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Market Time and Asset Price Movements Theory and Estimation*

Eric Ghysels[†], *Christian Gouriéroux*[‡], *Joanna Jasiak*[+]

Abstract / Résumé

Subordinated stochastic processes, also called time deformed stochastic processes, have been proposed in a variety of contexts to describe asset price behavior. They are used when the movement of prices is tied to the number of market transactions, trading volume or the more illusive concept of information arrival. The aim of the paper is to present a comprehensive treatment of the stochastic process theory as well as the statistical inference of subordinated processes. Numerous applications in finance are provided to illustrate the use of the processes to model market behavior and asset returns.

Nous étudions les mouvements de prix d'actifs financiers à l'aide de processus avec changement de temps. L'idée est que l'activité du marché, mesurée par des séries comme le volume de transactions, détermine l'échelle de temps intrinsèque du processus stochastique de prix ou de rendement. Les propriétés de ce type de processus, parfois aussi appelés subordonné, sont présentées en détail et illustrées par plusieurs applications à la théorie financière. On développe également les procédures d'inférence statistique correspondantes.

JEL: C13, C22, G12, C12

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* Correspondence address: Eric Ghysels, CIRANO, 2020, University Street, 25th floor, Montréal (Québec) CANADA H3A 2A5 Tel.: (514) 985-4025 Fax: (514) 985-4039 e-mail: eghysels@plgcn.umontreal.ca
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† C.R.D.E., Université de Montréal and CIRANO.

‡ CREST and CEPREMAP.

+ York University.

1. Introduction

Computer technology has not only changed the structure of trading, it has also made the collection, storage and retrieval of financial market data more widespread at levels of detail never seen before. Until only a few years ago most empirical studies involved daily, weekly or monthly time series. As high frequency data become more easily available it is now possible to study how financial markets evolve in real time. While data sets a researcher in microstructures would dream of involving the identity, motives and portfolio positions of those transacting, are not yet available it is clear that continuous record observations which are now easy to obtain contain already a vast amount of information. There are at least two key challenges one faces in modelling these newly available data sets. First, unlike daily, weekly or monthly series, quote or tick-based data are by their very nature irregularly spaced. The great majority of empirical asset pricing models, models of market volatility such as ARCH-type models, etc. are constructed on the basis of equally spaced data points such as daily observations. This simplification no longer suits high frequency data and therefore needs to be modified. It is particularly important to note that the spacing of time between quotes is not a purely technical issue, as indeed the recent vintage of microstructure models use the length of time elapsed between consecutive transactions as a signal revealing information known to market participants (see Easley and O'Hara (1992)). A second challenge one faces with the analysis of markets in real time is the sheer number of data points. A typical data set of daily observations spanning a number of years contains a couple of thousand observations. In contrast, there are an average of roughly between four to five thousand new quotes on a single market like the DM/US$ spot exchange recorded by the Reuters FXFX screen page every working day. Hence, data sets run into millions of records and are easier to measure in terms of the disk space they occupy rather than the number of observations. With such large data sets there is obviously also a great need to identify and summarize empirical regualrities in trading patterns and returns.

The concept of time deformed or subordinated process is a particularly apt to address some of the challenges we just described. The idea originated in the work by Mandelbrot and Taylor (1967), Clark (1973), among others, who argued that since the number of transactions in any time period is random, one may think of asset price movements as the realization of a process $Y_t = Y^*_{z_t}$ where Z_t is a directing process. This positive nondecreasing stochastic process Z_t can for instance be thought as related to the number of transactions or more fundamentally, to the arrival of information. This by now familiar concept of subordinated stochastic processes, originated by Bochner (1960), was used by Mandelbrot and Taylor (1967) and later refined by Clark (1970, 1973) to explain the behavior of speculative prices. Originally, it was mostly applied to daily observations since high frequency data were not available. A well known example in finance is the considerable amount of empirical evidence documenting nontrading day effects. Such phenomena can be viewed as time deformation due to market closure.[1] Obviously, as pointed out by Mandelbrot and Taylor (1967), time deformation also directly related to the mixture of distributions model of Tauchen and Pitts (1983), Harris (1987), Richardson and Smith (1993), Foster and Viswanathan (1993) among others. More to the point regarding high frequency data one should mention that in foreign exchange markets, there is also a tendency to rely on activity scales determined by the number of active markets around the world at any particular moment. Dacorogna et al. (1993a) describe explicitly a model of time deformation along these for intraday movements of foreign exchange rates. Besides these relatively simple examples, there are a number of more complex ones. Ghysels and Jasiak (1994) proposed a stochastic volatility model with the volatility equation evolving in an operational time scale. They use trading volume and leverage effects to specify the mapping between calendar and operational time. In Ghysels, Gouriéroux and Jasiak (1995) this framework is extended and applied to intraday foreign exchange data, providing an alternative to the Dacorogna et al. time scale transformation. Madan and Seneta (1990) and Madan and Milne (1991) introduced a Brownian motion evaluated at random (exogenous) time changes governed by independent gamma increments as an alternative martingale process for the uncertainty driving stock market returns. Geman and Yor (1993) also used time-changed Bessel processes to compute path-dependent option prices such as is the case with Asian options. It is also worth noting that there is some research specifically examining the time between trades, see Hausman and Lo (1990) and Han, Kolay and Rosenfeld (1994) for instance.

Despite the several examples just mentioned there is no comprehensive treat-

[1] Examples of such evidence include Lakonishok and Smidt (1988) and Schwert (1990) who argue that returns on Monday are systematically lower than on any other day of the week, while French and Roll (1986), French, Schwert and Stambaugh (1987) and Nelson (1991) demonstrate that daily return volatility on the NYSE is higher following nontrading days.

ment of the stochastic process theory and statistical estimation of subordinated processes. The aim of the paper is to describe some of the probabilistic and statistical properties of time deformed models. Such models are in principle defined in two steps. We first consider the process of interest with respect to intrinsic time Y_z^*, and the changing time process Z_t, which explains how to pass from calendar time to intrinsic time. Then the process of interest expressed in calendar time is the subordinated process: $Y_t = Y_{z_t}^*$. Clearly the observable model (the one corresponding to Y_t) is a dynamic factor model with Z_t as the underlying factor. As is typical in factor models we may distinguish different cases depending on whether Z is assumed to be observable (for instance when it relates a series like transactions volume or number of quotes) or unobservable. In the latter case, it is necessary to specify a latent factor process for Z_t (see Clark (1970, 1973) for such an approach) and to predict ex-post the values of the factor.

Sections 2 and 3 describe the stochastic behavior of time deformed processes and highlight their use in financial modelling. Sections 4 and 5 cover the empirical analysis of the processes. Besides estimation we also discuss diagnostic tests which help summarize the potentially vast amounts of data.

2. Properties of Subordinated Processes

In this section we will compare the properties of the process of interest, when it evolves in calendar time and in intrinsic (or operational) time. We first consider second order properties of the processes, namely: (1) second order stationarity, (2) the conservation of a unit root by time deformation and (3) the relation between the autocovariance functions of Y and Y^*. Next we study some distributional properties such as strong stationarity and examine when the subordinated process is Markovian. The section concludes with a description of a system of stochastic differential equations with at least two equations, a subordinated diffusion and a directing process. In a first subsection 2.1 we consider second order properties of a time deformed process. We study distributional properties in section 2.2 and focus on a system of diffusion processes in section 2.3. To set the scene we first introduce some notations:

i) the time changing process, called the directing process by Clark (1973), associates the operational scale with the calendar time. It is a positive strictly increasing process:

$$Z : t \in \Im \Leftrightarrow\!\!\!\rightarrow Z_t \in \mathcal{Z}. \tag{2.1}$$

ii) The process of interest evolving in the operational time is denoted by:

$$Y^* : z \in \mathcal{Z} \Leftrightarrow\!\!\!\rightarrow Y_z^* \in \mathcal{Y} \subset \mathbb{R}^M. \tag{2.2}$$

iii) Finally we may deduce the process in calendar time $t \in \Im$ by considering:

$$Y_t = Y^* \circ Z_t = Y_{z_t}^*. \tag{2.3}$$

The introduction of a time scaling process is only interesting if the probabilistic properties of the process of interest become simpler. It explains the introduction of the assumption below which ensures that all the links between the two processes $(Y_t), (Z_t)$ in calendar time come from the time deformation.

Assumption A.1: The two processes Z and Y^* are independent.

Assumption A.1 is not entirely innocent with respect to practical applications. Indeed, if Z is tied to trading volume and Y^* is a return process, for instance, it is clear that the two may not be independent in operational time. However, we would feel more comfortable with letting Y^* be the bivariate process of return and volume and Z being the (latent) process of information arrival. Hence, the use of Assumption A.1 has to be used judiciously. As noted before, we will proceed with this assumption as it makes the links between Z_t and Y_t result from subordination. It has also to be noted that our formalism allows for the treatment of both discrete and continuous time problems. Indeed one may consider: discrete calendar and operational times with $\Im = \mathcal{Z} = \mathbb{N}$, continuous calendar and operational times with $\Im = \mathcal{Z} = \mathbb{R}^+$ and finally $\Im = \mathbb{N}$, $\mathcal{Z} = \mathbb{R}^+$ for continuous operational time and discrete calendar time.

2.1. Second order properties

As usual for time series analysis we will first study the second order properties of the processes Y and Y^*. Assuming that both processes are second order integrable, we consider the first order moments:

$$
\begin{cases} m(t) = E(Y_t) \text{ defined on } \Im, \\ m^*(z) = E(Y_z^*) \text{ defined on } \mathcal{Z}, \end{cases} \tag{2.4}
$$

and the autocovariance functions:

$$
\begin{cases} \gamma(t,h) = E\left[(Y_t \Leftrightarrow EY_t)(Y_{t+h} \Leftrightarrow EY_{t+h})'\right], \; t \in \Im, h \in \Im, \\ \gamma^*(z_0, z) = E\left[\left[Y_{z_0}^* \Leftrightarrow E\left(Y_{z_0}^*\right)\right]\left[Y_{z+z_0}^* \Leftrightarrow E\left(Y_{z+z_0}^*\right)\right]'\right], \; z_0 \in \mathcal{Z}, z \in \mathcal{Z}. \end{cases} \tag{2.5}
$$

From the definition of the time deformed process, we obtain:

$$
m(t) = E(Y_t) = E\left[E\left(Y_{Z_t}^* \mid Z_t\right)\right],
$$

$$
\gamma(t,h) = E\left(Y_t Y_{t+h}'\right) \Leftrightarrow (EY_t)(EY_{t+h})' = E\left[E\left(Y_t Y_{t+h}' \mid Z_t, Z_{t+h}\right)\right] \Leftrightarrow (EY_t)(EY_{t+h})',
$$

$$
Cov(Y_t, Z_{t+h}) = E(Y_t Z_{t+h}) \Leftrightarrow EY_t EZ_{t+h} = E\left[E\left(Y_t^* \mid Z_t\right) Z_{t+h}\right] \Leftrightarrow EY_t EZ_{t+h}.
$$

Taking into account the independence assumption between the two processes Z and Y^*, we can establish the following result:

Property 2.1.1: Under Assumption A.1:

$$
m(t) = E\left[m^*(Z_t)\right],
$$

$$
\gamma(t,h) = E\left[\gamma^*(Z_t, Z_{t+h} \Leftrightarrow Z_t)\right] + Cov\left[m^*(Z_t), m^*(Z_{t+h})\right],
$$

$$
Cov(Y_t, Z_{t+h}) = Cov(m^*(Z_t), Z_{t+h}).
$$

It is possible now to discuss some sufficient conditions for the second order stationarity of the process Y. These conditions are *moment conditions* on the underlying process Y^*, and *distributional conditions* on the directing process Z.

Property 2.1.2: Let us assume the independence Assumption A.1. holds. Then the Y process in calendar time is second order stationary if the following assumptions are satisfied:

Assumption A.2: Y^* is second order stationary: $m^*(z) = m^*,\ \forall z,\ \gamma^*(z_0, z) = \gamma^*(z),\ \forall z_0, z.$

Assumption A.3: The directing process has strongly stationary increments: the distribution of $\tilde{\triangle}_h Z_t = Z_{t+h} \Leftrightarrow Z_t$ is independent of t, $\forall h, t$.

A consequence of Property 2.1.2 is that we can have second order stationarity of the processes Y and Y^* simultaneously, as can be seen from Property 2.1.2. In such a case we get $m(t) = m^*, \gamma(t,h) = E\left[\gamma^*\left(\tilde{\triangle}_h Z_t\right)\right], Cov(Y_t, Z_{t+h}) = 0,\ \forall h$, and in particular we observe no correlation between the series Y and Z, while Y is a (stochastic) function of Z.

Another case of interest is that of a unit root in the calendar time process Y_t. Considering the case $\mathcal{Z} = \mathbb{N}$ we first discuss sufficient conditions for the second order stationarity of the differentiated process $\tilde{\triangle} Y_t = Y_{t+1} \Leftrightarrow Y_t$. Let us examine the first and second order moments of the increments of the underlying process Y^*:

$$E\left(Y^*_{z_0+z} \Leftrightarrow Y^*_{z_0}\right) = \mu^*(z_0, z), \tag{2.6}$$

$$Cov\left(Y^*_{z_0+z_1} \Leftrightarrow Y^*_{z_0}, Y^*_{z_0+z_2} \Leftrightarrow Y^*_{z_0+z_3}\right) = c^*(z_0, z_1, z_2, z_3). \tag{2.7}$$

Provided the independence Assumption A.1 holds, the first and second order moments of the differentiated process $\tilde{\triangle} Y_t$ are:

$$\mu(t) = E(Y_{t+1} \Leftrightarrow Y_t) = E\mu^*\left(Z_t, \tilde{\triangle} Z_t\right), \tag{2.8}$$

$$
\begin{aligned}
c(t,h) &= Cov\left(Y_{t+1} \Leftrightarrow Y_t,\ Y_{t+h+1} \Leftrightarrow Y_{t+h}\right) \\
&= E\ c^*\left(Z_t, \tilde{\triangle} Z_t\ ,\ \tilde{\triangle}_{h+1} Z_t\ , \tilde{\triangle}_h Z_t\right) \\
&+ Cov\left[\mu^*\left(Z_t, \tilde{\triangle} Z_t\right),\ \mu^*\left(Z_{t+h}, \tilde{\triangle} Z_{t+h}\right)\right].
\end{aligned} \tag{2.9}
$$

Both equations yield the following result:

Property 2.1.3 Let us assume the independence Assumption A.1 to hold. Then the process Y_t in calendar time is integrated of order 1, henceforth I(1), and second order stationary in first differences under the following set of assumptions:

Assumption A.5: Y^* is I(1) and second order stationary in first differences:

$\mu^*(z_0, z) = \mu^*(z)$, $\forall z_0, z, c^*(z_0, z_1, z_2, z_3) = c^*(z_1, z_2, z_3)$, $\forall z_0, z_1, z_2, z_3$.[2]

Assumption A.6: The time changes have strongly stationary trivariate increments, i.e. the distribution of $\left(\tilde{\triangle} Z_t,\ \tilde{\triangle}_h Z_t,\ \tilde{\triangle}_{h+1} Z_t\right)$ is independent of t.

