  $c_p$  and  $c_\ell$  beginning with the QMLE results. Past volume has a positive impact on  $\Delta g(t)$  since  $c_v$  takes always positive values. This implies, as noted in section 2, that the marginal effect of increases in trading volume is a volatility process being less persistent in calendar time and more erratic. The leverage coefficient is also positive,

while past price change always enter with a negative coefficient in the  $\Delta g(t)$  specification. However, since  $|c_\ell| > |c_p|$  with  $c_\ell > 0$  and  $c_p < 0$  it follows that whenever  $\Delta \log p_{t-1}$  is negative, we find a greater positive effect of past returns on  $\Delta g(t)$  than when  $\Delta \log p_{t-1}$  is positive. Hence, bull markets tend to make volatility larger, less persistent and more erratic, while bear markets are associated with a lower volatility with smaller variance. Note that the EMM procedure yields larger values of the  $\Delta g(t)$  parameter estimates than QMLE. This is partly due to a different treatment of the time deformation function in the EMM framework, where we did not require  $\Delta g(t)$  to average to one in long term as we did in the Kalman filter, but we imposed instead an upper bound of  $1.394 \times 10^{65}$ . Joint tests, based on the QML results, have also been examined. Hence, we complement the Wald tests presented in Tables 4.1 and 4.2 with LR-type tests that appear in Tables 4.4 and 4.5. Tests regarding the time deformation hypothesis appear in the first table. The results indicate that when  $\Delta g(t)$  is determined by either one of the individual series, volume or prices the Wald and LR tests are not in agreement and there is also a difference depending on the process specification. However, prices combined with either a leverage effect or trading volume yield robust and strong results supporting significant time deformation. Finally, the three series combined again yield mixed results with the joint LR test favoring time deformation, though none of the coefficients are individually significant for the zero-drift model. In Table 4.5, we turn our attention to a number of LR tests regarding the functional specifications of time deformation. We test whether  $\Delta g(t)$  is determined by: (1) volume only against the alternative of volume and prices with leverage; (2) prices with leverage only against the same alternative; (3) prices only without leverage; and volume once again against all three series. In each case, the restricted model is rejected. We also test whether leverage should be introduced once prices and volume determine time deformation and find mixed results. In the zero-drift model, we observe a significant leverage effect, while the O-U process appears to have a very flat likelihood surface, making the marginal contribution of leverage to  $\Delta g(t)$  negligible.

We turn our attention now to the sample path of the time deformation process  $\Delta g(t)$  for a number of specifications. As we could not plot all possible combinations, since it would be quite repetitive, we selected a few representation cases. We first examine the path of time deformation for the AR(1) model with two alternative specifications of  $\Delta g(t)$ : one involving prices and volume, the other adding leverage effects. Four plots appear in Figure 4.2. Each  $\Delta g(t)$  specification yields a pair of plots, one for  $\Delta g(t)$ , the other for the innovation variance which also depends on  $\Delta g(t)$ . Figures 4.2a and 4.2b display the patterns of time deformation, both involving prices

Table 4.4  
Time Deformation Hypothesis Tests (LR)

Series in $\Delta g(t)$	AR(1)	Ornstein-Uhlenbeck
Volume only <sup>a</sup>	2.794	5.492
Prices only <sup>a</sup>	10.599	1.445
Prices with leverage <sup>b</sup>	6.455	8.961
Prices and volume <sup>b</sup>	9.635	11.947
Prices with leverage and volume <sup>c</sup>	15.994	5.299

Note : The likelihood ratio statistic is asymptotically distributed as  $\chi^2$  with respectively a = 1, b = 2 and c = 3 degrees of freedom.

Table 4.5  
Hypotheses Tests of the Time Deformation Function (LR) -  
The Continuous Time AR(1) Model

Hypotheses	AR(1)	Ornstein-Uhlenbeck
$H_0 : c_v \neq 0 \quad c_p = 0 \quad c_l = 0$	13.200	12.911
$H_A : c_v \neq 0 \quad c_p \neq 0 \quad c_l \neq 0$		
$H_0 : c_v = 0 \quad c_p \neq 0 \quad c_l \neq 0$	9.538	10.502
$H_A : c_v \neq 0 \quad c_p \neq 0 \quad c_l \neq 0$		
$H_0 : c_v = 0 \quad c_p \neq 0 \quad c_l = 0$	5.396	5.877
$H_A : c_v \neq 0 \quad c_p \neq 0 \quad c_l \neq 0$		
$H_0 : c_v \neq 0 \quad c_p \neq 0 \quad c_l = 0$	6.359	0.001
$H_A : c_v \neq 0 \quad c_p \neq 0 \quad c_l \neq 0$		

and volume with leverage effects included in the latter. They appear to be quite similar, though  $\Delta g(t)$  with leverage seems to be slightly less erratic. One key feature emerging from both figures, as well as the adjacent plots containing the innovation variance to the volatility process, is the infrequent appearance of sharp peaks in operational time acceleration. Since  $A$  is found to be negative, this means that the conditional variance function becomes locally extremely erratic, unattached to the previous period and subject to a large variance innovation shock. As noted in the introduction, this finding complements a competing specification of laws of motion via diffusion processes involving jumps. Such processes, proposed by Merton (1967a,b) were built on the premise that one would occasionally observe abnormal information leading to the incidence of a jump in asset prices. Through the time deformation specification, one can view such information arrival as extremely rapid acceleration of market time through the increased trading and price movement per unit of calendar time. The advantage of SV models with time deformation over jump-diffusion processes is that the former might be relatively easier to estimate, at least if one is satisfied with the asymptotically inefficient QMLE algorithm. Indeed, the ML estimation of jump-diffusion processes can be quite involved [see, for instance, Lo (1988) for details].<sup>13</sup>

[Insert Figure 4.2 here]