As noted in Property 2.1.2, the calendar time process Y is stationary if Y^* is second order stationary and A.6 is satisfied. From Property 2.1.3, however, we can also deduce that for Y to be nonstationary it is necessary that both Y^* and Z are nonstationary.

Finally we can note that Assumption A.6 is satisfied for changing time processes defined by:

$$
Z_t = \sum_{i=0}^{t} \eta_i, \tag{2.10}
$$

where η_t is a strongly stationary process with positive values.

[2] The conditions on the moments of the differentiated processes might also have been written in terms of the moments of the initial processes. For instance the condition: $\mu^*(z_0, z) = \mu^*(z) \quad \forall z_0, z$, is equivalent to: $m^*(z_0 + z) - m^*(z_0) = \mu^*(z)\ \forall z_0, z$. Whenever m^* is continuous, this means that m^* has a linear affine form: $m^*(z) = az + b$.

2.2. Distributional properties

While it is natural to consider first second order properties, it is obviously also of interest to study distributional properties of the two processes Y and Y^*, like strong stationarity or Markov properties. A subsection is devoted to each of the properties.

2.2.1. Strong stationarity

Property 2.2.1 Let us assume again Assumption A.1 holds. Then the process in calendar time is strongly stationary under the two following conditions:

Assumption A.2': Y^* is strongly stationary.

Assumption A.6': The changing time has strongly stationary multivariate increments, i.e. the distribution of $\left(\tilde{\Delta}_{t_1} Z_t, \ldots, \tilde{\Delta}_{t_n} Z_t\right)$ is independent of t for any $t_1, \ldots, t_n$.

In the remainder of this section let us sketch the proof of this property in the case where Y^* and Z have discrete values. Then we get:

$$P\left[Y_{t_1} = y_1, \ldots, Y_{t_n} = y_n\right]$$

$$= \sum_{z_1, \ldots, z_n} P\left[Y_{t_1} = y_1, \ldots, Y_{t_n} = y_n,\ Z_{t_1} = z_1, \ldots, Z_{t_n} = z_n\right]$$

$$= \sum_{z_1, \ldots, z_n} P\left[Y^*_{z_1} = y_1, \ldots, Y^*_{z_n} = y_n\right] P\left[Z_{t_1} = z_1, \ldots, Z_{t_n} = z_n\right]$$

$$= \sum_{z_1, \ldots, z_n} P^*\left(y_1, \ldots, y_n;\ z_2 \Leftrightarrow z_1, \ldots, z_n \Leftrightarrow z_1\right) P\left[Z_{t_1} = z_1, \ldots, Z_{t_n} = z_n\right],$$

where P^* is the elementary probability associated with $Y^*_{z_1}, \ldots, Y^*_{z_n}$ and taking into account the strong stationarity of Y^*. Therefore we obtain:

$$P\left[Y_{t_1} = y_1, \ldots, Y_{t_n} = y_n\right] = E\left[P^*\left(y_1, \ldots, y_n; Z_{t_2} \Leftrightarrow Z_{t_1}, \ldots, Z_{t_n} \Leftrightarrow Z_{t_1}\right)\right],$$

and the result follows from the property that Z has strongly stationary increments.

2.2.2. Markov properties

Let us now consider two independent underlying processes Y^* and Z, each of them being Markovian of order one. Continuing with processes with discrete values for tractability purpose, we get:

$$P\left[Y_{t_n} = y_n, Z_{t_n} = z_n \mid Y_{t_{n-1}} = y_{n-1},\ Z_{t_{n-1}} = z_{n-1}, \ldots, Y_{t_1} = y_1, Z_t = z_1\right]$$

$$= P\left[Y^*_{z_n} = y_n, Z_{t_n} = z_n \mid Y^*_{z_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}, \ldots, Y^*_{z_1} = y_1, Z_{t_1} = z_1\right]$$

$$= P\left[Y^*_{z_n} = y_n \mid Y^*_{z_{n-1}} = y_{n-1}, \ldots, Y^*_{z_1} = y_1\right] P\left[Z_{t_n} = z_n \mid Z_{t_{n-1}} = z_{n-1}, \ldots, Z_{t_1} = z_1\right]$$

where the latter follows from Assumption A.1. Then using the Markovian properties we obtain:

$$= P\left[Y^*_{z_n} = y_n \mid Y^*_{z_{n-1}} = y_{n-1}\right] P\left[Z_{t_n} = z_n \mid Z_{t_{n-1}} = z_{n-1}\right].$$

Therefore the conditional distribution depends on the past values through the most recent ones $Y_{t_{n-1}}, Z_{t_{n-1}}$.

Property 2.2.2 Under Assumption A.1, if Y^* and Z are Markov processes of order one, then the joint process (Y, Z) is also a Markov process of order one.

It is well known that while the joint process (Y, Z) is Markovian it does not necessarily imply that the marginal process Y is also Markovian of order one. However, this property is satisfied under the following additional conditions.

Property 2.2.3: Let the conditions of Property 2.2.2 hold then if

Assumption A7: the conditional distribution of $Y^*_{z+z_0}$ given $Y^*_{z_0} = y_0$ only depends on (z, z_0) through z.[3]

[3] In the discrete case we note that $P(Y^*_{z_n} = y_n \mid Y^*_{z_{n-1}} = y_{n-1}) = P^*_n(y_n, y_{n-1}\,;\, z_n - z_{n-1})$

Assumption A8: The directing process has independent increments,

then Y is a Markov process of order 1.

Proof: Again for convenience let us give the proof for processes with discrete values. Consider:

$$P\left[Y_{t_n} = y_n \mid Y_{t_{n-1}} = y_{n-1}, \ldots, Y_{t_1} = y_1\right] = \frac{P\left[Y^*_{Z_{t_n}} = y_n, Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1\right]}{P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1\right]}$$

$$= \frac{E\,P\left[Y^*_{Z_{t_n}} = y_n, Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1}, \ldots, Z_{t_n}\right]}{E\,P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1}, \ldots, Z_{t_{n-1}}\right]}$$

$$= E\left\{P\left[Y^*_{Z_{t_n}} = y_n \mid Y^*_{Z_{t_{n-1}}} = y_{n-1}, Z_{t_1}, \ldots, Z_{tn}\right] \times\right.$$

$$\left.P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1} \ldots Z_{t_{n-1}}\right]\right\} \Big/$$

$$E\left\{P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1} \ldots Z_{t_{n-1}}\right]\right\}$$

$$= \frac{E\left\{P^*_n\left(y_n, y_{n-1}\ ;\ Z_{t_n} \Leftrightarrow Z_{t_{n-1}}\right) P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1}, \ldots, Z_{t_{n-1}}\right]\right\}}{E\ P\left[Y^*_{Z_{t_{n-1}}} = y_{n-1}, \ldots, Y^*_{Z_{t_1}} = y_1 \mid Z_{t_1}, \ldots, Z_{t_{n-1}}\right]}$$

The latter follows from Assumption A.7 while Assumption A.8 in turn yields:

$$= \quad E\ P^*_n\ \left(y_n, y_{n-1}\ ;\ Z_{t_n} \Leftrightarrow Z_{t_{n-1}}\right).$$

Q.E.D.

A byproduct of the proof is a formula of the transition kernel for the process Y , namely:

$$P\left(Y_{t_n} = y_n \mid Y_{t_{n-1}} = y_{n-1}\right) = E\,P^*_n\left(y_n, y_{n-1}\ ;\ Z_{t_n} \Leftrightarrow Z_{t_{n-1}}\right), \qquad (2.11)$$

and as usual we can verify in this case that:

$$P\left[Y_{t_n} = y_n \mid Y_{t_{n-1}} = y_{n-1}\right] = \; P\left[Y_{t_n} = y_n \mid Y_{t_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}\right].$$

Indeed from the proof of Property 2.2.2 we have:

$$P\left[Y_{t_n} = y_n \mid Y_{t_{n-1}} = y_{n-1}, Z_{t_{n-1}} = z_{n-1}\right]$$

$$= \sum_{z_n} \quad P_n^*\left(y_n, y_{n-1} \;;\; z_n \Leftrightarrow z_{n-1}\right) \; P\left[Z_{t_n} = z_n \mid Z_{t_{n-1}} = z_{n-1}\right]$$

$$= \sum_{\tilde{z}} \quad P_n^*\left(y_n, y_{n-1} \;;\; \tilde{z}\right) \; P\left[Z_{t_n} \Leftrightarrow Z_{t_{n-1}} = \tilde{z} \mid Z_{t_{n-1}} = z_{n-1}\right]$$

$$= \sum_{\tilde{z}} \quad P_n^*\left(y_n, y_{n-1} \;;\; \tilde{z}\right) \; P\left[Z_{t_n} \Leftrightarrow Z_{t_{n-1}} = \tilde{z}\right] = E\, P_n^*\left(y_n, y_{n-1} \;;\; Z_{t_n} \Leftrightarrow Z_{t_{n-1}}\right).$$

2.3. Diffusion processes

We now examine cases where the bivariate process (Y^*, Z) is described by a stochastic differential system. Unlike in the previous subsection we now assume $\Im = \mathcal{Z} = \mathbb{R}^+$, and the system is defined by:

$$\left\{ \begin{array}{l} dY_z^* = a^*\left(Y_z^*\right) dz + b^*\left(Y_z^*\right) dW_z^*, \\ \\ dZ_t = \alpha\left(Z_t\right) dt + \beta\left(Z_t\right) d\tilde{W}_t, \end{array} \right. \tag{2.12}$$

where (W_z^*) and $\left(\tilde{W}_t\right)$ are two independent Brownian motions. In the Appendix, we prove the following result:

Property 2.3.1: When the condition $\left[b^{*2}\left(Y_t\right)\alpha\left(Z_t\right) \Leftrightarrow a^{*2}\left(Y_t\right)\beta^2\left(Z_t\right)\right] > 0, \quad \forall t,$ *holds, then the bivariate process* (Y_t, Z_t) *satisfies the stochastic differential system:*

$$\begin{pmatrix} dY_t \\ dZ_t \end{pmatrix} = \begin{pmatrix} a^* (Y_t) \, \alpha (Z_t) \\ \alpha (Z_t) \end{pmatrix} dt + \sum\nolimits^{\frac{1}{2}} (Y_t, Z_t) \begin{pmatrix} dW_t^1 \\ dW_t^2 \end{pmatrix}, \tag{2.13}$$

where $(W_t^1), (W_t^2)$ *are two independent Brownian motions, and where the matrix* $\sum^{\frac{1}{2}}$ *may be taken equal to:*

$$\sum\nolimits^{\frac{1}{2}} = \begin{bmatrix} \left(b^{*2} (Y_t) \, \alpha (Z_t) \Leftrightarrow a^{*2} (Y_t) \, \beta^2 (Z_t)\right)^{\frac{1}{2}} & a^* (Y_t) \, \beta (Z_t) \\ 0 & \beta (Z_t) \end{bmatrix}.$$

Comparing the last lines of system (2.12) and of the system given in Property 2.3.1, we note that $W_t^2 = \tilde{W}_t$. Therefore we can write (2.12) in a recursive form:

$$\begin{cases} dY_t = a^* (Y_t) \, \alpha (Z_t) \, dt + \left[b^{*2} (Y_t) \, \alpha (Z_t) \Leftrightarrow a^{*2} (Y_t) \, \beta (Z_t)\right]^{\frac{1}{2}} dW_t^1 \\ \qquad\qquad + a^* (Y_t) \, \beta (Z_t) \, d\tilde{W}_t, \\ dZ_t = \alpha (Z_t) \, dt + \beta (Z_t) \, d\tilde{W}_t. \end{cases} \tag{2.14}$$

The condition $\left[b^{*2} (Y_t) \, \alpha (Z_t) \Leftrightarrow a^{*2} (Y_t) \, \beta^2 (Z_t)\right] > 0$ introduces some restrictions between the characteristics of the two differential equations in (2.12). Indeed, it automatically requires a strictly positive trend $\alpha (\cdot) > 0$ for the time changes, and a not *too large* volatility effect. These conditions are natural to ensure that Z_t is a strictly increasing process. It is in particular satisfied when: $\beta (z) = 0 \quad \forall z$, i.e. when:

$$dZ_t = \alpha (Z_t) \, dt \Longleftrightarrow Z_t = \textstyle\int_0^t \alpha_s d_s, \text{ where } \alpha_s = \alpha (Z_s).$$

It is worth noting that the restriction $\beta (z){=}0$ is often encountered in financial applications, it is considered for instance in Conley-Hansen-Luttmer-Scheinkman (1994), who are interested in estimating subordinated diffusions, or in Yor (1992a,b), Leblanc (1994), who considered a setup where α_s is an exponential of the Brownian motion $\tilde{W}$ plus a drift.

One should observe that the drift and the volatility terms in the bivariate system (2.13) are with respect to the information generated by both processes

(Y_t, Z_t). It may therefore be useful to examine the evolution of Y_t with respect to its own filtration only. The characterization of this marginal evolution is in general quite difficult, but it may be discussed in specific cases, one such case is considered in the remainder of this section.

We should first point out that Property 2.3.1 remains valid if the drift and volatility parameters for the stochastic differential equations are functions of current as well as past values of the process instead of just the current ones. This will be denoted as $a^*\left(\underline{Y}_z^*\right), b^*\left(\underline{Y}_z^*\right), \alpha\left(\underline{Z}_t, \underline{\tilde{W}}_t\right)$ and $\beta\left(\underline{Z}_t, \underline{\tilde{W}}_t\right)$. Let us now consider the case where $\beta\left(\underline{Z}_t, \underline{\tilde{W}}_t\right) = 0$, and $\alpha\left(\underline{Z}_t, \underline{\tilde{W}}_t\right) = \tilde{\alpha}\left(\underline{\tilde{W}}_t\right)$.

Then we can write:

$$dY_t = a^*\left(\underline{Y}_t\right)\alpha\left(\underline{Z}_t, \underline{\tilde{W}}_t\right)dt + b^*\left(\underline{Y}_t\right)\alpha\left(\underline{Z}_t, \underline{\tilde{W}}_t\right)^{\frac{1}{2}} dW_t^1,$$

$$dY_t = a^*\left(\underline{Y}_t\right)\tilde{\alpha}\left(\underline{\tilde{W}}_t\right)dt + b^*\left(\underline{Y}_t\right)\tilde{\alpha}^{\frac{1}{2}}\left(\underline{\tilde{W}}_t\right)dW_t^1,$$

Since $\left(W_t^1\right), \left(\underline{\tilde{W}}_t\right)$ are independent Brownian motions, we obtain the following equality in distribution:

$$dY_t = a^*\left(\underline{Y}_t\right)\tilde{\alpha}\left(\underline{W}_t\right)dt + b^*\left(\underline{Y}_t\right)\tilde{\alpha}^{\frac{1}{2}}\left(\underline{W}_t\right)dW_t, \tag{2.15}$$

where $\left(\underline{W}_t\right)$ is another Brownian motion. As expected it is something like an "autoregressive-moving average" formulation of the process Y_t, where the drift and volatility parameters both depend on the past values $\underline{Y}_t$ and on the past values of the Brownian motion, whose "increments" are the analogous of the centered reduced innovations.

3. Examples

We noted in the introduction to Section 2 that time deformed processes are only interesting if we can tackle complex structures via simpler ones thanks to the rescaling of time. It is therefore important to have "workable" examples which can be used in mathematical finance or in empirical estimation of discrete and/or diffusion processes. The examples described in this section will also serve as illustrations of the results established in the previous section. In this section we will elaborate on several examples, beginning with time changed Bessel processes in Section 3.1, Ornstein-Uhlenbeck processes in Section 3.2 and last but not least the time deformed random walk with drift.

3.1. Time deformed Bessel processes

This first class of processes has been studied extensively by Yor (1992a,b). While we omit all the specific details here, as they are treated by Yor, we would like to use the example of Bessel processes to further clarify the relation between (2.13) and (2.15). The initial model is:

$$\begin{cases} dY_z^* = \left(\gamma + \frac{\sigma^2}{2}\right)(Y_z^*)^{-1}\, dz + \sigma dW_z^* \\ dZ_t = \exp 2\left(\sigma \tilde{W}_t + \gamma t\right) dt, \end{cases} \tag{3.1}$$

where Y_z^* follows a Bessel process. System (3.1) is similar to that defined in (2.12) except that there are parametric restrictions which will be exploited shortly. Then, using (2.15) we obtain:

$$dY_t = \left(\gamma + \frac{\sigma^2}{2}\right) \exp 2\left(\sigma W_t + \gamma t\right) Y_t^{-1} dt + \sigma \exp\left(\sigma W_t + \gamma t\right) dW_t.$$

A solution of this stochastic differential equation can be written as:

$$Y_t = \exp\left[\sigma W_t + \gamma t\right] \Longleftrightarrow dY_t = \left(\gamma + \frac{\sigma^2}{2}\right) Y_t dt + \sigma Y_t dW_t,$$

which corresponds to a geometric Brownian motion with drift.

This example illustrates how simplifications arise because of the strong links introduced between the parameters defining the evolution of Y^* and the evolution of Z in (3.1). As noted before, this process has some useful applications in finance in the pricing of options. See in particular Geman and Yor (1993) and Leblanc (1994). The former study the pricing of Asian options while the latter examines option pricing in a stochastic volatility context.

3.2. Time deformed Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is of course the simplest example of a stationary continuous time process satisfying a diffusion equation. It will therefore be ideal for illustrating the properties discussed in Section 2. Moreover, it is worth noting that this type of process appears in continuous time finance applications particularly in stochastic volatility models. Ghysels and Jasiak (1994) for instance used a subordinated Ornstein-Uhlenbeck process to analyze the Hull-White stochastic volatility model with a time deformed evolution of the volatility process. We will first examine the autocovariance structure of a subordinated Ornstein-Uhlenbeck process and show how time deformation affects the temporal dependence of the process. Typically, in discrete calendar time such processes have an ARMA representation with uncorrelated, and yet nonlinearly dependent, innovations. We therefore examine also the nonlinear dependencies.

3.2.1. Definition of the process

We consider the one dimensional case: $n = 1$. The process Y^* is defined as the stationary solution of the stochastic differential equation:

$$dY_z^* = k\left(m - Y_z^*\right) dz + \sigma dW_z^*,\ k > 0, \sigma > 0, \tag{3.2}$$

where W^* is a Brownian motion indexed by $\mathcal{Z} = \mathbb{R}^+$, independent of the directing process. It is well known that Y^* is a Markov process of order one, and that the conditional distribution of $Y^*_{z+z_0}$ given $Y^*_{z_0}$ has a Gaussian distribution, with conditional mean:

$$E\left(Y^*_{z+z_0} \mid Y^*_{z_0}\right) = m + \rho^z \left(Y^*_{z_0} - m\right), \tag{3.3}$$

and conditional variance:

$$V\left(Y^*_{z+z_0} \mid Y^*_{z_0}\right) = \sigma^2 \frac{1 - \rho^{2z}}{1 - \rho^2}, \tag{3.4}$$

with: $\rho = \exp -k$. Let us now assume again the independence Assumption A.1 holds and that $\Im = \mathbb{N}$. Then the previous properties may be rewritten in calendar time as:

$$Y_t = m + \rho^{\Delta Z_t} \left(Y_{t-1} - m\right) + \left\{\sigma^2 \frac{1 - \rho^{2\Delta Z_t}}{1 - \rho^2}\right\}^{\frac{1}{2}} \epsilon_t, \tag{3.5}$$

where $\epsilon_t \sim I.I.N\left(0,1\right)$ and independent of Z, with $\Delta Z_t = Z_t - Z_{t-1}$.

Moreover, we also have a similar relation for lag h:

$$Y_t = m + \rho^{\Delta_h Z_t} \left(Y_{t-h} - m\right) + \left\{\sigma^2 \frac{1 - \rho^{2\Delta_h Z_t}}{1 - \rho^2}\right\}^{\frac{1}{2}} \epsilon_{h,t}, \tag{3.6}$$

where $\epsilon_{h,t} \sim N\left(0,1\right)$ and is independent of Z.

3.2.2. The autocovariance function

Now that we have formally defined the process, let us study its second order properties. This entails of course a study of the temporal dependence of the process as measured by the autocovariance function. We will study several cases where we can compare the temporal dependence of Y^* and that of Y. From Property 2.1.1, and the fact that $m^*\left(z\right) = m, \gamma^*\left(z\right) = \left(\sigma^2\rho^z\right)/\left(1 - \rho^2\right)$, we directly obtain that:

$$m\left(t\right) = m,$$

$$\gamma\left(t,h\right) = \frac{\sigma^2}{1 - \rho^2} E\left(\rho^{\tilde{\Delta}_h Z_t}\right).$$

It should be noted, however, that the second order properties of the Y and Y^* processes may be rather different. To clarify this let us first examine a particular case in which they are similar. This is accomplished via the following result:

Property 3.2.1: If Z is a strong random walk, independent of the Ornstein-Uhlenbeck process Y^*, then Y has a linear autoregressive representation of order 1.

Proof: From the definition of the autocorrelation we have:

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = E\left(\rho^{\hat{\Delta}_h Z_t}\right) = E\left[\rho^{\Delta Z_{t+h}+\ldots+\Delta Z_{t+1}}\right] = E\left(\rho^{\Delta Z_{t+h}}\right)\ldots E\left(\rho^{\Delta Z_{t+1}}\right)$$

$$= \left[E\left(\rho^{\Delta Z_{t+1}}\right)\right]^h = r^h, \text{ where } r = E\left(\rho^{\Delta Z_{t+1}}\right). \qquad \text{Q.E.D.}$$

Moreover, from the convexity inequality and the restriction $0 < \rho < 1$, we have:

$$0 \leq r = E\left(\rho^{\Delta Z_{t+1}}\right) \leq \rho^{E(\Delta Z_{t+1})} < 1.$$

Hence, the process in calendar time is weakly stationary with an autoregressive coefficient which is smaller than ρ, if $E\rho^{\Delta Z_{t+1}} > 1$. In fact the value of r depends on the distribution of ΔZ_{t+1}. To illustrate this, let us consider increments with a Pascal distribution with parameter π, $0 < \pi < 1$:

$$P\left[\Delta Z_{t+1} = n\right] = (1 \Leftrightarrow \pi)\,\pi^{n-1}, \quad n \geq 1.$$

So, that:

$$r = \sum_{n \geq 1} \rho^n \left(1 \Leftrightarrow \pi\right) \pi^{n-1} = \frac{\rho \left(1 \Leftrightarrow \pi\right)}{1 \Leftrightarrow \pi \rho}.$$

The effect of changing time is summarized in figure 3.1, where the autoregressive coefficient in calendar time is given as a function of π.

So far we focussed on a situation where Z_t is a (strong) random walk, as assumed in Property 3.2.1. Let us now consider a situation where Z_t is no longer a strong random walk, but ΔZ_t is still strongly stationary. Then we can still characterize the asymptotic behavior of the autocorrelation coefficient $\rho\left(\cdot\right)$. To do so let us denote:

$$,^{2} = \gamma\left(\Delta Z_t\right) + 2 \sum_{h=1}^{\infty} Cov\left(\Delta Z_t, \Delta Z_{t+h}\right). \tag{3.7}$$

For h large, using a central limit argument we have:

$$\tilde{\Delta}_h Z_t = \Delta Z_{t+1} + \ldots + \Delta Z_{t+h} \simeq N\left[hE\left(\Delta Z_t\right), h, ^{2}\right],$$

Exploiting this property yields:

$$\begin{aligned} \rho\left(h\right) &= E\left[\rho^{\tilde{\Delta}_h Z_t}\right] \simeq E\left(\rho^{hE(\Delta Z_t) + \sqrt{h}\Gamma u}\right), \text{ where } u \sim N\left(0,1\right), \\ &= r_{\infty}^{h}, \end{aligned}$$

where:

$$r_{\infty} = \exp\left[E\left(\Delta Z_t\right) \log \rho + \frac{,^{2} \left(\log \rho\right)^2}{2}\right]. \tag{3.8}$$

Hence, for large h, the process Y has approximately the same properties as an autoregressive process of order 1, with autoregressive coefficient r_{∞}. In particular we have a larger long range dependence in calendar time than in operational time if:

$$r_\infty > \rho \Leftrightarrow E\left(\Delta Z_t\right) \Leftrightarrow 1 + \frac{,^{2}}{2} \log \rho < 0. \tag{3.9}$$

This condition is automatically satisfied for $E\left(\Delta Z_t\right) < 1$, but it may also hold for $E\left(\Delta Z_t\right) > 1$, in particular if the variance and covariances $Cov\left(\Delta Z_t, \Delta Z_{t+h}\right)$ are sufficiently large.

We conclude this section by noting that the behavior of the entire autocorrelation function can only be accomplished under some simplifying assumptions regarding the temporal dependence of the ΔZ_t process. Let us for instance consider that ΔZ_t is a Markov chain, with a transition matrix M whose elements are:

$$m_{ij} = P\left[\Delta Z_t = j \mid \Delta Z_{t-1} = i\right],\ i, j = 1, \ldots, J \tag{3.10}$$

Then we obtain a model with a qualitative factor whose alternatives define J regimes. This model is quite similar to the stochastic switching regime in Hamilton (1989), except that here the effect of the factor is nonlinear. Suppose we denote by μ the invariant probability distribution associated with M, then the autocorrelation function is as follows:

$$\begin{aligned}
\rho\left(h\right) &= E\left(\rho^{\tilde{\Delta}_h Z_t}\right) = E\left[\rho^{\Delta Z_{t+h} + \ldots + \Delta Z_{t+1}}\right] \\
&= \sum_{\upsilon_1, \ldots, \upsilon_h} \left(\rho^{\upsilon_1 + \ldots + \upsilon_h}\right) m_{\upsilon_{h-1}, \upsilon_h} m_{\upsilon_{h-2}, \upsilon_{h-1}} \ldots m_{\upsilon_1, \upsilon_2} \mu\left(\upsilon_1\right) \\
&= \sum_{\upsilon_1, \ldots, \upsilon_h} \rho^{\upsilon_1} \mu\left(\upsilon_1\right) \rho^{\upsilon_2} m_{\upsilon_1, \upsilon_2} \ldots \rho^{\upsilon_h} m_{\upsilon_{h-1}, \upsilon_h}.
\end{aligned}$$

Consider now the matrix $M\left(\rho\right)$ whose general element is of the form: $\left[M\left(\rho\right)\right]_{i,j} = \rho^j m_{i,j}$, while $\mu\left(\rho\right)$ is the vector whose general component is: $\rho^i \mu\left(i\right)$, then:

$$\rho\left(h\right) = M\left(\rho\right)^h \mu\left(\rho\right). \tag{3.11}$$

Whenever the matrix $M(\rho)$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_J$ we can write the autocorrelation as:

$$\rho(h) = \sum_{j=1}^{J} \alpha_j \lambda_j^h \ , \ \alpha_j, \lambda_j \in \mathbb{C}.$$

This implies that the process Y has a linear ARMA$[J, J-1]$ representation, with autoregressive polynominal:

$$\Phi(L) = \prod_{j-1}^{J} (1 - \lambda_j L) = \det\left[Id - M(\rho) L\right],$$

which implies that time deformed process has longer lags than the process Y^* expressed in intrinsic time.

3.2.3. The conditional distribution

In the previous subsection we described how under some circumstances the process Y may have a linear ARMA representation. Yet, the innovations corresponding to such representation are generally uncorrelated but not white noise. In such a case it is of interest to have some information on the conditional distribution of Y_t given $Y_{t-1}, Y_{t-2,\ldots.}$ to capture the nonlinear dependencies.

To do this we shall focus on a situation where Z is a strong random walk. Following Property 2.2.3 we know that Y will be both strongly stationary and Markovian of order one. The conditional pdf can then be written as:

$$l(Y_t \mid Y_{t-1} = y_{t-1}) = \int \frac{1}{\sigma\left(\dfrac{1-\rho^{2\delta}}{1-\rho^2}\right)^{\frac{1}{2}}} \Psi\left[\frac{y_t - m - \rho^{\delta}(y_{t-1} - m)}{\sigma\left(\dfrac{1-\rho^{2\delta}}{1-\rho^2}\right)^{\frac{1}{2}}}\right] g(\delta)\, d\delta, \tag{3.12}$$

where Ψ is the pdf of the standard normal, and g the pdf of the time increments ΔZ_t. Since the conditional distribution in (3.12) is a complicated mixture of

Gaussian distributions with different means and variances we shall examine the conditional moments which turn out to be simpler. Namely, let us first consider the conditional expectation:

$$E\left(Y_t \mid Y_{t-1}\right) = E\left[E\left(Y_t \mid Y_{t-1}, \Delta Z_t\right) \mid Y_{t-1}\right] = E\left[m + \rho^{\Delta Z_t}\left(Y_{t-1} \Leftrightarrow m\right) \mid Y_{t-1}\right]$$

$$= m + E\left(\rho^{\Delta Z_t}\right)\left(Y_{t-1} \Leftrightarrow m\right) = m + r\left(Y_{t-1} \Leftrightarrow m\right).$$

The latter implies that the optimal prediction coincides with the linear regression. However, let us study the conditional variance:

$$V\left(Y_t \mid Y_{t-1}\right) = V\left[E\left(Y_t \mid Y_{t-1}, \Delta Z_t\right) \mid Y_{t-1}\right] + E\left[V\left(Y_t \mid Y_{t-1}, \Delta Z_t\right) \mid Y_{t-1}\right]$$

$$= V\left[m + \rho^{\Delta Z_t}\left(Y_{t-1} \Leftrightarrow m\right) \mid Y_{t-1}\right] + E\left[\sigma^2 \, \frac{1 \Leftrightarrow \rho^{2\Delta Z_t}}{1 \Leftrightarrow \rho^2}\right]$$

$$= \left(Y_{t-1} \Leftrightarrow m\right)^2 V\left(\rho^{\Delta Z_t}\right) + \sigma^2 \, \frac{1 \Leftrightarrow E\left(\rho^{2\Delta Z_t}\right)}{1 \Leftrightarrow \rho^2}.$$

Hence we note that, contrary to the underlying process Y^*, the process in calendar time features conditional heteroskedasticity. This was first noted by Stock (1988), who compared the behavior of the time deformed Ornstein-Uhlenbeck process in discrete time with ARCH processes. This feature of the Ornstein-Uhlenbeck process makes it of course particularly attractive since financial time series are known to exhibit volatility clustering. In the next section we will in fact examine a related feature, namely that of leptokurtic asset return distributions as a result of time deformation.