We turn our attention now to the volatility process itself, i.e., the  $h_t$  process as extracted via the Kalman filter procedure. A first caveat to note is that the filtering algorithm we use, like the estimation procedure, is only an approximation of the true latent volatility process. Indeed, the Kalman filtering algorithm ignores all non-Gaussian features of the DGP, as noted in section 3. Jacquier, Polson and Rossi (1992) proposed a procedure that yields an exact extraction algorithm for the volatility process as a by-product of their Bayesian inference procedure for SV models. Their algorithm is numerically quite more involved in comparison to that described in section 3 and is probably not so easy to modify so that a time deformation SV model can be handled [see Ghysels and Jasiak (1993) for further discussion]. Figure 4.3a displays the approximate filter extraction of the volatility process  $h_t$ . The figure consists of two parts, namely, 4.3a displays stochastic volatility as extracted under the assumption of no time deformation. Hence, Figure 4.3a corresponds to a volatility process that one would obtain from the approach proposed by Nelson (1988)

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<sup>13</sup> There is, of course, a substantial difference between the stochastic process behavior of a jump process and a process with SV having occasionally very large volatility.

and Harvey, Ruiz and Shephard (1994). Figure 4.3b plots  $h_t$  extracted from a model with time deformation. In sharp contrast to the standard SV specification, we uncover a very smooth volatility process. This may not be as surprising, given the plots in Figure 4.2 where  $\Delta g(t)$  and the innovation variance appeared. Indeed, most of the erratic behavior of  $h_t$  obtained through a specification without time deformation is absorbed through the doubly stochastic random coefficient stochastic volatility specification. Once time deformation is taken into account, it appears that the underlying volatility process evolves smoothly in operational time. This yields an alternative interpretation. Indeed, the smooth evolution of  $h_t$  in operational time implies that the process is easier to predict over long horizons. This smooth and predictable component appears to be separated from the more erratic behavior of market time through  $\Delta g(t)$ . This separation into two components is interesting as it decomposes a volatility process that is itself latent.

[Insert Figure 4.3 here]

Let us now examine further the empirical results obtained from the EMM framework. A goodness of fit test can be performed by computing, a chi-square statistic  $N m_N'(\tilde{\alpha}_N, \tilde{\theta}_N) (\tilde{I}_N)^{-1} m_N(\tilde{\alpha}_N, \tilde{\theta})$ , which under a correct specification of the SV model is asymptotically distributed as  $\chi^2$  with degrees of freedom equal to the length of the SNP parameter vector  $\theta$  minus the number of parameters of the SV model, collected in the vector  $\alpha$ . All the three models under study failed the chi-square test (see Table 4.3). The value of the test statistic obtained for the nondeformation SV model falls far beyond the critical value. However, the objective function can be significantly improved upon, once we incorporate the time deformation determined by past price changes with leverage effects. A similar but weaker reduction in the value of the statistics can be observed when  $\Delta g(t)$  is defined as function of past returns only.

Figure 4.4 shows the improvement in terms of the t-statistics on the scores of the SNP model obtained by fitting a time deformation SV model. The parameters "psi" correspond to the AR coefficients of the SNP model, the "a's" indicate the quadratic and quartic terms, and finally the "tau's" are the coefficients on the ARCH components of the score generator. The time deformation model with leverage effects improves 28 among 34 t ratios, while the model without the leverage performs equally well, reducing 29 out of 34. It is interesting to note here that most of the improved moment

fits appear in the "tau" group representing ARCH components. This is of course a key issue, as time deformation should affect the volatility component first and foremost.

In Tables 4.4 and 4.5, we reported joint tests to examine the fit of the SV model with time deformation. In the context of the EMM estimation procedure, we can perform some model diagnostics more directly aimed at the restrictions imposed by the time deformation specification of the volatility equation. Returning to equation (2.10b), we notice that  $\Delta g(t)$  controls the intercept AR coefficient as well as the innovation variance. In a simulation context, we can ask ourselves whether we can improve the fit by breaking this link between the correlations and variance of the volatility. We do so by adding an extra term to the equation, namely :

$$(4.1.1) \quad h_t = [(1 - \exp(A\Delta g(t)))\beta + [\exp(A\Delta g(t)) + c_n \tilde{v}_t] h_{t-1} + v_t$$

where  $\tilde{v}_t$  is an i.i.d.  $N(0,1)$  sequence and  $c_n$  is an additional parameter. The results in Table 4.3 indicate that the fit deteriorates with the "noise added" specification. We also find larger t statistics on the score vector, is illustrated in Figure 4.5. Out of 34 t ratios, 17 went up in the case of model (3) and 13 out of 34 for model (5). These results suggest that breaking the restrictions obtained in equation (2.10b) do not improve the fit. Also, the "tau" group of moment conditions is the one where the deterioration proclaims itself as one would expect. We simply perturbed the specification by adding a noise term to the AR coefficient. Perturbations in other "directions" may perhaps yield other results. So, far however, we find the time deformation specification the best fit focused so far.

## 4.2 Empirical Volatility–Volume Models

In this section, we rely exclusively on EMM estimation guided by the bivariate return and volume SNP density described at the end of section 3. The parameter estimates of the bivariate model with time deformation are reported in Table 4.6. A total of 14 parameters are estimated with the empirical score of the bivariate SNP as a guidance of matching moments. In our specification of the time deformation, we did not include the absolute value of returns. We experimented with such a specification as well, but found the model reported in 4.6 a better fit. This does not mean the model does not produce leverage effects. Indeed, the bivariate structure is far more complex than the univariate one and asymmetric responses can arise without it appearing in  $\Delta g(t)$ . We shall in fact return to this issue shortly.

**Table 4.6**  
**EMM Parameter Estimates of Bivariate Stochastic**  
**Volatility-Volume Model with Time Deformation**

$\mu_P$	-0.1762	$a_{11}$	-0.4550
$\mu_V$	0.9511	$a_{12}$	-0.4267
$a_1$	0.4570	$a_{21}$	0.8026
$\beta_P$	-1.1440	$a_{22}$	-4.6646
$\beta_V$	0.9301	$\Sigma_{11}$	1.9558
$c_P$	-1.1204	$\Sigma_{21} = \Sigma_{12}$	-0.1298
$c_V$	0.3412	$\Sigma_{22}$	1.0171

Note : Moment matching using score function of bivariate SNP described by the following turning parameters :  $L_\mu = 2$ ,  $L_M = 18$ ,  $L_P = 2$ ,  $K_Z = 4$ ,  $I_Z = 1$ ,  $K_X = 2$ ,  $I_X = 1$ .