3.3. The subordinated random walk with drift

The last class of processes we would like to study as explicit example are random walks. Again to facilitate our discussion we divide the section in several subsections. Section 3.3.1 covers the continuous time case which is exploited in section 3.3.2 to illustrate applications in finance. Finally, section 3.3.3 deals with discrete time deformed random walks.

3.3.1. Definition of the process

We assume that the initial process is a (multivariate) random walk with drift:

$$dY_z^* = a^* dz + B^* dW_z, \tag{3.13}$$

where W_z is a standard Brownian motion. We immediately deduce from this that:

$$Y_t - Y_{t-1} = Y_{Z_t}^* - Y_{Z_{t-1}}^* = a^* \left(Z_t - Z_{t-1}\right) + B^* \left(W_{Z_t} - W_{Z_{t-1}}\right),$$

so that the first differenced process can be written as:

$$\Delta Y_t = a^* \Delta Z_t + \left(\Delta Z_t\right)^{\frac{1}{2}} B^* \epsilon_t, \tag{3.14}$$

where $\epsilon_t \sim I.I.N\left[0, Id\right]$.

Moreover, the first and second order moments of Y can directly be obtained from (3.14), namely:

$$E\left(\Delta Y_t\right) = a^* E\left(\Delta Z_t\right),$$

$$Cov\left(\Delta Y_t, \Delta Y_{t+h}\right) = a^* a^{*\prime} Cov\left(\Delta Z_t, \Delta Z_{t+h}\right) + E\left(\Delta Z_t\right) B^* B^{*\prime} \delta_0\left(h\right),$$

where $\delta_0\left(h\right)$ is the Kronecker symbol, and finally:

$$Cov\left(\Delta Y_t, \Delta Z_{t+h}\right) = a^* Cov\left(\Delta Z_t, \Delta Z_{t+h}\right).$$

3.3.2. A particular case: Z_t is a random walk with drift

The particular case where Z_t is a random walk is of special interest as it yields an easy characterization of leptokurtic features in the Y_t process. This can be put to use in the construction of optimal portfolios. We will first characterize the tail behavior before turning our attention to portfolio allocations. In particular it will be shown how optimal portfolio allocation depends on the information regarding the directing process Z_t. If Z_t is latent it will be shown that the optimal allocation rule will resemble one where there is no time deformation but where attitudes toward risk have been changed.

i) The leptokurtic effect

When Z_t is a strong random walk, with marginal pdf $g(\cdot)$, the process in calendar time is also a strong random walk with marginal pdf for the increments:

$$f(\Delta Y_t) = \int \frac{1}{(2\pi)^{-\frac{n}{2}}} v^{-\frac{n}{2}} \left(\det B^* B^{*\prime}\right)^{-\frac{1}{2}} \times$$

$$\exp \Leftrightarrow \frac{(\Delta Y_t \Leftrightarrow a^* v)' \left(B^* B^{*\prime}\right)^{-1} (\Delta Y_t \Leftrightarrow a^* v)}{2v} g(v)\, dv.$$

This pdf appears again as a mixture of Gaussian distributions, which will modify the tails of the distribution of (ΔY_t) compared to the tails of the distribution of (ΔY_t^*) (Mandelbrot (1963) among others stressed the importance of the tails of asset price distributions). In fact, Mandelbrot and Taylor (1967) used the framework of subordinated process to describe the fatness of the tails. Yet the result here may seem different from the usual one, where the introduction of heterogeneity in normal distributions implies heavier tails (see e.g. Clark (1970, 1973), Engle (1982)). The difference is a consequence of the fact that the time change not only affects the conditional variance, but also the conditional mean. It is important to note, however, that this observation also applies to examples with serial correlation and where one considers the conditional distribution $f(\Delta Y_t \mid \Delta Y_{t-1})$. Indeed it is known that after taking into account conditional linear dependence, conditional heteroskedasticity and the empirical kurtosis may be larger or smaller than that associated with the Gaussian distribution. To further explore this, let us compute the centered fourth order moment, for $n = 1$:

$$E\left[\Delta Y_t \Leftrightarrow E\Delta Y_t\right]^4 = E\left[a^* (\Delta Z_t \Leftrightarrow E\Delta Z_t) + (\Delta Z_t)^{\frac{1}{2}} B^* \epsilon_t\right]^4$$

$$= a^{*4} E \left(\Delta Z_t \Leftrightarrow E \Delta Z_t \right)^4 + 6 a^{*2} B^{*2} E \left[\left(\Delta Z_t \right) \left(\Delta Z_t \Leftrightarrow E \Delta Z_t \right)^2 \right] + 3 B^{*4} E \left(\Delta Z_t^2 \right).$$

It follows that:

$$E \left[\Delta Y_t \Leftrightarrow E \Delta Y_t \right]^4 \Leftrightarrow 3 \left[V \left(\Delta Y_t \right) \right]^2 = a^{*4} \left[E \left(\Delta Z_t \Leftrightarrow E \Delta Z_t \right)^4 \Leftrightarrow 3 V \left(\Delta Z_t \right)^2 \right]$$

$$+ 6 a^{*2} B^{*2} Cov \left[\Delta Z_t, \left(\Delta Z_t \Leftrightarrow E \Delta Z_t \right)^2 \right] + 3 B^{*4} V \left(\Delta Z_t \right).$$

Whenever the right hand side is nonnegative we have a kurtosis which is larger than 3. Such nonnegativity is immediate when $a^* = 0$, i.e. there is no heterogeneity in the conditional mean. In the general case the sign will depend on the kurtosis of the time increments ΔZ_t (in particular if a^* is large compared to B^*), and on the covariance term $Cov \left[\Delta Z_t, \left(\Delta Z_t \Leftrightarrow E \Delta Z_t \right)^2 \right]$.

The leptokurtic effect may be important as shown in the following example {see Feller (1957)}. Let us assume $n = 1$, $a^* = 0$, $B^* = 1$, and a time deformation with density :

$$g(v) = \frac{1}{\sqrt{2\pi} \sqrt{v^3}} \exp \Leftrightarrow \frac{1}{2v}.$$

Then the marginal pdf of Δy_t is :

$$f(\Delta y) = \int \frac{1}{2\pi} v^{-2} \exp \Leftrightarrow \frac{1}{2v} \left(1 + (\Delta y)^2 \right) dv = \frac{1}{\pi} \frac{1}{1 + (\Delta y)^2},$$

a Cauchy distribution, for which the first order moment does not exist.

ii) Comparison of optimal portfolios

In financial applications the subordinated random walk model may be used to facilitate the characterization of optimal portfolios. Let us assume that the components of Y_t are the log-prices of a set of financial assets, and that the short term interest rate is equal to zero.

We may determine two mean-variance optimal portfolios depending on whether we have or not information on time deformation. These optimal allocations are respectively:

$$\alpha_t(Z) = [V(\Delta Y_t \mid Z)]^{-1} E(\Delta Y_t \mid Z),$$

$$\alpha = V(\Delta Y_t)^{-1} E(\Delta Y_t).$$

Since the former is a function of Z_t it corresponds to the case where the portfolio allocation is an explicit function of an (observable) directing process. Replacing the moments by their explicit expressions, we have for the allocation rules using Z_t :

$$\alpha_t(Z) = \left(B^* B^{*'}\right)^{-1} a^*,$$

yielding a fixed composition of the optimal portfolio which is also equal to the composition in intrinsic time.

Now without the information on the directing time process we have the following allocation rule:

$$\begin{aligned}
\alpha &= \left[a^* a^{*'} V(\Delta Z_t) + E(\Delta Z_t) B^* B^{*'}\right]^{-1} a^* E(\Delta Z_t) \\
&= \left[a^* a^{*'} \frac{V(\Delta Z_t)}{E(\Delta Z_t)} + B^* B^{*'}\right]^{-1} a^* \\
&= \left[\left(B^* B^{*'}\right)^{-1} \Leftrightarrow \frac{V(\Delta Z_t)}{E(\Delta Z_t)} \frac{\left(B^* B^{*'}\right)^{-1} a^* a^{*'} \left(B^* B^{*'}\right)^{-1}}{1 + \frac{V(\Delta Z_t)}{E(\Delta Z_t)} a^{*'} \left(B^* B^{*'}\right)^{-1} a^*}\right] a^* \\
&= \left\{1 + \frac{V(\Delta Z_t)}{E(\Delta Z_t)} a^{*'} \left(B^* B^{*'}\right)^{-1} a^*\right]^{-1} \left(B^* B^{*'}\right)^{-1} a^*.
\end{aligned}$$

While this portfolio is proportional to $\alpha_t(Z)$ it can be seen that to correct for the lack of information, the agent has to modify his risk aversion. Suppose the risk aversion coefficient is η, when the information on Z is available. Then to obtain the same optimal portfolio allocation without information requires a risk aversion coefficient equal to:

$$\eta^* = \eta\left[1 + \frac{V(\Delta Z_t)}{E(\Delta Z_t)} a^{*\prime}\left(B^* B^{*\prime}\right)^{-1} a^*\right].$$

3.3.3. Subordinated random walk in discrete time

The two preceding sections dealt with continuous time models. Here we examine the discrete time case, hence $\Im = \mathcal{Z} = \mathbb{N}$. The subordinated random walk in discrete time can be defined as follows:

$$Y_z^* = \sum_{n=0}^{z} X_n, z \in \mathbb{N}, \tag{3.15}$$

where $(X_n, n \in \mathbb{N})$ is a sequence of i.i.d. random variables. It should be noted that, unlike the Gaussian innovations appearing in the continuous time model (3.13), we cover a much wider class of distributions in (3.15). The directing process is assumed to be with i.i.d. increments ΔZ_t, which amounts to a generalization in discrete time of the example considered in the previous section. We will first examine the general properties of such a process and then focus on a specific example yielding tractable formulas. The latter will be obtained for a Poisson mixture.

i) *General properties*

Let the subordinated process $Y_t = Y^*_{z_t}$ be defined following (3.15), namely:

$$Y_t = \sum_{n=0}^{Z_t} X_n = \sum_{n=0}^{Z_{t-1}} X_n + \sum_{n=Z_{t-1}+1}^{Z_t} X_n = Y_{t-1} + \sum_{n=Z_{t-1}+1}^{Z_t} X_n, \tag{3.16}$$

where we use the convention $\sum_{n=Z_{t-1}+1}^{Z_t} X_n = 0$, if $Z_t = Z_{t-1}$ or $\Delta Z_t = 0$. It follows from (3.16) that increments in Y_t, i.e. ΔY_t, are i.i.d. so that the subordinated process is also a strong random walk in calendar time.[4] Obviously, since both Y and Y^* are strong random walks we would like to compare the distributions governing the incremental processes in both cases. Such a comparison is typically performed via Laplace's second transforms.

For the univariate case they are defined by :

$$\Psi^*(u) = \log E(\exp uX_n) = \log E(\exp u\Delta Y_n^*),$$

$$\Phi(u) = \log E(\exp u\Delta Z_t),$$

$$\Psi(u) = \log E(\exp u\Delta Y_t).$$

Because of the structure of subordination, see for instance (2.3), we can also write:

$$\Psi(u) = \Phi[\Psi^*(u)].$$

To examine the relation between the increments of both random walks in operational and calendar time, we recall that Laplace's second transform can be expended as:

$$\Phi(u) = um + \frac{u^2}{2}m_2 + \frac{u^3}{6}m_3 + \frac{u^4}{24}(m_4 \Leftrightarrow 3m_2^2) + o(u^4),$$

where m is the mean and m_j denotes j^{th} centered moment. Replacing m and m_j by μ and μ_j or μ^* and μ_j^* yield similar expressions for $\Psi(\cdot)$ and $\Psi^*(\cdot)$. Moreover, the following holds:

$$\mu = m\mu^*,$$
$$\mu_2 = m\mu_2^* + m_2\mu^{*2},$$
$$\mu_3 = m\mu_3^* + 3m_2\mu^*\mu_2^* + m_3\mu^{*3},$$
$$\mu_4 \Leftrightarrow 3\mu_2^2 = m(\mu_4^* \Leftrightarrow 3\mu_2^{*2}) + 3m_2\mu_2^{*2} + 4m_2\mu^*\mu_3^* + 6m_3\mu^{*2}\mu_2^* + \mu^{*4}(m_4 \Leftrightarrow 3m_2^2).$$

The first term on the right hand side in each of the four expressions is obtained through $\sum_{n=1}^{E(\Delta Z_t)} X_n$, i.e. replacing Z_t by its expected values. The next terms specify the impact of the stochastic variation in Z_t on the moments. It is worth noting,

[4]This reaffirms the observation deduced from Property 2.1.3, that for Y to be nonstationary, it is necessary that both Y^* and Z are nonstationary.

for instance, that for a symmetric X_n process (i.e. $\mu_3^*=0$) one does not in general obtain a symmetrically distributed increment process with time deformation. Indeed, $\mu_3=3m_2\mu^*\mu_2^*+m_3\mu^{*3}$ is typically nonzero unless X has zero mean, i.e. $\mu^*=0$.

ii) Poisson mixtures

A specific case which is of practical interest is the Poisson mixture model where ΔZ_t has a Poisson$\mathcal{P}(\lambda)$ distribution [see Goovaerts et al. (1991)]. We also assume that the increments ΔY_t take positive integer values. Consider now the functions $\overline{\Psi}^*(u)=E\left(u^{\Delta Y_t^*}\right)$ and $\overline{\Psi}(u)=E\left(u^{\Delta Y_t}\right)$. It is easy to verify that:

$$\overline{\Psi}(u)=\exp\Leftrightarrow\lambda\left[1\Leftrightarrow\overline{\Psi}^*(u)\right], \tag{3.17}$$

and hence, differentiating (3.17) yields:

$$\frac{\partial\overline{\Psi}(u)}{\partial u}=\lambda\overline{\Psi}(u)\frac{\partial\overline{\Psi}^*(u)}{\partial u}. \tag{3.18}$$

The latter equation allows us to compute recursively the relationship between the elementary probabilities $p_i^*, i=0,1,\ldots$ associated with the distribution of increments ΔY_t^* and those associated with the distribution of ΔY_t, which will be denoted $p_i, i=0,1,\ldots$ Indeed, from (3.18) one obtains that:

$$\sum_{i=1}^{\infty}u^{i-1}ip_i=\lambda\sum_{j=0}^{\infty}u^jp_j\sum_{k=1}^{\infty}\mu^{k-1}kp_k^*. \tag{3.19}$$

Now, comparing the coefficients corresponding to the terms of the same order on both sides of (3.19) yields [See Panjer (1981)] :

$$p_i=\frac{\lambda}{i}\sum_{j=0}^{i-1}p_jp_{i-j}^*(i\Leftrightarrow j). \tag{3.20}$$

The formula (3.20) can be used for recursive calculations of the correspondence between p_i and p_i^*. This mapping is particularly useful in estimation since the likelihood function of $(\Delta Y_t, t=1,\ldots,T)$ has to be expressed in terms of the structural parameters λ and p_i^*.

4. Statistical inference for subordinated stochastic processes

In the previous two sections we discussed several elements of the theory of time deformed stochastic processes in discrete and continuous time. We also provided specific examples of processes that are potentially useful in financial time series modelling. We turn our attention now to statistical issues involving the estimation of subordinated stochastic processes. In a first subsection we describe the role played by the different parameters in a generic model with time deformation. The discussion of estimation is divided in two cases, a first one where the directing process Z_t is observable, which is treated in section 4.2, and a second situation where Z_t is latent. The latter is treated in section 4.3.

4.1. Parameters of interest

The analysis presented in the previous sections reveals that a generic model contains two types of parameters: (1) those characterizing the evolution of the directing process in intrinsic time, and (2) those corresponding to the time deformation. It is important to note that the knowledge of these two types of parameters is important in practice. Indeed let us for instance consider a problem of option pricing. Consider a European call in intrinsic time, with maturity H, strike price K and hence cash-flow $\left(Y^*_{z+H} \Leftrightarrow K\right)^+$. Furthermore assume that the complete model is given by the stochastic differential system (2.12). This system is driven by two independent Brownian motions, which will result in an incompleteness of the market, if only the price of the underlying asset Y^* is observed. To resolve this problem we may assume that the price of the option only depends on the current and past values of W^* , and not on the randomness specific to the time deformation. In such a case we have a unique price at z for this option: $P\left(z, Y^*_z, H, K\right)$, which will only depend on the parameters appearing in $a^{*\cdot}(\cdot)$ and $b^*(\cdot)$. Yet, we are interested of course in the pricing option in calendar time and not in intrinsic time. It is clear that the price of a European style call option $\left(Y^*_{t+H} \Leftrightarrow K\right)^+$ is necessarily $P\left[Z_t, Y_t, Z_{t+H} \Leftrightarrow Z_t, K\right]$. This price cannot be computed, however, when the directing process Z is unobserved. It will only be possible to approximate this price if we know the distribution of Z_t, Z_H, i.e. the parameters of the second equation in (2.12). In summary, this example stresses the importance of estimating all the parameters of the latent model and not just some subset.

It will be rather obvious that the estimation methods will depend on the information available regarding the process Z. We will distinguish two cases, in the first case the set of observable variables contain some variables in deterministic relationship with Z while in the second case no such variables will be available, resulting in Z being a completely unobservable factor.

4.2. Time deformation as a parametric function of observable processes

We will first look at processes where the time deformation is governed by a parametric function which is known up to some unknown parameters involving an observable process X_t. Namely, let us assume that:

$$Z_t = g_t\left(X_t; b\right), \tag{4.1}$$

where b is a parameter and X_t is a set of series like trading volume, bid-ask spreads, number of quotes, etc. Once the directing process is specified as in (4.1) we can proceed with estimating the vector b as well as the parameters governing the process Y_z^*. One can think of at least two estimation methods for estimating the parameters. A first one only exploits the second order properties of subordinated processes while the second one is based on a full characterization of the distributional properties via the maximum likelihood principle. A subsection is devoted to each method.

4.2.1. Estimation from empirical second order moments

In analogy with section 2 we first consider estimation only involving the second order moments of subordinated processes. Sufficient conditions for weak stationarity of subordinated processes where given in *Property 2.1.2* allowing us to estimate parameters through matching empirical and theoretical moments. To illustrate this, let us consider a time deformed Ornstein-Uhlenbeck process discussed in section 3.2. In particular, from section 3.2.2 we know that for the process defined by equation (3.2) with parameters m, γ and $\rho = \exp -k$, we have the following theoretical first and second moments for the marginal process Y_t in calendar time:

$$m\left(t\right) = m, \tag{4.2}$$

$$\gamma\left(t, h\right) = \sigma^2 \left(1 - \rho^2\right)^{-1} E\left(\rho^{\tilde{\Delta}_h g_t(X_t; b)}\right). \tag{4.3}$$

Hence, with a sufficient number of lags h we can identify the parameters m, σ, ρ as well as b. Consequently, using the empirical mean of Y_t and the empirical autocovariances, we can estimate the aformentioned parameters.

4.2.2. Maximum likelihood estimation

Let us suppose now that we provide a complete specification of the distributional properties to produce parameter estimates. In particular, let us assume that the two processes Y^* and X are independent and Markovian of order one. In such a case we have for discrete variables:

$$
\begin{aligned}
& P\left[Y_t = y_t, Z_t = z_t \mid Y_{t-1} = y_{t-1}, Z_{t-1} = z_{t-1}\right] \\
& = P\left[Y^*_{z_t} = y_t \mid Y^*_{z_{t-1}} = y_{t-1}\right] P\left[Z_t = z_t \mid Z_{t-1} = z_{t-1}\right],
\end{aligned}
$$

and a similar decomposition of the conditional pdf holds for continuous variables:

$$
\ell_t\left(y_t, z_t \mid y_{t-1}, z_{t-1}\right) = \ell^*_t\left(y_t \mid y_{t-1}; z_t, z_{t-1}\right) \tilde{\lambda}_t\left(z_t \mid z_{t-1}\right),
$$

where ℓ^* corresponds to the conditional distribution of Y^* and $\tilde{\lambda}$ to the conditional distribution of Z. Furthermore, we assume again that the available data are described by Y_t and X_t where the latter defines Z through (4.1). The process (Y_t, X_t) is Markovian of order one with its transition function given by:

$$
\ell^*_t\left(y_t \mid y_{t-1}; g_t\left(x_t\right), g_{t-1}\left(x_{t-1}\right)\right) \lambda_t\left(x_t \mid x_{t-1}\right),
$$

where λ_t is the conditional distribution of X. The model is completed by introducing a parametric specification for ℓ^*_t, λ_t and g. To characterize the likelihood function, let α, β and b denote the parameter vectors describing respectively ℓ^*_t, λ_t and g. Then, we have:

$$
L_T(\theta) = \prod_{t=1}^{T} \ell^*_t\left(y_t \mid y_{t-1}; g\left(x_t; b\right), g\left(x_{t-1}; b\right); \alpha\right) \prod_{t=1}^{T} \lambda_t\left(x_t \mid x_{t-1}; \beta\right). \tag{4.4}
$$

From (4.4) we note that the log likelihood function is the product of functions of (α, b) and of β. Therefore, the β parameter can be estimated using observation on X alone, and the ML estimators of the two subsets of parameters will be asymptotically independent.

We can proceed further with an illustrative example which, for the purpose of comparison, is the same as in subsection 4.2.1. Namely, consider again the Ornstein-Uhlenbeck process and suppose that :

$$g_t\left(X_t; b\right) = b_0 t + b_1 X_t. \tag{4.5}$$

Therefore $\Delta Z_t = b_0 + b_1 \Delta X_t$, and the evolution of Y_t conditional to X_t is summarized by:

$$Y_t = m + \rho^{b_0 + b_1 \Delta X_t}\left(Y_{t-1} \Leftrightarrow m\right) + \left[\sigma^2 \frac{1 \Leftrightarrow \rho^{2(b_0 + b_1 \Delta X_t)}}{1 \Leftrightarrow \rho^2}\right]^{\frac{1}{2}} \epsilon_t, \tag{4.6}$$

where ϵ_t is standard Gaussian white noise. We observe immediately that the parameters are not identifiable, and that we must impose some identifying constraint, such as $b_0 = 1$. Then the conditional likelihood becomes:

$$\ell_t^*\left(y_t \mid y_{t-1}; g_t\left(x_t; 1, b_1\right), g_{t-1}\left(x_{t-1}; 1, b_1\right), \alpha\right)$$

$$= (2\pi)^{-\frac{1}{2}} \left[\sigma^2 \ \frac{1 \Leftrightarrow \rho^{2(1+b_1 \Delta X_t)}}{1 \Leftrightarrow \rho^2}\right]^{-\frac{1}{2}} \exp \Leftrightarrow \frac{1}{2} \frac{\left[y_t \Leftrightarrow m \Leftrightarrow \rho^{1+b_1 \Delta X_t}\left(y_{t-1} \Leftrightarrow m\right)\right]^2}{\sigma^2 \ \dfrac{1 \Leftrightarrow \rho^{2(1+b_1 \Delta X_t)}}{1 \Leftrightarrow \rho^2}}.$$

Finally, it is also worth noting that the corresponding log likelihood function can easily be concentrated with respect to m, σ^2.

4.3. Estimation with latent directing processes

It should come as no surprise that the task of estimating subordinated stochastic processes with latent directing processes is considerably more difficult. We no longer assume that Z_t is observable through X_t via the parametric mapping $g_t(., b)$. Instead we must uncover Z_t through the sample behavior of Y_t. Once again we can draw a distinction between a method of moments approach, though not necessarily limited to second order properties, and a maximum likelihood approach. Since we are dealing with latent processes it might be more useful to organise our discussion on the basis of a different attribute. Indeed, we will first study a class of estimators which do not involve simulations of the latent Z process. Such is for example the case for the continuous time generalized method of moments (henceforth GMM) approach proposed by Hansen and Scheinkman (1994) and recently adapted by Conley et al. (1994) to subordinated diffusions. We shall review this method and in particular show the limitations it imposes to class of time deformed processes we can possibly estimate with such a method. In fact, the continuous time GMM procedure seems to only apply to a restrictive set of circumstances where Z is only governed by a deterministic drift. To estimate a wider class, containing many processes of interest in finance, we must entertain the possibility of simulating the process Z and use simulation-based methods discussed in Duffie and Singleton (1993), Gouriéroux, Monfort and Renault (1993), Gallant and Tauchen (1993) and Gouriéroux and Monfort (1994). A first subsection is devoted to the continuous time GMM estimator of Conley et al. (1994) while a second covers the simulation-based estimators for subordinated processes.

4.3.1. Method of Moments using Infinitesimal Operators

Hansen and Scheinkman (1994) proposed to estimate continuous time diffusions through the GMM principle. We will first discuss the principle of the estimation procedure and then elaborate on the identification of parameters. Finally, we will concentrate on a very special case where the directing process is predetermined, i.e. its path is not affected by the randomness of a Brownian motion. The discussion of identification issues will show that it is the latter rather restrictive case only which can be treated by the GMM.

(a) Moment Conditions for Diffusions

To describe the generic setup of the continuous time GMM estimator, let us consider the following multivariate system of diffusion equations:

$$dy_t = \mu_\theta (y_t)\, dt + \sigma_\theta (y_t)\, dW_t, \tag{4.7}$$

where W_t is a standard n-dimensional brownian motion and $Y_t \in \mathbb{R}^n$. The parameters in (4.7) are described by the vector $\theta \in \mathbb{R}^p$. Hansen and Scheinkman (1994) consider the infinitesimal operator A defined for a class of square integrable functions $\varphi:\mathbb{R}^n \to \mathbb{R}^d$ as follows:

$$A_\theta \varphi (y) = \frac{d\varphi (y)}{dy'} \mu_\theta (y) + \frac{1}{2} Tr \left(\sigma_\theta (y)\, \sigma_\theta' (y) \frac{d^2 \varphi (y)}{dy dy'} \right). \tag{4.8}$$

Because the operator is defined as a limit, namely :

$$A_\theta \varphi (y) = \lim_{t \to 0} t^{-1} [\mathbb{E} (\varphi(y_t) |\, y_o = y) \Leftrightarrow y],$$

it does not necessarily exist for all square integrable functions φ but only for a restricted domain D. A set of moment conditions can now be obtained for this class of functions $\varphi \in D$. Indeed, as shown for instance by Revuz and Yor (1991), the following equalities hold :

$$E A_\theta \varphi (y_t) = 0, \tag{4.9}$$

$$E \left[A_\theta \varphi (y_{t+1})\, \tilde{\varphi} (y_t) \Leftrightarrow \varphi (y_{t+1})\, A_\theta^* \tilde{\varphi} (y_t) \right] = 0, \tag{4.10}$$

where A_θ^* is the adjoint infinitesimal operator of A_θ for the scalar product associated with the invariant measure of the process y.[5] By choosing an appropriate set of functions, Hansen and Scheinkman exploit moment conditions (4.9) and (4.10) to construct a GMM estimator of θ.

Conley, Hansen, Luttmer and Scheinkman (1994) extended the previous approach to deal with subordinated processes. In particular let us consider the

[5]Please note that with A_θ^* is again associated a domain D^* so that $\varphi \in D$ and $\tilde{\varphi} \in D^*$ in (4.10).

system of diffusions described in section 2.3. To simplify the presentation let us only concentrate on the set of marginal moment conditions defined in (4.9), leaving aside those in (4.10). The infinitesimal operator argument applied to the joint process $y_t = (Y_t, Z_t)^{'}$ yields:

$$
\begin{aligned}
A_\theta \varphi \begin{pmatrix} y \\ z \end{pmatrix} &= \begin{bmatrix} a_\theta^* (y) \, \alpha_\theta (z) \\ \alpha_\theta (z) \end{bmatrix} \begin{bmatrix} \dfrac{\partial \varphi}{\partial y} (y, z) \\ \dfrac{\partial \varphi}{\partial z} (y, z) \end{bmatrix} \\
&+ \tfrac{1}{2} \, Tr \left\{ \begin{bmatrix} b_\theta^{*^2} (y) \, \alpha_\theta (z) & a_\theta^* (y) \, \beta_\theta^2 (z) \\ a_\theta^* (y) \, \beta_\theta^2 (z) & \beta_\theta^2 (z) \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \varphi}{\partial y^2} (y, z) & \dfrac{\partial^2 \varphi}{\partial y \partial z} (y, z) \\ \dfrac{\partial^2 \varphi}{\partial y \partial z} (y, z) & \dfrac{\partial^2 \varphi}{\partial z^2} (y, z) \end{bmatrix} \right\}
\end{aligned} \tag{4.11}
$$

For subordinated diffusions this is not the only infinitesimal operator we can (and should) introduce. Indeed, we can define an infinitesimal operator for the marginal process Y_t in calendar time as soon as it marginally satisfies a univariate diffusion equation (see for instance with $\beta(Z_t) = 0$ as in equation (2.15)) or even an operator associated with Y_z^*, i.e. with the operational time diffusion. From each of the infinitesimal operators associated with the joint process, as in (4.11), or the marginal process in calendar time, or the operational time diffusion Y_z^*, we can define a set of moment conditions similar to (4.9) (and of course also (4.10) not considered here) and all these conditions may be combined.