Judging the adequacy of the bivariate model is no longer as straightforward as judging that of the univariate of course. We mentioned in section 2 that we would therefore analyze the model via impulse response functions. We will follow a strategy proposed by Gallant, Rossi and Tauchen (1993) which consists of computing response profiles for the conditional mean and conditional volatility, where the profiles are defined by :

$$(4.2.1) \quad \hat{y}_j(x) = E(y_{t+j} \mid x_t = x) \quad j = 0, 1, \dots$$

for the conditional mean profile and :

$$(4.2.2) \quad \hat{V}_j(x) = E[\text{Var}(y_{t+j} \mid x_{t+j-1}) \mid x_t = x] \quad j = 0, 1, \dots$$

In both cases,  $y_{t+j}$  represents a component of the bivariate return-volume process. The conditioning vector  $x$  is the one which is perturbed to produce different response profiles. Gallant, Rossi and Tauchen consider three scenarios to compute the response profiles, namely :

$$(4.2.3) \quad x^+ = (y'_0, y'_{-1}, y'_{-2}, \dots)' + (\delta_y^+, 0, 0, \dots)$$

$$(4.2.4) \quad x^0 = (y'_0, y'_{-1}, y'_{-2}, \dots)'$$

$$(4.2.5) \quad x^- = (y'_0, y'_{-1}, \dots)' + (\delta_y^-, 0, 0, \dots)$$

where  $\delta_y^+$  and  $\delta_y^-$  are shocks to the nonlinear dynamic system. The vector  $x^0$  is called the baseline shock while  $x^+$  and  $x^-$  are respectively positive and negative shocks. Hence, the conditional profiles are predictions of  $y_{t+j}$  for the three different initial conditions listed in (4.2.3) through (4.2.5). The combination of equations (4.2.1) through (4.2.5) yield the conditional mean and volatility profiles  $(\hat{y}_j(x), \hat{V}_j(x))$  for  $x = x^+, x^0$  and  $x^-$ . These response profiles were computed for both stock returns and trading volume. Of course, so far we did not discuss how the conditional means and variances appearing in (4.2.1) and (4.2.2) are obtained. In Gallant, Rossi and Tauchen (1993), the empirical SNP, which yields an estimate of the conditional density was used to compute (4.2.1) and (4.2.2). This empirical density will serve as a benchmark against which we want to measure the success of the bivariate stochastic

volatility model with time deformation described in (2.12). In order to compare the impulse response profiles of the empirical SNP with those of the bivariate SV model with time deformation, we produced a simulated sample of data containing 20,000 observations (approximately the size of the empirical data using the estimated model). The simulated data generated by the model were then used to fit an SNP density from which the impulse profiles were computed. To assess the usefulness of the time deformation specification, we also estimated the bivariate stochastic volatility model defined in (2.12) through (2.14) restricting  $\Delta g(t) = 1 \forall t$ . We followed exactly the same procedure as above and obtained impulse response profiles under a bivariate model specification without time deformation. We have therefore three impulse response profiles to compare, a first from the empirical fit of the SNP from the data, a second from a SNP density fitted to data simulated from a bivariate model with time deformation using the EMM parameter estimates and finally a third which, like the second, is obtained from simulated data, yet without a time deformation specification.

Gallant, Rossi and Tauchen (1993) point out that impulse response profiles for a bivariate volatility-volume process need to take into account the widely documented contemporaneous relationship between price and volume movements [see, e.g., Karpoff (1987) or Tauchen and Pitts (1983)]. From a scatterplot of historical return-volume data, they define three types of impulses which described different scenarios of contemporaneous return-volume shocks. These shocks, called of type A, B and C were constructed to be consistent with the historical range of the data. We do not want to repeat all the details of the computations and findings based on the three types of shocks as they are reported in Gallant, Rossi and Tauchen. Instead, we will single out one particular case which the authors identified as one of the most interesting and novel findings emerging from the impulse response analysis. Namely, Gallant, Rossi and Tauchen found that the leverage effect is essentially a transient effect when analyzed in a bivariate system in sharp contrast to the univariate price shock models which shows a much more persistent wedge between effects of positive  $\delta_y^+$  and negative  $\delta_y^-$  price shocks. We reexamine this issue using both type A and B shocks, the former being a combined price-volume shock while the latter being a pure price shock.<sup>14</sup>

[Insert Figure 4.6a and b here]

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<sup>14</sup> Shocks of type C relate to volume only, see Gallant, Rossi and Tauchen (1993) for more details, in particular regarding the interpretation given to the three types of shocks.