(b) Moment Conditions and Parameter Identification

While parameter estimation via GMM is relatively straightforward there is the common and well-known point that moment conditions may pose identification problems. In a continuous time GMM framework we construct moment conditions via an appropriate choice of functions φ belonging to the domain of the operator. However, further restrictions on φ must be imposed when the diffusion y_t is only partly observable. As emphasized by Gouriéroux and Monfort (1994), for a large class of diffusions encountered in finance, particularly stochastic volatility models, one often cannot identify the latent parameters governing the dynamics of y.

Indeed to construct moment conditions with an empirical counterpart we must restrict the choice of φ to functions only involving observable transformations of y. Since we are dealing with a situation where Z_t is latent, this problem is of course encountered here as well. Consider the moment conditions :

$$E\left[A_\theta \varphi \binom{Y_t}{Z_t}\right] = 0, \tag{4.12}$$

where it is assumed that the functions φ are independent of the parameter θ. We may only consider the ones where $A_\theta \varphi \binom{Y_t}{Z_t}$ only depend on Y_t for any θ. As soon as the parameterization does not introduce links between the functions a_θ^*, b_θ^*, α_θ and β_θ defining the diffusions we deduce from (4.11) that we must restrict the class of functions to the one satisfying :

$$\begin{aligned}
&a_\theta^*(y)\,\alpha_\theta(z)\,\frac{\partial\varphi}{\partial y}(y,z)\,,\,\alpha_\theta(z)\,\frac{\partial\varphi}{\partial z}(y,z)\,,\,b_\theta^{*^2}(y)\,\alpha_\theta(z)\,\frac{\partial^2\varphi}{\partial y^2}(y,z)\,,\\
&a_\theta^*(y)\,\beta_\theta^2(z)\,\frac{\partial^2\varphi}{\partial y\partial z}(y,z)\,,\,\beta_\theta^2(z)\,\frac{\partial^2\varphi}{\partial z^2}(y,z) \text{ being all } \textit{independent of } z.
\end{aligned}$$

This yields the following restrictions on the class of admissible functions.

(1) Since $\left[b_\theta^{*^2}(y)\,\alpha_\theta(z)\,\dfrac{\partial^2\varphi}{\partial y^2}(y,z)\right] \Big/ \left[a_\theta^*(y)\,\alpha_\theta(z)\,\dfrac{\partial\varphi}{\partial y}(y,z)\right]$ has to be independent of z, we deduce that $\dfrac{\partial}{\partial y}\log\dfrac{\partial\varphi}{\partial y}(y,z)$ has also to satisfy this condition. Therefore :

$$\varphi(y,z) = G(y)\,f(z) + C(z)\,. \tag{4.13}$$

(2) Furthermore since $\alpha_\theta(z)\,\partial\varphi(y,z)/\,\partial y$ has to depend only on y one obtains from (4.13) that $f(z) = k\,(\alpha_\theta(z))^{-1}$ and therefore:

$$\varphi(y,z) = kG(y)\,(\alpha_\theta(z))^{-1} + C(z)\,, \tag{4.14}$$

(3) Similarly, $\alpha_\theta(z)\,\partial\varphi(y,z)/\,\partial z$ must be function of y only and hence:

$$\varphi(y,z) = \Leftrightarrow k d\alpha_\theta(z)/\,dz\;G(y)\,(\alpha_\theta(z))^{-1} + \alpha_\theta(z)\;dC(z)/\,dz. \tag{4.15}$$

Using the arguments in (1) through (3) one constraints the choice of φ. Two cases may be distinguished :

i) If $G(y)$ is not constant, it is necessary to choose $C(z)$ constant, and this choice is only valid if $[\alpha_\theta(z)]^{-1} d\alpha_\theta(z)/dz$ is constant.

ii) If $G(y)$ is a constant, it is necessary that $[\alpha_\theta(z)]^{-2} d\alpha_\theta(z)/dz$ is independent of θ.

These constraints are extremely restrictive since they impose conditions on the dynamics of the underlying processes. Therefore it seems difficult to construct moment conditions that will identify all elements of the parameter vector θ, except in some very special circumstances.

(c) Predetermined latent directing processes.

One special case, the one implicitly treated by Conley et al. (1994), is where the directing process Z_t satisfies:

$$dZ_t = \alpha_\theta(Z_t)\, dt, \tag{4.16}$$

and hence $\beta_\theta(Z_t) = 0$. Recall from the discussion in section 2.3 that in such a case one can also derive a diffusion for the marginal process (Y_t) as described by (2.15). Now the moment conditions (4.11) greatly simplify and amount to:

$$\begin{aligned} EA_\theta\varphi\binom{Y_t}{Z_t} = E\Big[\alpha_\theta(Z_t)\Big[a_\theta^*(Y_t)\tfrac{\partial\varphi}{\partial y}(Y_t, Z_t) + \tfrac{\partial\varphi}{\partial z}(Y_t, Z_t) \\ + \tfrac{1}{2}(b_\theta^*(Y_t))^2\tfrac{\partial^2\varphi}{\partial y^2}(Y_t, Z_t)\Big]\Big] = 0. \end{aligned} \tag{4.17}$$

Following Conley et al. (1994) let us consider now functions separable in y and z, i.e. $\varphi(y, z) = \varphi_0(y)\,\varphi_1(z)$. Then (4.16) further simplifies to:

$$
\begin{aligned}
EA_\theta\varphi\begin{pmatrix} Y_t \\ Z_t \end{pmatrix} = \;& E\left[\alpha_\theta\left(Z_t\right)\varphi_1\left(Z_t\right)\left\{a_\theta^*\left(Y_t\right)\frac{d\varphi_0\left(Y_t\right)}{dy}+\tfrac{1}{2}\left(b_\theta^*\left(Y_t\right)\right)^2\frac{d^2\varphi_0\left(Y_t\right)}{dy^2}\right]\right. \\
&+E\left[\alpha_\theta\left(Z_t\right)\frac{d\varphi_1\left(Z_t\right)}{dz}\varphi_0\left(Y_t\right)\right].
\end{aligned}
\tag{4.18}
$$

From the infinitesimal operator associated with the changing time process in (4.16) we obtain that

$$
E\left[\alpha_\theta\left(Z_t\right)\frac{d\varphi_1\left(Z_t\right)}{dz}\right]=0, \tag{4.19}
$$

for all φ_1 belonging to the appropriate domain.

Therefore we deduce from (4.19), that

$$
\begin{aligned}
&E\left[\alpha_\theta\left(Z_t\right)\frac{d\varphi_1\left(Z_t\right)}{dz}\varphi_0\left(Y_t\right)\right] \\
&= E\left\{\alpha_\theta\left(Z_t\right)\frac{d\varphi_1}{dz}\left(Z_t\right)E\left(\varphi_0\left(Y_t\right)\middle| Z_t\right)\right\}=0.
\end{aligned}
$$

Then the condition (4.18) implies :

$$
E\left[\alpha_\theta\left(Z_t\right)\varphi_1\left(Z_t\right)\left(a_\theta^*\left(Y_t\right)\frac{d\varphi_0\left(Y_t\right)}{dy}+\frac{1}{2}\left[b_\theta^*\left(Y_t\right)\right]^2\frac{d^2\varphi_0\left(Y_t\right)}{dy^2}\right)\right]=0, \forall\varphi_0,\varphi_1,
$$

which is equivalent to :

$$
E\left[a_\theta^*\left(Y_t\right)\frac{d\varphi_0\left(Y_t\right)}{dy}+\frac{1}{2}\left[b_\theta^*\left(Y_t\right)\right]^2\frac{d^2\varphi_0\left(Y_t\right)}{dy^2}\middle| Z_t\right]=0, \forall\varphi_0,
$$

and by integrating on Z_t :

$$
E\left[a_\theta^*\left(Y_t\right)\frac{d\varphi_0\left(Y_t\right)}{dy}+\frac{1}{2}\left[b_\theta^*\left(Y_t\right)\right]^2\frac{d^2\varphi_0\left(Y_t\right)}{dy^2}\right]=0, \forall\varphi_0. \tag{4.20}
$$

4.3.2. Simulation-based estimators

In general, the estimation problem is much more complicated with a latent directing process Z_t because the observable log likelihood, corresponding to $Y_1 \dots Y_r$ is now derived by integrating out the unobservable path of Z:

$$L_T^*(y_1, \dots, y_T \mid y_0, z_0) = \int \cdots \int \prod_{t=1}^{T} (\ell_t^*(y_t \mid y_{t-1}; z_t, z_{t-1}) \tilde{\lambda}_t(z_t \mid z_{t-1}) dz_t). \quad (4.21)$$

The presence of such multiple integrals inside the likelihood function is common in many empirical models for financial data. The best examples are stochastic volatility models. Statistical inference for such processes can be based on simulated estimation methods (Duffie-Singleton (1990), Gouriéroux-Monfort-Renault (1993), Gallant-Tauchen (1992), Gouriéroux-Monfort (1994)).

In recent years considerable advances were made in this area. Since simulation of a subordinated process with latent Z_t is a special case of the estimation problems treated by this class of simulation-based estimators it is a relatively straight forward application of the available theory. It may be noted here that Ghysels and Jasiak (1994) provide a specific example of such an estimator applied to a subordinated stochastic volatility model.

5. Testing the hypothesis of time deformation

In this final section we treat the problem of hypothesis testing, specifically focusing of course on testing for time deformation. We shall first consider diagnostic tests which are easy to apply. They are based on either a modified study of the correlogram in calendar time, or a direct estimation of the correlogram in deformed time. The methods assume some direct or indirect observations of the changing time process. Finally, a second section deals with the problem of testing restrictions regarding subordination in a parametric setting.

5.1. Descriptive diagnostic tools

5.1.1. Use of the calendar time correlogram

Before entertaining the possibility of modelling a time series via an operational time setup it is useful to have some simple diagnostic tests at hand designed to detect the need for such a specification. The first test we propose has features which are quite similar to tests for ARCH effects. Indeed, while the class of ARCH processes is quite large one typically constructs a diagnostic test for ARCH effects only on the basis of a simple ARCH(q) representation (see for instance Engle (1982)). Here we will also start from a simple structure to capture features belonging to a wider class of time deformed processes. The development of the test is based on the Ornstein-Uhlenbeck model, i.e. the first order autoregressive model in continuous time. From the discussion in section 3.2.2 we know that such a process, when subordinated to Z_t, has the following autocorrelation structure:

$$\rho(h) = E\left(\rho^{\tilde{\Delta}_h Z_t}\right). \tag{5.1}$$

To conduct the test we need to assume that the time deformation is related to an observable process X_t as in (4.1). In particular let us consider the linear function, as in (4.5):

$$\tilde{\Delta}_h Z_t = b_0 + b\ \tilde{\Delta}_h X_t. \tag{5.2}$$

To construct the test we will use an approximation to the expected value in (5.1), neglecting the randomness of $\tilde{\Delta}_h X_t$:

$$\log \rho(h) \approx c_0 + cE\tilde{\Delta}_h X_t, \text{ where } c_0 = b_o \log \rho, c = b \log \rho. \tag{5.3}$$

The result in (5.3) yields a formula which can be easily exploited once $\rho(h)$ is replaced by its sample analogue and $E\tilde{\Delta}_h X_t$ replaced by the corresponding sample average. It suggests to display graphically $\log \hat{\rho}(h)$, i.e. the log of the empirical autocorrelation, against empirical averages of $\tilde{\Delta}_h X_t$ for different lags $h = 1, 2, \ldots$ If $b \neq 0$ we should observe a slope pattern on the graph, as displayed in Figure 5.1. This deformed time correlogram extends the usual correlogram which corresponds to calendar time and for which $\tilde{\Delta}_h X_t$ is proportional to the lag h.

The significance of the slope coefficient b, which amounts to a time deformation scheme, might also be formally tested.

5.1.2. Estimation of the intrinsic time correlogram

In section 2.1 we discussed the second order properties of subordinated stochastic processes. We examined the autocovariance functions for Y and Y^*, which appear in (2.5). In this section we propose estimators for $\gamma(h)$ and $\gamma^*(z)$ under the assumption that Property 2.1.2 holds. Let us first recall that the empirical autocovariance function for a zero mean calendar time process can be written as:

$$\hat{\gamma}_T(h) \quad = \frac{1}{T}\sum_{t=1}^{T} Y_t Y_{t+h} = \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \mathbf{1}_{t-\tau=h}$$

$$\approx \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \mathbf{1}_{t-\tau=h}}{\sum_{t=1}^{\tau}\sum_{\tau=1}^{T} \mathbf{1}_{t-\tau=h}} \quad .$$

This formulation of the empirical autocovariance function would suggest an estimator for $\gamma^*(z)$. The difficulty is that the autocovariance function γ^* is defined on the real line, whereas we have only a finite number of observations Z_t, $t = 1$, ..., T, therefore a small number of pairs (Z_t, Z_τ) such that $Z_t \Leftrightarrow Z_\tau = z$ given. This forces us to rely on smoothing through a kernel namely:

$$\hat{\gamma}_T^*(z) = \frac{\sum_{t=1}^{T}\sum_{\tau=1}^{T} Y_t Y_\tau \frac{1}{h_T} K\left[\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T}\right]}{\sum_{t=1}^{T}\sum_{\tau=1}^{T} \frac{1}{h_T} K\left[\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T}\right]}, \tag{5.4}$$

where h_T is a bandwidth, depending on the size of the sample, and K is a kernel function.

A detailed analysis of the asymptotic properties of this nonparametric estimator is beyond the scope of the paper. We provide, however, some discussion of the form of the asymptotic first and second order moments of the estimator in Appendix 2.

The nonparametric estimator has, as shown in the appendix, the following asymptotic behavior :

i) $\hat{\gamma}_T^*(z)$ is a.s. consistent of $\gamma^*(z)$;

ii) It is asymptotically normal :

$$\sqrt{T h_T \left[\hat{\gamma}_T^*(z) \Leftrightarrow \gamma^*(z)\right]} \quad \overset{d}{\Leftrightarrow} \quad N\left(0, \quad \frac{\int K^2(\theta)\, d\theta}{2\sum_{n=1}^{\infty} f_n(z)} \qquad Var\left(Y_{z_0}^* Y_{z_0+z}\right)\right),$$

where f_n is the *pdf* of $Z_{t+n} \Leftrightarrow Z_t$. It is interesting to note that,

$$V\left(Y_{z_0}^* Y_{z+z_o}\right) = V\left[Y_{z_0}^* E\left(Y_{z+z_0}^* \mid Y_{z_0}^*\right)\right] + E\left[Y_{z_0}^{*2} V\left(Y_{z+z_0}^* \mid Y_z^*\right)\right],$$

and therefore the asymptotic precision of this estimator depend on the conditional first and second order moments of Y^*.

5.2. Testing for time deformation in parametric models

Besides diagnostic tests we turn our attention now to parametric models where the null hypothesis of subordination is being considered for testing. In section 4 we noted that there is an important distinction to be made between a situation where Z_t is latent and one where it isn't. We will therefore distinguish these two cases when discussing hypothesis testing.

5.2.1. Parametric models with observable directing processes

Let us consider the maximum likelihood estimator discussed in section 4.2.2. The likelihood function as formulated in (4.4) has a parameter vector $\theta = (\alpha, b, \beta)$ where b determines the mapping between the observable series X_t and the directing process Z_t. For the purpose of hypothesis testing, let us specify the time deformation (4.1) such that :

$$g_t\left(X_t; b\right)|_{b=0} = t. \tag{5.5}$$

It is for instance the case in the illustrative example given in (4.5) $g_t\left(X_t; b\right) = b_0 t + b_1 X_t$, with the identifying restriction $b_0 = 1$. The test of the hypothesis $Z_t = t$ may be performed by a Lagrange multiplier procedure based on the score: $\left[\partial \log L_T^*\left(\theta\right) / \partial b\right]_{\theta=\hat{\theta}_0}$, where $\hat{\theta}_o$ is the constrained ML estimator and $\theta \equiv (\alpha, b, \beta)$. As an illustration let us consider again the time deformed Ornstein-Uhlenbeck process and $g_t\left(X_t, b\right) = t + bX_t$. It can be shown that:

$$T^{-1} \frac{\partial \log L_T^*\left(\theta\right)}{\partial b}\bigg|_{\theta=\hat{\theta}_o} \approx \frac{1}{\hat{\gamma}_0^2} Cov_e\left(\Delta x_t, \hat{\epsilon}_{ot}^2\right) + \frac{1 \Leftrightarrow \hat{\rho}_0^2}{\hat{\rho}_0 \hat{\gamma}_0^2} Cov_e\left(\Delta x_t \left(y_{t-1} \Leftrightarrow \hat{m}_0\right), \hat{\epsilon}_{ot}\right), \tag{5.6}$$

where: $\hat{m}_0, \hat{\rho}_0,\quad \hat{\epsilon}_{ot} = y_t \Leftrightarrow \hat{m}_0 \Leftrightarrow \hat{\rho}_0\left(y_{t-1} \Leftrightarrow \hat{m}_0\right),\ \hat{\sigma}_0^2 = \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_{ot}^2$, are the constrained ML estimators and the constrained residuals. Consequently the score statistic contains two different terms: the first one $Cov_e\left(\Delta x_t, \hat{\epsilon}_{ot}^2\right)$ is useful for testing the presence of conditional heteroskedasticity in the direction Δx_t, the second one $Cov_e\left(\Delta x_t\left(y_{t-1} \Leftrightarrow \hat{m}_0\right), \hat{\epsilon}_{ot}\right)$ for testing the omission of $\Delta x_t\left(y_{t-1} \Leftrightarrow m\right)$ in the conditional mean. This double local effect of time deformation is easy to understand intuitively when we study the expansion of the regression model of Y_t given $Y_{t-1}, \Delta X_t$ in a neighborhood of the null hypothesis i.e., when b is small. Indeed we have, from (4.6) :

$$Y_t \approx m + \left(\rho + \rho b \Delta X_t \log\rho\right)\left(Y_{t-1} \Leftrightarrow m\right) + \left\{\frac{\sigma^2}{1 \Leftrightarrow \rho^2}\left(1 \Leftrightarrow \rho^2\left[1 + 2b\Delta X_t \log\rho\right]\right)\right\}^{\frac{1}{2}} \epsilon_t$$

$$\approx m + \rho\left(Y_{t-1} \Leftrightarrow m\right) + b\rho\log\rho\Delta X_t\left(Y_{t-1} \Leftrightarrow m\right) + \left[\sigma^2 \Leftrightarrow 2\frac{b\sigma^2\rho^2\log\rho}{1 \Leftrightarrow \rho^2}\Delta X_t\right]^{\frac{1}{2}} \epsilon_{t.}$$

Hence the test combines both effects due to time deformation in the case of an Ornstein-Uhlenbeck model.

5.2.2. Parametric models with latent directing processes

We will concentrate most of our attention on testing the hypothesis of time deformation when one uses the simulation-based estimators described in section 4.3.2. Some observations will also be made about testing when the continuous time GMM estimator is used. Since we discuss primarily simulation-based estimators, let us introduce an analogue to (4.1) to describe the dynamic of the changing time process in discrete time, namely:

$$Z_t = h_t\left(Z_{t-1}, \epsilon_{zt}; b\right), \tag{5.7}$$

where ϵ_{zt} is I.I.N (0,1). Again, for the purpose of discussion we assume that:

$$h_t\left(Z_{t-1}, \epsilon_{zt}; b\right)|_{b=0} = t \tag{5.8}$$

The score principle was advanced for testing $b = 0$ when the directing process was tied to an observable process X_t through g_t in (4.1). The same score principle can be applied to cases where Z_t is latent. Let us assume again that the parameter vector is $\theta = (\alpha, b)$ and that we estimate the model (via simulation) under the null restriction $b = 0$, yielding $\hat{\theta}_T^0 = (\hat{\alpha}_T^0, 0)$ for a sample of size T. Consider paths for the directing process simulated under the alternative $b \neq 0$. For any choice of b we obtain $\left[Z_t^s\left(b\right)_{t-1}^T\right]_{s=1}^S$. Taking the parameter estimates $\hat{\alpha}_T^0$ under the null we can simulate for any alternative b the process $\left[\left(y_t^s\left(\hat{\alpha}_T^0, b\right)\right)_{t=1}^T\right]_{s=1}^S$. For the Ornstein-Uhlenbeck example this would amount to:

$$y_t^s\left(\hat{\alpha}_T^0, b\right) = \hat{m}_{oT} + \hat{\rho}_{oT}^{\left(1+b_1 \Delta Z_t^s(b)\right)} \left(y_{t-1}^s\left(\hat{\alpha}_T^0, b\right) \Leftrightarrow \hat{m}_{0T}\right) + \hat{\sigma}_{0T} \left(\frac{1 \Leftrightarrow \hat{\rho}_{0T}^{2\left(1+b_1 \Delta Z_t^s(b)\right)}}{1 \Leftrightarrow \hat{\rho}_{0T}}\right)^{\frac{1}{2}} \varepsilon_t^s$$

where $\hat{\alpha}_T^0 = \left(\hat{m}_{0T}, \hat{\rho}_{0T}, \hat{\sigma}_{0T}\right)$ and b_1 is an element of the parameter vector b.

Appendix 1

The Stochastic differential system in calendar time

In this Appendix we provide a proof of *Property 2.3.1* in Section 2.3. The idea of the proof follows the approach of Stroock-Varadhan (1979). It consists of characterizing for the infinitesimal drift, volatilities and covolatilities. In particular, we have that:

$$a\left(y_t, z_t\right) = \lim_{h \to 0} E\left[\frac{Y_{t+h} \Leftrightarrow Y_t}{h} \mid Y_t = y_t, Z_t = z_t\right]$$

$$= \lim_{h \to 0} E\left[\frac{Y^*_{Z_{t+h}} \Leftrightarrow y_t}{h} \mid Y^*_{z_t} = y_t, Z_t = z_t\right]$$

$$= \lim_{h \to 0} \left[E\left[\frac{Y^*_{Z_{t+h}} \Leftrightarrow y_t}{h} \mid Y^*_{z_t} = y_t, Z_t = z_t, Z_{t+h}\right] \mid Y_t = y_t, Z_t = z_t\right]$$

$$= \lim_{h \to 0} E\left[\frac{Z_{t+h} \Leftrightarrow z_t}{h}\; E\left(\frac{Y^*_{Z_{t+h}} \Leftrightarrow y_t}{Z_{t+h} \Leftrightarrow z_t} \mid Y^*_{z_t} = y_t, Z_t = z_t, Z_{t+h}\right) \mid Y^*_{z_t} = y_t, Z_t = z_t\right]$$

$$= \lim_{h \to 0} E\left[\frac{Z_{t+h} \Leftrightarrow z_t}{h}\; a^*\left(Y^*_{z_t}\right) \mid Y^*_{z_t} = y_t, Z_t = z_t\right]$$

$$= a^*\left(y_t\right) \lim_{h \to 0}\; E\left(\frac{Z_{t+h} \Leftrightarrow z_t}{h} \mid Z_t = z_t\right) = a^*\left(y_t\right) \alpha\left(z_t\right).$$

Therefore the infinitesimal drift for the bivariate process is as follows:

$$\lim_{h \to 0}\; \tfrac{1}{h}\; \begin{pmatrix} E\left[Y_{t+h} \Leftrightarrow Y_t \mid Y_t, Z_t\right] \\ E\left[Z_{t+h} \Leftrightarrow Z_t \mid Y_t, Z_t\right] \end{pmatrix} = \begin{pmatrix} a^*\left(Y_t\right) \alpha\left(Z_t\right) \\ \alpha\left(Z_t\right) \end{pmatrix}.$$

The infinitesimal volatilities and covolatilities are determined in a similar way. Namely,

$$\lim_{h\to 0} E\left[\frac{(Y_{t+h} \Leftrightarrow Y_t)(Z_{t+h} \Leftrightarrow Z_t)}{h} \mid Y_t, Z_t\right]$$

$$= \lim_{h\to 0} E\left[\frac{(Z_{t+h} \Leftrightarrow Z_t)^2}{h} E\left[\frac{Y_{t+h} \Leftrightarrow Y_t}{Z_{t+h} \Leftrightarrow Z_t} \mid Y_t, Z_t, Z_{t+h}\right] \mid Y_t, Z_t\right]$$

$$= \lim_{h\to 0} E\left[\frac{(Z_{t+h} \Leftrightarrow Z_t)^2}{h} a^*(Y_t) \mid Y_t, Z_t\right] = a^*(Y_t)\,\beta^2(Z_t).$$

$$\lim_{h\to 0} E\left[\frac{(Y_{t+h} \Leftrightarrow Y_t)^2}{h} \mid Y_t, Z_t\right]$$

$$= \lim_{h\to 0} E\left[\frac{Z_{t+h} \Leftrightarrow Z_t}{h} E\left[\frac{(Y_{t+h} \Leftrightarrow Y_t)^2}{Z_{t+h} \Leftrightarrow Z_t} \mid Y_t, Z_t, Z_{t+h}\right] \mid Y_t, Z_t\right]$$

$$= \lim_{h\to 0} E\left[\frac{Z_{t+h} \Leftrightarrow Z_t}{h} b^{*2}(Y_t) \mid Y_t, Z_t\right] = b^{*2}(Y_t)\,\alpha(Z_t).$$

Therefore the bivariate infinitesimal volatility is given by:

$$\lim_{h\to 0} \tfrac{1}{h} V\left[\begin{pmatrix} Y_{t+h} \Leftrightarrow Y_t \\ Z_{t+h} \Leftrightarrow Z_t \end{pmatrix} \mid Y_t, Z_t\right] = \begin{bmatrix} b^{*2}(Y_t)\,\alpha(Z_t) & a^*(Y_t)\,\beta^2(Z_t) \\ a^*(Y_t)\,\beta^2(Z_t) & \beta^2(Z_t) \end{bmatrix}.$$

The matrix on the right hand side will be denoted $\sum(Y_t, Z_t)$ and it is positive definite as soon as:

$$b^{*2}(Y_t)\,\alpha(Z_t) \Leftrightarrow a^{*2}(Y_t)\,\beta^2(Z_t) > 0,\ \forall t.$$

With these infinitesimal drift and volatilities, we can immediately characterize the form of the stochastic differential system satisfied by the time deformated process. (It is easily checked that the matrix $\sum^{\frac{1}{2}}$given in (2.12) is such that $\sum^{\frac{1}{2}}\left(\sum^{\frac{1}{2}}\right)^{'}=\sum(Y_t,Z_t)$).

Appendix 2

The asymptotic moments of the functional estimator of the autocovariance function in intrinsic time

In this appendix we provide some ideas of the form of the asymptotic first and second order moments of $\hat{\gamma}_T^*(z)$, without discussing the regularity conditions for a.s. convergence and asymptotic normality of this estimator. As usual these properties are deduced from the properties of :

$$g_{1T}(z) = \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} Y_t \, Y_\tau \, \frac{1}{h_T} K \left[\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T} \right],$$

$$g_{2T}(z) = \frac{1}{T} \sum_{t=1}^{T} \sum_{\tau=1}^{T} \frac{1}{h_T} K \left[\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T} \right],$$

noting that : $\hat{\gamma}_T^*(z) = g_{1T}(z) / g_{2T}(z)$.

As soon as $[g_{1T}(z), g_{2T}(z)]$ is a.s. consistent to its asymptotic mean, we get :

$$\hat{\gamma}_T^*(z) \overset{a.s.}{\Leftrightarrow} \lim_T \frac{g_{1T}(z)}{g_{2T}(z)}.$$

Similarly as soon as :

$$\begin{bmatrix} g_{1T}(z) \Leftrightarrow \lim_T E g_{1T}(z) \\ g_{2T}(z) \Leftrightarrow \lim_T E g_{2T}(z) \end{bmatrix} \overset{d}{\Leftrightarrow} N \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \Omega_{11}(z) & \Omega_{12}(z) \\ \Omega_{21}(z) & \Omega_{22}(z) \end{bmatrix} \right],$$

we get :

$$\left[\hat{\gamma}_T^*(z) \Leftrightarrow \lim_T E g_{1T}(z) \Big/ \lim_T E g_{2T}(z) \right] \overset{d}{\Leftrightarrow}$$

$$N \left[0, \left(\frac{1}{\lim_T E g_{2T}(z)}, \Leftrightarrow \frac{\lim_T E g_{1T}(z)}{\lim_T E g_{2T}(z)^2} \right) \begin{pmatrix} \Omega_{11}(z) & \Omega_{12}(z) \\ \Omega_{21}(z) & \Omega_{22}(z) \end{pmatrix} \left(\frac{1}{\lim_T E g_{2T}(z)}, \Leftrightarrow \frac{\lim_T E g_{1T}(z)}{\lim_T E g_{2T}(z)^2} \right)' \right]$$

Below we derive the form of the various terms $\lim_T Eg_{1T}(z)$, $\lim_T Eg_{2T}(z)$, $\Omega_{11}(z)$, $\Omega_{12}(z)$, $\Omega_{22}(z)$, which characterize the asymptotic distribution.