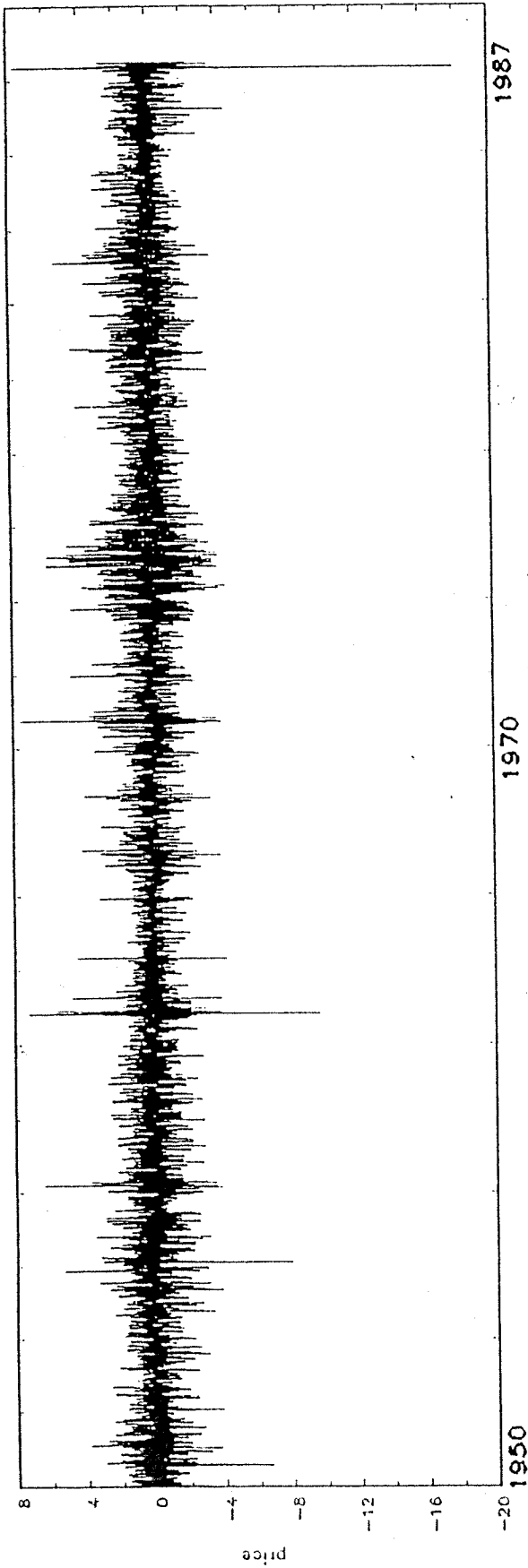
The impulse response profiles are summarized in Figures 4.6a and 4.6b. The former covers shocks of type A while the latter of type B. Each figure has six panels, the top three panels display conditional mean profiles of returns while the lower three panels exhibit conditional variance profiles. Each time, one has (1) impulse responses from the empirical SNP, (2) from the SNP generated by a time deformation bivariate model and (3) one without time deformation. We consider the empirical SNP as being the benchmark case where the data features are summarized. A first observation to make is that the response profiles for the model without time deformation appear to overstate the volatility responses considerably, both for shocks of type A and B. Besides being off track, we also notice that the baseline shock without time deformation shows a slight kink in both cases which does not appear in the deformation model nor the empirical SNP. While both models appear to confirm the transient nature of the leverage effect, we note the model without time deformation tends to slightly overstate, in relative terms, the initial response of a negative shock. The differences are minor, but the time deformation specification does show an edge, at least on the basis of these impulse response profiles. Other criteria may be found to better discriminate between models, but this we would rather leave for future research.

## 5. CONCLUSIONS

In this paper, we proposed an empirical class of time deformation stochastic volatility models that were fitted to daily return and volume data for the NYSE. Two estimation procedures were discussed, one involving a Kalman filter QMLE algorithm, the other involving a moment matching principle. A univariate as well as bivariate return-volume model specification were considered.

The framework can easily be extended to deal with high frequency data. For instance, Ghysels and Jasiak (1995) suggest a specification of a time deformed SV models involving arrivals of quotas and bid-ask spreads at 5 and 20 minutes intervals for foreign exchange markets. Last but not least, Ghysels, Gouriéroux and Jasiak (1995) provide a detailed discussion of the stochastic process theory for subordinated processes. They, as well as Conley, Hansen, Luttmer and Scheinkman (1994) discuss various estimation procedures not covered here.

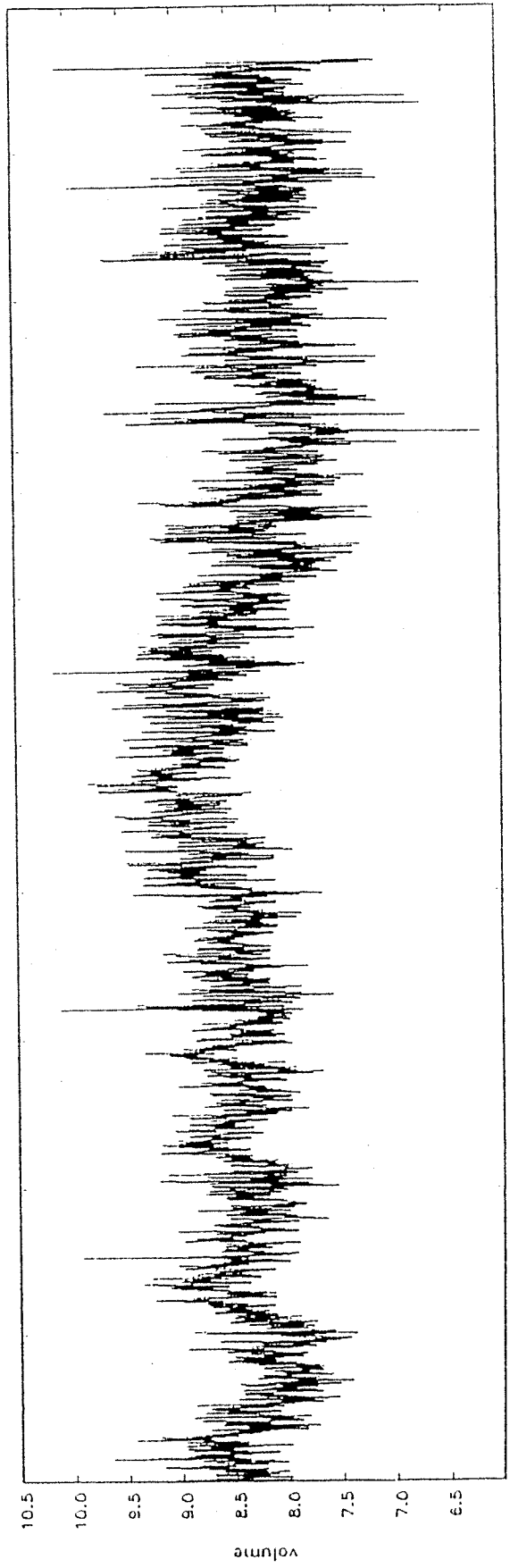
Fig 4.1a: S&P 500 (Adjusted)



1970

1987

Fig 4.1b: Volume NYSE (Adjusted)



10.5

10.0

9.5

9.0

8.5

8.0

7.5

7.0

6.5

Volume

Figure 4.2 : AR(1) SV Model with Time Deformation

Fig 4.2b: Volume and Prices Innovation Variance

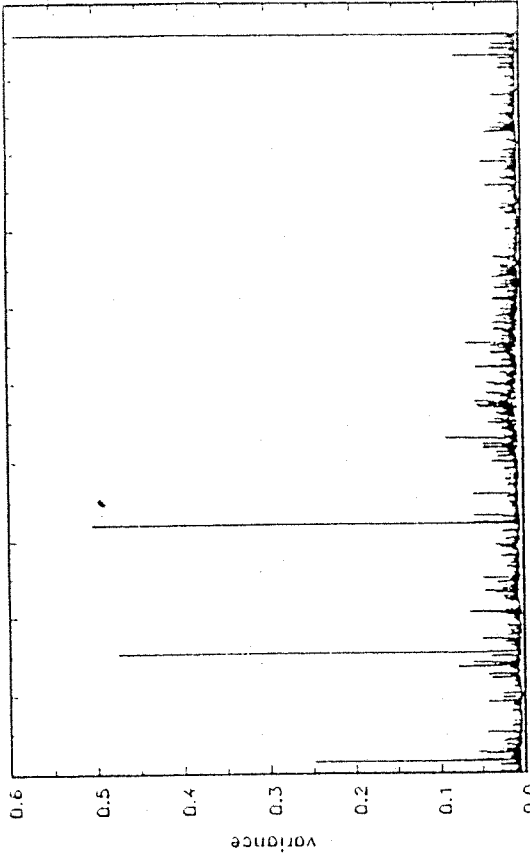


Fig 4.2d: Volume, Prices and Leverage Innovation Variance

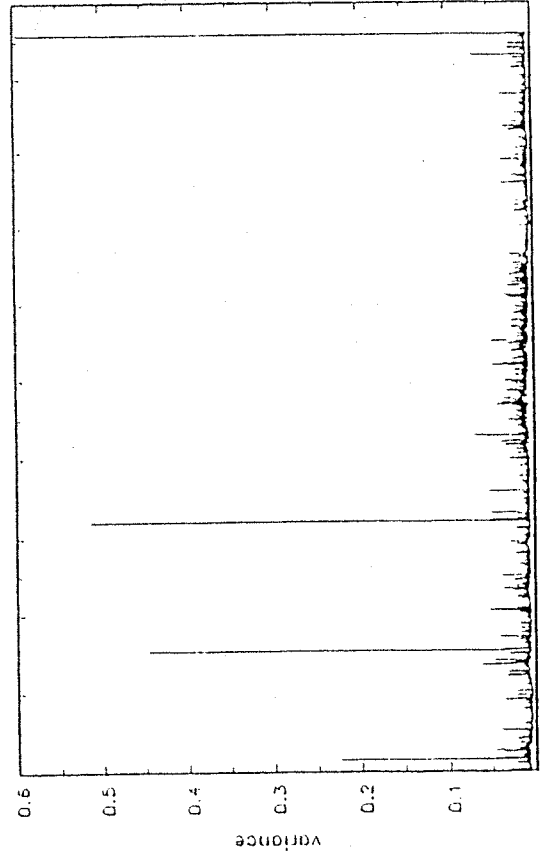