For this derivation, we assume that the processes (Y_z^*) and (Z_T) are independent, that (Y_z^*) is strongly stationary with zero mean and that (Z_T) is with iid increments.

a) **First order moments**

We get, for $z \neq 0$:

$$\begin{aligned}
A_T &= E\left\{\frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\begin{bmatrix}Y_t Y_\tau\\ 1\end{bmatrix}\frac{1}{h_T}K\left(\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T}\right)\right\}\\
&= \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}E\left\{\begin{bmatrix}Y^*_{Z_t}Y^*_{Z_t}\\ 1\end{bmatrix}\frac{1}{h_T}K\left(\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T}\right)\right\}\\
&= \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}E\left\{\begin{bmatrix}\gamma^*(Z_t \Leftrightarrow Z_\tau)\\ 1\end{bmatrix}\frac{1}{h_T}K\left(\frac{Z_t \Leftrightarrow Z_\tau \Leftrightarrow z}{h_T}\right)\right\}\\
&= \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\int\begin{bmatrix}\gamma^*(u)\\ 1\end{bmatrix}\frac{1}{h_T}K\left(\frac{u \Leftrightarrow z}{h_T}\right) f_{|\tau-t|}(u)\ du,
\end{aligned}$$

where $f_{|T-t|}(u)$ is the p.d.f. of $Z_t \Leftrightarrow Z_\tau$, which only depends on $|\tau \Leftrightarrow t|$ because of the assumption of strongly stationary increments. Therefore we obtain :

$$\begin{aligned}
A_T &= \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\int\begin{bmatrix}\gamma^*(vh_T + z)\\ 1\end{bmatrix} f_{|\tau-t|}(vh_T + z)\ K(v)\ dv\\
&\simeq \frac{1}{T}\sum_{t=1}^{T}\sum_{\tau=1}^{T}\begin{bmatrix}\gamma^*(z)\\ 1\end{bmatrix} f_{|\tau-t|}(z),
\end{aligned}$$

(since h_T tends to zero when T tends to infinity and $\int K(v)\,dv = 1$).

Then :

$$A_T \simeq \begin{bmatrix}\gamma^*(z)\\ 1\end{bmatrix}\sum_{n=-\infty}^{+\infty} f_{|n|}(z) = 2\begin{bmatrix}\gamma^*(z)\\ 1\end{bmatrix}\sum_{n=1}^{\infty} f_n(z).$$

This result is valid if the series $\sum_{n=1}^{\infty} f_n(z) = \sum_{n=1}^{\infty} f^{*n}(z)$ is convergent.

For instance if $Z_{t+1} \Leftrightarrow Z_t$ has an exponential distribution $\gamma(1, \lambda)$, we have :

$$\sum_{n=1}^{\infty} f_n(z) = \lambda \sum_{n=1}^{\infty} \frac{1}{,\ (n)} \exp \Leftrightarrow \lambda z (\lambda z)^{n-1} < +\infty.$$

This condition $\sum_{n=1}^{\infty} f_n(z) < \infty$ may be seen as the functional analogue of the similar condition on the covariance $\sum_{n=1}^{\infty} Cov[Z_t, Z_{t+n}] < +\infty$.

The form of the limit for $\hat{\gamma}_T^*(z)$ is immediately deduced :

$$\lim_T \hat{\gamma}_T^*(z) = \frac{\lim_T Eg_{1T}(z)}{\lim_T Eg_{2T}(z)} = \gamma^*(z),$$

which corresponds to the consistency property.

b) **Second order moments**

We have :

$$\begin{aligned} &B \\ &= V_{as}\left(\sqrt{Th_t}\left\{\begin{array}{l} g_{1T}(z) \Leftrightarrow \lim_T Eg_{1T}(z) \\ g_{2T}(z) \Leftrightarrow \lim_T Eg_{2T}(z) \end{array}\right\}\right) \\ &= V_{as}\left(\sqrt{Th_t}\left\{\tfrac{1}{T}\textstyle\sum_{t=1}^T \sum_{Z=1}^T \binom{y_t y_Z}{1} \frac{1}{h_T} K\left[\frac{Z_t - Z_Z - z}{h_T}\right] \Leftrightarrow \tfrac{1}{T}\sum_{t=1}^T \sum_{Z=1}^T E\binom{y_t y_Z}{1} \frac{1}{h_T} K\left[\frac{Z_t - Z_Z - z}{h_T}\right]\right\}\right) \end{aligned}$$

As usual for this kind of computation, it is possible to neglect in the developed expansion of this variance all the cross-terms, the one corresponding to different — (t, Z), (t', Z'). Then we get :

$$\begin{aligned} &B(z) \\ &= \lim_T \left\{\tfrac{1}{T}\textstyle\sum_{t=1}^T \sum_{Z=1}^T E\left(\begin{bmatrix} y_t y_Z \\ 1 \end{bmatrix}(y_t y_Z, 1) \frac{1}{h_T} K^2\left[\frac{Z_t - Z_Z - z}{h_T}\right]\right) \Leftrightarrow \frac{h_T}{T}\sum_{t=1}^T \sum_{Z=1}^T E\left[\binom{y_t y_Z}{1} \frac{1}{h_T} K\left(\frac{Z_t - Z_Z - z}{h_T}\right)\right.\right. \\ &= \lim_T \tfrac{1}{T}\textstyle\sum_{t=1}^T \sum_{Z=1}^T E\left(\binom{y_t y_Z}{1}[y_t y_Z, 1] \frac{1}{h_T} K^2\left[\frac{Z_t - Z_Z - z}{h_T}\right]\right) \end{aligned}$$

[Since the second term is asymptotically negligeable.]

$$B(z)$$
$$=\lim_{T} \tfrac{1}{T} \sum_{t=1}^{T} \sum_{Z=1}^{T} E\left[\begin{pmatrix} y_{Z_t}^{*2} y_{Z_z}^{*2} & y_{Z_t}^{*} y_{Z_z}^{*} \\ y_{Z_t}^{R} y_{Z_z}^{*} & 1 \end{pmatrix} \tfrac{1}{h_T} K^2\left(\tfrac{Z_t - Z_Z - z}{h_T}\right)\right]$$

Let us introduce the quantity :

$$\gamma^*(z) = E\left(y_{z_o}^{*2} y_{z_o+z}^{*2}\right) \qquad (A.1)$$

We directly deduce after a change of variable similar to the one of the previous subsection :

$$B(z) = \int K^2(\sigma)\, d\sigma \qquad 2\sum_{n=1}^{\infty} f_n(z) \left[\begin{smallmatrix} \gamma^*(z) & \gamma^*(z) \\ \gamma^*(z) & 1 \end{smallmatrix}\right].$$

Therefore the asymptotic variance of the estimated autocorrection will be :

$$V_{as}\sqrt{Th_t}\left[\widehat{\gamma}_T^*(z) \Leftrightarrow \gamma^Z(z)\right]$$
$$= \frac{\int K^2(v)\, d\sigma}{2\sum_{n=1}^{\infty} f_n(z)} \left[\gamma^*(z) \Leftrightarrow \gamma^*(z)^2\right],$$

where :

$$\gamma^*(z) \Leftrightarrow \gamma^*(z)^2$$
$$= E\left(y_{Z_z}^{*2} y_{z+z_o}^{*2}\right) \Leftrightarrow E\left(y_{z_o}^{*} y_{z+z_o}^{*}\right)^2$$
$$= V\left(y_{z_o}^{*} y_{z+z_o}^{*}\right)$$

References

[1] Allais, M. (1974): "The Psychological Rate of Interest", *Journal of Money, Credit and Banking*, 3, 285-331.

[2] Bessembinder, H. and P.J. Seguin (1993), "Price Volatility, Trading Volume and Market Depth : Evidence from Futures Markets," *Journal of Financial and Quantitative Analysis*, 28(1), 21-39.

[3] Bochner, S. (1960): "Harmonic Analysis and the Theory of Probability", University of California Press, Berkeley.

[4] Box, G. and G. Jenkins (1970): "Time Series Analysis, Forecasting and Control", San Francisco, Holden Day.

[5] Clark, P. (1970): "A Subordinated Stochastic Process Model of Cotton Futures Prices", Ph.D. Dissertation, Harvard University.

[6] Clark, P. (1973): "A Subordinated Stochastic Process Model with Finite Variance for Speculative Price", *Econometrica*, 41, 135-156.

[7] Conley, T., L. Hansen, E. Luttmer and J. Scheinkman (1994): "Estimating Subordinated Diffusions from Discrete Time Data", Presented at the CIRANO-C.R.D.E. conference on Stochastic Volatility, Montréal.

[8] Dacorogna, M., U. Müller, R. Nagler, R. Olsen and O. Pictet (1993a), "A Geographical Model for the Daily and Weekly Seasonal Volatility in the Foreign Exchange Market", *Journal of International Money and Finance*, 12, 413-438.

[9] Dacorogna, M., L. Gauvreau, U. Müller, R. Olsen and O. Pictet (1993 b), "Changing Time Scale for Short Term Forecasting in Financial Markets," Discussion paper, Olsen and Associates.

[10] Detemple, J. and S. Murthy (1993), "Intertemporal Asset Pricing with Heterogeneous Beliefs," Discussion Paper, School of Management, McGill University.

[11] Dimson, E. (ed.) (1988), "Stock Market Anomalies," Cambridge University Press, Cambridge.

[12] Duffie, D. and K. Singleton (1993): "Simulated Moments Estimation of Markov Models of Asset Prices", *Econometrica*, 61, 929-952.

[13] Easley, D. and M. O'Hara (1992), "Time and the Process of Security Price Adjustment," *Journal of Finance*, XLVII(2), 577-605.

[14] Engle, R. (1982): "Autoregressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflations", *Econometrica*, 50, 987-1008.

[15] Feller, W. (1957): "Introduction to Probability Theory and its Applications" (Wiley, New York).

[16] Forster, M. and George, T. (1992), "Can Speculative Trading Explain the Volume-Volatility Relation?", Working Paper, Duke University.

[17] French, K. (1980), "Stock Returns and the Weekend Effect," *Journal of Financial Economics*, 8, 55-69.

[18] French, K. and R. Roll (1986), "Stock Return Variances: The Arrival of Information and the Reaction of Traders," *Journal of Financial Economics*, 17, 5-26.

[19] French, K.R., G.W. Schwert and R.F. Stambaugh (1987), "Expected Stock Returns and Volatility," *Journal of Financial Economics*, 19, 3-29.

[20] Gallant, A.R., P.E. Rossi and G. Tauchen (1992), "Stock Prices and Volume," *Review of Financial Studies*, 5, 199-242.

[21] Gallant, A.R., P.E. Rossi and G. Tauchen (1993), "Nonlinear Dynamic Structures," *Econometrica*, 61, 871-908.

[22] Gallant, A.R. and G. Tauchen (1989), "Seminonparametric Estimation of Conditionally Constrained Heterogeneous Processes : Asset Pricing Applications," *Econometrica*, 57, 1091-1120.

[23] Gallant, A.R. and G. Tauchen (1994), "Which Moments to Match ?," *Econometric Theory* (forthcoming).

[24] Geman, H. and M. Yor (1993), "Bessel Processes, Asian Options and Perpetuities," *Mathematical Finance*, 3, 349-375.

[25] Ghysels, E., C. Gouriéroux and J. Jasiak (1995), "Trading Patterns, Time Deformation and Stochastic Volatility in Foreing Exchange Markets", Work in progress.

[26] Ghysels, E. and J. Jasiak (1994), "Stochastic Volatility and Time Deformation : An Application to Trading Volume and Leverage Effects", Paper presented at the 1994 Western Finance Association Meetings, Santa Fe.

[27] Gouriéroux, C. and A. Monfort (1994): "Simulation Based Econometric Methods", forthcoming CORE Lectures Series.

[28] Gouriéroux, C., A. Monfort and E. Renault (1993): "Indirect Inference", *Journal of Applied Econometrics*, 8, S85-S118.

[29] Goovaerts, M., R. Kaas, Van Heerwaarden and T. Bauwelincks (1990), "Effective Actuarial Methods", North Holland, Amsterdam.

[30] Hamilton, J. (1989), "A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle", *Econometrica*, 57, 357-384.

[31] Han, S., Kalay, A. and Rosenfeld, A. (1994), "The information Content of Non Tranding", Working paper, University of Utah.

[32] Hausman, J. and Lo, A. (1990), "A Continuous Time Discrete State Stochastic Process for Transactions Stock Prices : Theory and Empirical Evidence", Working paper, MIT.

[33] Hansen, L. and J. Scheinkman (1984): "Back to the Future: Generating Moment Implications for Continuous Time Markov Processes", Forthcoming in *Econometrica*.

[34] Harris, L. (1987), "Transaction Data Tests of the Mixture of Distributions Hypothesis", *Journal of Financial and Quantitative Analysis*. 22, 127-141.

[35] Hull, J. and A. White (1987), "The Pricing of Options on Assets with Stochastic Volatilities," *Journal of Finance*, 42, 281-300.

[36] Lakonishok, J. and S. Smidt (1988), "Are Seasonal Anomalies Real ? A Ninety Year Perspective," *Review of Financial Studies,* 1, 403-425.

[37] Leblanc, B. (1994): "Une Approche Unifiée pour une Forme Exacte du Prix d'une Option dans les Différents Modèles à Volatilité Stochastique", Discussion paper, CREST.

[38] Madan, D.B. and F. Milne (1991), "Option Pricing with V.G. Martingale Components," *Mathematical Finance*, 1, 39-55.

[39] Madan, D.B. and E. Seneta (1990), "The V.G. Model for Share Market Returns," *Journal of Business*, 63, 511-524.

[40] Mandelbrot, B. (1963): "The Variation of Certain Speculative Prices", *Journal of Business*, 36, 394-419.

[41] Mandelbrot, B. (1973): Comments on "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices", *Econometrica*, 41, 157-159.

[42] Mandelbrot, B. and H. Taylor (1967): "On the Distribution of Stock Prices Differences", *Operations Research*, 15, 1057-1062.

[43] Nelson, D.B. (1991), "Conditional Heteroskedasticity in Asset Returns : A New Approach," *Econometrica*, 59, 347-370.

[44] Nelson, D.B. and D.P. Foster (1993): "Asymptotic Filtering Theory for Univariate ARCH Models", *Econometrica*, 62, 1-41.

[45] Panjer, H. (1981), "Recursive Evaluation of a Family of Compound Distributions", *ASTIN Bulletin*, 12, 22-26.

[46] Revuz, A. and M. Yor (1991): "Continuous Martingales and Brownian Motion", Springer Verlag, Berlin..

[47] Richardson, M. and Smith, T. (1993), "A Direct Test of the Mixture of Distributions Hypothesis : Measuring the Daily Flow of Information", *Journal of Financial and Quantitative Analysis.* [volume and page numbers to come]

[48] Schwert, G.W. (1990): "Indexes of U.S. Stock Prices from 1802 to 1987", *Journal of Business*, 63, 399-438.

[49] Stock, J. (1983): "Dynamic Economic Models Subject to Time Deformation", Ph.D. Dissertation, University of California, Berkeley.

[50] Stock, J.H. (1988): "Estimating Continuous Time Processes Subject to Time Deformation", *Journal of the American Statistical Association*, 83(401), 77-84.

[51] Stroock, D. and S. Varadhan (1979): "Multidimensional Diffusion Processes", Springer Verlag, Berlin.

[52] Tauchen, G.E. and M. Pitts (1983), "The Price Variability Volume Relationship on Speculative Markets," *Econometrica*, 51(2), 485-505.

[53] Yor, M. (1992a): "Sur Certaines Fonctionnelles Exponentielles des Mouvements Browniens Réels", *Journal of Applied Probability*, 29, 202-208.

[54] Yor, M. (1992b): "On Some Exponential Functionals of Brownian Motion", Advances in Applied Probability.