Fig 4.2a: Volume and Prices Time Deformation

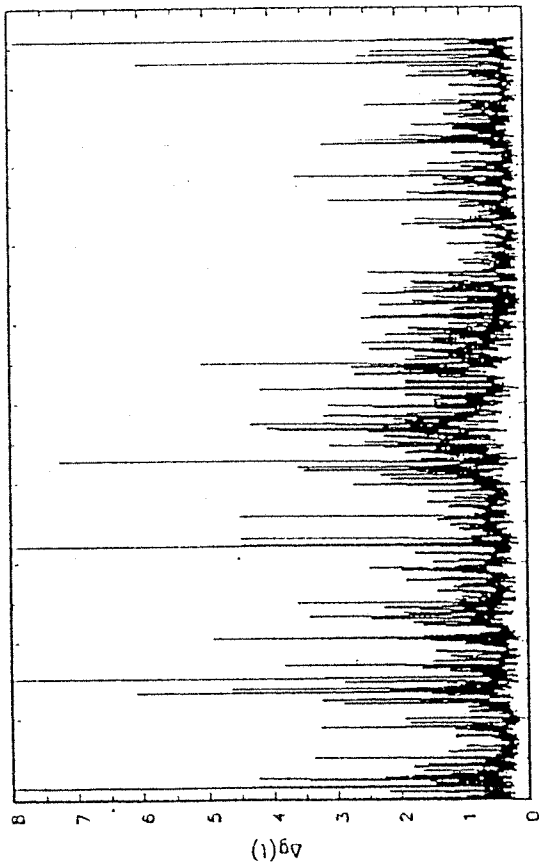


Fig 4.2c: Volume, Prices and Leverage Time deformation

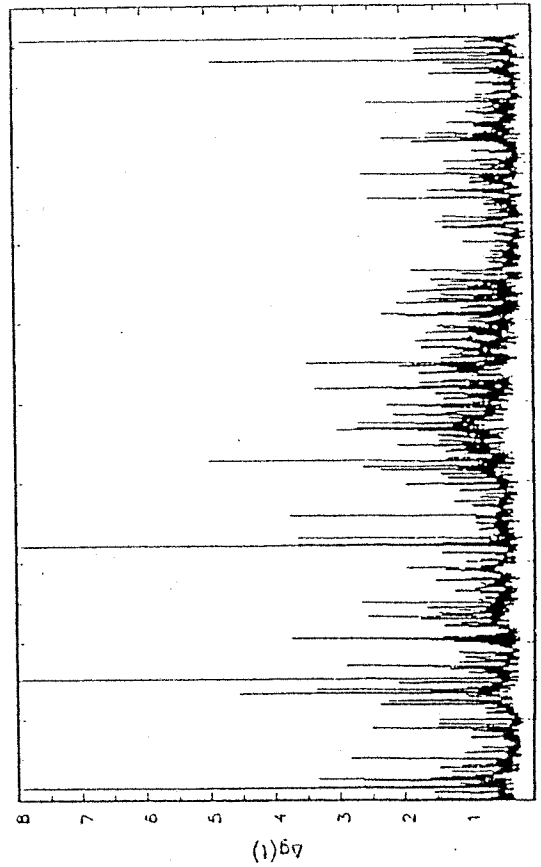


Fig 4.3a: Volatility without Time Deformation

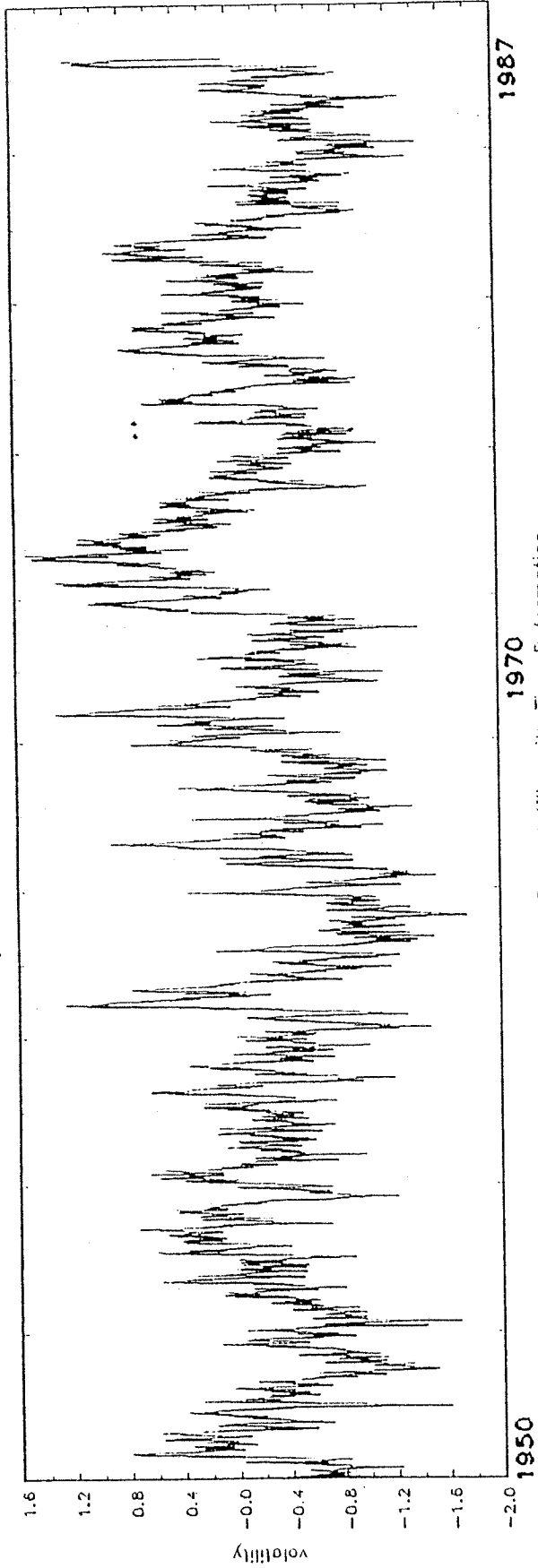
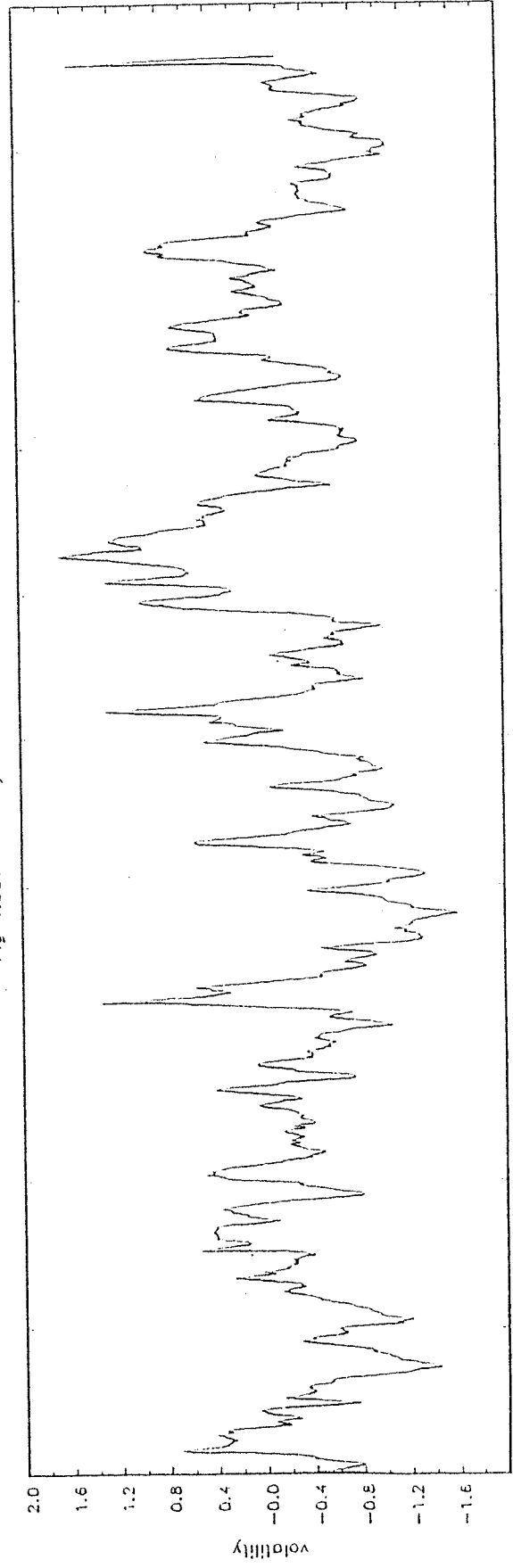


Fig 4.3b: Volatility with Time Deformation



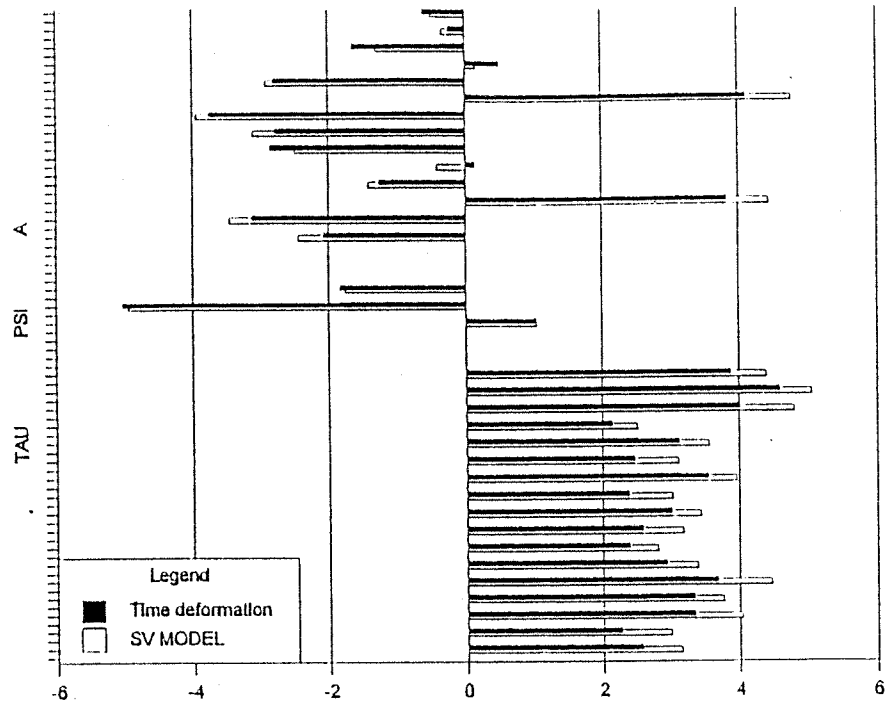


Figure 4.4 : SV MODEL vs TIME DEFORMATION, DETERMINED BY PAST RETURNS AND THEIR ABSOLUTE VALUE

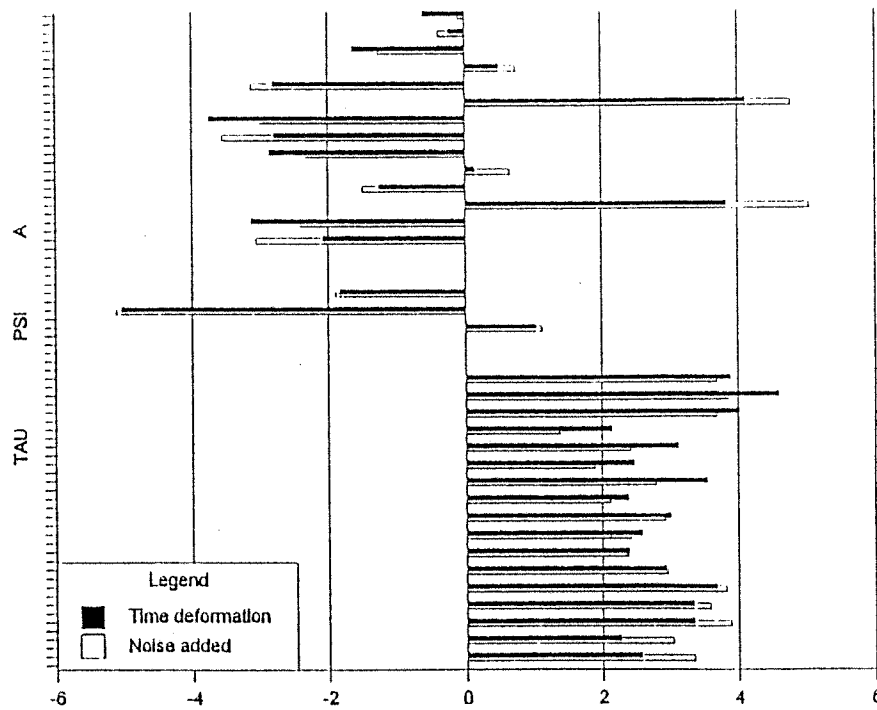


Figure 4.5 : TIME DEFORMATION vs MODEL WITH NOISE ADDED, DETERMINED BY PAST RETURNS AND THEIR ABSOLUTE VALUE

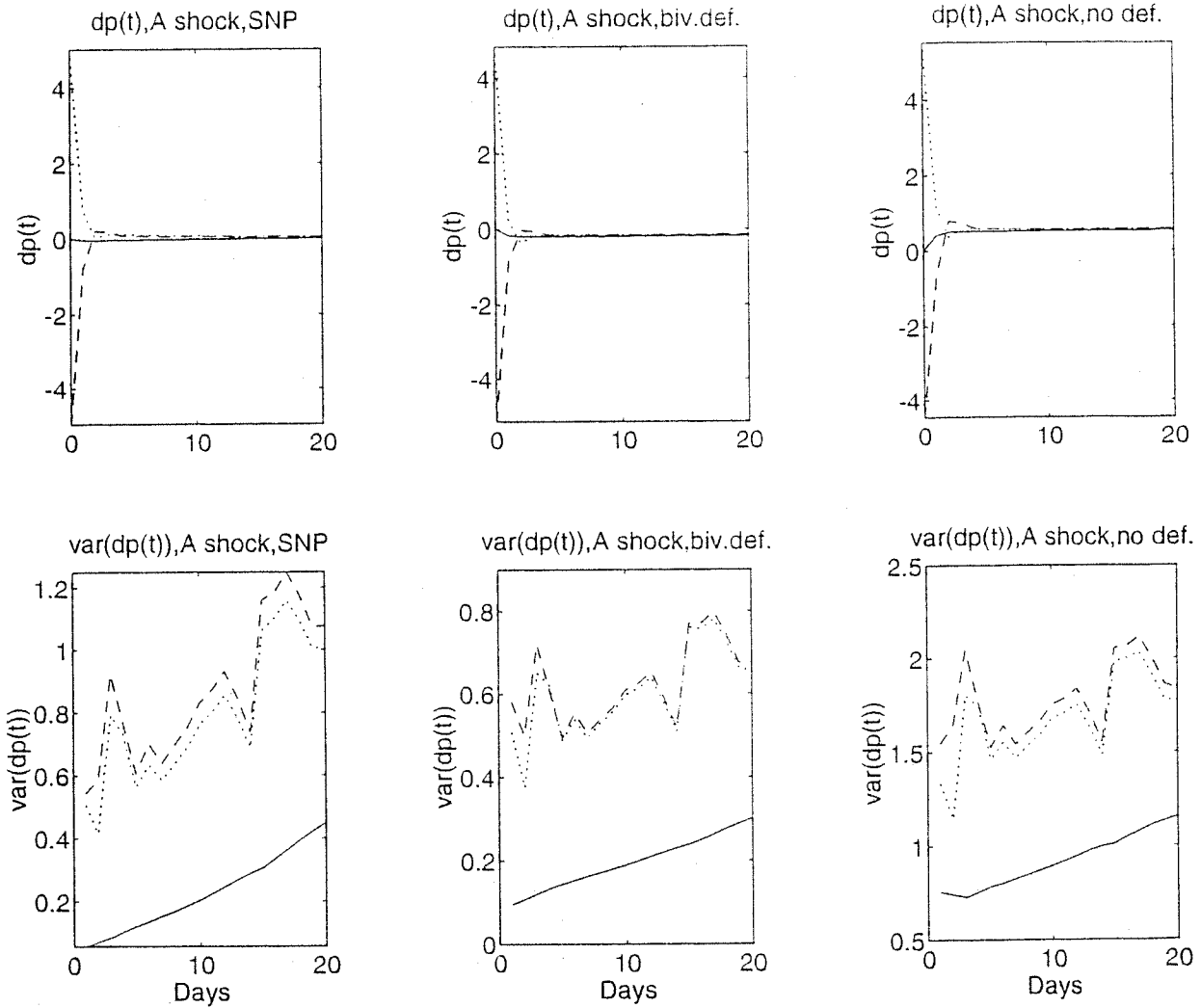


Figure 4.6a

Impulse responses of  $\Delta p$  and volatility to shock type A defined in Gallant, Rossi and Tauchen (1993, p. 892) from bivariate fit of SNP densities to empirical data and simulated data from bivariate SV models with and without time deformation. In each panel, the heavy solid line is the baseline where  $\hat{x}_\tau = x_\tau$  the solid line corresponds to a negative A shock where  $\hat{x}_\tau = x_\tau + (0, 0, \dots, \delta\bar{y}_A)'$ , and the dashed line corresponds to a positive A shock where  $\hat{x}_\tau = x_\tau + (0, 0, \dots, \delta\bar{y}_A^+)'$ .



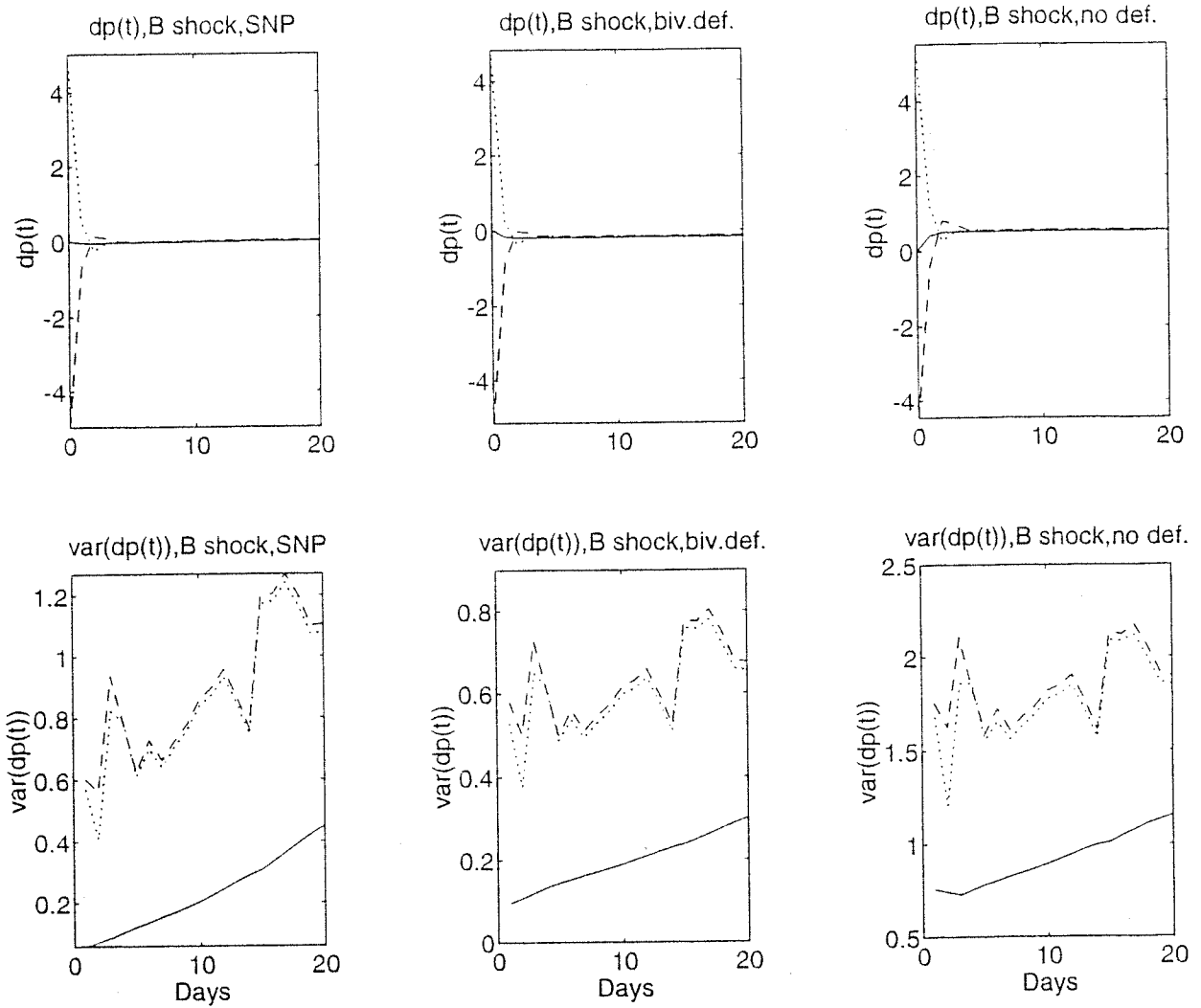


Figure 4.6b

Impulse responses of  $\Delta p$  and volatility to shock type B defined in Gallant, Rossi and Tauchen (1993, p. 892) from bivariate fit of SNP densities to empirical data and simulated data from bivariate SV models with and without time deformation. In each panel, the heavy solid line is the baseline where  $\hat{x}_\tau = x_\tau$ , the solid line corresponds to a negative B shock where  $\hat{x}_\tau = x_\tau + (0, 0, \dots, \delta \bar{y}_B)'$ , and the dashed line corresponds to a positive B shock where  $\hat{x}_\tau = x_\tau + (0, 0, \dots, \delta \bar{y}_B^+)'$ .

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